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The Grassmannian of an infinite dimensional separable Hilbert space

Diploma Thesis

by

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THE GRASSMANNIAN OF AN INFINITE DIMENSIONAL SEPARABLE HILBERT SPACE

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Abstract

The aim of this thesis is to describe the construction of the Sato Grassmannian on an infinite dimensional separable Hilbert space and to study some of its main geometric, analytic and functional properties. First infinite dimensional Grassmannian appears in work of M. Sato and Y. Sato published in 1982 as an inductive limit of a finite dimensional Grassmann manifold. The thesis is based on the work of G. Segal and A. Pressley and provides the careful and detailed description of the Sato Grassmannian in its most recent interpretation. We start by introducing classes of operators that will be used throughout of the thesis and discuss their main properties and relations. We consider linear spaces of Hilbert-Schmidt operators and the operators of trace class, that are analogous of L^2 and L^1, respectively, in mathematical analysis. Then we present the description of the class of Fredholm operators, that provides the class of invertible operators up to a compact operator and we end up with study of operators with a determinant, that used in the construction of the determinant bundle over the Sato Grassmannian. The introductory part also contains an overview of the restricted general linear group and provides the construction of its central extension.

In the main part of the thesis, we give the general definition of the Grassmannian $Gr(H)$ on an arbitrary infinite dimensional separable Hilbert space H and endow it with a natural Hilbert manifold structure, that is a consequence of the Hilbert structure of the space of Hilbert -Schmidt operators. After this, we focus on the Grassmannian over the Hilbert space $H = L^2(S^1, \mathbb{C})$, that widely used in physical applications. Then we discuss some particularly interesting dense submanifolds, given by real analytic, smooth and polynomial functions. The stratification and its cellular decomposition provide finer structure of the Grassmannian and it is also the subject of our thesis. Furthermore, we study an infinite dimensional analogue of the Plücker coordinates, and the action of one dimensional rotation group $\mathbb{T}$ on $Gr(H)$. The consideration of determinant bundle Det on $Gr(H)$, the Kähler metric and possible physical applications of the Sato Grassmannian in quantum mechanics finishes the thesis. We add a short Appendix collecting the fundamental definitions and theorems used in thesis.

To the best of the author's knowledge this is the first time in literature that the description of the Sato Grassmannian is presented in a detailed and expanded manner, collecting all the necessary preliminaries. The text of the thesis can be used by students and researchers as an introduction to this modern, highly used and rapidly developing subject.

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Contents

1. History of infinite Grassmannian

The history of infinite Grassmannians starts with the paper of M. Sato and Y. Sato published in 1982 [20]. They were interested in introducing the infinite Grassmannian in order to describe the structure of solutions to the Kadomtsev-Petviashvili equation

$$3u_{yy} + (-4u_t + u_{xxx} + 6uu_x)_x = 0.$$

They showed that the Kadomtsev-Petviashvili equation has a natural structure of Grassmann manifold of infinite dimension; that is e.g. generic points of the Grassmann manifold give generic solutions of special types.

They defined the infinite Grassmannian just by the limit of finite dimensional Grassmannians, which are well known objects in mathematics. His definition is cited as following: "the infinite dimensional Grassmannian (GM) and its standard line bundle ($\tilde{GM}$), which we need to parametrize the solutions of the Kadomtsev-Petviashvili hierarchy, are obtained as the topological closure of the inductive limit of $GM(m,n)$ and $\tilde{GM}(m,n)$ as m and n tend to ∞" [20]. Here $GM(m,n)$ is the standard finite dimensional Grassmannian of m-dimensional subspaces of a $(m+n)$-dimensional vector space and $\tilde{GM}(m,n)$ is its standard line bundle. Furthermore, he mentioned the role of the general linear group of infinite dimension

$$GL := \{A : GM \to GM \mid A \text{ linear, invertible, bounded}\}$$

as the automorphism group of the Grassmann manifold. It plays the role of group of transformations of Kadomtsev-Petviashvili equations. In his first definition Sato didn't state anything about the stratification, Schubert cells or Plücker coordinates of the infinite Grassmannian. For him it was more important to study the characteristics of the action of the general linear group on the infinite Grassmannian.

The structure and properties of infinite Grassmannian have been applied in a wide range of topics, such as microlocal analysis [6], loop groups [21], conformal and quantum field theories and string theory [11, 15, 23], representation theory of infinite dimensional lie algebras [10], Verlinde formula and Fock spaces [4], abelian and non-abelian reciprocity laws on curves [2, 14] and supersymmetric analogues [5, 13].

This thesis is based on the book "Loop groups" of A. Pressley and G. Segal [16], which in its turn is based on the paper of G. Segal and G. Wilson [21]. The aim of the paper was to describe a construction which assigns a solution of the KdV equation to each point of a certain infinite dimensional Grassmannian, to determine the class of solutions obtained by this method, to illustrate in detail how the geometry of the Grassmannian is reflected in properties of the solutions, and to show how the algebra-geometric solutions fit into the picture.

In the paper of E. Witten [23], the author described some aspects of the relation between Riemann surfaces and infinite Grassmannians making use of physical terminology. This relation is essential in recent studies of the Schottky problem and its relation with quantum field theory and string theory that have been the subject of recent discussions from a physical viewpoint. Furthermore, he pointed out the existence of a close analogy between conformal field theory on Riemann surfaces and the modern theory of automorphic representations.

In 1990 M. Mulase stated in his paper [12] the interesting equivalence between a category of arbitrary vector bundles on algebraic curves defined over a field of an arbitrary characteristic and a category of infinite dimensional vector spaces corresponding to certain points of Grassmannians together with their stabilizers. The contravariant functor between these categories gives a full generalization of the well-known Krichever map, which assigns points of Grassmannians to the geometric data consisting of curves and line bundles. We will find a similar construction idea in the chapter about determinant bundles.

Some of the above cited works are strongly based on the algebraic structure of the Sato Grassmannian, which is pointed out in all its particulars in the paper of A. Alvarez Azquez, J. M. Muñoz Porras and F. J. Plaza Martin [1]. They offered an algebraic construction of infinite dimensional Grassmannians and determinant bundles. Previously, G. Anderson [2] had constructed them by making use the theory of p-adic infinite determinants. A. Alvarez Azquez, J. M. Muñoz Porras and F. J. Plaza Martin changed this point of view completely and the formalism used by them is valid for an arbitrary base field. They begin by defining the functor of points $Gr(V, V^+)$ of the Grassmannian of a k-vector space V (with a fixed k-vector subspace $V^+ \subseteq V$) in such a way that the points $Gr(V, V^+)(Spec(k))$ are precisely the points of the Grassmannian defined by G. Segal and G. Wilson [21], although the points of an arbitrary k-scheme have not been considered previously by other authors.

We see that the construction of the infinite Grassmannians, which originally were constructed to handle the space of solutions of a special partial differential equation, rapidly developed into a helpful tool in a wide range of mathematical areas.

In this thesis we will have a look on one of the first special studies on Grassmannians, which are applied to a better understanding of special solutions of the Kadomtsev-Petviashvili equation. The main aim of the thesis is to provide a careful and detailed description of the Sato Grassmannian in its most recent interpretation, that is closer to the functional analysis approach, but nevertheless widely used algebraic and group theory language. We present proofs of numerous details, omitted in [16], that sometimes are very far from the trivial and that could take a lot of time to verify them. The structure of this thesis is the following. For the beginning in Sections 2 and 3 we remind the well-known definitions and main properties of Hilbert-Schmidts and Fredholm operators that are essential tools for the definition of the infinite dimensional Grassmannian. These chapters are mainly based on the lecture notes of B. K. Driver [8] and the books of W. Arveson [2] , R. G. Douglas [7], and M. Reed M, B. Simon [17]. The reader who is familiar with foundations of the operator theory can skip these two sections and proceed to Section 4 that is dedicated to the study of the general linear group of the infinite dimensional separable Hilbert space since it is a group of automorphisms of infinite dimensional Grassmannians. This is based on Chapter 6 in the book of A. Pressley and G. Segal [16] and a collection of books about group theory [18, 19, 22]. Section 5 is devoted to the careful definition and treatment of infinite dimensional Grassmannians themselves based on Chapter 7 in the book of A. Pressley and G. Segal [16].

2. Hilbert-Schmidt and trace class operators

2.1. **Hilbert-Schmidt operators.**

The notion of the Hilbert-Schmidt operator is one of necessary tools to define Sato Grassmannians, so it is important to be familiar with its definition and most important properties.

We assume from now on in Section 2 that R and H are separable Hilbert spaces. The space of all linear operators from H to R will be denoted by $L(H, R)$. The subspace of all linear bounded operators is denoted by $B(H, R)$. In the case $H = R$ we write $L(H)$ and $B(H)$ for the corresponding spaces.

Definition 1. *A bounded operator $K\colon H \to R$, i.e. $K \in B(H, R)$, is called* **compact** *operator if for all bounded sets $U \subseteq H$ the closure of the range $\overline{K(U)}$ is a compact set in R. It is equivalent to state that for all bounded sequences $\{x_n\}_{n=1}^{\infty} \subset H$ the sequence $\{Kx_n\}_{n=1}^{\infty} \subset R$ contains a convergent subsequence in R.*

The equivalence of both definitions is obvious. We also refer the reader to [17].

Lemma 1. *Let $K(H, R)$ denote the space of compact operators from H to R. Then $K(H, R)$ is closed subspace of $L(H, R)$ in the operator norm topology.*

Proof. We start by showing that $K(H, R)$ is a vector space. Consider two operators K, $T \in K(H, R)$, $\lambda \in \mathbb{C}$ and a bounded sequence $\{x_n\}_{n=1}^{\infty}$ in H. Then

$$(K + T)(x_n) = K(x_n) + T(x_n).$$

Since there exists convergent subsequences $\{Kx_j\}$ and $\{Tx_k\}$, we conclude that there exists a convergent subsequence $\{(K+T)x_n\}$, which is equivalent to say that the operator $K + T$ is compact.

We claim that λK is compact. This is true since λ is a complex number and so λK is also compact.

We finish to show that $K(H, R)$ is a vector space. Let $K_n\colon H \to R$ be compact operators and $K\colon H \to R$ be a linear operator such that

$$\lim_{n\to\infty} \|K_n - K\|_{op} = 0.$$

We need to show that K is compact. Let U be a bounded set in H. To prove that $K(U)$ is pre-compact we will use the equivalent definition of pre-compact sets in a Hilbert space and we will show that $K(U)$ can be covered by finitely many balls of fixed radius. Given $\varepsilon > 0$, choose $N = N(\varepsilon)$ such that $\|K_N - K\|_{op} < \varepsilon$. We can choose a finite subset V of U such that

$$\min_{\tilde{x}\in V} \|y - K_N\tilde{x}\|_R < \varepsilon \tag{1}$$

for all $y \in K_N(U)$, since $K_N(U)$ is pre-compact. Then for an arbitrary $\tilde{x} \in V$, for $z \in K(U)$, $z = Kx$, $x \in U$, we get

$$\begin{aligned}\|z - K\tilde{x}\| &= \|Kx - K\tilde{x}\| \\ &= \|Kx + (-K_N x + K_N x) + (-K_N\tilde{x} + K_N\tilde{x}) - K\tilde{x}\| \\ &= \|(K - K_N)x + K_N(x - \tilde{x}) + (K_N - K)\tilde{x}\| \\ &\leq \|(K - K_N)x\| + \|K_N(x - \tilde{x})\| + \|(K_N - K)\tilde{x}\| \\ &\leq 2\varepsilon + \|K_N x - K_N\tilde{x}\| + \varepsilon\end{aligned}$$

by using the triangle inequality. We conclude that $\min_{\tilde{x}\in V} \|z - K\tilde{x}\| < 3\varepsilon$ since $\|K_N x - K_N\tilde{x}\| < \varepsilon$ by (1). This shows that $K(U)$ can be covered by a finite number of balls of radius 3ε. We conclude that K is compact. □

We remind that a finite rank operator $F\colon H \to R$ is a linear operator, such that any vector $y \in \mathrm{im}(F)$, $\mathrm{im}(F)$ is the image of H under F, can be written as a finite sum $y = Fx = \sum_{i=1}^{N} \mu_i y_i$, where $\{y_i\}_{i=1}^N$ is some fixed family in R and $\mu_i \in \mathbb{C}$ for all $i \in \{1, ..., N\}$. We denote the space of finite rank operators from H to R by $FR(H, R)$.

We also recall the definition of the orthogonal projector. A linear operator $P \in B(H)$ such that $P^2 = P$ and $P = P^*$ is called an orthogonal projection. The range of P is always closed. The operator P acts as the identity operator on $\mathrm{im}(P)$ and as the null operator on $\mathrm{im}(P)^\perp = \mathrm{kern}(P)$. So there is a one-to-one correspondence between orthogonal projectors and closed subspaces of H.

Proposition 1. *A linear operator $K\colon H \to R$ is compact if and only if there exists a sequence $\{K_n\}_{n\in\mathbb{N}}$ of finite rank operators with $K_n\colon H \to R$, s.t. $\|K - K_n\|_{op} \to 0$ as $n \to \infty$.*

Proof. Suppose that $K\colon H \to R$ is a compact operator. Then $\overline{K(U)}$ is compact in R and it contains a countable dense subset for any bounded $U \in H$. It follows that $\overline{K(H)}$ is a separable subspace of R. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis for $\overline{K(H)} \subset R$ and

$$P_N y = \sum_{n=1}^{N} \langle y, e_n\rangle e_n$$

be the orthogonal projection of y onto the space $\mathrm{span}\{e_1, ..., e_N\}$. Then $\lim_{N\to\infty} \|P_N y - y\| = 0$ for all $y \in K(H)$. We define $K_n := P_n K$, which is a finite rank operator on H. If we suppose, on the contrary, that K is not a limit point of a sequence of finite rank operators, then $\limsup_{n\to\infty} \|K - K_n\|_{op} > \varepsilon$. In this case there exists a sequence $\{x_n\} \subset H$ such that $\|(K - K_n)x_n\| > \varepsilon$

for all big enough n. Since K is compact, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Kx_{n_k}\}_{k=1}^{\infty}$ is convergent in $K(H)$. Letting $y = \lim_{k\to\infty} Kx_{n_k}$, we get

$$\begin{aligned}\varepsilon &< \|(K - K_{n_k})x_{n_k}\| = \|(1 - P_{n_k})Kx_{n_k}\| \\ &= \|(1 - P_{n_k})Kx_{n_k} - (1 - P_{n_k})y + (1 - P_{n_k})y\| \\ &\leq \|(1 - P_{n_k})(Kx_{n_k} - y)\| + \|(1 - P_{n_k})y\| \\ &\leq \|Kx_{n_k} - y\| + \|(1 - P_{n_k})y\| \to 0\end{aligned}$$

as $k \to \infty$. This contradicts the assumption that ε is strictly positive. Hence we proved $\lim_{n\to\infty} \|K - K_n\|_{op} = 0$, i. e. K is an operator norm limit of finite rank operators $\{K_n\}_{n=1}^{\infty}$.

Conversely, we assume that a sequence $\{K_n\}_{n=1}^{\infty}$ of finite rank operators converges in operator norm to K. Since we know that every finite rank operator is compact, we have a sequence of compact operators $\{K_n\}_{n=1}^{\infty}$ converging to K. As $K(H, R)$ is a closed vector space we conclude that K is a compact operator. □

Corollary 1. *If K is compact, then so is K^*.*

Proof. Let $K_n := P_n K$ be as in the first part of the proof of Proposition 1. Then $K_n^* = K^* P_n$ is still of finite rank. Furthermore,

$$\|K^* - K_n^*\|_{op} = \|K - K_n\|_{op} \stackrel{n\to\infty}{\longrightarrow} 0$$

as $n \to \infty$, since $\|T^*\|_{op} = \|T\|_{op}$ for any compact operator T. We see that K^* is the limit of a sequence of finite rank operators and so it is compact by Proposition 1. □

After this short introduction to the compact operator theory, we define Hilbert-Schmidt operators.

Proposition 2. *Let $K\colon H \to R$ be a bounded linear operator, $\{e_n\}_{n=1}^{\infty}$ and $\{u_m\}_{m=1}^{\infty}$ be orthonormal basis for H and R. Then*

$$\sum_{n=1}^{\infty} \|Ke_n\|^2 = \sum_{m=1}^{\infty} \|K^* u_m\|^2. \tag{2}$$

Proof. We will use Parseval's identity, Pythagorean theorem and Fubini's theorem for sums with positive terms (which can be found in the Appendix) to get the following equation:

$$\begin{aligned}\sum_{n=1}^{\infty} \|Ke_n\|^2 &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\ \langle Ke_n, u_m\rangle\ |^2 \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |\ \langle e_n, K^* u_m\rangle\ |^2 = \sum_{m=1}^{\infty} \|K^* u_m\|^2\end{aligned}$$

for any orthonormal basis $\{u_m\}_{m=1}^{\infty}$ of R. Since the choice of an orthonormal basis $\{e_n\}_{m=1}^{\infty}$ of H was arbitrary, we deduce that the equality is true for any orthonormal basis of H and R. □

Corollary 2. *The equality* (2) *is independent of the choice of the orthonormal basis of H and R.*

This corollary is important for the well-defined property of the following definition.

Definition 2. *The* **Hilbert-Schmidt norm** *of K is defined by,*

$$\|K\|_{HS}^2 := \sum_{n=1}^{\infty} \|Ke_n\|^2$$

for any (and then for all) orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of H. We say that K is a **Hilbert-Schmidt operator** *(H-S operator) if $\|K\|_{HS} < \infty$. The space of Hilbert-Schmidt operators from H to R is denoted by $HS(H,R)$.*

Proposition 3. *Let $K\colon H \to R$ be a bounded linear operator. Then*

(1) $\|K\|_{HS} = \|K^*\|_{HS}$ *for any K and*

$$\|K\|_{HS} \geq \|K\|_{op}, \tag{3}$$

where $\|K\|_{op} := \sup\{\|Kh\| : h \in H \wedge \|h\| = 1\}$,

(2) *$HS(H,R)$ is a subspace of $K(H,R)$ with the norm $\|\cdot\|_{HS}$ and the inner product $\langle\cdot,\cdot\rangle_{HS}\colon HS(H,R) \times HS(H,R) \to \mathbb{C}$ defined by*

$$\langle K_1, K_2\rangle_{HS} := \sum_{n=1}^{\infty} \langle K_1 e_n, K_2 e_n\rangle \tag{4}$$

for some (and then for any) orthonormal basis $\{e_n\}_{n=1}^{\infty}$. The space $(HS(H,R), \langle\cdot,\cdot\rangle_{HS})$ gets the structure of a Hilbert space.

(3) *Let $P_N x := \sum_{n=1}^{N} \langle x, e_n\rangle e_n$ be the orthogonal projection onto the space* $\operatorname{span}\{e_1, .., e_N\} \subset H$ *and let $K_N := KP_N$ for $K \in HS(H,R)$. Then*

$$\|K - K_N\|_{op}^2 \leq \|K - K_N\|_{HS}^2 \ \to 0$$

as $N \to \infty$. We conclude that the space of finite rank operators $FR(H,R)$ is dense in $(HS(H,R), \|\cdot\|_{HS})$.

(4) *Suppose L is a Hilbert space, operators $A\colon L \to H$ and $C : R \to L$ are linear bounded, then*

$$\| KA \|_{HS} \leq \| K \|_{HS} \| A \|_{op}, \qquad \| CK \|_{HS} \leq \| K \|_{HS} \| C \|_{op} .$$

We can conclude by equation (3) of Proposition 3, that the space of Hilbert-Schmidt operators $HS(H)$ from H to H is a two-sided ideal in $B(H)$:

$$BK \in HS(H) \quad \text{and} \quad KB \in HS(H)$$

for all $B \in B(H)$ and $K \in HS(H)$.

Proof. We shall prove the proposition step by step.
(1) The equality $\|K\|_{HS} = \|K^*\|_{HS}$ follows from Proposition 2. To show (3) we take any $x \in H \setminus \{0\}$, normalize it by $x_1 := \frac{x}{\|x\|}$, and assume that it is an element of an orthonormal basis. Hence

$$\|Kx_1\| \leq \|K\|_{HS}.$$

We get $\frac{\|Kx\|}{\|x\|} \leq \|K\|_{HS}$ from the last inequality and hence $\|K\|_{op} \leq \|K\|_{HS}$ by taking the supremum.
(2) Let us show the triangle inequality. For $K_1, K_2 \in HS(H,R)$ we estimate

$$\begin{aligned} \|K_1 + K_2\|_{HS} &= \sqrt{\sum_{n=1}^{\infty} \|K_1 e_n + K_2 e_n\|^2} \leq \sqrt{\sum_{n=1}^{\infty} (\|K_1 e_n\| + \|K_2 e_n\|)^2} \\ &= \|\{\|K_1 e_n\| + \|K_2 e_n\|\}_{n=1}^{\infty}\|_{l_2} \\ &\leq \|\{\|K_1 e_n\|\}_{n=1}^{\infty}\|_{l_2} + \|\{\|K_2 e_n\|\}_{n=1}^{\infty}\|_{l_2} \\ &= \|K_1\|_{HS} + \|K_2\|_{HS}. \end{aligned}$$

Now we can conclude that $\|\cdot\|_{HS}$ is a norm, since all the other norm axioms are obvious.

By making use of the triangle inequality we can show that $K_n := P_n K$ converges to K in the H-S norm and since K_n is a finite rank operator, we conclude that

- finite rank operators from H to R are a dense subset in $HS(H,R)$,
- $HS(H,R)$ is a closed subspace of $K(H,R)$, since the convergence in H-S norm implies the convergence in the operator norm by (2).

Since $\{\|K_1 e_n\|\}_{n\in\mathbb{N}}, \{\|K_2 e_n\|\}_{n\in\mathbb{N}} \in l^2(\mathbb{N})$, we get the scalar product of $l^2(\mathbb{N})$ by

$$| \langle \{\|K_1 e_n\|\}_{n\in\mathbb{N}}, \{\|K_1 e_n\|\}_{n\in\mathbb{N}} \rangle_{l^2(\mathbb{N})} | = \sum_{n=1}^{\infty} \|K_1 e_n\| \|K_2 e_n\|.$$

Furthermore we know that $\| \{\|K_1 e_n\|\}_{n\in\mathbb{N}} \|_{l^2(\mathbb{N})} = \| K_1 \|_{HS}$. Now we use the Cauchy-Schwarz inequality to get

$$\begin{aligned} \sum_{n=1}^{\infty} | \langle K_1 e_n, K_2 e_n \rangle | \leq \sum_{n=1}^{\infty} \|K_1 e_n\| \|K_2 e_n\| &\leq \sqrt{\sum_{n=1}^{\infty} \|K_1 e_n\|^2} \sqrt{\sum_{n=1}^{\infty} \|K_2 e_n\|^2} \\ &= \|K_1\|_{HS} \|K_2\|_{HS}. \end{aligned}$$

This implies that the sum (4) is well defined. Furthermore, it is obvious that the inner product of $HS(H,R)$ is compatible with the H-S norm: $\|K\|_{HS}^2 = \langle K, K \rangle_{HS}$.

We claim that $HS(H,R)$ is complete with respect to the metric defined by its inner product. Suppose $\{K_m\}_{m=1}^{\infty}$ is a $\|\cdot\|_{HS}$-Cauchy sequence in

$HS(H,R)$. Since the space $B(H,R)$ is complete, there exists an operator $K \in B(H,R)$, such that $\|K_m - K\|_{op} \to 0$ as $m \to \infty$. Thus, we obtain

$$\sum_{n=1}^{N} \|(K - K_m)e_n\|^2 = \lim_{l\to\infty} \sum_{n=1}^{N} \|(K_l - K_m)e_n\|^2 \leq \limsup_{l\to\infty} \|K_l - K_m\|_{HS}$$

for any positive integer N and

$$\begin{aligned}\|K_m - K\|_{HS}^2 = \sum_{n=1}^{\infty} \|(K - K_m)e_n\|^2 &= \lim_{N\to\infty} \sum_{n=1}^{N} \|(K - K_m)e_n\|^2 \\ &\leq \limsup_{l\to\infty} \|K_l - K_m\|_{HS} \to 0 \quad \text{as} \quad m \to \infty.\end{aligned}$$

(3) We just notice that

$$\|K - K_N\|_{op}^2 \leq \|K - K_N\|_{HS}^2 = \sum_{n>N} \|Ke_n\|^2 \to 0 \quad \text{as} \quad N \to \infty.$$

(4) We observe

$$\|CK\|_{HS}^2 = \sum_{n=1}^{\infty} \|CKe_n\|^2 \leq \|C\|_{op}^2 \sum_{n=1}^{\infty} \|Ke_n\|^2 = \|C\|_{op}^2 \|K\|_{HS}^2$$

and

$$\|KA\|_{HS} = \|A^*K^*\|_{HS} \leq \|A^*\|_{op} \|K^*\|_{HS} = \|A\|_{op} \|K\|_{HS}.$$

□

2.2. Trace class operators.

Definition 3. *A bounded linear operator $A\colon H \to H$ is of* **trace class** *if and only if*

$$Ax = \sum_{k\in\mathbb{N}} \lambda_k \langle u_k, x\rangle w_k,$$

where $x \in H$ and $\{u_k\}_{k\in\mathbb{N}}$ and $\{w_k\}_{k\in\mathbb{N}}$ are orthonormal families of H, $\lambda_k \in \mathbb{C}$ such that $\sum_k | \lambda_k | < \infty$. The space of trace class operators is denoted by $L^1(H)$.

