

The complexity of dominating set reconfiguration

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Abstract. Suppose that we are given two dominating sets D_s and D_t of a graph G whose cardinalities are at most a given threshold k . Then, we are asked whether there exists a sequence of dominating sets of G between D_s and D_t such that each dominating set in the sequence is of cardinality at most k and can be obtained from the previous one by either adding or deleting exactly one vertex. This problem is known to be PSPACE-complete in general. In this paper, we study the complexity of this decision problem from the viewpoint of graph classes. We first prove that the problem remains PSPACE-complete even for planar graphs, bounded bandwidth graphs, split graphs, and bipartite graphs. We then give a general scheme to construct linear-time algorithms and show that the problem can be solved in linear time for cographs, trees, and interval graphs. Furthermore, for these tractable cases, we can obtain a desired sequence such that the number of additions and deletions is bounded by $O(n)$, where n is the number of vertices in the input graph.

1 Introduction

Consider the art gallery problem modeled on graphs: Each vertex corresponds to a room which has a monitoring camera and each edge represents the adjacency of two rooms. Assume that each camera in a room can monitor the room itself and its adjacent rooms. Then, we wish to find a subset of cameras that can monitor all rooms; the corresponding vertex subset D of the graph G is called a *dominating set*, that is, every vertex in G is either in D or adjacent to a vertex in D . For example, Fig. 1 shows six different dominating sets of the same graph. Given a graph G and a positive integer k , the problem of determining whether G has a dominating set of cardinality at most k is a classical NP-complete problem [4].

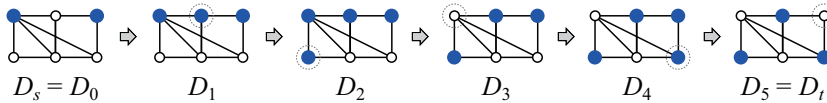


Fig. 1. A sequence $\langle D_0, D_1, \dots, D_5 \rangle$ of dominating sets in the same graph, where $k = 4$ and the vertices in dominating sets are depicted by large (blue) circles.

1.1 Our problem

However, the art gallery problem could be considered in more “dynamic” situations: In order to maintain the cameras, we sometimes need to change the current dominating set into another one. This transformation needs to be done by switching the cameras individually and we certainly need to keep monitoring all rooms, even during the transformation.

In this paper, we thus study the following problem: Suppose that we are given two dominating sets of a graph G whose cardinalities are at most a given threshold $k > 0$ (e.g., the leftmost and rightmost ones in Fig. 1, where $k = 4$), and we are asked whether we can transform one into the other via dominating sets of G such that each intermediate dominating set is of cardinality at most k and can be obtained from the previous one by either adding or deleting a single vertex. We call this decision problem the **DOMINATING SET RECONFIGURATION (DSR)** problem. For the particular instance of Fig. 1, the answer is yes as illustrated in Fig. 1.

1.2 Known and related results

Recently, similar problems have been extensively studied under the reconfiguration framework [8], which arises when we wish to find a step-by-step transformation between two feasible solutions of a combinatorial problem such that all intermediate solutions are also feasible. The reconfiguration framework has been applied to several well-studied problems, including **SATISFIABILITY** [5], **INDEPENDENT SET** [7, 8, 10, 13, 16], **VERTEX COVER** [8, 9, 12, 13], **CLIQUE**, **MATCHING** [8], **VERTEX-COLORING** [2], and so on. (See also a survey [15].)

Mouawad et al. [13] proved that **DOMINATING SET RECONFIGURATION** is $W[2]$ -hard when parameterized by $k + \ell$, where k is the cardinality threshold of dominating sets and ℓ is the length of a sequence of dominating sets.

Haas and Seyffarth [6] gave sufficient conditions for the cardinality threshold k for which any two dominating sets can be transformed into one another. They proved that the answer to **DOMINATING SET RECONFIGURATION** is **yes** for a graph G with n vertices if $k = n - 1$ and G has a matching of cardinality at least two; they also gave a better sufficient condition when restricted to bipartite or chordal graphs. Recently, Suzuki et al. [14] improved the former condition and showed that the answer is **yes** if $k = n - \mu$ and G has a matching of cardinality at least $\mu + 1$, for any nonnegative integer μ .

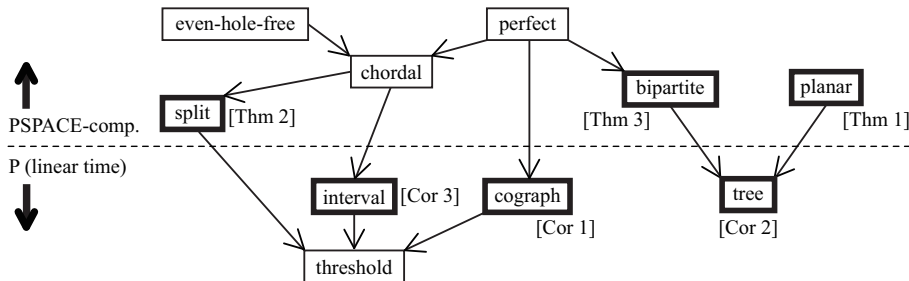


Fig. 2. Our results, where each arrow represents the inclusion relationship between graph classes: $A \rightarrow B$ represents that B is properly included in A [3]. We also show PSPACE-completeness on graphs of bounded bandwidth (Theorem 1).

1.3 Our contribution

To the best of our knowledge, no algorithmic results are known for the DOMINATING SET RECONFIGURATION problem and it is therefore desirable to obtain a better understanding of what separates “hard” from “easy” instances. To that end, we study the problem from the viewpoint of graph classes and paint an interesting picture of the boundary between intractability and polynomial-time solvability. (See also Fig. 2.)

We first prove that the problem is PSPACE-complete even on planar graphs, bounded bandwidth graphs, split graphs, and bipartite graphs. Our reductions for PSPACE-hardness follow from the classical reductions for proving the NP-hardness of DOMINATING SET. However, the reductions should be constructed carefully so that they preserve not only the existence of dominating sets but also the reconfigurability.

We then give a general scheme to construct linear-time algorithms for the problem. As examples of its application, we demonstrate that the problem can be solved in linear time on cographs (also known as P_4 -free graphs), trees, and interval graphs. Furthermore, for these tractable cases, we can obtain a desired sequence such that the number of additions and deletions (i.e., the length of a reconfiguration sequence) can be bounded by $O(n)$, where n is the number of vertices in the input graph.

Proofs of lemmas and theorems marked with a star can be found in the appendix.

2 Preliminaries

Graph notation and dominating set. We assume that each input graph G is a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. For a set $S \subseteq V(G)$ of vertices, the subgraph of

G induced by S is denoted by $G[S]$, where $G[S]$ has vertex set S and edge set $\{uv \in E(G) \mid u, v \in S\}$.

For a vertex v in a graph G , we let $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$. For a set $S \subseteq V(G)$ of vertices, we define $N_G[S] = \bigcup_{v \in S} N_G[v]$ and $N_G(S) = N_G[S] \setminus S$. We sometimes drop the subscript G if it is clear from the context.

For a graph G , a set $D \subseteq V(G)$ is a *dominating set* of G if $N_G[D] = V(G)$. Note that $V(G)$ always forms a dominating set of G . For a vertex $u \in V(G)$ and a dominating set D of G , we say that u is *dominated* by $v \in D$ if $u \notin D$ and $u \in N_G(v)$. A vertex w in a dominating set D is *deletable* if $D \setminus \{w\}$ is also a dominating set of G . A dominating set D of G is *minimal* if there is no deletable vertex in D .

Dominating set reconfiguration. We say that two dominating sets D and D' of the same graph G are *adjacent* if there exists a vertex $u \in V(G)$ such that $D \Delta D' = (D \setminus D') \cup (D' \setminus D) = \{u\}$, i.e. u is the only vertex in the *symmetric difference* of D and D' . For two dominating sets D_p and D_q of G , a sequence $\langle D_0, D_1, \dots, D_\ell \rangle$ of dominating sets of G is called a *reconfiguration sequence* between D_p and D_q if it has the following properties:

- (a) $D_0 = D_p$ and $D_\ell = D_q$; and
- (b) D_{i-1} and D_i are adjacent for each $i \in \{1, 2, \dots, \ell\}$.

Note that any reconfiguration sequence is *reversible*, that is, $\langle D_\ell, D_{\ell-1}, \dots, D_0 \rangle$ is also a reconfiguration sequence between D_p and D_q . We say a vertex $v \in V(G)$ is *touched* in a reconfiguration sequence $\sigma = \langle D_0, D_1, \dots, D_\ell \rangle$ if v is either added or deleted at least once in σ .

For two dominating sets D_p and D_q of a graph G and an integer $k > 0$, we write $D_p \overset{k}{\longleftrightarrow} D_q$ if there exists a reconfiguration sequence $\langle D_0, D_1, \dots, D_\ell \rangle$ between D_p and D_q in G such that $|D_i| \leq k$ holds for every $i \in \{0, 1, \dots, \ell\}$, for some $\ell \geq 0$. Note that $k \geq \max\{|D_p|, |D_q|\}$ clearly holds if $D_p \overset{k}{\longleftrightarrow} D_q$. Then, the DOMINATING SET RECONFIGURATION (DSR) problem is defined as follows:

Input: A graph G , two dominating sets D_s and D_t of G , and an integer threshold $k \geq \max\{|D_s|, |D_t|\}$

Question: Determine whether $D_s \overset{k}{\longleftrightarrow} D_t$ or not.

We denote by a 4-tuple (G, D_s, D_t, k) an instance of DOMINATING SET RECONFIGURATION. Note that DSR is a decision problem and hence it does not ask for an actual reconfiguration sequence. We always denote by D_s and D_t the *source* and *target* dominating sets of G , respectively.

3 PSPACE-completeness

In this section, we prove that DOMINATING SET RECONFIGURATION remains PSPACE-complete even for restricted classes of graphs; some of these classes show nice contrasts to our algorithmic results in Section 4. (See also Fig. 2.)

Theorem 1. *DSR is PSPACE-complete on planar graphs of maximum degree six and on graphs of bounded bandwidth.*

Proof. One can observe that the problem is in PSPACE [8, Theorem 1]. We thus show that it is PSPACE-hard for those graph classes by a polynomial-time reduction from VERTEX COVER RECONFIGURATION [8, 9, 12]. In VERTEX COVER RECONFIGURATION, we are given two vertex covers C_s and C_t of a graph G' such that $|C_s| \leq k$ and $|C_t| \leq k$, for some integer k , and asked whether there exists a reconfiguration sequence of vertex covers C_0, C_1, \dots, C_ℓ of G such that $C_0 = C_s$, $C_\ell = C_t$, $|C_i| \leq k$, and $|C_{i-1} \Delta C_i| = 1$ for each $i \in \{1, 2, \dots, \ell\}$.

Our reduction follows from the classical reduction from VERTEX COVER to DOMINATING SET [4]. Specifically, for every edge uw in $E(G')$, we add a new vertex v_{uw} and join it with each of u and w by two new edges uv_{uw} and $v_{uw}w$; let G be the resulting graph. Then, let $(G, D_s = C_s, D_t = C_t, k)$ be the corresponding instance of DOMINATING SET RECONFIGURATION. Clearly, this instance can be constructed in polynomial time.

We now prove that $D_s \overset{k}{\leftrightarrow} D_t$ holds if and only if there is a reconfiguration sequence of vertex covers in G' between C_s and C_t . However, the if direction is trivial, because any vertex cover of G' forms a dominating set of G and both problems employ the same reconfiguration rule (i.e., the symmetric difference is of size one). Therefore, suppose that $D_s \overset{k}{\leftrightarrow} D_t$ holds, and hence there exists a reconfiguration sequence of dominating sets in G between D_s and D_t . Recall that neither D_s nor D_t contain a newly added vertex in $V(G) \setminus V(G')$. Thus, if a vertex v_{uw} in $V(G) \setminus V(G')$ is touched, then v_{uw} must be added first. By the construction of G , both $N_G[v_{uw}] \subseteq N_G[u]$ and $N_G[v_{uw}] \subseteq N_G[w]$ hold. Therefore, we can replace the addition of v_{uw} by that of either u or w and obtain a (possibly shorter) reconfiguration sequence of dominating sets in G between D_s and D_t which touches vertices only in G' . Then, it is a reconfiguration sequence of vertex covers in G' between C_s and C_t , as needed.

VERTEX COVER RECONFIGURATION is known to be PSPACE-complete on planar graphs of maximum degree three [9, 12] and on graphs of bounded bandwidth [16]. Thus, the reduction above implies PSPACE-hardness on planar graphs of maximum degree six and on graphs of bounded bandwidth; note that, since the number of edges in G is only the triple of that in G' , the bandwidth increases only by a constant multiplicative factor. \square

We note that both pathwidth and treewidth of a graph G are bounded by the bandwidth of G . Thus, Theorem 1 yields that DOMINATING SET RECONFIGURATION is PSPACE-complete on graphs of bounded pathwidth and treewidth.