We state properties of the trace class operators, which will be used later in this thesis.

Proposition 4.
(1) *The space of trace class operators $L^1(H)$ in $B(H)$ forms a two sided ideal, i.e. for $A \in B(H)$ and $C \in L^1(H)$:*

$$AC \in L^1(H) \text{ and } CA \in L^1(H).$$

(2) *If $A, B \in HS(H)$, then $AB \in L^1(H)$.*

(3) *The trace class operators are also Hilbert-Schmidt operators, i.e.*

$$L^1(H) \subset HS(H).$$

Definition 4. *An operator* $A\colon H \to H$ *has a* **determinant** *if and only if* $A - 1$ *is of trace class. The determinant is defined by*

$$\det(A) := \prod_{k \in \mathbb{Z}} (1 + \lambda_k(A - 1))$$

where $\lambda_k(A-1)$ *is the* k*-th eigenvalue of the operator of trace class* $A - 1$*.*

Proposition 5.

(1) *If* A *has a determinant then it is invertible if and only if* $\det(A) \neq 0$*.*

(2) *If* A *and* B *have determinants, then so does* AB *and* $\det(AB) = \det(A)\det(B)$*.*

(3) *If* A *has a determinant and* q *is bounded and invertible, then* qAq^{-1} *and* $q^{-1}Aq$ *have determinants.*

3. Fredholm operators

The Fredholm operators are another necessary tool to define the Grassmannians, and to be familiar with properties of Fredholm operators is essential for the comprehension of the Grassmannians.

For the beginning we state two propositions about properties of compact operators, whose proofs can be found in [7, 17].

Proposition 6. *The space $K(H)$ is a minimal closed two-sided ideal in $B(H)$ and for the separable Hilbert space H the space $K(H)$ is the only proper closed two-sided ideal in $B(H)$.*

Proposition 7. *An operator K belongs to $K(H)$ if and only if the range of K contains no closed infinite dimensional subspaces.*

Remind that only for finite dimensional Hilbert spaces we have the coincidence $K(H) = B(H)$. In the case of infinite dimensional Hilbert spaces, the quotient algebra $B(H)/K(H)$ is not trivial and is called Calkin algebra. It has numerous applications in mathematical physics. The natural homomorphism from $B(H)$ onto $B(H)/K(H)$ is denoted by $\pi\colon B(H) \to B(H)/K(H)$.

Definition 5. *An operator $T \in B(H)$ is called the* **Fredholm operator** *if $\pi(T)$ is an invertible element of $B(H)/K(H)$, i. e. there exists an operator $A \in B(H)/K(H)$ such that $AT = TA = \mathrm{Id} + K$ with $K \in K(H)$.*

The space of Fredholm operators from H to H is denoted by $F(H)$. We give an equivalent definition of Fredholm operators. The equivalence of the two definitions is the statement of the Atkinson theorem and can be found, for instance, in [7].

Definition 6. *An operator $T \in B(H)$ is called a* **Fredholm operator** *if the kernel and the cokernel of T are finite dimensional , i. e.*

$$\dim(\mathrm{kern}(T)) < \infty \quad \textit{and} \quad \dim(H/\operatorname{im}(T)) < \infty.$$

Definition 7. *We define the index of a Fredholm operator T by*

$$\mathrm{ind}(T) := \dim(\mathrm{kern}(T)) - \dim(\mathrm{cokern}(T)).$$

Proposition 8. *Let $T, L \in F(H)$ and $K \in K(H)$, then*

1. *$F(H)$ is an open subset of $B(H)$.*
2. *TL and LT are Fredholm operators.*
3. *$T + K \in F(H)$.*
4. *the adjoint operator T^* of T is also a Fredholm operator.*

Note that every H-S operator is a compact operator. We can conclude that the sum of a Fredholm operator and a H-S operator is a Fredholm operator from Proposition 8. This fact will play an important role in Section 5 where the definition of Grassmannians will be given.

Proof. We proceed step by step.
(1) We denote the group of all invertible elements of $B(H)/K(H)$ by Δ, i. e.

$$\begin{aligned}\Delta := \{A \in B(H)/K(H) \quad | \quad \exists B \in B(H)/K(H) \quad \text{such that} \\ AB \;=\; BA \;=\; \mathrm{Id}\; \in B(H)/K(H)\}.\end{aligned}$$

This group is open since the set of invertible elements of a Banach space is open. The natural projection π is continuous, therefore the space $F(H) = \pi^{-1}(\Delta)$ is open.
(2) The function π^{-1} is multiplicative since

$$\begin{aligned}\pi^{-1}((T_1 + K_1)(T_2 + K_2)) &= \pi^{-1}(T_1T_2 + T_2K_1 + T_1K_2 + K_1K_2) \\ &= T_1T_2 = \pi^{-1}(T_1 + K_1)\pi^{-1}(T_2 + K_2)\end{aligned}$$

where T_1,$T_2 \in B(H)$ and K_1,$K_2 \in K(H)$. Furthermore, Δ is a group and we can conclude that $F(H)$ is closed under multiplication.
(3) That $F(H)$ is closed under addition of compact operators follows easily from

$$(T + K)A = TA + KA = \mathrm{Id} + K_1 + K_2 = \mathrm{Id} + K_3, \quad \text{since} \quad TA = \mathrm{Id} + K_1,$$

and $KA = K_2$. Here $T \in F(H)$, $A \in B(H)$, and $K, K_1, K_2, K_3 \in K(H)$ by using Proposition 6. The conclusion is that $\pi(T + K)$ is invertible in $B(H)/K(H)$ and so it is an element of $F(H)$.
(4) Suppose $T \in F(H)$. Then there exist $S \in B(H)$ and K_1,$K_2 \in K(H)$ such that

$$ST = \mathrm{Id} + K_1, \qquad TS = \mathrm{Id} + K_2$$

with

$$(ST)^* = T^*S^* = \mathrm{Id} + K_1^*, \qquad (TS)^* = S^*T^* = \mathrm{Id} + K_2^*.$$

We conclude that $\pi(T^*)$ is invertible by Proposition 6 and hence, $T^* \in F(H)$. □

Proposition 9. *If A and B are Fredholm operators, then*

$$\mathrm{ind}(AB) = \mathrm{ind}(A) + \mathrm{ind}(B).$$

Proof. We observe that

$$\begin{aligned}\dim(\mathrm{kern}(AB)) \;&=\; \dim(\mathrm{kern}(A)) + \dim(\mathrm{kern}(B)) \\ &-\; \dim(\mathrm{kern}(A) \cap H/\,\mathrm{im}(B))\end{aligned}$$

and

$$\begin{aligned}\dim(\mathrm{cokern}(AB)) \;&=\; \dim(\mathrm{cokern}(A)) + \dim(\mathrm{cokern}(B)) \\ &-\; \dim(\mathrm{kern}(A) \cap H/\,\mathrm{im}(B)).\end{aligned}$$

We calculate

$$\begin{aligned}\operatorname{ind}(AB) &= \dim(\operatorname{kern}(AB)) - \dim(\operatorname{cokern}(AB)) \\ &= \dim(\operatorname{kern}(A)) + \dim(\operatorname{kern}(B)) - \dim(\operatorname{kern}(A) \cap H/\operatorname{im}(B)) \\ &- \dim(\operatorname{cokern}(A)) - \dim(\operatorname{cokern}(B)) + \dim(\operatorname{kern}(A) \cap H/\operatorname{im}(B)) \\ &= \dim(\operatorname{kern}(A)) - \dim(\operatorname{cokern}(A)) \\ &+ \dim(\operatorname{kern}(B)) - \dim(\operatorname{cokern}(B)) \\ &= \operatorname{ind}(A) + \operatorname{ind}(B).\end{aligned}$$

□

One additional property of the index is its invariance under the addition of a compact operator.

Corollary 3. *For every Fredholm operator $A|$ and compact operator K*

$$\operatorname{ind}(A + K) = \operatorname{ind}(A).$$

We will not present the proof here and refer the interested reader to the book of W. Arveson [3], chapter ”3.4 The Fredholm index”.

Theorem 1. *The index of an adjoint operator T^* of a Fredholm operator T is the negative index of T, i.e.*

$$\operatorname{ind}(T^*) = -\operatorname{ind}(T).$$

4. Groups of operators in Hilbert space

4.1. **The restricted general linear group of Hilbert space.**

We suppose from now on that a separable Hilbert space H is equipped with a polarization $H_+ \oplus H_-$.

Definition 8. *The* **general linear group** $GL(H)$ *consists of all bounded invertible linear operators from H to H. Its norm is defined as the operator norm in the space $B(H)$, i. e.*

$$\| T \|_{B(H)} = \sup\{\| Tx \|_H \mid x \in H \wedge \| x \|_H = 1\} = \|T\|_{op}.$$

The **restricted general linear group** GL_{res} *consists of all elements A of the general linear group $GL(H)$, whose commutator $[J, A] = JA - AJ$ is a H-S operator, where $J\colon H \to H$ is defined by*

$$J|_{H_+} = \mathrm{Id}\colon H_+ \to H_+, \qquad J|_{H_-} = -\mathrm{Id}\colon H_- \to H_-.$$

An equivalent definition of $GL_{res}(H)$ can be given by using (2×2)-matrix representation. This definition will be very useful in the following sections.

Definition 9. *Let us write $A \in GL(H)$ as*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5}$$

with respect to the polarization by making use of linear bounded operators

$$a\colon H_+ \to H_+, \qquad b\colon H_- \to H_+$$

$$c\colon H_+ \to H_-, \qquad d\colon H_- \to H_-.$$

Then $GL_{res}(H)$ consists of all (2×2)-matrices $A \in GL(H)$ such that b and c are H-S operators.

Proposition 10. *Definitions 8 and 9 are equivalent.*

Proof. Suppose $A \in GL(H)$ is given by (5) with H-S operators b and c. Then, since

$$[J, A] = \begin{pmatrix} a & b \\ -c & -d \end{pmatrix} + \begin{pmatrix} -a & b \\ -c & d \end{pmatrix} = \begin{pmatrix} 0 & 2b \\ -2c & 0 \end{pmatrix},$$

we get that $[J, A]$ is a H-S operator in H.

Now suppose $A \in GL(H)$ and that the commutator $[J, A]$ is a H-S operator. Thus, if we write A as in (5), then

$$[J, A] = \begin{pmatrix} 0 & 2b \\ -2c & 0 \end{pmatrix} \quad \text{is a H-S operator in H.}$$

Taking restrictions of $[J, A]$ on H_+ and H_- we conclude that $b\colon H_- \to H_+$ and $c\colon H_+ \to H_-$ are H-S operators. □

Proposition 11. *The restricted general linear group $GL_{res}(H)$ is a group with respect to the composition "$\circ$".*

Proof. We omit the symbol "$\circ$" writing the composition as a product. We verify the group axioms.
(1) We show that if $A, B \in GL_{res}(H)$, then $AB \in GL_{res}(H)$. Note that

$$[J, A] = JA - AJ = A(A^{-1}JA - J) = (J - AJA^{-1})A.$$

Then we get

$$\begin{aligned}[J, AB] &= JAB - ABJ = A(A^{-1}JAB - BJ) \\ &= A(A^{-1}JA - BJB^{-1})B = A(A^{-1}JA + J - J - BJB^{-1})B \\ &= A(A^{-1}JA - J)B + A(J - BJB^{-1})B = [J, A]B + A[J, B].\end{aligned}$$

Since operators A and B are bounded and $[J, A]$ and $[J, B]$ are H-S operators, it follows that $[J, AB]$ is a H-S operator by Proposition 2.
(2) The product is associative by (1) and the associativity of the product in $GL(H)$.
(3) As an identity element in $GL_{res}(H)$ we can take the identity element of $GL(H)$ because of

$$[J, \mathrm{Id}] = J\,\mathrm{Id} - \mathrm{Id}\,J = J - J = 0 \in HS(H).$$

(4) All elements of $GL(H)$ are invertible operators, therefore for all $A \in GL_{res}(H)$ there exists $A^{-1} \in GL(H)$. A^{-1} is an element of $GL_{res}(H)$ by

$$[J, A^{-1}] = JA^{-1} - A^{-1}J = A^{-1}(AJ - JA)A^{-1} = -A^{-1}[J, A]A^{-1}$$

and by Proposition 2, since the product of the bounded operator A^{-1} by a H-S operator $[J, A]$ is a H-S operator. □

Corollary 4. *$GL_{res}(H)$ is a subgroup of $GL(H)$.*

Proposition 12. *If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A \in GL_{res}(H)$, then the operators a and d are Fredholm.*

Proof. We proved in Proposition 11 that

$$A^{-1} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in GL_{res}(H)$$

with f and g being H-S operators. Then

$$\begin{pmatrix} \mathrm{Id}\,|_{H_+} & 0 \\ 0 & \mathrm{Id}\,|_{H_-} \end{pmatrix} = \mathrm{Id}_H = AA^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}.$$

Thus

$$ae + bg = \mathrm{Id}\,|_{H_+} \implies ae = \mathrm{Id}\,|_{H_+} - bg = \mathrm{Id}\,|_{H_+} + K, \quad K \in K(H_+),$$

since bg is a H-S operator and any H-S operator is compact. Therefore both of the operators a, e belong to $B(H_+)/K(H_+)$, they are Fredholm by Definition 5 and, moreover, they are mutually inverse in H_+. By similar arguments and

$$cf + dh = \mathrm{Id}\,|_{H_-} \quad \Longrightarrow \quad dh = \mathrm{Id}\,|_{H_-} - cf = \mathrm{Id}\,|_{H_-} + K, \quad K \in K(H_-),$$

we conclude that d and h are mutually inverse Fredholm operators in H_-. □

Definition 10. *We define the Banach algebra $B_J(H)$ by*

$$B_J(H) := \{A \in B(H) \mid [J, A] \text{ is a H-S operator } \},$$

where the multiplication is the composition of operators. The norm $\| \cdot \|_J$ is defined by

$$\| A \|_J := \| A \|_{op} + \| [J, A] \|_{HS} .$$

Remark 1. (1) *We note that $GL_{res}(H)$ is a subset of $B_J(H)$, because any $A \in GL_{res}(H)$ is a bounded operator and the commutator $[J, A]$ is a H-S operator by definition of $GL_{res}(H)$.*

(2) *Remind that a unit of an algebra is defined as an invertible, with respect to the multiplication, element of the algebra.*

Proposition 13. *The group of units of $B_J(H)$ is $GL_{res}(H)$.*

Proof. We know that $GL_{res}(H) \subset B_J(H)$. We want to prove that

$$A \in GL_{res}(H) \quad \Leftrightarrow \quad \exists \;\; B \in B_J(H) \quad \text{such that} \quad BA = AB = \mathrm{Id}\,.$$

Suppose that $A \in GL_{res}(H)$. As $A^{-1} \in GL_{res}(H) \subset B_J(H)$, we completed the proof in one direction.

Conversely, assume that A is a unit of $B_J(H)$: there exists $B \in B_J(H)$ with $AB = BA = \mathrm{Id}$. We see that A is a invertible bounded linear operator whose commutator $[J, A]$ is a H-S operator. It follows that $A \in GL_{res}(H)$. □

Definition 11. *The subgroup of $GL_{res}(H)$, which consists of its unitary operators, is denoted by $U_{res}(H)$:*

$$U_{res}(H) := \{A \in GL_{res}(H) \mid A \text{ is an unitary operator}\}.$$

4.2. **Sequences and extensions.**

This subsection collects some auxiliary algebraic notions such as short sequences, exact sequences, central extensions, and others that we need to define the central extension of GL_{res} in Subsection 4.3.

Definition 12. *A subgroup H of a group G is called* **normal** *subgroup if and only if*

$$gHg^{-1} \subseteq H \quad \text{for all} \quad g \in G.$$

Definition 13. *If H and F are groups, then an* **extension** *of F by H is a group G having a normal subgroup $H_1 \subset G$ such that $H_1 \cong H$ and $G/H_1 \cong F$. We used the symbol $\cong$ to denote an isomorphism of groups.*

Definition 14. *A* **sequence** *(G_i, f_i) is defined as a pair of sequences $\{G_i\}$ of groups and sequences $\{f_i\}$ of homomorphisms from G_i to G_{i+1}, i. e.*

$$\dots \xrightarrow{f_{i-1}} G_i \xrightarrow{f_i} G_{i+1} \xrightarrow{f_{i+1}} \dots .$$

A sequence is called **exact** *if and only if $\operatorname{im}(f_{i-1}) = \operatorname{kern}(f_i)$ for each i.*

Now we can introduce an equivalent definition of an extension of a group.

Proposition 14. *A group G is an* **extension** *of F by H if and only if the following sequence*

$$1 \xrightarrow{f_0} H \xrightarrow{f_1} G \xrightarrow{f_2} F \xrightarrow{f_3} 1$$

is exact, where the map f_1 from H to G is an injective homomorphism and the map f_2 from G to F is a surjective homomorphism.

Proof. Suppose that G is an extension of F by H and a group H_1 is the normal subgroup of G such that $H_1 \cong H$ and $F \cong G/H_1$. Denote by $\tilde{f}_1$ and $\tilde{f}_2$ the corresponding isomorphisms $\tilde{f}_1 \colon H_1 \to H$ and $\tilde{f}_2 \colon G/H_1 \to F$. We aim to find an exact sequence

$$1 \xrightarrow{f_0} H \xrightarrow{f_1} G \xrightarrow{f_2} F \xrightarrow{f_3} 1. \tag{6}$$

We define the homomorphisms $f_0 : 1 \to H$ and $f_3 : F \to 1$ by

$$f_0(1) = 1_H \qquad \text{and} \qquad f_3(x) = 1 \text{ for all } x \in F.$$

Furthermore, we define $f_1 \colon H \to G$ by

$$f_1(H) = H_1, \quad f_1(x) = \tilde{f}_1^{-1}(x).$$

The map f_1 is injective as $\tilde{f}_1$ is bijective and hence its kernel is $\{1_H\}$, so it is equal to the image of f_0. Furthermore, we define $f_2 \colon G \to F$ by

$$f_2(x) = \tilde{f}_2(x \mod (H_1)).$$

We see that it is surjective since

$$\operatorname{im} f_2 = \tilde{f}_2(G/H) = \tilde{f}_2(G/H_1) = F.$$

We also see that the kernel of f_2 is H, which is the image of f_1. Then the sequence (6) is exact.

To prove the proposition in the other direction we suppose that the sequence (6) is exact with injective map f_1 and surjective map f_2. We want to show that $H \cong H_1 \subset G$, where H_1 is a normal subgroup of G and $G/H_1 \cong F$. Define H_1 by $H_1 := f_1(H)$. Then H_1 is isomorphic to H. Furthermore, we know that $f_1(H) = \operatorname{kern}(f_2)$ and $f_2(G) = F$. We see that the restriction of f_2 on G/H_1 is an isomorphism and so F is isomorphic to G/H_1. As H_1 is the kernel of a homomorphism on G, we conclude that H_1 is normal. We completed to show that G is an extension of H by F. $\square$

Corollary 5. *Given a short exact sequence*

$$1 \xrightarrow{f_0} A \xrightarrow{f_1} B \xrightarrow{f_2} C \xrightarrow{f_3} 1.$$

It is equivalent to say that the group B is an extension of C by $f_1(A)$.

Proof. We know that for an exact sequence the kernel of f_{i-1} and the image of f_i have to be equal. We note that $\text{kern}(f_0) = \{1\}$ because f_0 is a homomorphism and its domain is just $\{1\}$. It gives $\text{kern}(f_1) = \{1_A\}$ and so f_1 is an injective homomorphism from A to $f_1(A)$.

On the other hand we know that the image of f_3 is $\{1\}$ and so $f_3(C) = \{1\}$. We conclude that the kernel of f_3 have to be C and so the image of f_2 has to be C. It follows that f_2 is surjective.

As $f_1(A)$ is the kernel of the homomorphism f_2 on B, we conclude that $f_1(A)$ is normal in B. It is not known whether A is a subset of B so we just affirme that B is an extension of C by $f_1(A) \cong A$. $\square$

Definition 15. *A* **central extension** *H of a group G by Z is an exact sequence*

$$1 \longrightarrow Z \longrightarrow H \longrightarrow G \longrightarrow 1$$

such that Z (or, more precisely, the image of Z in H) belongs to the center of H. We say that the group H is a central extension of G by Z.

Remark 2. *If H is a central extension of G, then we remind that Z is a normal subgroup of H and that H/Z is isomorphic to G.*

Proof. The remark obviously follows from Definition 14 and properties of short exact sequences. $\square$

Definition 16. *Let K be a subgroup of a group G. Then a subgroup $Q \subseteq G$ is called the* **complement** *of K in G if $K \cap Q = 1$ and $KQ = G$.*

Example 1. *Suppose K is a normal subgroup of G. If we define $Q := G/K$, then it follows that*

$$K \cap Q = K \cap G/K = 1 \qquad \text{and} \qquad KQ = K(G/K) = G.$$

So we see that G/K is a complement of K.

Definition 17. *A group G is a* **semidirect product** *of K by Q, denoted by $G = K \rtimes Q$, if K is a normal subgroup of G and K has the complement $Q_1 \cong Q$.*

Lemma 2. *If K is a normal subgroup of a group G, then the following statements are equivalent:*

(1) *G is a semidirect product of K by G/K.*

(2) *There is a subgroup $Q \subseteq G$ such that every element $g \in G$ has a unique expression $g = ax$, where $a \in K$ and $x \in Q$.*
(3) *There exists a homomorphism $s\colon G/K \to G$ with $vs = 1_{G/K}$, where $v : G \to G/K$ is the natural projection.*
(4) *There exists a homomorphism $\pi\colon G \to G$ with $\operatorname{kern}(\pi) = K$ and $\pi(x) = x$ for all $x \in \operatorname{im}(\pi)$.*

Proof. We proceed step by step.
(1) ⇒ (2) Let Q be a complement of K in G and $g \in G$. Since $G = KQ$, there exists $a \in K$ and $x \in Q$ with $g = ax$. If $g = by$ is another factorization of g by $b \in K$ and $y \in Q$, then

$$Q \ni xy^{-1} = a^{-1}b \in K \quad \Rightarrow \quad xy^{-1} = a^{-1}b \in K \cap Q = \{1\}.$$

Therefore $xy^{-1} = 1$ and $a^{-1}b = 1$, and hence $b = a$ and $y = x$.
(2) ⇒ (3) It is given that any $g \in G$ has an unique expression $g = ax$, where $a \in K$ and $x \in Q$. If $Kg \in G/K$, then $Kg = Kax = Kx$. Define $s\colon G/K \to G$ by $s(Kg) = x$. This defines a group homomorphism since K is a normal subgroup ($Kg = gK$) and

$$\begin{aligned} s(Kg_1Kg_2) &= s(Kx_1Kx_2) = s(K(x_1K)x_2) = s(K(Kx_1)x_2) \\ &= s(Kx_1x_2) = x_1x_2 = s(Kg_1)s(Kg_2). \end{aligned}$$

If we define $v\colon G \to G/K$ with $v(g) = v(ax) := Kx$, then we can conclude that it is the identity of G/K, i. e. $vs = 1_{G/K}$ by

$$v(s(Kg)) = v(x) = Kx = Kg.$$

(3) ⇒ (4) Define $\pi\colon G \to G$ by $\pi = sv$. For all $x \in \operatorname{im}(\pi)$ there exists $g \in G$ such that $x = \pi(g)$. Then

$$\pi(x) = \pi(\pi(g)) = svsv(g) = sv(g) = \pi(g) = x$$

as vs is the identity of G/K. If $a \in K$, then $\pi(a) = sv(a) = 1$ because $K = \operatorname{kern}(v)$ implies $K \subset \operatorname{kern}(sv)$.

To show the reverse inclusion, assume that

$$1 = \pi(g) = sv(g) = s(Kg).$$

Now s is an injection by set theory. It follows that $Kg = 1$ and we conclude $g \in K$. Therefore, $K \supset \operatorname{kern}(sv)$. We completed to show $K = \operatorname{kern}(sv)$.
(4) ⇒ (1) Define $Q := \operatorname{im}(\pi)$. If $g \in Q$, then $\pi(g) = g$. If $g \in K$, then $\pi(g) = 1$. If $g \in K \cap Q$, then $g = 1$. If $g \in G$, then $g\pi(g^{-1}) \in K = \operatorname{kern}(\pi)$ for $\pi(g\pi(g^{-1})) = 1$. Since $\pi(g) \in Q$, we have $g = [g\pi(g^{-1})]\pi(g) \in KQ$. Therefore, Q is a complement of K in G and G is a semidirect product of K by Q. □

We denote by $\operatorname{Aut}(K)$ the group of automorphisms of K.

Lemma 3. *If $G = K \rtimes Q$ is a semidirect product of K by Q, then there is a homomorphism $\theta\colon Q \to \mathrm{Aut}(K)$, defined by*

$$\theta_x(a) = xax^{-1} \quad \text{for all} \quad x \in Q, \quad a \in K.$$

Thus

$$\theta_{1_Q}(a) = a \quad \text{and} \quad \theta_x(\theta_y(a)) = \theta_{xy}(a)$$

for all $x, y \in Q$ and $a \in K$.