Adapting known techniques from NP-hardness proofs for the DOMINATING SET problem [1], we also show PSPACE-completeness of DOMINATING SET RECONFIGURATION on split graphs and on bipartite graphs; a graph is *split* if its vertex set can be partitioned into a clique and an independent set [3].

Theorem 2 (*). *DSR is PSPACE-complete on split graphs.*

Theorem 3 (*). *DSR is PSPACE-complete on bipartite graphs.*

4 General scheme for linear-time algorithms

In this section, we show that DOMINATING SET RECONFIGURATION is solvable in linear time on cographs, trees, and interval graphs. Interestingly, these results can be obtained by the application of the same strategy; we first describe the general scheme in Section 4.1. We then show in Sections 4.2–4.4 that the problem can be solved in linear time on those graph classes.

4.1 General scheme

The general idea is to introduce the concept of a “canonical” dominating set for a graph G . We say that a minimum dominating set C of G is *canonical* if $D \overset{k}{\rightsquigarrow} C$ holds for every dominating set D of G and $k = |D| + 1$. Then, we have the following theorem.

Theorem 4. *If a graph G has a canonical dominating set, then DOMINATING SET RECONFIGURATION can be solved in linear time on G .*

We note that proving the existence of a canonical dominating set is sufficient for solving the decision problem. Therefore, we do not need to find an actual canonical dominating set in linear time. In Sections 4.2–4.4, we will show that cographs, trees, and interval graphs admit canonical dominating sets, and hence the problem can be solved in linear time on those graph classes. Note that, however, Theorem 4 can be applied to any graph which has a canonical dominating set. In the remainder of this subsection, we prove Theorem 4 starting with the following lemma.

Lemma 1. *Suppose that a graph G has a canonical dominating set. Then, an instance (G, D_s, D_t, k) of DOMINATING SET RECONFIGURATION is a yes-instance if $k \geq \max\{|D_s|, |D_t|\} + 1$.*

Proof. Let C be a canonical dominating set of G . Then, $D_s \overset{k'}{\rightsquigarrow} C$ holds for $k' = |D_s| + 1$. Suppose that $k \geq \max\{|D_s|, |D_t|\} + 1$. Since $k \geq |D_s| + 1 = k'$, we clearly have $D_s \overset{k}{\rightsquigarrow} C$. Similarly, we have $D_t \overset{k}{\rightsquigarrow} C$. Since any reconfiguration sequence is reversible, we have $D_s \overset{k}{\rightsquigarrow} C \overset{k}{\rightsquigarrow} D_t$, as needed. \square

Lemma 1 implies that if a graph G has a canonical dominating set C , then it suffices to consider the case where $k = \max\{|D_s|, |D_t|\}$. Note that there exist no-instances of DOMINATING SET RECONFIGURATION in such a case but we show that they can be easily identified in linear time, as implied by the following lemma.

Lemma 2. *Let (G, D_s, D_t, k) be an instance of DOMINATING SET RECONFIGURATION, where G is a graph admitting a canonical dominating set and $k = \max\{|D_s|, |D_t|\}$. Then, (G, D_s, D_t, k) is a yes-instance if and only if D_i is not minimal for every $i \in \{s, t\}$ such that $|D_i| = k$.*

Lemma 2 can be immediately obtained from the following lemma.

Lemma 3. *Suppose that a graph G has a canonical dominating set C . Let D be an arbitrary dominating set of G and let $k = |D|$. Then, $D \overset{k}{\rightsquigarrow} C$ holds if and only if D is not a minimal dominating set.*

Proof. Necessity. Suppose that D is not minimal. Then, D contains at least one vertex x which is deletable from D , that is, $D \setminus \{x\}$ forms a dominating set of G . Since $k = |D| = |D \setminus \{x\}| + 1$, we have $D \setminus \{x\} \overset{k}{\rightsquigarrow} C$. Therefore, $D \overset{k}{\rightsquigarrow} D \setminus \{x\} \overset{k}{\rightsquigarrow} C$ holds.

Sufficiency. We prove the contrapositive. Suppose that D is minimal. Then, no vertex in D is deletable and hence any dominating set D' which is adjacent to D must be obtained by adding a vertex to D . Therefore, $|D'| = k + 1$ for any dominating set D' which is adjacent to D . Hence, $D \overset{k}{\rightsquigarrow} C$ does not hold. \square

We note again that Lemmas 1 and 2 imply that an actual canonical dominating set is not required to solve the problem. Furthermore, it can be easily determined in linear time whether a dominating set of a graph G is minimal or not. Thus, Theorem 4 follows from Lemmas 1 and 2.

Before constructing canonical dominating sets in Sections 4.2–4.4, we give the following lemma showing that it suffices to construct a canonical dominating set for a connected graph.

Lemma 4 (*). *Let G be a graph consisting of p connected components G_1, G_2, \dots, G_p . For each $i \in \{1, 2, \dots, p\}$, suppose that C_i is a canonical dominating set for G_i . Then, $C = C_1 \cup C_2 \cup \dots \cup C_p$ is a canonical dominating set for G .*

4.2 Cographs

We first define the class of cographs (also known as P_4 -free graphs) [3]. For two graphs G_1 and G_2 , their *union* $G_1 \cup G_2$ is the graph such that $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$, while their *join* $G_1 \vee G_2$ is the graph such that $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{vw \mid v \in V(G_1), w \in V(G_2)\}$. Then, a *cograph* can be recursively defined as follows:

- (1) a graph consisting of a single vertex is a cograph;
- (2) if G_1 and G_2 are cographs, then the union $G_1 \cup G_2$ is a cograph; and
- (3) if G_1 and G_2 are cographs, then the join $G_1 \vee G_2$ is a cograph.

In this subsection, we show that DOMINATING SET RECONFIGURATION is solvable in linear time on cographs. By Theorem 4, it suffices to prove the following lemma.

Lemma 5. *Any cograph admits a canonical dominating set.*

As a proof of Lemma 5, we will construct a canonical dominating set for any cograph G . By Lemma 4, it suffices to consider the case where G is connected

and we may assume that G has at least two vertices, because otherwise the problem is trivial. Then, from the definition of cographs, G must be obtained by the join operation applied to two cographs G_a and G_b , that is, $G = G_a \vee G_b$. Notice that any pair $\{w_a, w_b\}$ of vertices $w_a \in V(G_a)$ and $w_b \in V(G_b)$ forms a dominating set of G . Let C be a dominating set of G , defined as follows:

- If there exists a vertex $w \in V(G)$ such that $N[w] = V(G)$, then let $C = \{w\}$.
- Otherwise choose an arbitrary pair of vertices $w_a \in V(G_a)$ and $w_b \in V(G_b)$ and let $C = \{w_a, w_b\}$.

Clearly, C is a minimum dominating set of G . We thus prove the following lemma, which completes the proof of Lemma 5.

Lemma 6 (*). *For every dominating set D of G , $D \overset{k}{\rightsquigarrow} C$ holds, where $k = |D| + 1$.*

We have thus proved that any cograph has a canonical dominating set. Then, Theorem 4 gives the following corollary.

Corollary 1. *DSR can be solved in linear time on cographs.*

4.3 Trees

In this subsection, we show that DOMINATING SET RECONFIGURATION is solvable in linear time on trees. As for cographs, it suffices to prove the following lemma.

Lemma 7. *Any tree admits a canonical dominating set.*

As a proof of Lemma 7, we will construct a canonical dominating set for a tree T . We choose an arbitrary vertex r of degree one in T and regard T as a rooted tree with root r .

We first label each vertex in T either 1, 2, or 3, starting from the leaves of T up to the root r of T , as in the following steps (1)–(3); intuitively, the vertices labeled 2 will form a dominating set of T , each vertex labeled 1 will be dominated by its parent, and each vertex labeled 3 will be dominated by at least one of its children (see also Fig. 3(a)):

- (1) All leaves in T are labeled 1.
- (2) Pick an internal vertex v of T , which is not the root, such that all children of v have already been labeled. Then,
 - assign v label 1 if all children of v are labeled 3;
 - assign v label 2 if at least one child of v is labeled 1; and
 - otherwise assign v label 3.
- (3) Assign the root r (of degree one) label 3 if its child is labeled 2, otherwise assign r label 2.

For each $i \in \{1, 2, 3\}$, we denote by V_i the set of all vertices in T that are assigned label i . Then, $\{V_1, V_2, V_3\}$ forms a partition of $V(T)$.

We will prove that V_2 forms a canonical dominating set of T . We first prove, in Lemmas 8 and 9, that V_2 is a minimum dominating set of T and then prove, in Lemma 10, that $D \overset{k}{\rightsquigarrow} V_2$ holds for every dominating set D of T and $k = |D| + 1$.

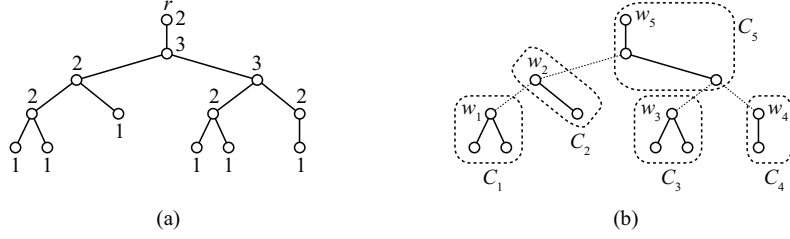


Fig. 3. (a) The labeling of a tree T , and (b) the partition of $V(T)$ into C_1, C_2, \dots, C_5 .

Lemma 8. V_2 is a dominating set of T .

Proof. It suffices to show that both $V_1 \subseteq N(V_2)$ and $V_3 \subseteq N(V_2)$ hold.

Let v be any vertex in V_1 , and hence v is labeled 1. Then, by the construction above, v is not the root of T and the parent of v must be labeled 2. Therefore, $v \in N(V_2)$ holds, as claimed.

Let u be any vertex in V_3 , and hence u is labeled 3. Then, u is not a leaf of T . Notice that label 3 is assigned to a vertex only when at least one of its children is labeled 2. Thus, $u \in N(V_2)$ holds. \square

We now prove that V_2 is a minimum dominating set of T . To do so, we introduce some notation. Suppose that the vertices in V_2 are ordered as $w_1, w_2, \dots, w_{|V_2|}$ by a post-order depth-first traversal of the tree starting from the root r of T . For each $i \in \{1, 2, \dots, |V_2|\}$, we denote by T_i the subtree of T which is induced by w_i and all its descendants in T . Then, for each $i \in \{1, 2, \dots, |V_2|\}$, we define a vertex subset C_i of $V(T)$ as follows (see also Fig. 3(b)):

$$C_i = \begin{cases} V(T_i) \setminus \bigcup_{j < i} V(T_j) & \text{if } i \neq |V_2|; \\ V(T) \setminus \bigcup_{j < i} V(T_j) & \text{if } i = |V_2|. \end{cases}$$

Note that $\{C_1, C_2, \dots, C_{|V_2|}\}$ forms a partition of $V(T)$. Furthermore, notice that

$$V_2 \cap C_i = \{w_i\} \tag{1}$$

holds for every $i \in \{1, 2, \dots, |V_2|\}$. Then, Eq. (1) and the following lemma imply that V_2 is a minimum dominating set of T .

Lemma 9 (*). *Let D be an arbitrary dominating set of T . Then, $|D \cap C_i| \geq 1$ holds for every $i \in \{1, 2, \dots, |V_2|\}$.*

We finally prove the following lemma, which completes the proof of Lemma 7.

Lemma 10 (*). *For every dominating set D of T , $D \overset{k}{\leftrightarrow} V_2$ holds, where $k = |D| + 1$.*

We have thus proved that V_2 forms a canonical dominating set for any tree T . Then, Theorem 4 gives the following corollary.

Corollary 2. *DSR can be solved in linear time on trees.*

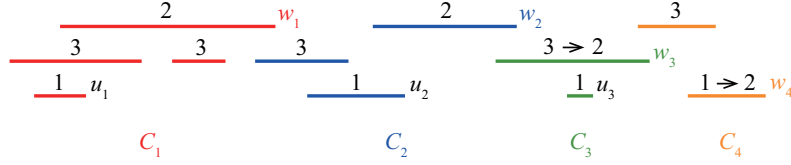


Fig. 4. The labeling of an interval graph in the interval representation.

4.4 Interval graphs

A graph G with $V(G) = \{v_1, v_2, \dots, v_n\}$ is an *interval graph* if there exists a set \mathcal{I} of (closed) intervals I_1, I_2, \dots, I_n such that $v_i v_j \in E(G)$ if and only if $I_i \cap I_j \neq \emptyset$ for each $i, j \in \{1, 2, \dots, n\}$. We call the set \mathcal{I} of intervals an *interval representation* of the graph. For a given graph G , it can be determined in linear time whether G is an interval graph, and if so obtain an interval representation of G [11]. In this subsection, we show that DOMINATING SET RECONFIGURATION is solvable in linear time on interval graphs. As for cographs, it suffices to prove the following lemma.