Proof. Normality of K gives us the fact that $\theta_x(K) = K$ and so it is an automorphism of K. The other claims follow from $(xy)^{-1} = y^{-1}x^{-1}$ and the following equations

$$\theta_1(a) = 1a1^{-1} = a1 = a$$

$$\theta_x(\theta_y(a)) = \theta_x(yay^{-1}) = xyay^{-1}x^{-1} = xya(xy)^{-1} = \theta_{xy}(a).$$

□

Definition 18. *Let Q and K be groups and let $\theta\colon Q \to \mathrm{Aut}(K)$ be a homomorphism. We say that the semidirect product G of K by Q* **realizes** *θ if for all $x \in Q$ and $a \in K$,*

$$\theta_x(a) = xax^{-1}.$$

Definition 19. *Given groups Q and K and a homomorphism $\theta\colon Q \to \mathrm{Aut}(K)$, define the semidirect product $G = K \rtimes_\theta Q$ with respect to θ to be the set of all ordered pairs $(a, x) \in K \times Q$ equipped with the operation*

$$(a, x)(b, y) = (a\theta_x(b), xy).$$

In the following theorem we show that any semidirect product with respect to some homomorphism realizes this homomorphism.

Theorem 2. *Given groups Q and K and a homomorphism $\theta\colon Q \to \mathrm{Aut}(K)$, then $G = K \rtimes_\theta Q$ is a semidirect product of K by Q that realizes θ.*

Proof. First we have to prove that G is a group.

We start by showing that the multiplication on G is associative.

$$\begin{aligned}[(a, x)(b, y)](c, z) &= (a\theta_x(b), xy)(c, z) = (a\theta_x(b)\theta_{xy}(c), xyz) \\ &= (a\theta_x(b\theta_y(c)), xyz) = (a, x)(b\theta_y(c), yz) \\ &= (a, x)[(b, y)(c, z)].\end{aligned}$$

The identity element of G is $(1, 1)$ by

$$(1, 1)(a, x) = (1\theta_1(a), 1x) = (a, x).$$

The inverse of (a, x) is $((\theta_{x^{-1}}(a))^{-1}, x^{-1})$, since

$$((\theta_{x^{-1}}(a))^{-1}, x^{-1})(a, x) = ((\theta_{x^{-1}}(a))^{-1}\theta_{x^{-1}}(a), x^{-1}x) = (1, 1).$$

We conclude that G is a group.

Define a map $\pi\colon G \to Q$ by $(a,x) \mapsto x$. The map π is obviously surjective. The homomorphism property of π follows from

$$\pi((a,x)(b,y)) = \pi((a\theta_x(b), xy)) = xy = \pi((a,x))\pi((b,y)).$$

As $\pi((a,1)) = 1$ for all $a \in K$, the kernel of π is $\{(a,1) \mid a \in K\}$. Recall that the kernel of a homomorphism is a normal subgroup.

We identify K with $\mathrm{kern}(\pi)$ via the isomorphism $a \mapsto (a,1)$. We also identify Q with $\{(1,x) \mid x \in Q\} \subset G$ by the isomorphism $x \mapsto (1,x)$. We can see that $KQ = G$ as $(a,1)(1,x) = (a,x)$ for all $a \in K, \quad x \in Q$ and that $K \cap Q = \{1\}$ as $(a,1) = (1,x)$ if and only if $a = 1 \wedge x = 1$. We conclude that G is a semidirect product of K by Q.

Finally we see that G does realize θ:

$$(1,x)(a,1)(1,x)^{-1} = (\theta_x(a), x)(1, x^{-1}) = (\theta_x(a), 1).$$

□

Now we can assert that actually any semidirect product is isomorphic to a semidirect product with respect to some homomorphism.

Theorem 3. *If G is a semidirect product of K by Q, then there exists $\theta\colon Q \to \mathrm{Aut}(K)$ such that $G \cong K \rtimes_\theta Q$.*

Proof. Define $\theta_x(a) = xax^{-1}$. We know from Lemma 2 that every $g \in G$ has an unique expression $g = ax$ with $a \in K$ and $x \in Q$. Since multiplication in G satisfies

$$(ax)(by) = a(xbx^{-1})xy = a\theta_x(b)xy,$$

we can see that the map $f\colon K \rtimes_\theta Q \to G$ by $(a,x) \mapsto ax$ is an isomorphism:

$$f((a,x)(b,y)) = f((a\theta_x(b), xy)) = a\theta_x(b)xy = (ax)(by) = f((a,x))f((b,y)).$$

The map f is surjective by $KQ = G$.

We will prove the injective property by contradiction. Let us assume that f is not injective, then the kernel is non-trivial and thus there exists $a \in K$ and $x \in Q$ such that $f((a,x)) = 1$ with $a \neq 1$ and $x \neq 1$. Then $x = a^{-1} \in K$ and $a \in Q$ implies $a^{-1} \in Q$ by the group property of Q. We conclude that $a^{-1} \in K \cap Q = \{1\}$ and so $a^{-1} = 1$ leads to $a = 1$. This is a contradiction to the assumption that the kernel is non-trivial. We deduce that f is injective. □

4.3. The central extension of $GL_{res}(H)$.

The motivation of this subsection comes from the last subsection of Section 5, where we aim to define an action of the central extension of $GL_{res}(H)$ on the determinant bundle of the Grassmannian that covers the action of the $GL_{res}(H)$ on the Grassmannian. We start from the construction of the central extension of the identity component $GL_{res,0}(H)$ of $GL_{res}(H)$.

Operator $A \in GL_{res}(H)$ will be written as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a, d are Fredholm operators and b, c are H-S operators for the rest of the subsection.

Definition 20. *We define the identity component $GL_{res,0}(H)$ of $GL_{res}(H)$ by*

$$GL_{res,0}(H) := \{A \in GL_{res}(H) \mid \mathrm{ind}(a) = 0\}.$$

We define the set τ by

$$\tau := \{q \in GL(H_+) \mid q \text{ has a determinant}\}$$

and τ_1 by

$$\tau_1 := \{q \in \tau \mid \det(q) = 1\}.$$

We define $\mathfrak{E}$ by

$$\mathfrak{E} := \{(A, q) \in GL_{res,0}(H) \times GL(H_+) \mid aq^{-1} - 1 \text{ is of trace class}\}.$$

Corollary 6. *The set $\mathfrak{E}$ is a group.*

Proof. We define the group operation of $\mathfrak{E}$ canonically by the group operations of $GL_{res,0}(H)$ and $GL(H_+)$:

$$(A, q)(B, p) = (AB, qp).$$

We define $(1, \mathrm{Id}_{H_+}) \in \mathfrak{E}$ as the neutral element. This is true since

$$(1, \mathrm{Id}_{H_+})(A, q) = (1A, \mathrm{Id}_{H_+} q) = (A, q) = (A1, q\,\mathrm{Id}_{H_+}) = (A, q)(1, \mathrm{Id}_{H_+}).$$

We have to check whether $(A, q)(B, p) = (AB, qp) \in \mathfrak{E}$ for (A, q),$(B, p) \in \mathfrak{E}$. We know that $qp \in GL(H_+)$ and that $AB \in GL_{res,0}(H)$. We write AB by

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}. \tag{7}$$

Finally we have to check if $(ae+bg)(qp)^{-1} - \mathrm{Id}_{H_+}$ is of trace class. We know that b and g are H-S operators such that bg is trace class operator. We further know that $aq^{-1} - \mathrm{Id}_{H_+}$ and $ep^{-1} - \mathrm{Id}_{H_+}$ are of trace class, a, q^{-1} and $-a^{-1}q$ are bounded and $a^{-1}q$ has a determinant, since $-a^{-1}(aq^{-1} - \mathrm{Id}_{H_+})q = a^{-1}q - \mathrm{Id}_{H_+}$ is of trace class. It follows that $(ep^{-1} - \mathrm{Id}_{H_+}) - (a^{-1}q - \mathrm{Id}_{H_+})$ is of trace class and so also

$$\begin{aligned} a((ep^{-1} - \mathrm{Id}_{H_+}) - (a^{-1}q - \mathrm{Id}_{H_+}))q^{-1} &= a(ep^{-1} - a^{-1}q - \mathrm{Id}_{H_+} + \mathrm{Id}_{H_+})q^{-1} \\ &= a(ep^{-1} - a^{-1}q)q^{-1} \\ &= aep^{-1}q^{-1} - \mathrm{Id}_{H_+} \\ &= (ae)(qp)^{-1} - \mathrm{Id}_{H_+}. \end{aligned}$$

This implies that $(AB, qp) \in \mathfrak{E}$.

Notice that for all $(A,q) \in \mathfrak{E}$ there exists its inverse $(A,q)^{-1} = (A^{-1}, q^{-1}) \in \mathfrak{E}$ because of

$$(A,q)(A^{-1},q^{-1}) = (AA^{-1}, qq^{-1}) = (1, \mathrm{Id}_{H_+}) = (A^{-1}A, q^{-1}q) = (A^{-1},q^{-1})(A,q).$$

The operators

$$A^{-1} := \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in GL_{res,0} \quad \text{and} \quad q^{-1} \in GL(H_+)$$

exist, as both are groups. We know that $AA^{-1} = 1$ implies

$$ae + bg = \mathrm{Id}_{H_+} \quad \text{and} \quad ae - \mathrm{Id}_{H_+} = bg.$$

This yields that $ae - \mathrm{Id}_{H_+}$ is of trace class as b and g are H-S operators. It follows that $a^{-1}(ae - \mathrm{Id}_{H_+})q$ is of trace class as a^{-1} and q are bounded. We already proved in the first part of this proof that $a^{-1}q - \mathrm{Id}_{H_+}$ is of trace class. Then $(a^{-1} - \mathrm{Id}_{H_+}) + a^{-1}(ae - \mathrm{Id}_{H_+})q$ is of trace class and so $eq - \mathrm{Id}_{H_+}$ is by

$$a^{-1}(ae - \mathrm{Id}_{H_+})q = eq - a^{-1}q = eq - a^{-1}q + \mathrm{Id}_{H_+} - \mathrm{Id}_{H_+}$$
$$(a^{-1}q - \mathrm{Id}_{H_+}) + a^{-1}(ae - \mathrm{Id}_{H_+})q = eq - \mathrm{Id}_{H_+}.$$

This implies that $(A,q)^{-1} \in \mathfrak{E}$. □

Proposition 15. *We note that τ is a normal subgroup of $GL(H_+)$.*

Proof. We claim that τ is a group. As the neutral element of τ we choose Id_{H_+}. Every element p in τ is invertible by definition. This inverse p^{-1} has a determinant, as $p - \mathrm{Id}_{H_+}$ is of trace class and so also

$$-p^{-1}(p - \mathrm{Id}_{H_+}) = -\mathrm{Id}_{H_+} + p^{-1},$$

as $-p^{-1}$ is bounded. We conclude that p^{-1} has a determinant, which implies that $p^{-1} \in \tau$.

We prove that the multiplication of two elements of τ is an element of τ. Suppose q,$p \in \tau$, then the product $qp \in GL(H_+)$. Furthermore, it has an inverse $p^{-1}q^{-1}$, as

$$qpp^{-1}q^{-1} = qq^{-1} = \mathrm{Id}_{H_+}.$$

The operators $q - \mathrm{Id}_{H_+}$ and $p - \mathrm{Id}_{H_+}$ are of trace class and since the space of trace class operators is a two-sided ideal in the space of bounded operators we get that $(q - \mathrm{Id}_{H_+})(p - \mathrm{Id}_{H_+})$ is a trace class operator. Furthermore, we know that the finite sum of trace class operators is again a trace class operator. It follows that

$$\begin{aligned} qp - \mathrm{Id}_{H_+} &= qp - q - p + \mathrm{Id}_{H_+} + q - \mathrm{Id}_{H_+} + p - \mathrm{Id}_{H_+} \\ &= (q - \mathrm{Id}_{H_+})(p - \mathrm{Id}_{H_+}) + (q - \mathrm{Id}_{H_+}) + (p - \mathrm{Id}_{H_+}) \end{aligned}$$

is a trace class operator, as $q - \mathrm{Id}_{H_+}$ and $p - \mathrm{Id}_{H_+}$ are trace class operators as q and p have determinants. We conclude that $qp - \mathrm{Id}_{H_+}$ is a trace class. Thus qp is an operator with a determinant and it is an element of τ.

To prove that τ is a normal subgroup in $GL(H_+)$, we have to show that

$$BqB^{-1} \in \tau \quad \text{for all} \quad q \in \tau, \quad B \in GL(H_+).$$

We know that $q - \mathrm{Id}_{H_+}$ is of trace class and that

$$B(q - \mathrm{Id}_{H_+})B^{-1} = BqB^{-1} - BB^{-1} = BqB^{-1} - \mathrm{Id}_{H_+}$$

is of trace class since B and B^{-1} are bounded. Then BqB^{-1} has a determinant. It is obviously linear, bounded and has the inverse $Bq^{-1}B^{-1}$, as

$$Bq^{-1}B^{-1}BqB^{-1} = Bq^{-1}qB^{-1} = BB^{-1} = \mathrm{Id}_{H_+}.$$

We conclude that $BqB^{-1} \in \tau$ and so τ is a normal subgroup of $GL(H_+)$. □

We introduced all necessary sets to define the central extension of $GL_{res,0}(H)$. Before we do this, we examine relations between the sets in more detail.

Proposition 16. (1) *The quotient space τ/τ_1 is isomorphic to $\mathbb{C}^\times$.*

(2) *To every Fredholm operator $a\colon H_+ \to H_+$ of index zero one can add a finite rank operator $t\colon H_+ \to H_+$ such that the sum $q := a + t$ is an invertible operator in H_+.*

(3) *The set*

$$\mathfrak{E}_1 := \{(1, q) \in GL_{res,0} \times GL(H_+) \mid 1q^{-1} - 1 \text{ is of trace class}\} \subset \mathfrak{E}$$

is isomorphic to τ, i.e. $\mathfrak{E}_1 = \{\mathrm{Id}_H\} \times \tau$.

(4) *The set $\mathfrak{E}_1$ is a normal subgroup in $\mathfrak{E}$.*

Proof. We argue as follows.

(1) Consider the determinant function $\det\colon \tau/\tau_1 \to \mathbb{C}^\times$ that defines a group homomorphism by

$$\det(q_1 q_2) = \det(q_1)\det(q_2).$$

It is obviously surjective and since the kernel of $\det\colon \tau \to \mathbb{C}$ is τ_1, we conclude that det is an isomorphism.

(2) We know that $\dim(\mathrm{kern}(a)) = \dim(\mathrm{cokern}(a)) < \infty$. Choose the orthonormal basis $(e_1, ..., e_n)$ of $\mathrm{kern}(a)$ and $(b_1, ..., b_n)$ of $\mathrm{cokern}(a)$ where $n \in \mathbb{N}$ is finite. Then we define $t\colon H_+ \to H_+$ by

$$t(e_i) := b_i \text{ and}$$

$$t(x) := 0 \text{ if } x \in H_+ \setminus \mathrm{span}(e_1, ..., e_n)$$

and get a finite rank operator.

If we write $q := a + t$, then we see that the kernel of q is empty by

$$H_+ = (H_+ \setminus \mathrm{span}(e_1, ..., e_n)) \oplus \mathrm{span}(e_1, ..., e_n)$$

and moreover

$$a(x) \neq 0 \text{ if } x \in H_+ \setminus \operatorname{span}(e_1, ..., e_n)$$
$$t(x) = 0 \text{ if } x \in H_+ \setminus \operatorname{span}(e_1, ..., e_n)$$
$$a(x) = 0 \text{ if } x \in \operatorname{span}(e_1, ..., e_n)$$
$$t(x) \neq 0 \text{ if } x \in \operatorname{span}(e_1, ..., e_n),$$

which shows the injectivity of q.

The surjectivity of q follows from

$$a(H_+ \setminus \operatorname{span}(e_1, ..., e_n)) = H_+ \setminus \operatorname{span}(b_1, ..., b_n)$$
$$t(H_+ \setminus \operatorname{span}(e_1, ..., e_n)) = \{0\}$$
$$a(\operatorname{span}(e_1, ..., e_n)) = \{0\}$$
$$t(\operatorname{span}(e_1, ..., e_n)) = \operatorname{span}(b_1, ..., b_n)$$

and

$$q(H_+) = a(H_+) \oplus t(H_+) = a(H_+ \setminus \operatorname{span}(e_1, ..., e_n)) \oplus t(\operatorname{span}(e_1, ..., e_n)) = \\ = H_+ \setminus \operatorname{span}(b_1, ..., b_n) \oplus \operatorname{span}(b_1, ..., b_n) = H_+.$$

It follows that the operator q is bijective and so invertible.

(3) From the definition of $\mathfrak{E}_1$

$$\mathfrak{E}_1 = \{(1, q) \in GL_{res,0} \times GL(H_+) \mid 1q^{-1} - 1 \text{ is of trace class}\}$$

we know that $q^{-1} - 1$ is a trace class operator and we get that q^{-1} has a determinant and so q has a determinant. Moreover $q \in GL(H_+)$ implies $q \in \tau$. We get that $\mathfrak{E}_1 = \{1\} \times \tau$. Then the map $y \colon \tau \to \mathfrak{E}_1$ defined by

$$y(q) := (1, q)$$

is surjective. The kernel of y is obviously trivial which yields to the bijectivity of y. The conclusion is that $\mathfrak{E}_1$ is isomorphic to τ.

(4) Suppose $(A, q) \in \mathfrak{E}$ and $(1, p) \in \mathfrak{E}_1$. Then

$$(A, q)^{-1}(1, p)(A, q) = (A^{-1}, q^{-1})(1, p)(A, q) = (A^{-1}A, q^{-1}pq) = (1, q^{-1}pq)$$

is an element of $\mathfrak{E}_1 = \{(1, q) \in GL_{res,0} \times \tau\}$, as $q^{-1}pq \in \tau$ and τ is normal in $GL(H_+)$. This implies that $\mathfrak{E}_1$ is normal in $\mathfrak{E}$. □

To give the statement of the following proposition we introduce the notation

$$\mathfrak{E}_{11} := \{(1, q) \in GL_{res,0}(H) \times \tau_1\}.$$

It is obvious that $\mathfrak{E}_{11}$ is isomorphic to τ_1. We also remind that the quotient τ/τ_1 is isomorphic to multiplicative group $\mathbb{C}^\times$.

Proposition 17. *A central extension $GL^{\sim}_{res,0}(H)$ of $GL_{res,0}(H)$ by $\mathbb{C}^\times$ is $\mathfrak{E}/\mathfrak{E}_{11}$. We write it as the exact sequence:*

$$\mathbb{C}^\times \to \mathfrak{E}/\mathfrak{E}_{11} \to GL_{res,0}(H).$$

Proof. We first construct the central extension of $GL_{res,0}(H)$ by τ, i. e.

$$\tau \xrightarrow{f_1} \mathfrak{E} \xrightarrow{f_2} GL_{res,0}(H).$$

We define $f_1 \colon \tau \to \mathfrak{E}$ by

$$f_1(q) = (1, q),$$

which is injective since the kernel of f_1 is trivial. We set $f_2 \colon \mathfrak{E} \to GL_{res,0}(H)$ by $f_2((A, q)) = A$, which is obviously surjective and its kernel consists of elements $(1, q) \in \mathfrak{E}$. So we see that the kernel of f_2 is $\{(1, q) \mid q \in \tau\}$. Thus we got a central extension of $GL_{res,0}(H)$ by τ.

Notice that we get the same result if we take $\tau/\tau_1 \cong \mathbb{C}^\times$ instead of τ and $\mathfrak{E}/\mathfrak{E}_{11}$ instead of $\mathfrak{E}$ and modify f_1, f_2 correspondingly. □

We are interested in the central extension $GL^\sim_{res,0}(H)$ and not in the central extension $\mathfrak{E}$, as we are not able to construct $GL^\sim_{res}(H)$ as a semidirect product of $\mathfrak{E}$. More precisely, the automorphism of $GL_{res,0}$ which generates the semidirect product $GL_{res}(H)$ from $GL_{res,0}(H)$ can not be covered by an automorphism of $GL^\sim_{res,0}(H)$, which could generate a semidirect product $GL^\sim_{res}(H)$ from $GL^\sim_{res,0}(H)$. There only exists an endomorphism.

Recall that the unilateral shift $\sigma \colon H \to H$ is defined by

$$\sigma|_{H_+}(z_k) = z_{k+1} \qquad \sigma|_{H_-} := \mathrm{Id}\,|_{H_-},$$

where $H = \mathrm{span}\,\{z_k\}_{k\in\mathbb{Z}}$. Roughly speaking the shift operator is a proper isometry of H_+ with range equal to all vectors which vanish in the first coordinate. If we fix a basis $z_0, z_1, \dots,$ of H_+, it is easy to see that $\mathrm{kern}\,\sigma = \{0\}$, while $\mathrm{cokern}\,\sigma = \{z_0\}$. Since σ is an isometry, its range is closed, and thus σ is a Fredholm operator and $\mathrm{ind}(\sigma) = -1$.

Now define

$$\sigma^n = \begin{cases} (\sigma)^n & \text{if} \quad n \geq 0 \\ (\sigma^*)^{-n} & \text{if} \quad n < 0 \end{cases},$$

where σ^* is the adjoint operator of σ. Since for $n \geq 0$ we have $\mathrm{kern}\,\sigma^n = \{0\}$ and

$$\mathrm{cokern}\,\sigma^n = \mathrm{kern}(\sigma^*)^n = \mathrm{span}(z_0, \dots, z_{n-1}),$$

it follows that $\mathrm{ind}(\sigma^n) = -n$. Similarly, since $(\sigma^n)^* = \sigma^{-n}$ for $n < 0$, we have $\mathrm{ind}(\sigma^n) = -n$ for all $n \in \mathbb{Z}$.

For $\mathbb{Z}$ we define the isomorphic group Q, which is generated by the shift operator σ by $Q := \{q \in GL(H_+) \mid \exists \;\; n \in \mathbb{Z} : q = \sigma^n\}$.

Proposition 18. *The restricted general linear group $GL_{res}(H)$ is the semidirect product of its identity component $GL_{res,0}(H)$ by Q.*

Proof. We have to show that $GL_{res}(H)$ is equal to all ordered pairs $(A, q) \in GL_{res,0}(H) \times Q$ equipped by the operation

$$(A, q_1)(B, q_2) = (A\theta_{q_1}(B), q_1 q_2)$$

by using Lemma 2.

We start from the inclusion $GL_{res}(H) \supseteq GL_{res,0}(H) \times Q$. Consider $(A, \sigma^n) \in GL_{res,0}(H) \times Q$. If we write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \sigma = \begin{pmatrix} \sigma|_{H_+} & 0 \\ 0 & 1 \end{pmatrix},$$

then the index of $A\sigma$ is equal by definition to

$$\mathrm{ind}(A\sigma) = \mathrm{ind}(a\sigma|_{H_+}) = \mathrm{ind}(a) + \mathrm{ind}(\sigma|_{H_+}) = \mathrm{ind}(a) - 1.$$

To identify elements of $GL_{res,0}(H) \times Q$ with $GL_{res}(H)$ we define the map $t \colon GL_{res,0}(H) \times Q \to GL_{res}(H)$ by $t(A, \sigma^n) = A\sigma^n$. We claim that t is injective and it is true if the kernel of t is trivial. If $A\sigma^n = 1$, then $A = (\sigma^n)^{-1}$ and since $A \in GL_{res,0}(H)$ then σ^n also have to be from $GL_{res,0}(H)$. It is true only if $n = 0$, but then $\sigma^n = 1$ and $A = 1$ and the kernel of t is trivial.

Let us show the inverse inclusion $GL_{res}(H) \subseteq GL_{res,0}(H) \times Q$. Consider $A \in GL_{res}(H)$ with $\mathrm{ind}(a) = n$, then $\mathrm{ind}(A\sigma^n) = 0$. We know that $A\sigma^n \in GL_{res,0}(H)$. We continue to use the identification given by the map t. Thus any $A \in GL_{res}(H)$ can be identified with $(A\sigma^n, \sigma^{-n}) \in GL_{res,0}(H) \times Q$ and so

$$GL_{res}(H) \subseteq GL_{res,0}(H) \times Q.$$

□

Proposition 19. *The central extension of* $GL_{res}(H)$ *by* $\mathbb{C}^\times$ *is* $GL^\sim_{res}(H) := \mathbb{Z}\widetilde{\ltimes} GL^\sim_{res,0}(H)$*, i.e.*

$$\mathbb{C}^\times \to GL^\sim_{res}(H) \to GL_{res}(H).$$

Proof. We know that

$$\mathbb{C}^\times \to GL^\sim_{res,0}(H) \to GL_{res,0}(H)$$

is a central extension. We can conclude that

$$\mathbb{C}^\times \to \mathbb{Z}\widetilde{\ltimes} GL^\sim_{res,0}(H) \to \mathbb{Z}\widetilde{\ltimes} GL_{res,0}(H)$$

is a central extension. We know that $\mathbb{Z}\widetilde{\ltimes} GL_{res,0}(H)$ is equal $GL_{res}(H)$ and so it is enough to show that $\mathbb{Z}\widetilde{\ltimes} GL^\sim_{res,0}(H)$ is a semidirect product and to define it as $GL^\sim_{res}(H)$.