Lemma 11. *Any interval graph admits a canonical dominating set.*

As a proof of Lemma 11, we will construct a canonical dominating set for any interval graph G . By Lemma 4 it suffices to consider the case where G is connected. Let \mathcal{I} be an interval representation of G . For an interval $I \in \mathcal{I}$, we denote by $l(I)$ and $r(I)$ the left and right endpoints of I , respectively; we sometimes call the values $l(I)$ and $r(I)$ the *l-value* and *r-value* of I , respectively. As for trees, we first label each vertex in G either 1, 2, or 3, from left to right; the vertices labeled 2 will form a dominating set of G (see Fig. 4 as an example):

- (1) Pick the unlabeled vertex v_i which has the minimum r -value among all unlabeled vertices and assign v_i label 1.
- (2) Let v_j be the vertex in $N[v_i]$ which has the maximum r -value among all vertices in $N[v_i]$. Note that v_j may have been already labeled and $v_j = v_i$ may hold. We (re)label v_j to 2.
- (3) For each unlabeled vertex in $N(v_j)$, we assign it label 3.

We execute steps (1)–(3) above until all vertices are labeled. For each $i \in \{1, 2, 3\}$, we denote by V_i the set of all vertices in G that are assigned label i . Then, $\{V_1, V_2, V_3\}$ forms a partition of $V(G)$.

By the construction above, it is easy to see that V_2 forms a dominating set of G . We thus prove that V_2 is canonical in Lemmas 12 and 13, that is, V_2 is a minimum dominating set of G (in Lemma 12) and $D \overset{k}{\leftrightarrow} V_2$ holds for every dominating set D of G and $k = |D| + 1$ (in Lemma 13).

We now prove that the dominating set V_2 of G is minimum. To do so, we introduce some notation. Assume that the vertices in V_2 are ordered as $w_1, w_2, \dots, w_{|V_2|}$ such that $r(w_1) < r(w_2) < \dots < r(w_{|V_2|})$. For each $i \in \{1, 2, \dots, |V_2|\}$, we define the vertex subset C_i of $V(G)$ as follows (see Fig. 4 as an example):

$$C_i = \begin{cases} \{v \mid r(v) \leq r(w_1)\} & \text{if } i = 1; \\ \{v \mid r(w_{i-1}) < r(v) \leq r(w_i)\} & \text{if } 2 \leq i \leq |V_2| - 1; \\ \{v \mid r(w_{|V_2|-1}) < r(v)\} & \text{if } i = |V_2|. \end{cases} \quad (2)$$

Note that $\{C_1, C_2, \dots, C_{|V_2|}\}$ forms a partition of $V(G)$ such that

$$V_2 \cap C_i = \{w_i\} \quad (3)$$

holds for every $i \in \{1, 2, \dots, |V_2|\}$. Then, Eq. (3) and the following lemma imply that V_2 is a minimum dominating set of G .

Lemma 12 (*). *Let D be an arbitrary dominating set of G . Then, $|D \cap C_i| \geq 1$ holds for every $i \in \{1, 2, \dots, |V_2|\}$.*

We finally prove the following lemma, which completes the proof of Lemma 11.

Lemma 13 (*). *For every dominating set D of G , $D \overset{k}{\rightsquigarrow} V_2$ holds, where $k = |D| + 1$.*

Combining Lemma 11 and Theorem 4 yields the following corollary.

Corollary 3. *DSR can be solved in linear time on interval graphs.*

5 Concluding remarks

In this paper, we delineated the complexity of the DOMINATING SET RECONFIGURATION problem restricted to various graph classes. As shown in Fig. 2, our results clarify some interesting boundaries on the graph classes lying between tractability and PSPACE-completeness: For example, the structure of interval graphs can be seen as a path-like structure of cliques. As a super-class of interval graphs, the well-known class of chordal graphs has a tree-like structure of cliques. We have proved that DOMINATING SET RECONFIGURATION is solvable in linear time on interval graphs, while it is PSPACE-complete on chordal graphs.

We note again that our linear-time algorithms for cographs, trees, and interval graphs employ the same strategy. We also emphasize that this general scheme can be applied to any graph which admits a canonical dominating set. It is easy to modify our algorithms so that they actually find a reconfiguration sequence for a **yes**-instance (G, D_s, D_t, k) on cographs, trees, or interval graphs. Observe that each vertex is touched at most once in the reconfiguration sequence from D_s (or D_t) to the canonical dominating set. Therefore, for a **yes**-instance on an n -vertex graph belonging to one of those classes, there exists a reconfiguration sequence between D_s and D_t which touches vertices only $O(n)$ times. In other words, the length of a shortest reconfiguration sequence between D_s and D_t can be bounded by $O(n)$.

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Appendix

A Details omitted from Section 3

Proof of Theorem 2

Proof. We again give a polynomial-time reduction from VERTEX COVER RECONFIGURATION. We extend the idea developed for the NP-hardness proof of DOMINATING SET on split graphs [1].

Let (G', C_s, C_t, k) be an instance of VERTEX COVER RECONFIGURATION, where $V(G') = \{v_1, v_2, \dots, v_n\}$ and $E(G') = \{e_1, e_2, \dots, e_m\}$. We construct the corresponding split graph G , as follows. (See also Fig. 5(a) and (b).) Let $V(G) = A \cup B$, where $A = V(G')$ and $B = \{w_1, w_2, \dots, w_m\}$; each vertex $w_i \in B$ corresponds to the edge e_i in $E(G')$. We join all pairs of vertices in A so that A forms a clique in G . In addition, for each edge $e_i = v_p v_q$ in $E(G')$, we join $w_i \in B$ with each of v_p and v_q in G . Let G be the resulting graph, and let $(G, D_s = C_s, D_t = C_t, k)$ be the corresponding instance of DOMINATING SET RECONFIGURATION. Clearly, this instance can be constructed in polynomial time. Thus, we will prove that $D_s \overset{k}{\rightsquigarrow} D_t$ holds if and only if there is a reconfiguration sequence of vertex covers in G' between C_s and C_t .