We define the endomorphism $\tilde{\sigma} \colon \mathfrak{E} \to \mathfrak{E}$ by

$$(A, q) \mapsto (\sigma A \sigma^{-1}, q_\sigma) = \begin{cases} (\sigma A \sigma^{-1}, \sigma q \sigma^{-1}) \text{ on } \sigma(H_+) \\ (\sigma A \sigma^{-1}, 1) \text{ on } H_+ \ominus \sigma(H_+) \end{cases}.$$

It is not an automorphism as $q \mapsto q_\sigma$ is obviously not an automorphism. But we see that $\det(q_\sigma) = \det(q)$ and so it is an automorphism of $\mathfrak{E}/\tau_1 = GL_{res,0}(H)$. So we got that $\mathbb{Z}\widetilde{\ltimes} GL^\sim_{res,0}(H)$ is a semidirect product. □

5. Grassmannian

5.1. Definition of $Gr(H)$.

First of all we introduce basic notations and definitions, which will be used in this subsection.

Suppose that H is an infinite dimensional separable Hilbert space, with a given polarization $H = H_+ \oplus H_-$. We define H_+ and H_- by an orthonormal basis $\{z_k\}_{k\in\mathbb{Z}}$ of H:

$$\text{span}(\{z_k\}_{k\in\mathbb{N}}) := H_+ \quad \text{and} \quad \text{span}(\{z_k\}_{k\in\mathbb{Z}\setminus\mathbb{N}}) := H_-.$$

The explicit choice of $\{z_k\}_{k\in\mathbb{Z}}$ will be introduced later. We note that 0 is an element of $\mathbb{N}$ in our notation. So H_+ and H_- are infinite dimensional closed subspaces of H. It is well known that a finite dimensional Grassmanian $Gr_k(\mathbb{C}^n)$, that is a set of k-dimensional subspaces of n-dimensional complex vector space $\mathbb{C}^n$, has the structure of differentiable manifold. We want to show the same for infinite dimensional Grassmannians over an infinite dimensional separable Hilbert space. After the definition we will see that a Grassmannian could be locally identified with a Hilbert manifold modelled over the space of Hilbert-Schmidt operators from H_+ to H_-. This convenient topological and manifold structure will help us to work with its elements and to solve interesting physical problems.

Definition 21. *The infinite Grassmannian $Gr(H)$ is the set of closed subspaces W of H such that*
(i) the orthogonal projection $pr_+ \colon W \to H_+$ is a Fredholm operator,
(ii) the orthogonal projection $pr_- \colon W \to H_-$ is a Hilbert-Schmidt operator.

Definition 22. *$W \in Gr(H)$ if and only if there is $w \in B(H_+, H)$, such that*

(1) *$w(H_+) = W$,*
(2) *$pr_+ \circ w$ is a Fredholm operator,*
(3) *$pr_- \circ w$ is a H-S operator.*

Notice that w have to be bounded if we require $pr_+ \circ w$ to be a Fredholm operator and $pr_- \circ w$ to be a H-S operator, since both $pr_+ \circ w$ and $pr_- \circ w$ are bounded operators and

$$w = pr_+ \circ w + pr_- \circ w.$$

Now we prove that these definitions are equivalent.

Proposition 20. *Definitions 21 and 22 are equivalent.*

Proof. Definition 21 $\Longrightarrow$ Definition 22. Suppose $W \subset H$ closed, $pr_+ \colon W \to H_+$ is Fredholm and $pr_- \colon W \to H_-$ is H-S operator. Set $\dim(\text{kern}(pr_+)) = n$ and $\dim(\text{cokern}(pr_+)) = m$ with m, $n \in \mathbb{N}$. We conclude that there exists an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ of W. We define the bijective linear bounded operator $w \colon H_+ \to W$ by

$$w(z_l) = e_l \qquad \text{with} \quad l \in \mathbb{N}.$$

We conclude that $w(H_+) = W$. It follows that $pr_+ \circ w$ is Fredholm as w is bijective and pr_+ is Fredholm. Moreover, $pr_- \circ w$ is H-S as w is bounded and pr_- is H-S.

Definition 21 $\Longleftarrow$ Definition 22. Suppose that $w \in B(H_+, H)$, $w(H_+) = W$, $pr_+ \circ w$ is Fredholm and $pr_- \circ w$ is H-S. We know that $w\colon H_+ \to W$ is surjective and that the restriction of w on $V := H_+/\operatorname{kern}(w)$ is bijective. We get the bijective linear bounded operator $w\mid_V^{-1}$. We conclude that W is closed as $(w\mid_V)^{-1}(W) = H_+$, H_+ is closed and $(w\mid_V)^{-1}$ is continuous. We also know that the restriction $pr_+ \circ w\mid_V$ is Fredholm as the image does not change and the dimension of the kernel will be less or equal to the dimension of the kernel of $pr_+ \circ w$. We conclude that

$$pr_+ \circ w\mid_V \circ w\mid_V^{-1} = pr_+\colon W \to H_+$$

is Fredholm as $w\mid_V^{-1}$ is bijective. It is easy to see that the restriction of a H-S operator is also H-S, that implies that $pr_- \circ w\mid_V$ is H-S operator and then

$$pr_- \circ w\mid_V \circ w\mid_V^{-1} = pr_-\colon W \to H_-$$

is H-S as $w\mid_V^{-1}$ is bounded. □

There are some properties, that follow directly from the definition.

Lemma 4. *The orthogonal complement $W^\perp$ of an element of $W \in Gr(H)$ has the following "mirrored" properties:*

- *$pr_-^\perp\colon W^\perp \to H_-$ is a Fredholm operator.*
- *$pr_+^\perp\colon W^\perp \to H_+$ is a H-S operator.*

Proof. We know that

(a) $pr_+\colon W \to H_+$ is a Fredholm operator,

(b) $pr_-\colon W \to H_-$ is a H-S operator.

We also know that H_+ and H_- have the orthogonal decomposition

$$H_+ = \operatorname{im}(pr_+) \oplus H_+/\operatorname{im}(pr_+) = \operatorname{im}(pr_+) \oplus \operatorname{cokern}(pr_+);$$
$$H_- = \operatorname{im}(pr_-^\perp) \oplus H_-/\operatorname{im}(pr_-^\perp) = \operatorname{im}(pr_-^\perp) \oplus \operatorname{cokern}(pr_-^\perp).$$

We claim that

$$\operatorname{kern}(pr_+) = \operatorname{cokern}(pr_-^\perp) \tag{8}$$

and

$$\operatorname{cokern}(pr_+) = \operatorname{kern}(pr_-^\perp). \tag{9}$$

The proofs of both statements are analogous, therefore we do it only for (8).

We start from the inclusion $\operatorname{kern}(pr_+) \subseteq \operatorname{cokern}(pr_-^\perp)$. Suppose $v \in \operatorname{kern}(pr_+) = W \cap H_-$, i.e. $v \in W$ and $v \in H_-$. From this it follows that $v \notin W^\perp$, which together implies that $v \notin \operatorname{im}(pr_-^\perp)$. Making use of the above decomposition of H_-, we conclude that $v \in \operatorname{cokern}(pr_-^\perp)$.

We continue and show $\text{kern}(pr_+) \supseteq \text{cokern}(pr_-^\perp)$. Suppose $v \in \text{cokern}(pr_-^\perp) = H_-/\,\text{im}(pr_-^\perp)$, $v \neq 0$, which implies that $v \in H_-$, $v \notin \text{im}(pr_-^\perp)$, and $v \notin \text{kern}(pr_-^\perp)$. Then $v \notin W^\perp$, that implies $v \in W \cap H_- = \text{kern}(pr_+)$.

After we proved our claim we can see that

$$\begin{aligned}\dim(W \cap H_-) &= \dim(\text{kern}(pr_+)) = \dim(\text{cokern}(pr_-^\perp))\\ &= \dim(H_-/\,\text{im}(pr_-^\perp));\end{aligned}$$

$$\begin{aligned}\dim(H_+/\,\text{im}(pr_+)) &= \dim(\text{cokern}(pr_+)) = \dim(\text{kern}(pr_-^\perp))\\ &= \dim(W^\perp \cap H_+).\end{aligned}$$

Our assumption tells us that pr_+ is a Fredholm operator, i.e. the dimension of the kernel and the cokernel of pr_+ is finite. The last equations gives that the dimensions of the kernel and cokernel of $pr_-^\perp$ are finite, i.e. $pr_-^\perp$ is a Fredholm operator.

We define the two bijective isometries $t_W\colon W^\perp \to W$ and $t_{H_-}\colon H_- \to H_+$ by

$$t_W(w_j^\perp) = w_j$$
$$t_{H_-}(z_i) = z_k, \quad \text{with} \quad k \geq 0, i < 0$$

such that

$$t_{H_-} \circ pr_- \circ t_W = pr_+^\perp\colon W^\perp \to H_+,$$

where $\{w_j\}$ and $\{w_j^\perp\}$ are orthonormal basis of W and $W^\perp$, respectively, and $\{z_i\}_{i\in\mathbb{Z}}$ is the standard orthonormal basis of H, formed of the standard orthonormal basis $\{z_i\}_{i\geq 0}$ and $\{z_i\}_{i<0}$ of H_+ and H_-.

We know that pr_- is a H-S operator and that the composition with the two above defined linear bounded operators is also a H-S operator, that leads to the conclusion that $pr_+^\perp$ is a H-S operator. □

Definition 23. *We define the graph of an operator $T\colon W \to H$ by*

$$graph(T) := (W, T(W)) = W_T = \{x \oplus y \mid x \in W \wedge y = T(x)\}.$$

We define the orthogonal projection from A to $H_\pm$ by

$$(pr_\pm)_A\colon A \to H_\pm.$$

Proposition 21. *The graph of every Hilbert-Schmidt operator $T\colon W \to W^\perp$ with $W \in Gr(H)$ is an element of $Gr(H)$.*

Proof. We have to prove that $pr_+\colon (W, T(W)) \to H_+$ is a Fredholm operator and $pr_-\colon (W, T(W)) \to H_-$ is a H-S operator. Notice that $pr_+\colon W \to H_+$ is Fredholm as $W \in Gr(H)$ by definition. The composition of the bounded operator $(pr_+)_{T(W)}$ and the H-S operator T, which is bounded, is a H-S operator. We define $V := W/\,\text{kern}(T)$ and it follows that $T\mid_V\colon V \to T(W)$ is bijective and a H-S operator, $T\mid_V^{-1}$ is bounded, and that

$$(pr_+)_{T(W)} = (pr_+)_{T(W)} \circ T\mid_V \circ T\mid_V^{-1}$$

is a H-S operator and therefore a compact operator. We write

$$(pr_+)_{(W,T(W))} = (pr_+)_W \, + \, (pr_+)_{T(W)} \colon (W, T(W)) \to H_+.$$

As the sum of a Fredholm operator and a compact operator is a Fredholm operator, we get that $(pr_+)_{(W,T(W))}$ is a Fredholm operator.

The projection $(pr_-)_{(W,T(W))}$ is the sum of $(pr_-)_W$ and $(pr_-)_{T(W)}$, i.e.

$$(pr_-)_{(W,T(W))} = (pr_-)_W \, + \, (pr_-)_{T(W)}.$$

Both summands are H-S operators since

$$W \in Gr(H) \quad \text{implies} \quad (pr_-)_W \text{ is a H-S operator}$$

and as $(pr_-)_{T(W)}$ is bounded, $T\mid_V$ is a bijective H-S operator and $T\mid_V^{-1}$ is bounded implies that

$$(pr_-)_{T(W)} = (pr_-)_{T(W)} \circ T\mid_V \circ T\mid_V^{-1}$$

is a H-S operator. So we can conclude that $(pr_-)_{(W,T(W))}$ is a H-S operator as the finite sum of H-S operators is a H-S operator. □

Definition 24. *Define the set U_W by*

$$U_W := \{W' \in Gr(H) \mid \text{ there is an orthogonal projection } \pi_W \colon W' \to W \text{ that is an isomorphism}\}$$

It is clear that $U_W \subset Gr(H)$. We define now another subset in $Gr(H)$.

Definition 25.

$$\widetilde{U}_W := \{(W, T(W)) \in Gr(H) \mid T \colon W \to W^\perp \text{ is a H-S operator}\}$$

Proposition 22. *The set U_W is equal to the set $\widetilde{U}_W$.*

Proof. First we want to show that $\widetilde{U}_W \subset U_W$. Let $(W, T(W)) \in \widetilde{U}_W$ for some H-S operator T. We define the projection $(\pi_W) \colon (W, T(W)) \to W$ by

$$\pi_W = \mathrm{Id}\,|_W \, + \, (\pi_W)|_{T(W)}, \quad (\pi_W)|_{T(W)} \colon T(W) \to W.$$

Since $T(W) \subset W^\perp$, the operator $(\pi_W)|_{T(W)}$ is just the zero operator. We conclude that the operator (π_W) is surjective with the image W.

It is injective since $\mathrm{kern}(\pi_W) = \{(0,0)\} \in (W, T(W))$.

Now we show that $U_W \subset \widetilde{U}_W$. Let $W' \in U_W$. We need to find a H-S operator $T \colon W \to W^\perp$ such that $W' = (W, T(W))$.

It follows that

$$W' = (\pi_W)\mid_{W'} (W') \oplus (\pi_{W^\perp})\mid_{W'} (W')$$

and $(\pi_W)\mid_{W'}$ is an isomorphism. We obtain $W' = (W, (\pi_{W^\perp})\mid_{W'} (W'))$. With the fact that $(\pi_W)\mid_{W'}$ is an isomorphism, we can conclude that

$$(\pi_{W^\perp})\mid_{W'} (W') = (\pi_{W^\perp})\mid_{W'} \circ (\pi_W)\mid_{W'}^{-1} (W),$$

where $(\pi_W)\mid_{W'}^{-1}$ is the inverse of $(\pi_W)\mid_{W'}$. If we define the bounded operator $T\colon W \to W^{\perp}$ by

$$T := (\pi_{W^{\perp}})\mid_{W'} \circ (\pi_W)\mid_{W'}^{-1},$$

then $W' = (W, T(W))$.

If we are able to show that T is a H-S operator, then we are done. As $W' \in Gr(H)$, we can conclude that $(pr_+)\mid_{(W,T(W))} = (pr_+)\mid_{W'}$ is a Fredholm operator and $(pr_-)\mid_{(W,T(W))} = (pr_-)\mid_{W'}$ is a H-S operator. It is well known that

$$(pr_+)\mid_{(W,T(W))} = (pr_+)\mid_W + (pr_+ \circ T)\mid_W,$$

$$(pr_-)\mid_{(W,T(W))} = (pr_-)\mid_W + (pr_- \circ T)\mid_W .$$

We see that

$$(pr_- \circ T)\mid_W = (pr_-)\mid_{(W,T(W))} - (pr_-)\mid_W$$

is a H-S operator, as $(pr_-)\mid_{(W,T(W))}$ and $(pr_-)\mid_W$ are H-S operators as $(W, T(W)) = W'$ and W are elements of $Gr(H)$. We know from Lemma 4 that $(pr_+)\mid_{W^{\perp}}$ is a H-S operator and so is $(pr_+)\mid_{W^{\perp}} \circ T = (pr_+ \circ T)\mid_W$, as T is a bounded operator. So T is a H-S operator as it is the sum of two H-S operators,

$$(pr_+ \circ T)\mid_W + (pr_- \circ T)\mid_W = T.$$

It follows that for all $W' \in U_W$ there exists a H-S operator $T\colon W \to W^{\perp}$ such that $W' = (W, T(W))$ and it gives $U_W \subset \widetilde{U}_W$. □

In the following Corollary we will bring a topology from the space of H-S operators from W to $W^{\perp}$ to U_W by proving the existence of a bijective map between the two spaces.

Corollary 7. *There exists a bijective map from U_W to $HS(W, W^{\perp})$.*

Proof. An element of U_W can be identified with $(W, T(W))$ where $T \in HS(W, W^{\perp})$. We define the map φ_W from U_W to $HS(W, W^{\perp})$ by

$$\varphi_W((W, T(W)) := T.$$

It is injective as if the graphs of two operators from $HS(W, W^{\perp})$ are different, then the two operators have to be different.

The surjectivity is obvious, since we know that the graph of every H-S operator $T\colon W \to W^{\perp}$ is an element of $U_W \subset Gr(H)$. □

Now we can identify every element of $Gr(H)$ with at least one H-S operator which maps an element $W \in Gr(H)$ to $W^{\perp}$. Furthermore, we will see that one can identify locally elements of $Gr(H)$ with elements of $HS(H_+, H_-)$. To prove this, we previously need the following auxiliary statements.

Definition 26. *We define the subset S of $\mathbb{Z}$ with finitely many negative elements and a finite difference from the positive integers by*

$$S := \{s_1, s_2, ..., s_n < 0\} \cup \mathbb{N} \setminus \{i_1, \dot{;} i_m \geq 0\},$$

i.e. the cardinality of $\mathbb{N} \setminus S$ and $S \setminus \mathbb{N}$ is finite. The subset S defines $H_S \in Gr(H)$ and U_S by

$$H_S := \mathrm{span}(\{z_s\}_{s \in S}), \qquad U_S := U_{H_S}.$$

Denote $\mathbb{Z} \setminus S := \overline{S}$. The set of all such $S \subset \mathbb{Z}$ is defined by

$$\mathfrak{S} := \{S \subset \mathbb{Z} \mid\mid S \backslash \mathbb{N} \mid < \infty \wedge \mid \mathbb{N} \backslash S \mid < \infty\}.$$

We define the virtual cardinality of S by

$$\mathrm{virtcard}(S) = \mid S \setminus \mathbb{N} \mid - \mid \mathbb{N} \setminus S \mid .$$

Remark 3. *If we index $S \in \mathfrak{S}$ such that*

$$S = \{s_{-d}, s_{-d+1}, ...\} \quad \textit{with } s_i < s_j \textit{ for } i < j$$

where $d := virtcard(S)$, then there exists $N \in \mathbb{N}$ such that

$$s_n = n \textit{ for all } n \geq N.$$

Proposition 23. *For any W of $Gr(H)$ exists a $S \in \mathfrak{S}$ such that the orthogonal projection $pr_{H_S} \colon W \to H_S$ is an isomorphism. We conclude that*

$$Gr(H) = \bigcup_{S \in \mathfrak{S}} U_S.$$

Proof. We know that the dimension of the kernel and the cokernel of the orthogonal projection $pr_+ \colon W \to H_+$ is finite. Denote them by

$$\dim(\mathrm{kern}(pr_+)) = n \quad \text{and} \quad \dim(\mathrm{cokern}(pr_+)) = m.$$

Suppose that $\mathrm{kern}(pr_+) := \mathrm{span}(e_{m-1}, ..., e_{m-n})$. Then we define $S_0 := \mathbb{N} \cup \{s_{m-1}, ..., s_{m-n} < 0\}$, such that $S_0 \in \mathfrak{S}$ and the kernel of $pr_{H_{S_0}} \colon W \to H_{S_0}$ is trivial, which implies the injectivity $pr_{H_{S_0}}$.

Since $\dim(H_+/\,\mathrm{im}(pr_+)) < \infty$, we write $H_+/\,\mathrm{im}(pr_+) := \mathrm{span}(e_{l_1}, ..., e_{l_m})$ with $l_1, ..., l_m \in \mathbb{N}$. Then we define

$$S_1 := S_0 \setminus \{i_{l_1}, ..., i_{l_m} \geq 0\} \in \mathfrak{S},$$

where $\{s_{m-1}, ..., s_{m-n} < 0\} \subset S_1$. It's obvious that the orthogonal projection $pr_{H_{S_1}} \colon W \to H_{S_1}$ is surjective. Combining this result with the injectivity, proved above, we see that $pr_{H_{S_1}}$ is an isomorphism.

We conclude that for all W in $Gr(H)$ there exists a S in $\mathfrak{S}$ such that the orthogonal projection $pr_{H_S} \colon W \to H_S$ is bijective. That implies that every W in $Gr(H)$ is an element of at least one U_S, and therefore $Gr(H)$ is covered by the union of U_S, $S \in \mathfrak{S}$. ∎

Proposition 24. *The subgroup $U_{res}(H)$ of $GL_{res}(H)$ acts transitively on $Gr(H)$: for any W_0, $W \in Gr(H)$ there is $A \in U_{res}$ such that $A(W_0) = W$.*

The stabilizer of H_+ is $U(H_+) \times U(H_-)$, i.e. we have

$$A(H_+) = H_+ \quad \forall \;\; A \in U(H_+) \times U(H_-).$$

Proof. Suppose $W \in Gr(H)$, and we will construct an element $A \in U_{res}$ such that $A(H_+) = W$. Thus, if $B \in U_{res}$ with $B(H_+) = W_0$, then $AB^{-1} \in U_{res}$ and $AB^{-1}(W_0) = W$.

We need isometries $w \colon H_+ \to W$ and $w^{\perp} \colon H_- \to W^{\perp}$. They can be constructed as follows. Let $\{z_k\}_{k=0}^{\infty}$ be a canonical basis of H_+ and $\{z_k\}_{k=-1}^{-\infty}$ be a canonical basis of H_-. Choose any orthonormal basis $\{w_k\}_{k=0}^{\infty}$ and $\{w_k\}_{k=-1}^{-\infty}$ of W and $W^{\perp}$, respectively. Now we define the isometries w and $w^{\perp}$ on basis by

$$w(z_k) = w_k, \quad k = 0, 1, 2, \dots \qquad w^{\perp}(z_k) = w_k, \quad k = -1, -2, \dots,$$

and continue them by linearity on H_+ and H_-, respectively.

Then we define

$$A := w \oplus w^{\perp} \colon H_+ \,\oplus\, H_- \to H_+ \oplus H_-$$

$$A := \begin{pmatrix} w_+ & w_+^{\perp} \\ w_- & w_-^{\perp} \end{pmatrix}, \qquad \begin{aligned} w_{\pm} &:= \; pr_{\pm} \circ w \\ w_{\pm}^{\perp} &:= \; pr_{\pm}^{\perp} \circ w^{\perp} \end{aligned}$$

with $A(H_+) = W$, as

$$w_+(H_+) \oplus w_-(H_+) = pr_+ \circ w(H_+) \oplus pr_- \circ w(H_+) = w(H_+) = W.$$

A is an unitary bijective transformation since its both components are isometries.

Finally we need to show that $A \in GL_{res}(H)$. We know that $A \in GL_{res}(H)$ if and only if w_- and $w_+^{\perp}$ are H-S operators by Definition 9.

Since $W \in Gr(H)$, the projection $pr_- \colon W \to H_-$ is the H-S operator, that yields that $pr_- \circ w = w_-$ is also the H-S operator as a composition with the bounded operator w. Making use of "mirrored" properties, we summarize the following properties:

- $pr_+ \circ w^{\perp} = w_+^{\perp}$ is the H-S operator and
- $pr_- \circ w^{\perp}$ is the Fredholm operator,

because $pr_+^{\perp}$ is the H-S operator, $pr_-^{\perp}$ is the Fredholm operator and $w^{\perp}$ is bounded and bijective. The conclusion is that $A \in GL_{res}(H)$.

A stabilizer of H_+ has the following properties:

- if $A \in U_{res}(H)$, then $A := \begin{pmatrix} w_+ & w_+^{\perp} \\ w_- & w_-^{\perp} \end{pmatrix}$ with

  $w \colon H_+ \to H, \quad w(H_+) = H_+, \quad w^{\perp} \colon H_- \to H_-, \quad w^{\perp}(H_-) = H_-,$
- $A(H_+) = H_+$.

It follows that $w_+(H_+) = H_+$, $w_- = 0$, $w_+^\perp = 0$ and $w_-^\perp(H_-) = H_-$. Furthermore, $w_+ \in U(H_+)$ and $w_-^\perp \in U(H_-)$. We conclude that the stabilizer of H_+ in $U_{res}(H)$ is $U(H_+) \times U(H_-)$. $\square$

Remark 4. *We should note that the group theory implies that*

$$U_{res}(H)/U(H_+) \times U(H_-) \cong \{A(H_+) \mid A \in U_{res}(H)\} = Gr(H).$$

Now we are able to prove that $Gr(H)$ has locally the structure of a Hilbert space.

Proposition 25. *$Gr(H)$ is a Hilbert manifold modelled on $HS(H_+, H_-)$.*

Proof. First we will see that U_W is an open subset of $Gr(H)$. We constructed the bijective map

$$\varphi_W \colon U_W \to HS(W, W^\perp)$$

from the proof of Corollary 7. The topology $\mathfrak{T}_{HS}$ in $HS(W, W^\perp)$ is given by the norm $\|\cdot\|_{HS}$. We define topology $\mathfrak{T}_{U_W}$ in U_W by

$$\begin{aligned} \mathfrak{T}_{U_W} &:= \{E \subset U_W : \varphi_W(E) \in \mathfrak{T}_{HS}\} \\ &= \{E \subset U_W : E = \varphi_W^{-1}(F) \text{ with } F \in \mathfrak{T}_{HS}\}. \end{aligned}$$

Then, by definition, φ_W became a continuous map.