We first prove the if direction. Because both problems employ the same reconfiguration rule, it suffices to prove that any vertex cover C of G' forms a dominating set of G . Since $C \subseteq V(G') = A$ and A is a clique, all vertices in A are dominated by the vertices in C . Thus, consider a vertex w_i in B , which corresponds to the edge $e_i = v_p v_q$ in $E(G')$. Then, since C is a vertex cover of G' , at least one of v_p and v_q must be contained in C . This means that w_i is dominated by the endpoint v_p or v_q in G . Therefore, C is a dominating set of G .

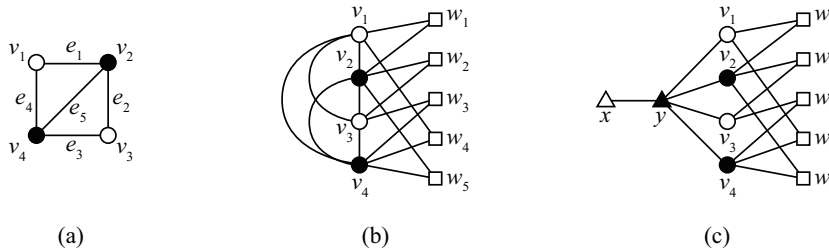


Fig. 5. (a) Vertex cover $\{v_2, v_4\}$ of a graph, (b) dominating set $\{v_2, v_4\}$ of the corresponding split graph, and (c) dominating set $\{v_2, v_4, y\}$ of the corresponding bipartite graph.

We now prove the only-if direction. Notice that, for each vertex $w_i \in B$ corresponding to the edge $e_i = v_p v_q$ in $E(G')$, we have $N_G[w_i] \subseteq N_G[v_p]$ and $N_G[w_i] \subseteq N_G[v_q]$. Therefore, if $D_s \overset{k}{\rightsquigarrow} D_t$ holds, then we can obtain a reconfiguration sequence of dominating sets in G between D_s and D_t which touches vertices only in $A = V(G')$; recall the arguments in the proof of Theorem 1.

Observe that any dominating set D of G such that $D \subseteq A = V(G')$ forms a vertex cover of G' , because each vertex $w_i \in B$ is dominated by at least one vertex in $C \subseteq V(G')$. We have thus verified the only-if direction. \square

Proof of Theorem 3

Proof. We give a polynomial-time reduction from DOMINATING SET RECONFIGURATION on split graphs to the same problem restricted to bipartite graphs. The same idea is used in the NP-hardness proof of DOMINATING SET for bipartite graphs [1].

Let (G', D'_s, D'_t, k') be an instance of DOMINATING SET RECONFIGURATION, where G' is a split graph. Then, $V(G')$ can be partitioned into two subsets A and B which form a clique and an independent set in G' , respectively. Furthermore, by the reduction given in the proof of Theorem 2, the problem for split graphs remains PSPACE-complete even if both $D'_s \subseteq A$ and $D'_t \subseteq A$ hold.

We now construct the corresponding bipartite graph G , as follows. (See also Fig. 5(b) and (c).) First, we delete any edge joining two vertices in A , and make A an independent set. Then, we add a new edge consisting of two new vertices x and y and join y with each vertex in A . The resulting graph G is bipartite. Let $D_s = D'_s \cup \{y\}$, $D_t = D'_t \cup \{y\}$, $k = k' + 1$, and we obtain the corresponding DOMINATING SET RECONFIGURATION instance (G, D_s, D_t, k) , where G is bipartite. Clearly, this instance can be constructed in polynomial time. Thus, we will prove that $D_s \overset{k}{\rightsquigarrow} D_t$ holds if and only if $D'_s \overset{k'}{\rightsquigarrow} D'_t$ holds.

We first prove the if direction. Suppose that $D'_s \overset{k'}{\rightsquigarrow} D'_t$ holds. Hence, there exists a reconfiguration sequence in G' between D'_s and D'_t . Consider any dominating set D' of G' in this sequence. Then, $B \subset N_{G'}[D']$ holds because $B \subset N_{G'}[D']$ and we have deleted only the edges such that both endpoints are in A . Since $N_G(y) = A \cup \{x\}$, we can conclude that $D' \cup \{y\}$ is a dominating set of G . Furthermore, $|D' \cup \{y\}| \leq k' + 1 = k$. Thus, $D_s \overset{k}{\rightsquigarrow} D_t$ holds.

We then prove the only-if direction. Suppose that $D_s \overset{k}{\rightsquigarrow} D_t$ holds, and hence there exists a reconfiguration sequence in G between $D_s = D'_s \cup \{y\}$ and $D_t = D'_t \cup \{y\}$. Notice that any dominating set of G contains at least one of x and y . Since $N_G[x] \subset N_G[y]$ and $y \in D_s, D_t$, we can assume that y is contained in all dominating sets in the reconfiguration sequence. Recall that both $D'_s \subseteq A$ and $D'_t \subseteq A$ hold. Thus, if a vertex $w_i \in B$ is touched, then it must be added first. Since $N_G(y) = A \cup \{x\}$, we have $N_G[\{w_i, y\}] = N_G[\{v_p, y\}] = N_G[\{v_q, y\}]$, where $N_G(w_i) = \{v_p, v_q\}$. Therefore, we can replace the addition of w_i by that of either v_p or v_q and obtain a reconfiguration sequence in G between D_s and D_t which touches vertices only in A . Consider any dominating set D of G in such a reconfiguration sequence. Since $y \in D$, we have $|D \cap V(G')| \leq k - 1 = k'$. Furthermore, since $D \cap V(G') \subseteq A$ and A forms a clique in G' , we have $A \subseteq N_{G'}[D \cap V(G')]$. Since there is no edge joining y and a vertex in B , each vertex in B is dominated by some vertex in $D \cap V(G')$. Therefore, $D \cap V(G')$ is a dominating set of G' of cardinality at most k' and $D'_s \overset{k'}{\rightsquigarrow} D'_t$ holds. \square

B Details omitted from Section 4

Proof of Lemma 4

Proof. Let D be any dominating set of G . For each $i \in \{1, 2, \dots, p\}$, since C_i is canonical for G_i , we have $D \cap V(G_i) \overset{k_i}{\rightsquigarrow} C_i$ for $k_i = |D \cap V(G_i)| + 1$. Therefore, we can independently transform $D \cap V(G_i)$ into C_i for each $i \in \{1, 2, \dots, p\}$. Clearly, this is a reconfiguration sequence from D to $C = C_1 \cup C_2 \cup \dots \cup C_p$. Furthermore, since C_i is a minimum dominating set of G_i , we have $|D \cap V(G_i)| \geq |C_i|$ for each $i \in \{1, 2, \dots, p\}$. Thus, any dominating set appearing in the sequence is of cardinality at most $|D| + 1$. \square

C Details omitted from Section 4.2

Proof of Lemma 6

Proof. We construct a reconfiguration sequence from D to C such that each intermediate dominating set is of cardinality at most $|D| + 1$.

Case (i): $|C| = 1$.

In this case, C consists of a universal vertex w , that is, $N[w] = V(G)$. Therefore, we first add w to D if $w \notin D$, and then delete the vertices in $D \setminus \{w\}$ one by one. Since $N[w] = V(G)$, all intermediate vertex subsets are dominating sets of G . Since the addition is applied only to w , we have $D \overset{k}{\rightsquigarrow} C$ for $k = |D| + 1$.

Case (ii): $|C| = 2$.

In this case, C consists of two vertices $w_a \in V(G_a)$ and $w_b \in V(G_b)$. Since C is a minimum dominating set of G , we have $|D| \geq 2$. Note that, however, $D \subseteq V(G_a)$ or $D \subseteq V(G_b)$ may hold. We assume without loss of generality that $|D \cap V(G_a)| \geq |D \cap V(G_b)|$. Then, we construct a sequence of vertex subsets of G , as follows:

- (1) Add w_b to D if $w_b \notin D$; let $D_1 = D \cup \{w_b\}$.
- (2) If $|D_1 \cap V(G_a)| = |D \cap V(G_a)| \geq 2$, then delete one vertex in $D_1 \cap (V(G_a) \setminus \{w_a\})$; otherwise delete a vertex in $D_1 \cap (V(G_b) \setminus \{w_b\}) = D \cap (V(G_b) \setminus \{w_b\})$ if it exists. Let D_2 be the resulting vertex subset of G .
- (3) Add w_a to D_2 if $w_a \notin D_2$; let $D_3 = D_2 \cup \{w_a\}$.
- (4) Delete from D_3 all vertices in $D \setminus \{w_a, w_b\}$ one by one.

We will prove that each vertex subset appearing above is a dominating set of G with cardinality at most $|D| + 1$. Indeed, it suffices to show that D_2 is a dominating set of G such that $|D_2| \leq |D|$; note that D_3 contains both $w_a \in V(G_a)$ and $w_b \in V(G_b)$ and hence any vertex subset appearing in Steps (3) and (4) above is a dominating set of G with cardinality at most $|D_2| + 1$.

We first consider the case where $|D_1 \cap V(G_a)| \geq 2$. In this case, $D_1 \cap (V(G_a) \setminus \{w_a\}) \neq \emptyset$, and hence we can delete one vertex $u (\neq w_a)$ from D_1 . We thus have $|D_2| = |D_1| - 1 \leq |D|$, as required. Since $|D_1 \cap V(G_a)| \geq 2$, $D_2 (= D_1 \setminus \{u\})$ contains at least one vertex in $V(G_a)$. Furthermore, $w_b \in D_2$ and hence D_2 is a dominating set of G .

We then consider the case where $|D_1 \cap V(G_a)| \leq 1$. Note that, since $|D| \geq 2$ and $|D_1 \cap V(G_a)| = |D \cap V(G_a)| \geq |D \cap V(G_b)|$, we have $|D \cap V(G_a)| = |D \cap V(G_b)| = 1$ in this case. Let $D \cap V(G_b) = \{z\}$. If $w_b \notin D$ (and hence $z \neq w_b$) then $|D_1| = |D| + 1$ and $D_1 \cap (V(G_b) \setminus \{w_b\}) = \{z\}$. Therefore, $D_2 = D_1 \setminus \{z\}$ and $|D_2| = |D_1| - 1 = |D|$. Furthermore, since $w_b \in D_2$ and $|D_2 \cap V(G_a)| = |D_1 \cap V(G_a)| = 1$, D_2 is a dominating set of G . On the other hand, if $w_b \in D$, then we have $D \cap V(G_b) = \{w_b\}$. Consequently, $D_2 = D_1 = D$ and hence D_2 is a dominating set of G of cardinality $|D_2| = |D|$. \square

D Details omitted from Section 4.3

Proof of Lemma 9

Proof. Suppose for a contradiction that $D \cap C_i = \emptyset$ holds for some index $i \in \{1, 2, \dots, |V_2|\}$. We will prove that C_i contains at least one vertex u such that $N[u] \subseteq C_i$. Then, since $D \cap C_i = \emptyset$, the vertex u is not dominated by any vertex in D ; this contradicts the assumption that D is a dominating set of T . Recall that all leaves in T are labeled 1, and hence w_i is an internal vertex.

First, consider the case where w_i has a child u which is a leaf of T . Then, $N[u] \subseteq C_i$ holds for the leaf u ; a contradiction.

Second, consider the case where $i = |V_2|$, that is, $C_i (= C_{|V_2|})$ contains the root r of T . Recall that r is of degree one and is labeled either 2 or 3; we will prove that $N[r] \subseteq C_{|V_2|}$ holds. If r is labeled 3, then its (unique) child v is labeled 2 and hence $v = w_{|V_2|}$. Therefore, $C_{|V_2|}$ contains both r and v and hence $N[r] \subseteq C_{|V_2|}$ holds; a contradiction. On the other hand, if r is labeled 2 and hence $r = w_{|V_2|}$, then its child v is labeled either 1 or 3. Therefore, $C_{|V_2|}$ contains both r and v , and hence $N[r] \subseteq C_{|V_2|}$ holds; a contradiction.

Finally, consider the case where $i \neq |V_2|$ and w_i is an internal vertex such that all children of w_i are also internal vertices in T . Since w_i is labeled 2, there exists at least one child u of w_i which is labeled 1. Then, since u is an internal vertex, all children of u (and hence all ‘‘grandchildren’’ of w_i) are labeled 3. Therefore, $N[u] \subseteq C_i$ holds for the child u of w_i ; a contradiction. \square

Proof of Lemma 10

Proof. We construct a reconfiguration sequence from D to V_2 such that each intermediate dominating set is of cardinality at most $|D| + 1$.

Let $D_0 = D$. For each i from 1 to $|V_2|$, we focus on the vertices in C_i and transform $D_{i-1} \cap C_i$ into $V_2 \cap C_i$ as follows:

- (1) add the vertex $w_i \in V_2 \cap C_i$ to D_{i-1} if $w_i \notin D_{i-1}$;
- (2) delete the vertices in $D_{i-1} \cap (C_i \setminus \{w_i\})$ one by one; and
- (3) let D_i be the resulting vertex set.