Moreover we claim that $\varphi_W^{-1} : HS(W, W^\perp) \to U_W$ is also continuous. Since for any $E \subset U_W$ the preimage $(\varphi_W^{-1})^{-1}(E)$ of E is equal to $\varphi_W(E)$ and we know that $\varphi_W(E) \in \mathfrak{T}_{HS}$, it follows that φ_W^{-1} is continuous.

Thus, we get a continuous, bijective and continuous invertible map

$$\hat{\varphi}_W \colon U_W \to HS(H_+, H_-),$$

$$\hat{\varphi}_W((W, T(W))) = (w^\perp)^{-1} \circ T \circ w.$$

As $Gr(H) = \bigcup_{S \in \mathfrak{S}} U_S$ we have an atlas of $Gr(H)$ by $\{(U_S, \varphi_S)\}$.

Finally, we just have to show that the change of coordinates of open subsets is smooth. Consider the intersection $U_{W_0} \cap U_{W_1} \subset Gr(H)$, which corresponds to $I_{01} \subset HS(W_0, W_0^\perp)$ and $I_{10} \subset HS(W_1, W_1^\perp)$, i.e. $\varphi_{W_0}(U_{W_0} \cap U_{W_1}) = I_{01}$ and $\varphi_{W_1}(U_{W_0} \cap U_{W_1}) = I_{10}$. Consider the identity transformation $\mathrm{Id} \colon H \to H$. As $W_0 \oplus W_0^\perp = H = W_1 \oplus W_1^\perp$, we can write it as a change of coordinates of H by

$$\mathrm{Id} \colon W_0 \oplus W_0^\perp \to W_1 \oplus W_1^\perp$$

$$\mathrm{Id} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \begin{matrix} a \colon W_0 \to W_1, & b \colon W_0^\perp \to W_1 \\ c \colon W_0 \to W_1^\perp & d \colon W_0^\perp \to W_1^\perp. \end{matrix}$$

More precisely, $a = pr_{W_1} \mid_{W_0}$, $b = pr_{W_1} \mid_{W_0^\perp}$, $c = pr_{W_1^\perp} \mid_{W_0}$ and $d = pr_{W_1^\perp} \mid_{W_0^\perp}$. We have to show that b and c are H-S operators.

We know that $pr_{W_1}\mid_{W_0^\perp} = pr_{W_1}\mid_{W_0^\perp}(pr_+\mid_{W_0^\perp} + pr_-\mid_{W_0^\perp})$. Lemma 4 shows that $pr_+\mid_{W_0^\perp}$ is a H-S operator and so $pr_{W_1}\circ pr_+\mid_{W_0^\perp}$, as pr_{W_1} is bounded. We finally have to show that $pr_{W_1}\circ pr_-\mid_{W_0^\perp}$ is a H-S operator. Since $pr_-\mid_{W_1}$ is a H-S operator as $W_1\in Gr(H)$ the operator $pr_-\mid_{W_1}\circ pr_{W_1}$ is H-S as pr_{W_1} is bounded. Furthermore, we know that $(pr_{W_1}\circ pr_-\mid_{W_0^\perp})^* = pr_-\mid_{W_1}\circ pr_{W_1}$ as for arbitraries $x\in W_0^\perp$ and $y\in W_1$ we have

$$\langle pr_{W_1}\circ pr_-(x); y\rangle = \langle pr_-(x); pr_{W_1}(y)\rangle = \langle x; pr_-\circ pr_{W_1}(y)\rangle.$$

We conclude that $pr_{W_1}\circ pr_-\mid_{W_0^\perp}$ is a H-S operator and so $pr_{W_1}\mid_{W_0^\perp}$. Analogously one can show that $pr_{W_1^\perp}\mid_{W_0}$ is a H-S operator. We conclude that b and c are H-S operators and so that a and d are Fredholm operators .

Choose a point $W'\in U_{W_0}\cap U_{W_1}$. Then

$$(W_0, T_0(W_0)) = W' = (W_1, T_1(W_1))$$

where $T_0\in I_{01}$ and $T_1\in I_{10}$. We get the following identities

$$a(W_0)\oplus b(T_0(W_0)) = W_1,$$

$$c(W_0)\oplus d(T_0(W_0)) = T_1(W_1),$$

since

$$\begin{pmatrix} W_1 \\ T_1(W_1)\end{pmatrix} = W' = AW' = A\begin{pmatrix} W_0 \\ T_0(W_0)\end{pmatrix} = \begin{pmatrix} a(W_0)\oplus b(T_0(W_0)) \\ c(W_0)\oplus d(T_0(W_0))\end{pmatrix}.$$

Furthermore, there exists an isomorphism $q\colon W_0\to W_1$ such that

$$\begin{pmatrix} a+bT_0 \\ c+dT_0\end{pmatrix} = \begin{pmatrix} q \\ T_1 q\end{pmatrix} : W_0\to W'.$$

We know that the images of both operators coincide, the dimension of W_0 and W_1 are equal and that the operators are injective. So q is just a permutation of the basis elements such that both operators coincide as operators. So there exists a change of coordinates which is defined by $\phi\colon I_{01}\to I_{10}$

$$\phi(T_0) = (c+dT_0)(a+bT_0)^{-1} = T_1$$

in a set where $(a+bT_0)^{-1}$ exists. But we know that $a+bT_0 = q$, so we conclude that

$$I_{01} = \{T_0\in HS(W_0, W_0^\perp)\mid a+bT_0 \text{ is invertible}\}$$

is an open set. This set is not empty since for every Fredholm operator a the operator $a+bT_0$ is also invertible, since the operator bT_0 is compact.

This change of coordinates is holomorphic in the sense of the definition 59 given in the appendix. We get a smooth change of coordinates by ϕ, and since $HS(H_+, H_-)$ is a Hilbert space, the Grassmannian $Gr(H)$ is the manifold modelled on a Hilbert space $HS(H_+, H_-)$. □

Definition 27. *A subset of H is* **commensurable with** H_+ *if and only if*

$$\dim(H_+/W \cap H_+) < \infty \quad \textit{and} \quad \dim(W/W \cap H_+) < \infty.$$

The **virtual dimension of W relative to** H_+ *is defined as*

$$\operatorname{virtdim}(W)_{H_+} := \dim(W/W \cap H_+) - \dim(H_+/W \cap H_+).$$

The commensurable elements are dense in $Gr(H)$.

Lemma 5. $\overline{\{W \subset H \mid W \textit{ commensurable with } H_+\}} = Gr(H)$

Proof. We can assert that W is commensurable with H_+ if and only if $W = (H_+, T(H_+))$ where T is a finite rank operator. Indeed

$$\dim(W^\perp \cap H_+) < \infty \quad \Longleftrightarrow \quad \dim(H_+/W \cap H_+) < \infty$$

$$\dim(W \cap H_-) < \infty \wedge \dim(T(H_+)) < \infty \quad \Longleftrightarrow \quad \dim(W/W \cap H_+) < \infty.$$

The closure of commensurable subsets of H is equal to $Gr(H)$, since the finite rank operators are dense in the space of H-S operators and since $Gr(H) = \bigcup_{S \in \mathfrak{S}} U_S$. □

Corollary 8. *$W \subset H$ is commensurable if and only if it is the graph of a H-S operator $T\colon H_S \to H_S^\perp$ of finite rank.*

Now we will define a similar notion of the dimension for $Gr(H)$.

Definition 28. *The* **virtual dimension of** $W \in Gr(H)$ *is defined by the Fredholm index of pr_+:*

$$\operatorname{virtdim}(W) := \dim(\operatorname{kern}(pr_+)) - \dim(\operatorname{cokern}(pr_+)).$$

Corollary 9. *The virtual dimension of H_S is*

$$\operatorname{virtdim}(H_S) = \mid S \setminus \mathbb{N} \mid - \mid \mathbb{N} \setminus S \mid .$$

Now we can realize that the elements of the same virtual dimension form a connected set and that the union of all spaces with the same virtual dimension separates $Gr(H)$ into disconnected pieces.

Lemma 6. *The virtual dimension of all $W \in U_S$ is equal to the virtual dimension of H_S.*

Proof. Suppose that the virtual dimension of H_S is d and

$$\operatorname{kern}((pr_+)_{H_S}) = \operatorname{span}(z_{i_1}, ..., z_{i_n}), \quad \operatorname{cokern}((pr_+)_{H_S}) = \operatorname{span}(z_{i_{n+1}}, ..., z_{i_{n+m}})$$

which implies $d = n - m$. We consider $W = (H_S, T(H_S)) \in U_{H_S}$. The virtual dimension of W is

$$\begin{aligned} \operatorname{virtdim}(W) &= \dim(\operatorname{kern}((pr_+)_{H_S} \oplus (pr_+)_{T(H_S)} \circ T)) \\ &- \dim(\operatorname{cokern}((pr_+)_{H_S} \oplus (pr_+)_{T(H_S)} \circ T)), \end{aligned}$$

where we used the notation $(pr_\pm)_A \colon A \to H_\pm$. We have to make a case-by-case analysis of the change of dimension between the (co)kernel of $(pr_+)_{H_S}$ and the (co)kernel of $(pr_+)_{H_S} \oplus (pr_+)_{T(H_S)} \circ T$. We fix the notation

$$(pr_+)_{H_S}(z_s) := e_s \text{ and } (pr_+)_{T(H_S)} \circ T(z_s) := e_s^\perp.$$

As $T(z_s) \in H_S^\perp$, it follows that $e_s + e_s^\perp \neq 0$, if $e_s \neq 0$ or $e_s^\perp \neq 0$.

Case 1. Suppose $z_s \notin \mathrm{kern}((pr_+)_{H_S})$ and $z_s \notin \mathrm{kern}((pr_+)_{T(H_S)} \circ T)$. It follows that

$$(pr_+)_{H_S}(z_s) + (pr_+)_{T(H_S)} \circ T(z_s) = e_s + e_s^\perp \neq 0.$$

We conclude that the dimension of the kernel and the cokernel of $(pr_+)_{H_S} \oplus (pr_+)_{T(H_S)} \circ T$ is equal to the dimension of the kernel and the cokernel of $(pr_+)_{H_S}$. So the virtual dimension of W and H_S are the same.

Case 2. Suppose $z_s \in \mathrm{kern}((pr_+)_{H_S})$ and $z_s \notin \mathrm{kern}((pr_+)_{T(H_S)} \circ T)$. It follows that

$$(pr_+)_{H_S}(z_s) + (pr_+)_{T(H_S)} \circ T(z_s) = 0 + e_s^\perp = e_s^\perp \neq 0.$$

We conclude that the dimension of the kernel of $(pr_+)_{H_S} \oplus (pr_+)_{T(H_S)} \circ T$ is reduced by one compared with the dimension of the kernel of $(pr_+)_{H_S}$, i.e.

$$\dim(\mathrm{kern}((pr_+)_{H_S})) - 1 = \dim(\mathrm{kern}((pr_+)_{H_S} \oplus (pr_+)_{T(H_S)} \circ T)).$$

Furthermore, the dimension of the image of $(pr_+)_{H_S} \oplus (pr_+)_{T(H_S)} \circ T$ grows by one compared with $(pr_+)_{H_S}$, which is equivalent to the fact that the dimension of cokernel of $(pr_+)_{H_S} \oplus (pr_+)_{T(H_S)} \circ T$ decreases by one compared with the dimension of the cokernel of $(pr_+)_{H_S}$, i.e.

$$\dim(\mathrm{cokern}((pr_+)_{H_S})) - 1 = \dim(\mathrm{cokern}((pr_+)_{H_S} \oplus (pr_+)_{T(H_S)} \circ T)).$$

It follows that

$$\begin{aligned} \mathrm{virtdim}(W) &= \dim(\mathrm{kern}((pr_+)_{H_S} \oplus (pr_+)_{T(H_S)} \circ T))) \\ &\quad - \dim(\mathrm{cokern}((pr_+)_{H_S} \oplus (pr_+)_{T(H_S)} \circ T))) \\ &= \dim(\mathrm{kern}((pr_+)_{H_S})) - 1 - (\dim(\mathrm{cokern}((pr_+)_{H_S})) - 1) \\ &= \dim(ker((pr_+)_{H_S})) - \dim(\mathrm{cokern}((pr_+)_{H_S})) \\ &= \mathrm{virtdim}(H_S). \end{aligned}$$

Case 3. Suppose $z_s \notin \mathrm{kern}((pr_+)_{H_S})$ and $z_s \in \mathrm{kern}((pr_+)_{T(H_S)} \circ T)$. It follows that

$$(pr_+)_{H_S}(z_s) + (pr_+)_{T(H_S)} \circ T(z_s) = e_s + 0 = e_s \neq 0.$$

We conclude that the dimension of the kernel and the cokernel of $(pr_+)_{H_S} \oplus (pr_+)_{T(H_S)} \circ T$ will be the same as the dimension of the kernel and the cokernel of $(pr_+)_{H_S}$. So the virtual dimension of W and H_S are equal.

Case 4. Suppose $z_s \in \mathrm{kern}((pr_+)_{H_S})$ and $z_s \in \mathrm{kern}((pr_+)_{T(H_S)} \circ T)$. It follows that

$$(pr_+)_{H_S}(z_s) + (pr_+)_{T(H_S)} \circ T(z_s) = 0 + 0 = 0.$$

Thus the dimension of the kernel and the cokernel of $(pr_+)_{H_S} \oplus (pr_+)_{T(H_S)} \circ T$ does not change compared to the dimension of the kernel and the cokernel of $(pr_+)_{H_S}$. This yields the equality of the virtual dimensions of W and H_S.

We know that the dimension of the kernel of $(pr_+)_{H_S}$ is finite and so we can conclude that the virtual dimension of all elements of U_{H_S} coincides with the virtual dimension of H_S. □

Proposition 26. *The set U_0 is the closure of all graphs of H-S operators T from H_+ to H_-.*

Proof. We know that every graph of $T \in HS(H_+, H_-)$ is an element of $U_{\mathbb{N}}$ and as $H_{\mathbb{N}} = H_+$, we conclude that the virtual dimension of all the graphs of operators $T \in HS(H_+, H_-)$ is zero.

Now we take any $S \in \mathfrak{S}$ with virtual cardinal zero which is not $\mathbb{N}$. Then there exists at least one $s \in S$ such that $s < 0$ and one orthonormal basis element z_s of H_S such that $z_s \in H_S \cap H_-$. We take a graph W_T of an operator $T \in HS(H_+, H_-)$, which has virtual dimension zero, then there exists a basis element $w_k = z_k + z_s$ with $k > 0$, $s \leq 0$. Choose a sequence $a_n \in \mathbb{C}$ converging to zero and construct a sequence of H-S operators such that the sequence

$$(w_k)_n = a_n z_k + z_s$$

converges to z_s. The limit element of $Gr(H)$ has the virtual dimension zero, as we add one dimension to the kernel of pr_+ and add one dimension to the cokernel of pr_+.

After this construction we are able to approximate every space of virtual dimension zero by a sequence of spaces which are given by graphs of H-S operators from H_+ to H_-. We can conclude that U_0 is the closure of the graphs of all operators $T \in HS(H_+, H_-)$. □

Corollary 10. *The set U_S with $S = \{-d, -d+1, ...\}$ is dense in U_d.*

Proposition 27. *The set of elements of $Gr(H)$ with the same virtual dimension*

$$U_d := \{W \in Gr(H) \mid \mathrm{virtdim}(W) = d\}$$

is a connected component of $Gr(H)$ and $Gr(H) = \bigcup_{d\in\mathbb{Z}} U_d$.

Proof. Denote $\mathrm{virtdim}(H_S) = d$. We know that U_S with $S = \{-d, -d+1, ...\}$ is dense in U_d. So it is enough to show that U_S is a connected set and as U_S and $HS(H_S, H_S^\perp)$ are homeomorphic, it is enough to we show that $HS(H_S, H_S^\perp)$ is connected. We claim that $HS(H_S, H_S^\perp)$ is path connected.

Consider two operators $T_0, T_1 \in HS(H_S, H_S^\perp)$ and define the path $p\colon [0,1] \to HS(H_S, H_S^\perp)$ by $t \mapsto (1-t)T_0 + tT_1$ with $p(0) = T_0$ and $p(1) = T_1$. It is clear that

$$\|(1-t)T_0 + tT_1\|_{HS} \leq \|T_0\|_{HS} + \|T_1\|_{HS} < \infty.$$

We conclude that $HS(H_S, H_S^\perp)$ is path connected and so connected.

It is clear that $U_d \cap U_k = \emptyset$ for $d \neq k$, which implies that U_d is a connected component. □

Proposition 28. *If $W \in Gr(H)$ and $A \in GL_{res}(H)$ with*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then $A(W) \in Gr(H)$ and $\operatorname{virtdim}(A(W)) = \operatorname{virtdim}(W) + \operatorname{ind}(a)$.

Proof. We note that $A\colon H_+ \oplus H_- \to H_+ \oplus H_-$. So if we say that A maps W to H, then this means that A from $pr_+(W) \oplus pr_-(W) = W$ to $H_+ \oplus H_- = H$. We have to show that $(pr_+)_{A(W)}\colon A(W) \to H_+$ is a Fredholm operator and that $(pr_-)_{A(W)}\colon A(W) \to H_-$ is a H-S operator. But

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} pr_+ \\ pr_- \end{pmatrix} = \begin{pmatrix} apr_+ + bpr_- \\ cpr_+ + dpr_- \end{pmatrix}.$$

It gives

$$pr_+ \circ A = apr_+ + bpr_-\colon W \to H_+$$

and

$$pr_- \circ A = cpr_+ + dpr_-\colon W \to H_-.$$

We know that a and pr_+ are Fredholm operators and so $a \circ pr_+$ is. Furthermore, b and pr_- are H-S operators and thus we get that $b \circ pr_-$ is a H-S operator and therefore a compact operator, which implies that $pr_+ \circ A$ is Fredholm. We conclude that $(pr_+)_{A(W)}\colon A(W) \to H_+$ is the Fredholm operator, as A^{-1} is bijective and

$$(pr_+)_{A(W)} \circ A \circ A^{-1} = (pr_+)_{A(W)}.$$

Since Fredholm operators are bounded we get that pr_+ and d are bounded operators. As c and pr_- are H-S operators, we conclude that $c \circ pr_+$ and $d \circ pr_-$ are H-S operators and so $(pr_-)_{A(W)} \circ A\colon W \to H_-$ is a H-S operator. This implies that

$$(pr_-)_{A(W)} \circ A \circ A^{-1} = (pr_-)_{A(W)}\colon A(W) \to H_-$$

is a H-S operator as A^{-1} is bounded. Thus $A(W)$ is an element of $Gr(H)$.

We know from Definition 28 of the virtual dimension that

$$\operatorname{virtdim}(A(W)) = \dim(\operatorname{kern}(pr_+)_{A(W)}) - \dim(\operatorname{cokern}(pr_+)_{A(W)}).$$

Since $(pr_+)_{A(W)} = a \circ pr_+ + b \circ pr_- \colon W \to H_+$, where aw_+ is a Fredholm operator and bw_- is a compact operator we have

$$\mathrm{ind}((pr_+)_{A(W)}) = \mathrm{ind}(a \circ pr_+ \,+\, b \circ pr_-) = \mathrm{ind}(a \circ pr_+).$$

Furthermore, we know that

$$\begin{aligned}\mathrm{ind}((pr_+)_{A(W)}) &= \dim(\mathrm{kern}(pr_+)_{A(W)}) - \dim(\mathrm{cokern}(pr_+)_{A(W)}) \\ &= \mathrm{virtdim}(A(W)),\end{aligned}$$

and thus

$$\mathrm{virtdim}(A(W)) = \mathrm{ind}(a \circ pr_+).$$

As the index is a group homomorphism, we get that

$$\mathrm{ind}(a \circ pr_+) = \mathrm{ind}(a) + \mathrm{ind}(pr_+).$$

Taking into account

$$\mathrm{ind}(pr_+) = \dim(\mathrm{kern}(pr_+)_W) - \dim(\mathrm{cokern}(pr_+)_W) = \mathrm{virtdim}(W),$$

we get that

$$\begin{aligned}\mathrm{virtdim}(A(W)) &= \mathrm{ind}(a \circ pr_+) = \mathrm{ind}(a) + \mathrm{ind}(pr_+) \\ &= \mathrm{ind}(a) + \mathrm{virtdim}(W).\end{aligned}$$

□

5.2. **Dense submanifolds of $Gr(H)$.**

Starting from this subsection we will identify H with $L^2(S^1, \mathbb{C})$ and z_k with

$$\exp(i\theta k) \colon S^1 \to \mathbb{C}, \qquad \exp(i\theta k) = z^k.$$

In this section we want to consider four dense submanifolds of $Gr(H)$. They are important as it is often easier to prove a property for a dense subset and then extend it over the density property to the entire space. Furthermore, they will help us to understand the structure of $Gr(H)$ and so we will receive a better imagination of the Sato Grassmannian. Three of the dense submanifolds will give us interesting information about the functions contained in the elements of the dense submanifolds.

Case 1. First we want to introduce a dense submanifold which consists of elements W of the infinite dimensional Grassmannian $Gr(H)$ and which can be identified with elements of finite dimensional Grassmannians.

Suppose that a closed subset W of H which lies in $Gr(H)$ is such that

$$z^k H_+ \subset W \subset z^{-k} H_+.$$

This W can be written as a subset of the quotient space

$$H_{-k,k} := \; z^{-k} H_+ / z^k H_+.$$

A generic element f of W has the form

$$f = \sum_{j=-k}^{\infty} f_j z^j = \sum_{j=-k}^{k-1} f_j z^j + \sum_{j=k}^{\infty} f_j z^j \tag{10}$$

and it is equivalent to the elements of an equivalence class

$$[g] := \{f \in W \mid \sum_{j=-k}^{k-1} g_j z^j = \sum_{j=-k}^{k-1} f_j z^j\}.$$

As the space of the equivalence classes is of finite dimension (exactly of dimension 2k), W can be considered as a point in

$$Gr(H_{-k,k}) := \bigcup_{n=1}^{2k} Gr_{2k}^n(H_{-k,k}),$$

where $Gr_{2k}^n(H_{-k,k})$ is the finite dimensional Grassmannian of n-dimensional planes of $2k$-dimensional linear spaces $H_{-k,k}$. We notice that a point in $Gr(H_{-k,k})$ is of infinite dimension, although it can be identified with a finite dimensional point.

We define $Gr_0(H) := \bigcup_{k=0}^{\infty} Gr(H_{-k,k})$. So if $W \in Gr_0(H)$, then there exists $k \in \mathbb{N}$ such that $W \in Gr(H_{-k,k})$.

Now we want to describe $Gr_0(H)$ in terms of coordinate charts. Suppose $W \in Gr_0(H) \subset Gr(H)$. It follows that there exists $S \in \mathfrak{S}$ such that $W = (H_S, T(H_S)) \in U_S$. The inclusion $z^k H_+ \subset W \subset z^{-k} H_+$ and the form (10) of f imply that $\{k, k+1, k+2, ...\} \subset S$ and if $s \in S$, then $s \geq -k$. Furthermore, it follows that for all $s \in S$ one have

$$\begin{cases} Tz^s = 0, & \text{if } s \geq k, \\ Tz^s = \sum\limits_{j \in \{-k,-k+1,\dots,k-1\} \setminus S} T_{js} z^j, & \text{if } -k \leq s < k. \end{cases}$$

We conclude that there exist at most k^2 non-vanishing matrix entries in T. Such kind of operators are dense in the space of H-S operators. We can conclude that $Gr_0(H)$ is dense in $Gr(H)$ in L^2-norm.

Case 2. We already studied the dense submanifold $Gr_1(H) := \{W \subset H \mid W \text{ commensurable with } H_+\}$ in Lemma 5. By corollary 8 we know that the elements of this dense submanifold are the graphs of H-S operators from H_S to $H_S^{\perp}$ with finite rank.

Case 3. We will now define $Gr_\omega(H)$ as the real-analytic Grassmannian manifold, which consists of graphs of all H-S operators $T \colon H_S \to H_S^{\perp}$ with matrix entries T_{pq}, $p \in \bar{S}$, $q \in S$, such that $r^{p-q} T_{pq}$ is bounded, i.e. $\|r^{p-q} T_{pq}\|_{\mathbb{C}} < \infty$ for some r with $0 < r < 1$.

The set of H-S operators with a finite number of non-vanishing entries in the matrix is dense subset of set of operators satisfying the condition

$|r^{p-q}T_{pq}|_{\mathbb{C}} < M$ for all p, q. It is also dense in the entire space of H-S operators. This allows us to conclude that the set of operators described in the Case 3 is dense subset of the space of H-S operators and therefore the real-analytic Grassmannian $Gr_\omega(H)$ is dense submanifold in $Gr(H)$.

Why the set is called real-analytic and the idea of the definition will be evident after the next remark. The definition of a real-analytic function on the loop group will already now give us an intuition.

Definition 29. *A function* $f\colon S^1 \to \mathbb{C}$ *is called* **real-analytic** *if and only if* f *can be written as*

$$f(z) = \sum_{k=-\infty}^{\infty} f_k z^k,$$

such that the series converges in some annulus $r \leq \|z\|_{\mathbb{C}} \leq r^{-1}$ *for* $0 < r < 1$*, i.e.* $\|f_k r^{-|k|}\| < \infty$ *for all* $k \in \mathbb{Z}$*.*

Proof. We want to prove that the series $f(z) = \sum_{k=-\infty}^{\infty} f_k z^k$ converges in some annulus $r \leq \|z\| \leq r^{-1}$ if and only if $\|f_k r^{-|k|}\| < \infty$. If the series

$$f(z) = \sum_{k=-\infty}^{\infty} f_k z^k = \sum_{k=0}^{\infty} f_k z^k + \sum_{j=1}^{\infty} f_{-j} z^{-j}$$

converges, the first sum denoted as f_1 and the second sum denoted as f_2 also have to converge. This is equivalent to the fact that there exists $q_1 \in (0,1)$ and $k_1 > 0$ such that for any $k \geq k_1$ we have

$$\sqrt[k]{\mid f_k \mid} \mid z \mid \leq \sqrt[k]{\mid f_k \mid} r^{-1} < q_1 < 1$$

and there exists $q_2 \in (0,1)$ and $j_1 > 0$ such that for any $j \geq j_1$ we have

$$\sqrt[j]{\mid f_{-j} \mid} \mid z^{-1} \mid \leq \sqrt[j]{\mid f_{-j} \mid} r^{-1} < q_2 < 1.$$

We can reformulate the statement asking for the existence of $M > 0$ such that

$$\mid f_{\pm k} \mid r^{-k} \leq M < \infty.$$

Then $\mid f_{\pm k} \mid (\sqrt[k]{2M} r)^{-k} < 1$ and this just leads to the different choice of $r \in (0,1)$. ☐

Case 4. The following dense submanifold $Gr_\infty(H)$ of $Gr(H)$ is called the smooth Grassmannian manifold. It consists of all operators $T \in HS(H_S, H_S^\perp)$ whose entries T_{pq} are "rapidly decreasing", i.e. $\mid p - q \mid^m \ T_{pq}$ is bounded i.e. $\| \mid p - q \mid^m T_{pq}\|_{\mathbb{C}} < \infty$ for all $(p, q) \in \bar{S} \times S$ and for each m.

The arguments of density of $Gr_\infty(H)$ in $Gr(H)$ are the same as in the Case 3. Why the set is called smooth and the idea of the definition will be evident after the next remark.

Remark 5. *In this remark the basis of* $W = (H_S, T(H_S)) \in Gr(H)$ *is defined as a loop group function from* S^1 *to* $\mathbb{C}$ *by*

$$w_q = z^q + \sum_{p \in \bar{S}} T_{pq} z^p.$$

The following statements hold:

1. *If* W *belongs to* $Gr_\infty(H)$*, then every basis element of* W *is smooth in the sense of a loop group function.*
2. *If* W *belongs to* $Gr_\omega(H)$*, then every basis element of* W *is real-analytic in the sense of a loop group function.*
3. *If* W *belongs to* $Gr_0(H)$*, then every basis element of* W *is a trigonometric polynomial in the sense of a loop group function.*

We conclude that a finite linear combination of smooth, real-analytic or trigonometric basis elements is also smooth, real-analytic or is a trigonometric polynomial. Moreover smooth functions, real-analytic functions and trigonometric polynomials are dense in $W \in Gr_\infty(H)$*,* $Gr_\omega(H)$*, and* $Gr_0(H)$*, respectively.*

Proof. (1) We write $\sum_p := \sum_{p \in \bar{S}}$ and

$$w_q = z^q + \sum_p T_{pq} z^p = z^q(1 + \sum_p T_{pq} z^{p-q}),$$

where w_q is the function $w_q(\theta)$ from S^1 to $\mathbb{C}$. Since $z^q = \exp(i\theta q) \in L^2(S^1, \mathbb{C})$ is smooth, it is enough to prove that

$$h(z(\theta)) := 1 + \sum_p T_{pq} z^{p-q}(\theta)$$

is smooth with respect to θ. The m-th derivative of $h(z(\theta))$ with respect of θ is $\sum_p i^m (p-q)^m T_{pq} z^{p-q}(\theta)$, by

$$(z^k(\theta))^{(m)} = (\exp(ik\theta))^{(m)} = i^m k^m \exp(ik\theta) = i^m k^m z^k(\theta).$$

We need to show that $\sum\limits_p i^m (p-q)^m T_{pq} z^{p-q}(\theta)$ is continuous for each m. Consider a convergent sequence $\{\theta_n\}_{n \in \mathbb{N}} \subset S^1$ with $\theta_n \to \theta \in S^1$ if $n \to \infty$. We define $z_n := \exp(i\theta_n)$ and $z := \exp(i\theta)$. We have to show that if

$$\sum_{k=-\infty}^{\infty} \|z_n^k - z^k\|_{\mathbb{C}} \to 0,$$

then

$$\sum_p i^m (p-q)^m T_{pq} z_n^{p-q} \to \sum_p i^m (p-q)^m T_{pq} z^{p-q}$$

in $L^2(S^1,\mathbb{C})$. Since $\|(p-q)^m T_{pq}\|_{\mathbb{C}} \leq M < \infty$ for all $p \in \bar{S}$, $q \in S$, and for each $m \in \mathbb{N}$, and $\|z_n^{p-q} - z^{p-q}\|_{\mathbb{C}} \to 0$ as $n \to \infty$, it follows that

$$\begin{aligned}\|h^{(m)}(z_n) - h^{(m)}(z)\|_{\mathbb{C}} &\leq \frac{1}{2\pi}\sum_p \||p-q|^m T_{pq}\|_{\mathbb{C}}\|z_n^{p-q} - z^{p-q}\|_{\mathbb{C}} \\ &\leq \frac{M}{2\pi}\sum_p \|z_n^{p-q} - z^{p-q}\|_{\mathbb{C}} \to 0\end{aligned}$$

as $n \to \infty$. We showed that all derivatives are continuous functions.

The existence of $\sum_p i^m(p-q)^m T_{pq} z^{p-q}$ follows from the fact that

$$\| \mid p-q \mid^m T_{pq}\|_{\mathbb{C}} \leq C_m < \infty$$

for all p, q and for each $m \geq 0$. We conclude that

$$\|T_{pq}\|_{\mathbb{C}} \leq \mid p-q \mid^{-m-2} C_{m+2}$$

and so

$$\begin{aligned}\sum_p \| \mid p-q \mid^m T_{pq}\|_{\mathbb{C}} &\leq \sum_p \mid p-q \mid^m \mid p-q \mid^{-m-2} C_{m+2} \\ &= C_{m+2}\sum_p \mid p-q \mid^{-2} < \infty.\end{aligned} \tag{11}$$

(2) A basis element w_q of $Gr_\omega(H)$ has the form

$$w_q = z^q(1 + \sum_p T_{pq} z^{p-q}).$$

It is well known that z^q is real-analytic, such that it is enough to prove that $\sum_p T_{pq} z^{p-q}$ is real-analytic, i.e. $\|r^{-|p-q|}T_{pq}\| < \infty$ for some $r \in (0,1)$. But if $W \in Gr_\omega(H)$, then the H-S operator with matrix entries T_{pq} satisfies this condition. We conclude that w_q is real-analytic.

It is clear that w_q exists as $\|r^{-|p-q|}T_{pq}\|_{\mathbb{C}} \leq C$ and so $\|T_{pq}\|_{\mathbb{C}} \leq r^{|p-q|}C$ for some $0 < r < 1$. It follows that

$$\sum_p \|T_{pq}\|_{\mathbb{C}} \leq C \sum_p r^{|p-q|} < \infty.$$

(3) The basis element of $Gr_0(H)$ is the finite sum of complex exponential functions with a factor in front of it. As every trigonometric polynomial has the form

$$f(x) = \sum_{k=-N}^{N} c_k \exp(ikx),$$

we can see that w_q of $W \in Gr_0(H)$ is a trigonometric polynomial.

The finite linear combination of smooth, real-analytic or trigonometric functions is also smooth, real-analytic or trigonometric, respectively. It is obvious that the space of all finite linear combinations of a basis of a separable vector space is dense subset of this vector space. The conclusion is that the smooth, real-analytic or trigonometric functions are dense in any W of $Gr_\infty(H)$, $Gr_\omega(H)$ or $Gr_0(H)$, respectively. □

Example 2. *We will show that the condition of Remark 5* **does not** *characterize $Gr_0(H), Gr_\omega(H)$ and $Gr_\infty(H)$. Consider the graph $W_T \in Gr(H)$ of the H-S operator $T\colon H_+ \to H_-$ defined by*

$$Tz^0 = 0, \qquad Tz^k := \frac{1}{k} z^{-k} \qquad \text{for } k \geq 1.$$

It follows that

$$T_{pq} = \begin{cases} 0 & p \neq -q \\ \frac{1}{q} & p = -q. \end{cases}$$

We claim that $W_T \notin Gr_0(H), Gr_\omega(H), Gr_\infty(H)$, though the trigonometric polynomials, smooth and real-analytic functions are dense in W_T.

- *Since $T_{-qq} = \frac{1}{q} \neq 0$ for all $q \in \mathbb{N} \setminus \{0\}$, the operator T has infinitely many non-zero matrix entries. Therefore $W_T \notin Gr_0(H)$.*
- *Because*

$$\mid -q - q \mid^m T_{-qq} = (2q)^m \frac{1}{q} = 2^m q^{m-1}$$

is not bounded for $m \geq 2$ and $q \to \infty$, it follows that $W_T \notin Gr_\infty(H)$.

- *It is well known from calculus that $r^{-q-q} T_{-qq} = r^{-2q} \frac{1}{q}$ is not bounded if $r \in (0,1)$ and $q \to \infty$. It follows that $W_T \notin Gr_\omega(H)$.*
- *We consider the canonical basis $\{w_q\}_{q \in \mathbb{N}}$ which looks like*

$$w_q = \begin{cases} z^0 & q = 0 \\ z^q + \frac{1}{q} z^{-q} & q \neq 0 \end{cases}.$$

This basis element is smooth as $z^{\pm k}$ is smooth. It is a trigonometric polynomial by definition. It is also real-analytic as $\mid \frac{1}{q} r^{-q} \mid < \infty$ and $\mid r^{-q} \mid < \infty$ for fixed $q \in \mathbb{N} \setminus \{0\}$ and some fixed $r \in (0,1)$. It follows that every finite linear combination of the canonical basis is smooth, real-analytic and trigonometric. We can conclude that the smooth, real-analytic functions and trigonometric polynomials are dense in W_T.

So it follows that the density of smooth, real-analytic functions or trigonometric polynomials does not characterize $Gr_\infty(H)$, $Gr_\omega(H)$ or $Gr_0(H)$, respectively.

Proposition 29. *Every element W of $Gr(H)$ is an element of $Gr_\infty(H)$ if and only if the images of the orthogonal projections*

$$pr_-\colon W \to H_- \quad \text{and} \quad pr_+^\perp\colon W^\perp \to H_+$$

consist of smooth functions, i.e.

$$\textit{if}\ \ f \in im(pr_-), \quad \textit{then}\ \ f\ \textit{is smooth}$$

and

$$\textit{if}\ \ f \in im(pr_+^\perp), \quad \textit{then}\ \ f\ \textit{is smooth.}$$

Proof. First we suppose that $W \in Gr_\infty(H)$. We know that the above projections pr_- and $pr_+^\perp$ are H-S operators by Definition 21, Lemma 4, and that the projection of the smooth normalized basis elements w_q are smooth. We conclude that the sum

$$\sum_q \|pr_-(w_q)\|^2$$

converges and so the infinite sum of smooth functions $pr_-(w_q)$ converges uniformly on S^1. So it is also smooth. As the span of $pr_-(w_q)$ is the entire image of pr_-, we conclude that the image consists of smooth functions. For $pr_+^\perp$ the proof is analogous.

To show the reciprocal statement we suppose that $W \in Gr(H)$ and that the image of the orthogonal projections pr_- and $pr_+^\perp$ consists of smooth functions. We also suppose that W is the graph of a H-S operator $T\colon H_S \to H_S^\perp$. Then

$$T = pr_- \circ T + pr_+ \circ T.$$

The image of $pr_- \circ T$ consists of smooth functions as the image of pr_- does

$$f \in pr_-(W) = pr_-((H_S, T(H_S))) : f \text{ smooth}$$

and so

$$f \in pr_{H_S^\perp}(pr_-(H_S, T(H_S))) = pr_-(T(H_S)) : f \text{ smooth.}$$

The image of $pr_+ \circ T$ is of finite dimension as the intersection of $H_S^\perp \cap H_+$ is of finite dimension and $\text{im}(T) \subset H_S^\perp$. The basis elements of the intersection $H_S^\perp \cap H_+$ are of the form z^q and are smooth as we know. The finite linear combination of them will be smooth and so every element of the intersection is smooth. We conclude that the image of $pr_+ \circ T$ is smooth.

So an element f of the image of T is the sum of two smooth functions $f_+ \in \text{im}(pr_+ \circ T)$ and $f_- \in \text{im}(pr_- \circ T)$ and so f is smooth. We conclude that the image of T consists of smooth functions.

Mention that T maps H_S to the space of smooth functions on the circle. As the graph of T is an element of $Gr(H)$ it is closed by definition. By the closed-graph theorem (see Appendix) we conclude that T is continuous.

We define $f \in H_S$ by

$$f := \sum_{q \in S} f_q z^q \text{ with } f_q \in \mathbb{C}.$$

Then $Tf \in T(H_S) = \mathrm{im}(T)$ is a smooth function from S^1 to $\mathbb{C}$, as it is an element of the image of T. For fixed $f \in H_S$ and for all $x \in S^1$ we write Tf as

$$(Tf)(x) = \sum_p \sum_q T_{pq} f_q z^p(x). \tag{12}$$

The m-th derivative with respect to x exists, is continuous and looks like

$$\frac{d^m}{dx^m}(Tf)(x) = \sum_p \sum_q p^m i^m T_{pq} f_q z^p(x) \in \mathbb{C} \quad \text{for all} \quad f \in H_S. \tag{13}$$

To continue the proof we define a linear continuous functional $T\colon S^1 \to H_S^*$ as the map of T from S^1 to the dual space of H_S defined by

$$(T\cdot)(x) := \sum_p \sum_q (T_{pq}\cdot) z^p(x) \in H_S^* \tag{14}$$

and its m-th derivative

$$\frac{d^m}{dx^m}(T\cdot)(x) = \sum_p \sum_q p^m i^m (T_{pq}\cdot) z^p(x) \in H_S^*. \tag{15}$$

The two functionals in the equations (14), (15) are defined for all $f \in H_S$ by (12) (13). They are also continuous as T is continuous in respect to $f \in H_S$. It follows that $T\colon S^1 \to H_S^*$ is smooth with respect to x.

We claim that the smoothness of $T\colon S^1 \to H_S^*$ is equivalent to the fact that

$$|p|^m [\sum_q \mid T_{pq} \mid^2]^{\frac{1}{2}} \tag{16}$$

is bounded as $p \to \infty$ for each $m \geq 0$, i.e. we consider $\mid p \mid^m [\sum_q \mid T_{pq} \mid^2]^{\frac{1}{2}}$ as a sequence over p such that for each $m \geq 0$ there exists a constant $C(m) \in \mathbb{R}$ depending on m such that

$$\mid p \mid^m [\sum_q \mid T_{pq} \mid^2]^{\frac{1}{2}} \leq C(m).$$

To show the statement we suppose that for each $m \geq 0$ it exists a constant $C(m) \in \mathbb{R}$ depending only on m such that $\mid p \mid^m [\sum_q \mid T_{pq} \mid^2]^{\frac{1}{2}} \leq C(m)$. As T is continuous with respect to f it is enough to show that (15) exists for $m \geq 0$. It follows that

$$\Big(\sum_q \mid T_{pq} \mid^2\Big)^{\frac{1}{2}} \leq C(m+2) \mid p \mid^{-m-2}.$$

We conclude

$$\begin{aligned}\sum_p \mid p\mid^m [\sum_q \mid T_{pq}\mid^2]^{\frac{1}{2}} &\leq \sum_p \mid p\mid^m \mid p\mid^{-m-2} C(m+2) \\ &= C(m+2)\sum_p \mid p\mid^{-2} \leq 2C(m+2) < \infty, \end{aligned} \tag{17}$$

which guarantees the existence of $\sum_p \mid p\mid^m [\sum_q \mid T_{pq}\mid^2]^{\frac{1}{2}}$.

Consider the operator norm of H_S^*, which is

$$\|(T\cdot)(x)\|_{op} = \sup_{\|f\|=1} \mid (Tf)(x)\mid$$

with $f \in H_S \subset L^2(S^1, \mathbb{C})$. We note that $\|f\|^2 = \sum_q \mid f_q\mid^2 = 1$. We get that

$$\begin{aligned}\mid \frac{d^m}{dx^m}(Tf)(x)\mid &= \mid \sum_p \sum_q p^m i^m T_{pq} f_q z^p(x)\mid \\ &= \mid \sum_p p^m i^m z^p(x) \sum_q T_{pq} f_q \mid = \sum_p |p|^m |\sum_q T_{pq} f_q|\end{aligned}$$

for each $m \geq 0$ as $\mid z^p(x)\mid = 1$, $\mid i^m\mid = 1$ and $\{z^p\}_p$ is an orthonormal system and so $\|\sum_p a_p z^p\| = \sum_p \|a_p z^p\|$.

Furthermore, since $\{T_{pq}\}_q, \{f_q\}_q \in l^2$, we get

$$\mid \sum_q T_{pq} f_q\mid \leq [\sum_q \mid T_{pq}\mid^2]^{\frac{1}{2}} [\sum_q \mid f_q\mid^2]^{\frac{1}{2}} = [\sum_q \mid T_{pq}\mid^2]^{\frac{1}{2}}$$

by the Cauchy-Schwarz inequality. From this it follows that

$$\begin{aligned}\mid \frac{d^m}{dx^m}(Tf)(x)\mid &\leq \sum_p \mid p\mid^m \mid \sum_q T_{pq} f_q\mid \\ &\leq \sum_p \mid p\mid^m [\sum_q \mid T_{pq}\mid^2]^{\frac{1}{2}} \leq 2C(m+2) < \infty\end{aligned}$$

for each $m \geq 0$ and we finish the proof in one direction.

To show the reciprocal statement we suppose that $T\colon S^1 \to H_S^*$ is smooth. It implies that $\mid \frac{d^m}{dx^m}(Tf)(x)\mid$ is bounded for each $m \geq 0$. We know that

$$|\frac{d^m}{dx^m}(Tf)(x)| = |\sum_p p^m i^m z^p(x) \sum_q T_{pq} f_q| = \sum_p |p|^m |\sum_q T_{pq} f_q|$$

for all $f \in H_S$. We define $g \in H_S$ by $g(x) := \sum_q \overline{T_{pq}} z^q$ and denote $g_q := \overline{T}_{pq}$. Then

$$| \frac{d^m}{dx^m}(Tg)(x) | = \sum_p | p |^m | \sum_q T_{pq} g_q | = \sum_p | p |^m | \sum_q T_{pq} \overline{T_{pq}} |$$
$$= \sum_p | p |^m | \sum_q | T_{pq} |^2 | = \sum_p | p |^m \sum_q | T_{pq} |^2.$$

We will prove our claim by contradiction. Suppose $| p |^m \, [\sum_q | T_{pq} |^2]^{\frac{1}{2}}$ is unbounded, then $| p |^{2m} \, [\sum_q | T_{pq} |^2]^{\frac{1}{2}}$ is unbounded. So we get that

$$| p |^{2m} \, [\sum_q | T_{pq} |^2] =| p |^{2m} \, [[\sum_q | T_{pq} |^2]^{\frac{1}{2}}]^2 = [| p |^m \, [\sum_q | T_{pq} |^2]^{\frac{1}{2}}]^2$$

is unbounded as $| p |^m \, [\sum_q | T_{pq} |^2]^{\frac{1}{2}}$ is unbounded. It implies that the expression $\sum_p | p |^{2m} \, [\sum_q | T_{pq} |^2]$ is unbounded and so $| \frac{d^{2m}}{dx^{2m}}(Tg)(x) |$ do as

$$| \frac{d^{2m}}{dx^{2m}}(Tg)(x) |= \sum_p | p |^{2m} \sum_q | T_{pq} |^2.$$

But this is a contradiction to the fact that $| \frac{d^m}{dx^m}(Tf)(x) |$ is bounded for all $f \in H_S$ for each $m \geq 0$. The conclusion is that $| p |^m \, [\sum_q | T_{pq} |^2]^{\frac{1}{2}}$ is bounded for $p \to \infty$ for each m. Finally, we can write

$$\| \sum_q T_{pq} \| = |p|^m (\sum_q |T_{pq}|^2)^{\frac{1}{2}} < \infty$$

for $p \to \infty$ for each m.

For $pr_+^{\perp}$ we can use the same arguments as above, because of the mirrored properties of W and $W^{\perp}$ and the same smoothness of the image. It gives

$$\| \sum_p T_{pq} \| = |q|^m [\sum_p |T_{pq}|^2]^{\frac{1}{2}} < \infty \quad \text{as} \quad q \to \infty \quad \text{for each} \quad m.$$

We conclude that $| p - q |^m \, T_{pq}$ is bounded for all $(p, q) \in \bar{S} \times S$ and each m that yields to $W \in Gr_\infty(H)$.

□

Remark 6. *Mention that the last proposition also holds for $Gr_0(H)$ and $Gr_\omega(H)$ if we replace smooth functions by real-analytic functions or trigonometric polynomials. The only change in the proofs is the corresponding definition of smoothness with the certain property. We will not present proofs for $Gr_0(H)$ or $Gr_\omega(H)$ here since they are mostly literally repeats the proof above.*

Remark 7. *The smooth manifold* $Gr_\infty(H)$ *can be identified with the elements of the H-S operators from* H_+ *to* H_- *with the property that* $| p-q |^m T_{pq}$ *is bounded for all* $(p,q) \in (\mathbb{Z}\backslash\mathbb{N})\times\mathbb{N}$ *and each* m*. It gets its own topology defined by the sequence of seminorms*

$$\rho_m := \sup_{p,q} | p-q |^m | T_{pq} |$$

which is well defined since $| p-q |^m T_{pq}$ *is bounded for all* $(p,q) \in (\mathbb{Z}\backslash\mathbb{N})\times\mathbb{N}$ *and each* m*.*

Proposition 30. *Every holomorphic function* $f\colon Gr(H) \to \mathbb{C}$ *is constant on each connected component.*

Proof. As f is continuous it is enough to prove the proposition for the dense subset $Gr_0(H)$. Remember that $Gr_0(H)$ is the union of the finite dimensional Grassmannians $Gr(H_{-n,n})$. These are compact algebraic varieties. We also know from function theory that every holomorphic function is constant on a compact algebraic variety. So we can conclude that f is constant on $Gr(H)$. □

The proof, particularly, shows why it is useful to introduce the dense submanifolds.

5.3. The stratification of $Gr(H)$.

We already saw in Proposition 27 that we can divide $Gr(H)$ into parts of the same virtual dimension. For a more accurate consideration of the Grassmannians we will need a finer stratification of $Gr(H)$, which is the main goal of the present subsection.

Definition 30. *We define a* **generic** *element* $W \in Gr(H)$ *of virtual dimension zero by relations:*

$$W \cap H_- = 0 \quad \textit{and} \quad W \oplus H_- = H.$$

Proposition 31. *The generic elements form a dense open subset* V *of* U_0*.*

Proof. We know by Proposition 26 that the closure of all H-S operators $T\colon H_+ \to H_-$ coincides with the connected set of virtual dimension zero. If we can show that the generic elements can be identified with the H-S operators $T\colon H_+ \to H_-$, then we get that V is a dense subset.

As $W \cap H_- = \emptyset$, every basis element of W has to contain at least one element of H_+. As $W \oplus H_- = H$ we can conclude that without loss of generality every basis element of W contains exactly one element of H_+ and that every basis element z^s of H_+ with $s \geq 0$ is contained in a basis element of W. So we can build up the basis of W canonicly with a H-S operator $T\colon H_+ \to H_-$ by

$$w_s = z^s + \sum_{p<0} T_{ps} z^p.$$

This also implies that V is an open set. □

We define some additional properties of elements of the Grassmannian to be able to construct a finer stratification of it.

Definition 31. *An element f of $H = L^2(S^1, \mathbb{C})$ is of* **finite order s** *with $s \in \mathbb{Z}$, if it is of the form*

$$f = \sum_{k=-\infty}^{s} f_k z^k$$

with $f_s \neq 0$ and $f_k \in \mathbb{C}$ for any $k \leq s$.

We should note that an element f of finite order s is holomorphic in the hemisphere $\infty >| z |> 1$, if $s > 0$. If $s \leq 0$, it is holomorphic for all z with $| z |> 1$.

Definition 32. *Let W^{fin} be defined as the set of elements f of $W \in Gr(H)$ with finite order, i.e.*

$$W^{fin} := \{f \in W \mid f \text{ is of finite order } s, \text{with } s < \infty\}.$$

Proposition 32. *The set W^{fin} is dense in W.*

Proof. We know that there exists H_S, such that the orthogonal projection $pr_{H_S} \colon W \to H_S$ is an isomorphism, and that the elements of finite order are dense in H_S, because they contain any finite linear combination of the orthonormal basis elements z^s with $s \in S$. These finite linear combinations approximate any element from H_S. Therefore, any element $f \in W$, $f = pr_{H_S}^{-1} g$, $g \in H_S$, can also be approximated by inverse images $pr_{H_S}^{-1}(g_n)$ of g_n, such that $g_n \to g$ as $n \to \infty$. □

It could be helpful to introduce a space W_m containing only elements of W of finite order less or equal than m.