We first claim that D_i forms a dominating set of T for each $i \in \{1, 2, \dots, |V_2|\}$. Notice that $D_i \cap V(T_i) = V_2 \cap V(T_i)$ for the resulting vertex set D_i . Moreover, only the root w_i of T_i is adjacent to a vertex in $V(T) \setminus V(T_i)$. Since $w_i \in V_2$ and

both V_2 and D_{i-1} form dominating sets of T , we can conclude that D_i forms a dominating set of T . Then, all vertex subsets appearing in Steps (1) and (2) above also form dominating sets of T , because each of them is a superset of D_i .

We then claim that $|D_{i-1}| \geq |D_i|$ for each $i \in \{1, 2, \dots, |V_2|\}$. If $w_i \in D_{i-1}$, then the claim clearly holds because we only delete vertices in Step (2) without adding the vertex w_i in Step (1). We thus consider the case where $w_i \notin D_{i-1}$. Since D_{i-1} is a dominating set of T , Lemma 9 implies that $D_{i-1} \cap (C_i \setminus \{w_i\}) \neq \emptyset$ in this case. Therefore, we have $|D_{i-1}| \geq |D_i|$.

Note that, since addition is executed only in Step (1), the maximum cardinality of any dominating set in the reconfiguration sequence from D_{i-1} to D_i is at most $|D_{i-1}| + 1$. Since $|D_{i-1}| \geq |D_i|$ for each $i \in \{1, 2, \dots, |V_2|\}$, the maximum cardinality of any dominating set in the reconfiguration sequence from D_0 ($= D$) to $D_{|V_2|}$ ($= V_2$) is at most $|D| + 1$. Therefore, there exists a reconfiguration sequence from D to V_2 such that all intermediate dominating sets are of cardinality at most $|D| + 1$. \square

E Details omitted from Section 4.4

Proof of Lemma 12

Proof. Suppose for a contradiction that $D \cap C_i = \emptyset$ holds for some index $i \in \{1, 2, \dots, |V_2|\}$. Assume that the vertices in V_1 are ordered as $u_1, u_2, \dots, u_{|V_1|}$ such that $r(u_1) < r(u_2) < \dots < r(u_{|V_1|})$. Then, observe that $V_1 \cap C_i = \{u_i\}$ holds for every $i \in \{1, 2, \dots, |V_1|\}$. In addition, $V_1 \cap C_{|V_2|} = \emptyset$ holds if $|V_2| = |V_1| + 1$.

First, we consider the case where both $i = |V_2|$ and $|V_2| = |V_1| + 1$ hold; in this case, both $V_1 \cap C_{|V_2|} = \emptyset$ and $V_2 \cap C_{|V_2|} = \{w_{|V_2|}\}$ hold. Since $D \cap C_{|V_2|} = \emptyset$, $w_{|V_2|} \in V_2$ must be dominated by some vertex v in $C_- = C_1 \cup C_2 \cup \dots \cup C_{|V_2|-1}$. Then, $vw_{|V_2|} \in E(G)$ and hence we have $l(w_{|V_2|}) \leq r(v)$. Since $v \in C_-$, by Eq. (2) we have $r(v) \leq r(w_{|V_2|-1})$ and hence $l(w_{|V_2|}) \leq r(w_{|V_2|-1}) < r(w_{|V_2|})$. Therefore, $w_{|V_2|} \in N(w_{|V_2|-1})$ holds and $w_{|V_2|}$ must be labeled 3. This contradicts the assumption that $w_{|V_2|}$ is labeled 2.

We now consider the other case, that is, both $V_1 \cap C_i = \{u_i\}$ and $V_2 \cap C_i = \{w_i\}$ hold for index i . Since $D \cap C_i = \emptyset$, $u_i \in V_1$ must be dominated by at least one vertex in $C_- = C_1 \cup C_2 \cup \dots \cup C_{i-1}$ or $C_+ = C_{i+1} \cup C_{i+2} \cup \dots \cup C_{|V_2|}$. If u_i is dominated by some vertex in C_- , then the same arguments given above yield a contradiction, i.e. u_i must be labeled 3 even though u_i is in V_1 . Therefore, u_i must be dominated by some vertex v in C_+ . Then, since $vu_i \in E(G)$, we have $v \in N(u_i) \subset N[u_i]$. Furthermore, since $v \in C_+$, by Eq. (2) we have $r(w_i) < r(v)$. However, recall that $w_i \in V_2$ is chosen as the vertex in $N[u_i]$ which has the maximum r -value among all vertices in $N[u_i]$. This contradicts the assumption that w_i is labeled 2. \square

Proof of Lemma 13

Proof. We construct a reconfiguration sequence from D to V_2 such that each intermediate dominating set is of cardinality at most $|D| + 1$.

Let $D_0 = D$. For each i from 1 to $|V_2|$, we focus on the vertices in C_i , and transform $D_{i-1} \cap C_i$ into $V_2 \cap C_i$ as follows:

- (1) add the vertex $w_i \in V_2 \cap C_i$ to D_{i-1} if $w_i \notin D_{i-1}$;
- (2) delete the vertices in $D_{i-1} \cap (C_i \setminus \{w_i\})$ one by one; and
- (3) let D_i be the resulting vertex set.

For each $i \in \{1, 2, \dots, |V_2|\}$, let $C_{-i} = C_1 \cup C_2 \cup \dots \cup C_i$ and $C_+ = C_{i+1} \cup C_{i+2} \cup \dots \cup C_{|V_2|}$. We claim that D_i forms a dominating set of G :

- Consider a vertex v such that $r(v) \leq r(w_i)$. Since $D_i \cap C_{-i} = V_2 \cap C_{-i}$ holds, v is dominated by some vertex in $V_2 \cap C_{-i}$.
- Consider a vertex v such that $r(w_i) \leq l(v)$. Since $D_i \cap C_+ = D \cap C_+$ holds, v is dominated by some vertex in $D \cap C_+$.
- Finally, consider a vertex v such that $l(v) < r(w_i) < r(v)$. Then, $vw_i \in E(G)$ and hence v is dominated by $w_i \in D_i$.

Thus, D_i forms a dominating set of G . Since each vertex subset appearing in Steps (1) and (2) above is a superset of D_i , it also forms a dominating set of G .

By the same arguments as in the proof of Lemma 10, we can conclude that the reconfiguration sequence from D to V_2 above consists only of dominating sets of cardinality at most $|D| + 1$. \square