Corollary 11. *The space of all elements of $W \in Gr(H)$ which are of finite order less or equal than $m \in \mathbb{Z}$ is given by $W_m := W \cap z^{m+1} H_-$ and it is of finite dimension.*

Proof. Since the intersection $W \cap H_-$ is of finite dimension for any $W \in Gr(H)$, we conclude

$$\dim(W_m) = \dim(W \cap z^{m+1} H_-) \leq \dim(W \cap H_-) + m < \infty.$$

The fact that only elements of finite order less or equal than m of W are in W_m is obvious since

$$\begin{aligned} f \in W \cap z^{m+1} H_- \quad &\iff \quad f = \sum_{k=-\infty}^{m} f_k z^k \\ &\iff \quad f \text{ is of finite order } \leq m \\ &\iff \quad f \in W_m. \end{aligned}$$

□

Note that W_m collects elements of finite order less or equal m, but it is not able to give us any concrete information about what kind of finite orders exist in W.

Definition 33. *The set S_W consists of all elements $s \in \mathbb{Z}$ such that there exists an element of finite order s in $W \in Gr(H)$, i.e.*

$$S_W := \{s \in \mathbb{Z} \mid \exists \;\; f \in W : f \textit{ is of finite order } s\}.$$

Remark 8. *The set S_W is an element of $\mathfrak{S}$ and its virtual cardinal is the virtual dimension of W.*

Proof. Suppose that $W = (H_S, T(H_S))$ is a graph of virtual dimension d. We consider the basis $\{w_s\}_{s\in S}$ of W defined by

$$w_s := z^s + \sum_{p\in \overline{S}} T_{ps} z^p.$$

We know that there exists a maximum of $\overline{S}$ which we denote by $l_1 := \max_{s\in\overline{S}}(s)$. It follows that the finite order of every basis element w_s lies between s and l_1. We conclude that for all $s > l_1$ the finite order of the basis elements w_s are s and so we get that S_W contains all elements of S greater than l_1.

Furthermore, we know that the set S has a minimum, which we denote by $l_0 := \min_{s\in S}(s)$. So we get that the finite order of w_{l_0} lies between l_0 and l_1. As the finite order of a linear combination of the basis elements can not be smaller than l_0, we conclude that does not exist an element of finite order smaller than l_0, which is equivalent to the fact that S_W is bounded from below and therefore S_W is an element of $\mathfrak{S}$.

Note that S_W can differ from S only on a number lying in between l_0 and l_1.

We can transform the basis $\{w_s\}_{s\in S}$ by multiplying its elements by $\mu_s \in \mathbb{C}$ and combine them linearly such that

$$w_{s_W} = z^{s_W} + \sum_{p\in\overline{S}_W} L_{ps_W} z^p$$

where $(L_{pq})_{\overline{S}_W \times S_W}$ is a H-S operator from H_{S_W} to $H^{\perp}_{S_W}$, such that we get a basis $\{w_{s_W}\}_{s_W\in S_W}$ of W. This implies that $W \in U_{S_W}$ and so

$$\text{virtdim}(W) = \text{virtdim}(H_{S_W}) = \text{virtcard}(S_W). \tag{18}$$

□

The identity (18) allows to express the dimension of W_m in terms of the set S_W.

Corollary 12. *The dimension of W_m is the number of elements of S_W which are smaller or equal m.*

Proof. We know from the proof of Corollary 11 that W_m is of finite dimension and that two elements of different finite order have to be linear independent. So a basis of W_m contains all elements of different order. We consider the basis $\{w_{s_W}\}_{s_W \in S_W}$ of W

$$w_{s_W} = z^{s_W} + \sum_{p \in \overline{S_W}} L_{p s_W} z^p$$

and conclude that a basis of $W \cap z^{m+1} H_-$ is $\{w_{s_W}\}_{s_W \leq m}$. This implies that the dimension of W_m equals the cardinality of $\{s_W \in S_W \mid s_W \leq m\}$. □

Proposition 33. *If the orthogonal projection from W to $z^{m+1}H_+$ is surjective, then the dimension of W_m is equal $m+1+d$, where $d = \mathrm{virtdim}(W)$.*

Proof. The dimension of W_m is the dimension of the intersection between W and H_-, plus the dimension of $pr_+(W) \cap \mathrm{span}\{z^0, z^1, ..., z^m\}$. The surjectivity of the projection from W to $z^{m+1}H_+$ implies that the cokernel of $(pr_+)_W$ lies in $\mathrm{span}\{z^0, z^1, ..., z^m\}$, and we can write the dimension of the last intersection by $m + 1 - \dim(\mathrm{cokern}((pr_+)_W))$. We mention that

$$\begin{aligned} d = \mathrm{virtdim}(W) &= \dim(\mathrm{kern}((pr_+)_W)) - \dim(\mathrm{cokern}((pr_+)_W)) \\ &= \dim(W \cap H_-) - \dim(\mathrm{cokern}((pr_+)_W)). \end{aligned}$$

This implies

$$\dim(W_m) = \dim(W \cap H_-) + m + 1 - \dim(\mathrm{cokern}((pr_+)_W)) = d + m + 1.$$

□

We aim to define a basis of W which consists of elements of finite order.

Definition 34. *A* **canonical basis** *$\{w_s\}_{s \in S_W}$ of W consists of elements $w_s \in W$, which have different finite order s and it is written in the form*

$$w_s = z^s + \sum_{k<s,\ k \in \overline{S_W}} f_k z^k.$$

Now we can define a subset Σ_S of $Gr(H)$ such that all elements W of it are isomorphic to the same element H_S and have the same finite orders.

Definition 35. *The* **stratum** *of $Gr(H)$ with respect to $S \in \mathfrak{S}$ is defined by*

$$\Sigma_S := \{W \in Gr(H) \mid S_W = S\}.$$

We give an equivalent definition of the stratum which helps us to understand the connection of its elements with S.

Definition 36. *The stratum consists of all points $W \in Gr(H)$ such that $\dim(W_m) = d_m(S)$ for all m, where $d_m(S)$ is defined as the number of elements of S which are less or equal than m.*

Proof. The proof of the equivalence is trivial if we remined that $\dim(W_m) = d_m(S_W)$ for all m by Corollary 12.

If W is an element of the stratum, then $S = S_W$ and so

$$d_m(S) = d_m(S_W) = \dim(W_m).$$

If $\dim(W_m) = d_m(S)$ for all m, then $d_m(S) = d_m(S_W)$ for all m as $\dim(W_m) = d_m(S_W)$ for all m. This implies that $S = S_W$, therefore W is an element of the stratum. □

Now we want to introduce an index of S which will give us the possibility to order the elements of $\mathfrak{S}$. It will be helpful to order and to describe elements of $Gr(H)$ with the same properties.

Definition 37. *Every $S \in \mathfrak{S}$ of virtual cardinal d can be indexed by the following way:*

$$S = (s_{-d}, s_{-d+1}, ...)$$

with $s_{-d} < s_{-d+1} < ...$ and $s_k = k$ for all large enough k.

Remark 9. *Let show that the claim "$s_k = k$ for large k" is true. We will prove it by induction. Consider $S = \{s_{-d}, ..., s_{-d+n-1} < 0\} \cup \mathbb{N} \setminus \{i_1, ..., i_m \geq 0\}$ and $d = \mid S \setminus \mathbb{N} \mid - \mid \mathbb{N} \setminus S \mid = n - m$.*

Base cases:
$n = 0$, $m = 0$: $\Rightarrow S = \mathbb{N}$ ✓.
$n = 1$, $m = 0$: $\Rightarrow S = \{s_{-1} < 0\} \cup \mathbb{N} \Rightarrow s_0 = 0,\ s_1 = 1\ ,...$ ✓.
$n = 0$, $m = 1$: $\Rightarrow S = \mathbb{N} \setminus \{i_1 \geq 0\} \Rightarrow s_1 = 1\ ,...$ *if* $i_1 = 0$. *If* $i_1 > 0$, *then* $s_k = k - 1$ *for* $s_k < i_1$ *and* $s_k = k$ *for* $s_k > i_1$ ✓.

Inductive steps:

$n \to n+1$, *m fixed* $\Rightarrow d \to d+1$, $S = \{s_{-d-1}, ..., s_{-d+n-1}\} \cup \mathbb{N} \setminus \{i_1, ..., i_m\}$. *We know that for $S' = \{s_{-d}, ..., s_{-d+n-1}\} \cup \mathbb{N} \setminus \{i_1, ..., i_m\}$ there exists $N \in \mathbb{Z}$ such that $s_k = k$ for all $k \geq N$. As the index of S is the same as the index of S' for $k > -d-1$, we conclude $s_k = k$ for all $k \geq N$.* ✓

n fixed, $m \to m+1 \Rightarrow d \to d-1$, $S = \{s_{-d+1}, ..., s_{-d+n}\} \cup \mathbb{N} \setminus \{i_1, ..., i_{m+1}\}$. *We know that for $S' = \{s_{-d+1}, ..., s_{-d+n}\} \cup \mathbb{N} \setminus \{i_1, ..., i_m\}$ there exists $N \in \mathbb{Z}$ such that $s'_k = k$ for all $k \geq N$. We changed in S compared with S' the index by plus one, such that $s_k = k - 1$ for $s_N \leq s_k < i_{m+1}$. It follows that for $s_k = i_{m+1} + 1 = k$, and so $s_k = k$ for all $s_k \geq i_{m+1}$.* ✓

Now we can order sets of the same virtual cardinal.

Definition 38. *If* $\mathrm{virtcard}(S) = \mathrm{virtcard}(S') = d$ *with $d \in \mathbb{Z}$ and $S, S' \in \mathfrak{S}$, then S is less than S' if and only if the number of elements which are less or equal than m in S are smaller than the number of the same elements in S' for all m. This is equivalent to the condition that s_k is greater or equal* *than* $s_k^{'}$ *for all* $k \in \{-d, -d+1, ...\}$*, i.e.*

$$S \leq S^{'} \iff s_k \geq s_k^{'} \quad \forall\, k \in \{-d, -d+1, ...\}$$
$$\iff d_m(S) \leq d_m(S^{'}) \quad \forall\, m.$$

Proof. First we want to show that

$$s_k \geq s_k^{'} \quad \forall\, k \in \{-d, -d+1, ...\} \quad \Longrightarrow \quad d_m(S) \leq d_m(S^{'}) \quad \forall\, m.$$

Without loss of generality, we suppose that S and $S^{'}$ could be only different in the points s_k and $s_k^{'}$.

If $s_k = s_k^{'}$, then it is obvious that $d_m(S) = d_m(S^{'})$ for all m.

If $s_k > s_k^{'}$, then it follows that $d_m(S) < d_m(S^{'})$ for $s_k^{'} \leq m < s_k$, $d_m(S) = d_m(S^{'})$ for $m < s_k^{'} < s_k$ and $s_k^{'} < s_k \leq m$. This implies that $d_m(S) \leq d_m(S^{'})$ for all m, and this proves the claim.

The other direction is

$$s_k \geq s_k^{'} \quad \forall\, k \in \{-d, -d+1, ...\} \quad \Longleftarrow \quad d_m(S) \leq d_m(S^{'}) \quad \forall\, m.$$

Suppose there exists $k_0 \in \{-d, -d+1, ...\}$ such that $s_{k_0} < s_{k_0}^{'}$. Then it follows that $d_{s_{k_0}}(S) > d_{s_{k_0}}(S^{'})$. This contradicts the fact that $d_m(S) \leq d_m(S^{'})$ for all m. □

Definition 39. *The* **length** $l(S)$ *of* S *is defined by* $l(S) := \sum_{k \geq -d} (k - s_k)$*, where* d *is the virtual cardinal of* S*.*

This allows us to define an "absolut" order of $\mathfrak{S}$.

Definition 40. *An element* S *of* $\mathfrak{S}$ *is "absolutely" less than* $S^{'}$ *if and only if the length of* S *is "absolutely" less than the length of* $S^{'}$*, i.e.*

$$S < S^{'} \iff l(S) < l(S^{'}).$$

Now we need one more definition to understand the following proposition.

Definition 41. *The* **strictly lower triangular** *subgroup* $N_- \subset GL_{res}(H)$ *contains all elements* $A \in GL_{res}(H)$ *such that* $A(z^k H_-) = z^k H_-$ *and* $(A - 1)(z^k H_-) \subset z^{k-1} H_-$ *for all* k*, where* 1 *is defined as the identity operator from* H *to* H*.*

Proposition 34.

(1) *The stratum of* S *is a contractible closed submanifold of the open set* U_S *with codimension* $l(S)$*.*
(2) *The stratum of* S *is the orbit of* H_S *under* N_-*, i.e.*

$$\Sigma_S = \{A(H_S) \mid A \in N_-\}.$$

(3) *If* $W \in U_S$*, then* $S \geq S_W$*.*

(4) *The closure of the stratum of S is the union of the strata of S' with $S' \geq S$, i.e.*

$$\overline{\Sigma}_S = \bigcup_{S' \geq S} \Sigma_{S'}.$$

We should mention that the meaning of the word closed in (1) is different from the meaning of the word closure in (4). In (1) we want to study the closeness of the manifold, in particular a closed space in the space of H-S operators. In (4) we want to study the closure of the stratum in a different way. We will consider elements of the stratum as subspaces of H, i.e. as vector spaces. That means we can study the closure by sequences of orthonormal basis $\{w_s^n\}_{s\in S}$ of W_n of the stratum. That both spaces are different can be seen in the proof, where we will show that in the first case the stratum is closed and in the second case the stratum has a closure which is different from the stratum itself.

Proof. (1) First we have to show that Σ_S is a subset of U_S. A canonical basis $\{w_s\}_{s\in S_W}$ of $W \in \Sigma_S$ with $s \in S_W = S$ was defined as

$$w_s = z^s + \sum_{p<s, p\in \overline{S_W}} T_{ps} z^p.$$

It yields that the orthogonal projection of $\operatorname{span}\{w_s\}_{s\in S_W}$ on $H_{S_W} = H_S$ is an isomorphism. We conclude that $W \in U_{S_W} = U_S$.

For the contractibility property of Σ_S we have to define

$$f\colon U_S \to H_S$$
$$f((H_S, T(H_S))) := H_S$$

and

$$H\colon \Sigma_S \times [0,1] \to \Sigma_S$$
$$H(W,t) := (1-t)W \oplus tf(W).$$

We see that

$$H(W,0) = W$$
$$H(W,1) = f(W) = H_S$$
$$H(W,t) = (1-t)W \oplus tf(W) \in \Sigma_S.$$

We claim $H(W,t) \in \Sigma_S$. Indeed

$$\begin{aligned}(1-t)W \oplus tf(W) &= (1-t)(H_S, T(H_S)) \oplus tH_S \\ &= (1-t)(H_S \oplus T(H_S)) \oplus tH_S \\ &= (1-t)H_S \oplus (1-t)T((1-t)H_S) \oplus tH_S \\ &= H_S \oplus (1-t)T((1-t)H_S) \in \Sigma_S.\end{aligned}$$

This implies that Σ_S is contractible to H_S.

Since the inclusion map $j\colon \Sigma_S \to U_S$ is smooth and has non degenerate differential, the stratum Σ_S gets a manifold structure and it is a submanifold of U_S.

Now we claim that Σ_S has the co-dimension $l(S)$. Suppose $W = (H_S, T(H_S)) \in U_S$ with $T \in HS(H_S, H_S^\perp)$. The orthogonal projection from W to H_S is an isomorphism and there exists an unique canonical basis $\{w_s\}_{s\in S}$, such that the projection of w_s is z^s for all $s \in S$. We mentioned at the beginning of this proof that $W \in \Sigma_S$ if and only if the projection of each canonical basis element has the same finite order as before the projection. Since $W = (H_S, T(H_S))$ is the graph, we can write the basis w_s as $w_s = z^s + Tz^s$. The finite order s of w_s implies that Tz^s has to be of the finite order less than s. This is equivalent to the fact that $T_{pq} = 0$ if $p > q$.

We claim that the number of pairs (p, q) with $p > q$ is $l(S)$. Remind that

$$\mathrm{virtcard}(S) = \mathrm{card}(S \setminus \mathbb{N}) - \mathrm{card}(\mathbb{N} \setminus S) = n - m = d.$$

The number of pairs (p, s_{-d}) such that $p > s_{-d}$ can be count as follows:

- If $s_{-d} < 0$, then we count every element between s_{-d} and 0, minus all negative elements of S which are bigger than s_{-d}, plus all non-negative elements of $\bar{S}$. It gives
$$-s_{-d} - 1 - (n-1) + m = -s_{-d} - n + m$$
- If $s_{-d} \geq 0$, then $n = 0$ and we count all non-negative elements of $\bar{S}$, minus all non-negative elements of $\bar{S}$ which are smaller than s_{-d}. In total it is $m - s_{-d}$.

It follows that the number of pairs (p, s_{-d}) such that $p > s_{-d}$ is

$$-s_{-d} - n + m = -s_{-d} - d.$$

The number of pairs (p, s_{-d+1}) is $-s_{-d+1} - d + 1$ and we conclude that the number of pairs (p, s_{-d+j}) is $-s_{-d+j} - d + j$ for all $j \geq 0$. We get that the length $l(S)$ is equal to the sum of this pairs:

$$l(S) = \sum_{k \geq -d} (k - s_k) = \sum_{j \geq 0} (-d + j - s_{-d+j}).$$

As every entry T_{pq} of the matrix $T = (T_{pq})_{\bar{S}\times S}$ is a basis element of the space $HS(H_S, H_S^\perp)$ and we fixed $l(S)$ of this basis elements equal to zero to get the space Σ_S, it follows that the codimension of Σ_S is $l(S)$.

We already showed that all elements in the stratum can be identified with a matrix T with $T_{pq} = 0$ for $p > q$. Stratum Σ_S is closed since any sequence of matrices $T^{(n)}$ with $T^{(n)}_{pq} = 0$ for $p > q$ converges to a matrix of the same type.

(2) We aim to show that the orbit of H_S under N_- is a subset of the stratum of S. Remind that every $A \in N_- \subset GL_{res}(H)$ is invertible and so it is

injective. So we know that $A(V) \cap A(U) = A(V \cap U)$. Therefore

$$\begin{aligned} A(H_S) \cap z^k H_- &= A(H_S) \cap A(z^k H_-) = A(H_S \cap z^k H_-) \\ &= H_S \cap z^k H_-. \end{aligned}$$

We also know that $H_S \in \Sigma_S$. Together with Definition 36 it gives the equality $d_{k-1}(S) = \dim(H_S \cap z^k H_-)$. It follows that

$$\dim(A(H_S) \cap z^k H_-) = \dim(H_S \cap z^k H_-) = d_{k-1}(S),$$

which implies $A(H_S) \in \Sigma_S$ by Definition 36.

To show that Σ_S is a subset of the orbit we assume that $W = (H_S, T(H_S))$. Furthermore, we define an operator $A \colon H \to H$ by

$$A := 1 + T \circ pr_S,$$

where $pr_{H_S} \colon H \to H_S$ is an orthogonal projection and $1 \colon H \to H$ is the identity operator. The condition $T_{pq} = 0$ if $p > q$ says that the finite order of $T(z^k)$ is less than k. It is obvious that the finite order of the projection is less or equal than the finite order of the original element. This implies

$$(A - 1)(z^k H_-) = T(pr_S(z^k H_-)) \subset z^{k-1} H_-.$$

Furthermore,

$$A(z^k H_-) = z^k H_- \oplus T(pr_S(z^k H_-)) = z^k H_-$$

as $T(pr_S(z^k H_-)) \subset z^{k-1} H_- \subset z^k H_-$. From this it follows that $A \in N_-$ and

$$A(H_S) = H_S \oplus T(H_S) = (H_S, T(H_S)) = W.$$

The stratum of S is a subset of the orbit of H_S under N_-.

(3) If W is an element of U_S, then there exists an orthogonal projection from W to H_S that is an isomorphism. We mentioned in item (2) of this proof that the projection of an element of finite order k is smaller or equal than k. We can conclude that the number of elements $d_m(S_W)$ of finite order $\leq m$ in W is less or equal than the number $d_m(S)$ of elements of finite order $\leq m$ in H_S for all $m \in \mathbb{Z}$, i.e.

$$\dim(W_m) = d_m(S_W) \leq d_m(S) = \dim((H_S)_m).$$

It follows that $S_W \leq S$ by Definition 38.

(4) We start to show that the closure is a subset of the union $\overline{\Sigma}_S = \bigcup_{S' \geq S} \Sigma_{S'}$.

To obtain the closure of the stratum we have to add all limit points of every convergent sequence. These sequences can be identified with sequences of the orthonormal bases and they can only decrease the order of the basis elements. Thus S_W is smaller or equal than S_{W_0} where W_0 is a limit point of a convergent sequence in the stratum.

Now we show that every stratum of S' with $S' > S$ is a subset of the closure of the stratum of S. Notice that there exists at least one $k \geq -d$

such that $s_k > s_k^{'}$ with $s_k \in S$, $s_k^{'} \in S^{'}$. Then we define the space W_t which is spanned by

$$(1-t)z^{s_k} + tz^{s_k^{'}}, \qquad k \geq -d.$$

If $0 \leq t < 1$, then W_t belongs to the stratum of S. If $t = 1$, then $W_t = H_{S'}$ is an element of the stratum of $S^{'}$. This shows that the closure of Σ_S meets $\Sigma_{S'}$ or, in other words, the closure of this orbit meets another orbit. We claim that in this case the closure of the first orbit has to contain the other one.

Proof. Let G be a group and Gx, Gy be orbits of the elements $x \in X$ and $y \in X$. Suppose that $(Gx) \cap \overline{(Gy)} \neq \emptyset$. Then $gx \in (Gx) \cap \overline{(Gy)}$ if there exists a sequence $\{g_n\} \subset G$ such that $g_n y \to gx$ as $n \to \infty$. This is equivalent to the convergence of $g^{-1} g_n y \to x \in \overline{(Gy)}$ as $n \to \infty$. Then

$$\text{for all} \quad g^{'} \in G \quad \text{one has} \quad g^{'} g^{-1} g_n y \to g^{'} x \in \overline{(Gy)}.$$

On the other hand $g^{'} x \in (Gx)$. So we can conclude that $(Gx) \subset \overline{(Gy)}$. □

We proved $\Sigma_{S'} \subset \overline{\Sigma}_S$, that shows that the union of the strata is a subset of the closure.

□

5.4. **The cellular decomposition of** $Gr_0(H)$**.**

Now we will construct the "dual" of the stratification of $Gr(H)$ which we introduced in Subsection 5.3. It is the analogue of the decomposition of the finite dimensional Grassmannians $Gr(H_{-n,n})$ into Schubert cells. As the union of these Grassmannians is $Gr_0(H)$, we can use them to decompose $Gr_0(H)$ into Schubert cells. The name "dual" of the stratification will become clear at the end of this section. We start from the definition of an order, which is quite similar to the order in Subsection 5.3.

Definition 42. *An element $f \in H$ is of* **co-order** *k if and only if it has the form*

$$f = \sum_{j=-N}^{N} f_j z^j$$

with $f_k \neq 0$, $f_j = 0$ for all j with $-N \leq j < k$ and $f_j \in \mathbb{C}$ for $-N \leq j \leq N$.

Now we can define a set analogous to S_W.

Definition 43. *The set of all $s \in \mathbb{Z}$ for which there exists an element of co-order s in $W \in Gr_0(H)$ is defined by S^W, i.e.*

$$S^W := \{s \in \mathbb{Z} \mid W \textit{contains an element of co-order } s\}.$$

Notice that the co-order (order) is a "lower" ("upper") bound of the polynomial f, we put symbol W in the exponent (sub-index) of S.

Proposition 35. $S^W \in \mathfrak{S}$.

Proof. Let $W \in Gr_0(H)$. Write $W = (H_S, T(H_S))$ where $T \in HS(H_S, H_S^\perp)$ with only finitely many non-vanishing matrix entries. We consider the basis $\{w_s\}_{s\in S}$ defined by $w_s := z^s + Tz^s$. Since S has a minimum element and T has finitely many non-vanishing entries, we conclude that every element of $\{w_s\}_{s\in S}$ has a minimum co-order. This implies that S^W is bounded from below. On the other hand, the set S contains all sufficiently large integers. Thus $w_s = z^s$ for large enough s because T_{pq} vanish for big enough indices, that explains why S^W contains all sufficiently large integers. Conclusion is: $S^W \in \mathfrak{S}$. □

Remark 10. *Since every element f of $W \in Gr_0(H)$ has a co-order and two elements of different co-order are linear independent, the orthogonal projection from W to H_{S^W} is an isomorphism. It implies $W \in U_{S^W}$. We can define a basis $\{w_s\}_{s\in S^W}$ of W by $w_s := z^s + \sum_{p\notin S^W} f_p z^p$, where every basis element w_s is of co-order s, i.e. the co-order of the finite sum $\sum_{p\notin S^W} f_p z^p$ is greater than s.*

Now we are ready to define a counterpart of the Schubert cells.

Definition 44. *The Schubert cell with respect to $S \in \mathfrak{S}$ is defined by*

$$C_S = \{W \in Gr_0(H) \mid S^W = S\}.$$

Definition 45. *The* **strictly triangular subgroup** $N_+ \subset GL_{res}(H)$ *consists of all $B \in GL_{res}(H)$ such that $B(z^k H_+) = z^k H_+$ and $(B-1)(z^k H_+) \subset z^{k+1} H_+$ for all k.*

The following proposition is the "dual" of Proposition 34 of the stratification.

Proposition 36.

(1) *C_S is a submanifold of the open set U_S of $Gr(H)$ and it is diffeomorphic to $\mathbb{C}^{l(S)}$.*
(2) *C_S is the orbit of H_S under N_+, i.e. $C_S = \{B(H_S) \mid B \in N_+\}$.*
(3) *If $W \in Gr_0(H)$ and $W \in U_S$, then $S \leq S^W$.*
(4) *The closure of C_S is the union of the $C_{S'}$ with $S' \leq S$, i.e.*

$$\overline{C}_S = \bigcup_{S' \leq S} C_{S'}.$$

(5) *C_S intersects $\Sigma_{S'}$ if and only if $S \geq S'$.*
(6) *C_S intersects Σ_S transversally in the single point H_S, i.e.*

$$C_S \cap \Sigma_S = \{H_S\}.$$

The proof is quite similar to the proof of Proposition 34, which is not very surprising as Definitions 42 and 43 are analogous to Definitions 31 and 36 of the last section.

Proof. (1) Suppose $W \in C_S$. Then $W \in U_S$ by Remark 10. We argue as in Proposition 34 to show that C_S is a submanifold.

If $W \in U_S$, then there exists a basis $\{w_s\}_{s\in S}$ of W with basis elements $w_s = z^s + Tz^s$. We claim that W is an element of the Schubert cell of S if and only if all w_s are of co-order s.

Proof. Suppose $W \in C_S$, i.e. $S = S^W$. The sum of two elements f_1, f_2 of W with co-order m, n has the co-order m or n, that depends on which of both is smaller. This conclusion can be extended to an arbitrary sum of elements of co-order. Therefore, the co-orders of the basis elements w_s are elements of $S^W = S$. As $w_s = z^s + \sum_{p\in\overline{S}} T_{ps}z^p$ is a basis of W and the co-order of $\sum_{p\in\overline{S}} T_{ps}z^p$ is an element of $\overline{S}$, we conclude that the co-order of w_s is s.

Conversely, suppose that the co-order of w_s is s for all $s \in S$, i.e. $s \in S^W$ for all $s \in S$. Since the sum of two basis elements have the co-order of one of both basis elements, it follows that $S = S^W$. □

We get that $W \in C_S$ if and only if $T_{pq} = 0$ for indices $p < q$, where $T \in HS(H_S; H_{\bar{S}}^{\perp})$. So there are only $l(S)$ entries of $T = (T_{pq})_{\bar{S}\times S}$ with $p > q$. We conclude that the dimension of C_S is $l(S)$ and as $T_{pq} \in \mathbb{C}$, we can conclude that C_S is diffeomorphic to $\mathbb{C}^{l(S)}$.

The closeness of C_S is proved by the same argument as in the proof of Proposition 34.

(2) Our first claim is that C_S is a subset of the orbit. Suppose $W \in C_S$. Write $W = (H_S, T(H_S))$ and observe that $T(pr_{H_S}(z^kH_+)) \subset z^{k+1}H_+$. We define an operator $B := 1+T\circ pr_{H_S}$. It follows that $(B-1)(z^kH_+) \subset z^{k+1}H_+$ and $B(z^kH_+) = z^kH_+$ such that $B \in N_+$. As $B(H_S) = W$, we can conclude that C_S is a subset of the orbit of H_S under N_+.

Now we show that the orbit is a subset of the Schubert cell of S. We know that $B(z^kH_+) = z^kH_+$, $(B-1)(z^kH_+) \subset z^{k+1}H_+$, and $\mathrm{span}\{B(z^s)\}_{s\in S} = B(H_S)$. Suppose $f \in H_S$ of co-order $k \in S$ is written as $f = f_kz^k + \sum_{p>k} f_pz^p$ with $f_k \neq 0$. It follows that $B(f) - f \in z^{k+1}H_+$, i.e. $B(f) - f = \sum_{p>k} g_pz^p$. Thus

$$B(f) = f_kz^k + \sum_{p>k} g_pz^p + \sum_{p>k} f_pz^p.$$

This implies that $B(f)$ is of co-order k and we conclude that $B(H_S)$ is spanned by the basis $\{B(z^s)\}_{s\in S}$, where the basis elements $B(z^s)$ are of co-order s. Therefore, $S^{B(H_S)} = S$, $B(H_S) \in C_S$ and the orbit is a subset of C_S.

(3) The orthogonal projection from W to H_S increases or holds the co-order of an element. Then

$$\mathrm{card}\{s \in S : s \leq m\} \leq \mathrm{card}\{s \in S^W : s \leq m\} \iff S \leq S^W.$$

(4) Let us show that $\overline{C_S} \subset \bigcup_{S' \leq S} C_{S'}$. The limit point of a sequence $\{W_n\}_{n\in\mathbb{N}} \subset C_S$, where W_n is spanned by the orthonormal bases with basis elements

$$(w_s)_n = (a_0)_n z^s + (a_1)_n T z^s$$

having co-order s. These basis elements converge to elements with co-order greater or equal s. This implies that if W is spanned by these basis elements, then it is an element of $C_{S'}$ with $S' \leq S$. We conclude that the closure is an element of the union. The proof in the other direction is a copy of the proof of Proposition 34 under the observation that $s_k' < s_k$.

(5) Suppose $C_S \cap \Sigma_{S'} \neq \emptyset$, i.e. there exists $W \in C_S \cap \Sigma_{S'}$. Then $W \in Gr_0(H)$ with $S^W = S$ and $S_W = S'$. If we write the basis of W in the form of $w_{s_k'} = z^{s_k'} + Tz^{s_k'}$ with the finite order s_k', then it is obvious that T has only finitely many non-vanishing matrix entries, because $W \in Gr_0(H)$ and every basis element $w_{s_k'}$ is of co-order less than s_k'. As the co-order of every basis element $w_{s_k'}$ also has to be an element of S^W, it follows that $S^W = S$ and S' have the same virtual cardinal, and therefore we are able to compare them. We denote the co-order of $w_{s_k'}$ by c_k. Then $c_k \leq s_j'$ for all $j \geq k$. If we suppose that $c_{k-n} > c_k$ with $n > 0$, then $c_k < c_{k-n} \leq s_{j-n}' \leq s_k'$. We define $c_{k-n} := s_k$, if there exists no c_j which is greater than c_{k-n} for $j \leq k$. So we get S^W and $s_k \leq s_k'$ for all k, which implies $S \geq S'$.

Conversely, suppose $s_k \leq s_k'$ for all k, i.e. $S \geq S'$ with virtual cardinal d. We know that there exists $N \in \mathbb{Z}$ such that $s_k = k = s_k'$ for all $k \geq N$. We define the basis $\{w_k\}_{k\in\{-d,-d+1,\ldots\}}$ by $w_k := z^{s_k'} + z^{s_k}$, i.e. $w_k = 2z^{s_k}$ for all $k \geq N$ and $W := \text{span}\{w_k\}_{k\in\{-d,-d+1,\ldots\}} \in Gr(H)$. Furthermore, it is easy to see that $W \in Gr_0(H)$, $W \in C_S$ and $W \in \Sigma_{S'}$ such that $W \in C_S \cap \Sigma_{S'} \neq \emptyset$.

(5) If C_S intersects the stratum of S, then $S^W = S = S_W$. Writing $W = (H_S, T(H_S)) = (H_S, T_0(H_S))$ and observing that $T_{pq} = 0$ for all $p < q$ and $(T_0)_{pq} = 0$ for all $p > q$, we get $T = 0 = T_0$. We conclude that $W = (H_S, 0) = H_S$. □

Remark 11. *We are now able to understand why the Schubert cells can be called the "dual" of the stratification of $Gr(H)$.*

1. *The same set $\mathfrak{S}$ indexes the cells $\{C_S\}$ and the strata $\{\Sigma_S\}$.*
2. *The dimension of C_S is the co-dimension of Σ_S.*
3. *C_S meets Σ_S transversally in a single point and meets no other stratum of the same codimension.*

At the end of these two sections we observe that every W with its H-S operator T can be decomposed in an operator T^{up} and T_{down} such that T^{up} generates a $W^{up} \in C_S$ and T_{down} generates a $W_{down} \in \Sigma_S$.

5.5. **The Plücker embedding.**

In finite dimensional Grassmannians it is common to use Plücker coordinates to describe the embedding of the Grassmannian into a bigger space. In this section we will introduce the Plücker coordinates for the infinite dimensional Grassmannian. We need a new type of basis, which we will call the admissible basis.

Definition 46. *Suppose that the virtual dimension of $W \in Gr(H)$ is d and that $pr_{z^{-d}H_+} : W \to z^{-d}H_+$ is the orthogonal projection from W to $z^{-d}H_+$. A sequence $\{w_k\}_{k\geq -d} \subset W$ is called an* **admissible basis** *for $W \in Gr(H)$ if and only if*

(1) *the linear map $w \colon z^{-d}H_+ \to W$ defined by $w(z^k) = w_k$ is a continuous isomorphism, and*
(2) *the composition $pr_{z^{-d}H_+} \circ w$ is an operator with a determinant.*

Remark 12.

(1) *In the following, when we mention an admissible basis, we mean the linear map w.*
(2) *The canonical basis for W is admissible. More precisely, the composition $pr_{z^{-d}H_+} \circ w$ differs from the identity by an operator of finite rank.*

Let us prove statement (2). The canonical basis of $W \in U_S$ is given by

$$w_q = z^q + \sum_{p\in\bar{S}} T_{pq} z^p.$$

From $\dim(H_S^\perp \cap z^{-d}H_+) < \infty$ follows that $\dim(\mathrm{im}(T) \cap z^{-d}H_+) < \infty$ since $\mathrm{im}(T) \subset H_S^\perp$. This implies that $pr_{z^{-d}H_+} \circ T$ is of finite rank.

Furthermore, we know that $z^{-d}H_+$ differs from H_S in only finitely many basis elements as $\{-d, -d+1, ...\}$ differs from S just in finitely many points. This allows us to define a permutation operator $p \colon z^{-d}H_+ \to H_S$, which differs from the identity by an operator of finite rank. Since $w = p + Tp$, we see that $pr_{z^{-d}H_+} \circ w$ differs from the identity by an operator of finite rank.

The above defined permutation $p \colon z^{-d}H_+ \to H_S$, $\mathrm{virtcard}(S) = d$, is given by $p(z^k) = z^{s_k}$. This is obviously a linear bounded invertible map. We define $w \colon z^{-d}H_+ \to W$ by $w := (1 + T)p$. Its inverse map is the orthogonal projection pr_{H_S} on H_S composed with p^{-1}, i.e. $w^{-1} := p^{-1} \circ pr_{H_S}$:

$$\begin{aligned} w^{-1}w(z^k) &= (p^{-1} \circ pr_{H_S})((1 + T)p(z^k)) \\ &= (p^{-1} \circ pr_{H_S})(p(z^k) + Tp(z^k)) = p^{-1}(p(z^k)) = z^k = \mathrm{Id}_{z^{-d}H_+}(z^k), \end{aligned}$$

$$ww^{-1}(w_{s_k}) = (p + Tp)(p^{-1} \circ pr_{H_S})(w_{s_k}) = (1 + T)(z^k) = w_{s_k} = \mathrm{Id}_W(w_{s_k}).$$

We conclude that the canonical basis is an admissible basis.

Corollary 13. *Suppose w is an admissible basis of $W \in Gr(H)$ with virtual dimension d. Assume that $S \in \mathfrak{S}$ is of virtual cardinal d and $pr_S\colon W \to H_S$ is an orthogonal projection. Then the composition $pr_S \circ w\colon z^{-d}H_+ \to H_S$ has a determinant.*

Proof. Definition 46 of the admissible basis implies that $pr_{z^{-d}H_+} \circ w$ has a determinant. We can write

$$pr_{z^{-d}H_+} \circ w := \mathrm{Id} + A_0,$$

where A_0 is a trace class operator. The intersection $H_S \cap z^{-d}H_-$ is of finite dimension as S is bounded from below. We conclude that the composition $pr_S \circ pr_{z^{-d}H_-} =: B_0\colon W \to H_S$ is of finite rank and therefore

$$pr_S \circ pr_{z^{-d}H_-} \circ w = B_0 w := B_1$$

is of finite rank. Since $z^{-d}H_+$ differs from H_S by finitely many basis elements we know that

$$pr_S \circ pr_{z^{-d}H_+} \circ w = \mathrm{Id} + A_1\colon z^{-d}H_+ \to H_S,$$

has a determinant since A_1 is an operator of trace class. From this it follows that

$$\begin{aligned} pr_S \circ w &= pr_S \circ (pr_{z^{-d}H_+} + pr_{z^{-d}H_-}) \circ w \\ &= pr_S \circ pr_{z^{-d}H_+} \circ w + pr_S \circ pr_{z^{-d}H_-} \circ w \\ &= pr_S(\mathrm{Id} + A_0) + B_1 = \mathrm{Id} + A_1 + B_1. \end{aligned}$$

Since w is an isomorphism and $A_1 + B_1$ is a trace class operator, the operator $pr_S \circ w$ has a determinant. □

Since the projection of an admissible basis on a "similar" S has a determinant, we are ready to define the Plücker coordinates.

Definition 47. *Suppose $W \in Gr(H)$ has virtual dimension d and w is an admissible basis. The* **Plücker coordinate** *$\pi_S(w)$ of w by S is defined by*

$$\pi_S(w) := \begin{cases} \det(pr_S \circ w) & \text{if } \operatorname{virtcard}(S) = d \\ 0 & \text{if } \operatorname{virtcard}(S) \neq d \end{cases}.$$

Definition 48. *Suppose w_0 and w_1 are admissible basis of $W \in Gr(H)$. We define $\Delta_{w_0w_1} := w_1^{-1}w_0$ as the matrix relating w_0 and w_1 or relation matrix of w_0 and w_1.*

Proposition 37. *Suppose w_0 and w_1 are admissible basis of $W \in Gr(H)$ and $\Delta_{w_0w_1}$ is the matrix relating w_0 and w_1. Then*

$$\pi_S(w_0) = \det(\Delta_{w_0w_1})\pi_S(w_1).$$

Proof. The proposition follows easily from

$$\begin{aligned}\pi_S(w_0) &= \det(pr_S \circ w_0) = \det(pr_S \circ w_1 \circ w_1^{-1} \circ w_0)\\ &= \det(pr_S \circ w_1)\det(w_1^{-1} \circ w_0) = \det(\Delta_{w_0 w_1})\pi_S(w_1).\end{aligned}$$

□

Definition 49. *We define an equivalence relation on $\mathfrak{l}^2(\mathfrak{S})$. We say $a, b \in \mathfrak{l}^2(\mathfrak{S})$ are equivalent and write $a \sim b$ if there exists $\lambda \in \mathbb{C}$ such that $a = \lambda b$. Then the projective space $P(\mathfrak{l}^2(\mathfrak{S}))$ of $\mathfrak{l}^2(\mathfrak{S})$ is defined as the set of equivalence classes on $\mathfrak{l}^2(\mathfrak{S})$.*

Proposition 38. *The Plücker coordinates $\{\pi_S\}_{S\in\mathfrak{S}}$ define a holomorphic embedding*

$$\pi\colon Gr(H) \to P(\mathbb{H})$$

into the projective space of the Hilbert space $\mathbb{H} := \mathfrak{l}^2(\mathfrak{S})$.

Proof. First we check that π is well defined, more precisely we claim that the image under π is an element of $\mathbb{H} := \mathfrak{l}^2(\mathfrak{S})$ and can be calculated by:

$$\| \pi(W) \|_{\mathfrak{l}^2} = \| \pi(w) \|_{\mathfrak{l}^2} = \sum_{S\in\mathfrak{S}} | \pi_S(w) |^2 < \infty,$$

where w is an admissible basis of $W \in Gr(H)$ and w^* is its adjoint operator. We claim that w^*w has a determinant and that

$$\| \pi(W) \|_{\mathfrak{l}^2} = \det(w^*w).$$

To prove this we notice that $H = z^{-d}H_+ \oplus z^{-d}H_-$. We use the following notations in the proof: $pr_\pm\colon W \to z^{-d}H_\pm$ for the orthogonal projection from H to $z^{-d}H_\pm$ and $w_\pm := pr_\pm \circ w\colon z^{-d}H_+ \to z^{-d}H_\pm$. So we can write $w\colon z^{-d}H_+ \to H$ as $w = w_+ + w_-$. Then we get the equation

$$w^*w = w_+^*w_+ + w_-^*w_-.$$

We know from Definition 46 that w_+ has a determinant. Since the adjoint operator of an operator with determinant also has a determinant, we conclude that w_+^* has a determinant and so the product of both $w_+^*w_+$ has a determinant. Furthermore, w_- is a H-S operator as $pr_- |_W$ is a H-S operator. Proposition 3 implies that w_-^* is also a H-S operator and as the product of two H-S operators is an operator of trace class, we conclude that $w_-^*w_-$ is of trace class. It follows that the following operator is of trace class

$$w^*w - \mathrm{Id} = w_+^*w_+ + w_-^*w_- - \mathrm{Id} = (w_+^*w_+ - \mathrm{Id}) + w_-^*w_-,$$

as it is the sum of the two trace class operators $(w_+^*w_+ - \mathrm{Id})$ and $w_-^*w_-$. This implies that w^*w has a determinant.

To prove the equation $\| \pi(W) \|_{\mathfrak{l}^2} = \det(w^*w)$ of the claim, it is enough to prove it for any admissible basis of $W \in Gr(H)$, since any two admissible bases differ by the multiplication by a complex number and all Plücker coordinates differ by the multiplication of the same complex number $\Delta_{w_0 w_1}$. Furthermore, we can prove the equation for a dense subset of $Gr(H)$, as we know that the determinant is a continuous function. We choose the dense subset $Gr_0(H)$ of $Gr(H)$ for our purpose.

The matrix of the map w_+ differs from the identity matrix by only finitely many non-zero matrix entries and the matrix of w_- has finitely many non-zero matrix entries. From this it follows that the sum of both differs from the identity by only finitely many non-vanishing matrix entries. The same is obviously true for w^*. So we get that

$$w^* w = (\mathrm{Id} + f_1)(\mathrm{Id} + f_2) = \mathrm{Id} + f_1 + f_2 + f_1 f_2 = \mathrm{Id} + f_3,$$

where f_1, f_2, f_3 are operators with finitely many non-zero matrix entries such that

$$\mathrm{Id} + f_1 = w^*, \quad \mathrm{Id} + f_2 = w, \quad f_3 := f_1 + f_2 + f_1 f_2.$$

Furthermore, we know that the determinant of $w^* w$ is

$$\det(w^* w) = \prod_{i=1}^{\infty} [1 + \lambda_i (w^* w - \mathrm{Id})],$$

where $\lambda_i \neq 0$ only for finitely many indices i. This implies

$$\det(w^* w) = \prod_{j=1}^{k} [1 + \lambda_{i_j} (w^* w - \mathrm{Id})]$$

for some $k \in \mathbb{N}$. Thus the determinant of an operator coincides with the determinant of the finite submatrix consisting of non-zero entries out of the diagonal. Assume without loss of generality, that $\mathrm{Id}|_{n\times m} + f_1$ is a $n \times m$ matrix and that $\mathrm{Id}|_{m\times n} + f_2$ is a $m \times n$ matrix, such that

$$\det(w^* w) = \det((\mathrm{Id}|_{n\times m} + f_1)(\mathrm{Id}|_{m\times n} + f_2)),$$

where the determinant on the left hand side of the equation is an infinite-dimensional determinant and the determinant on the right hand side of the equation is a finite-dimensional determinant.

With the following proposition our claim follows.

Proposition 39. *If P and Q are $n \times m$ and $m \times n$ matrices, with $n \leq m$, then*

$$\det(PQ) = \sum_{J \in U} \det(P_J) \det(Q_J),$$

where $U := \{J \subset \{1, ..., m\} \mid \mathrm{card}(J) = n\}$ and P_J, Q_J are the corresponding $n \times n$ submatrices of P and Q.

We use this proposition by identifying $(\mathrm{Id}|_{n\times m} + f_1)(\mathrm{Id}|_{m\times n} + f_2)$ with PQ and P_J, Q_J with the corresponding operators $pr_{S_J} \circ (\mathrm{Id}|_{n\times m} + f_1)$ and

$pr_{S_J} \circ (\mathrm{Id}\mid_{n\times m} + f_2)$. Remind the following properties of the determinant

$$\det(A^T) = \det(A), \quad \det(\overline{A}) = \overline{\det(A)} = \overline{\det(A^T)} = \det(A^*),$$

where $A^* = \overline{A^T}$. We use this and the fact that $\pi_S(w) = \det(pr_S \circ w) = \det(pr_S \circ (\mathrm{Id}\mid_{n\times m} + f_1))$ and get

$$\begin{aligned}\det((\mathrm{Id}\mid_{n\times m} + f_1)(\mathrm{Id}\mid_{m\times n} + f_2)) &= \sum_{S\in C} \Big(\det(pr_S \circ (\mathrm{Id}\mid_{n\times m} + f_1)) \\ &\quad \times \det(pr_S \circ (\mathrm{Id}\mid_{n\times m} + f_2))\Big) \\ &= \sum_{S\in C} \overline{\pi_S(w)}\pi_S(w) = \sum_{S\in C} \mid \pi_S(w) \mid^2,\end{aligned}$$

where we define

$$C := \Big\{ k \in \mathbb{Z} \mid \min\{s_0, -d\} \le k \le \min\{i \ge -d \mid s_j = j \quad \forall \ \ j \ge i\} \Big\}.$$

Now we prove that π is an embedding. Let W_T be the graph of an operator $T\colon H_S \to H_S^{\perp} = H_{\overline{S}}$. The canonical basis $\{w_k\}_{k\ge -d}$ for W_T is given by

$$w_k = z^{s_k} + \sum_{p\in\overline{S}} T_{ps_k} z^p.$$

Suppose that S and $S^{'} \in \mathfrak{S}$ have virtual cardinal d, $S \ne S^{'}$ and write $A := S^{'} \setminus S$ and $B := S \setminus S^{'}$. We know from Remark 12 that the composition $pr \circ w$ of the orthogonal projection $pr\colon H \to z^{-d}H_+$ with w differs from the identity by an operator of finite rank. Since $S^{'}$ and S differ from each other by finitely many elements, we conclude that the composition of $pr_{S'}$ and w differs from the identity by an operator of finite rank. The matrix of the operator $pr_{S'} \circ w$ has the form

$$\left(\begin{array}{ccc|c} 1 & \cdots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & 1 & \\ \hline & P & & T_{A\times B} \end{array}\right),$$

where the identity submatrix is a $(S^{'} \cap S) \times (S \cap S^{'})$ matrix and P is a $(S^{'} \setminus S) \times (S \setminus S^{'})$ submatrix of T. This implies that the finite rank operator is the restriction of T on the rows A and the columns B. From this it follows

$$\pi_{S'}(w) = \det(T \mid_{A\times B}).$$

If $S^{'} = S$, then $pr_S(w) = \mathrm{Id}_{H_S}$ and so $\pi_S(w) = \det(\mathrm{Id}_{H_S}) = 1 \ne 0$. Therefore, $\pi(w) \ne 0$ for all $W \in Gr(H)$ and so π is injective. Its continuity is obvious. So we get an embedding in the coordinate patch U_S. $\square$

Proposition 40.

(1) $W \in U_S \Leftrightarrow \pi_S(W) \neq 0$
(2) $W \in \Sigma_S \Leftrightarrow \pi_S(W) \neq 0$ *and* $\pi_{S'}(W) = 0$ *when* $S' < S$
(3) $W \in C_S \Leftrightarrow \pi_S(W) \neq 0$ *and* $\pi_{S'}(W) = 0$ *unless* $S' \leq S$
(4) $W \in Gr_0(H) \Leftrightarrow \pi_S(W) = 0$ *except for finitely many* S
(5) $W \in Gr_\omega(H) \Leftrightarrow r^{-l(S)}\pi_S(W)$ *is bounded for* $S \in \mathfrak{S}$, *for some* $r < 1$
(6) $W \in Gr_\infty(H) \Leftrightarrow l(S)^m \pi_S(W)$ *is bounded for* $S \in \mathfrak{S}$, *for each* m

Proof. We suppose that $W \in Gr(H)$, $T\colon H_S \to H_S^\perp$, $W = graph(T)$, virtcard$(S) = d$ and $w\colon z^{-d}H_+ \to W$ is the canonical basis of W and so an admissible basis of W.

(1) If $W \in U_S$, then the orthogonal projection $pr_S\colon W \to H_S$ is bijective. Since w is bijective by Definition 46, we conclude that $pr_S \circ w$ is bijective and therefore invertible. From this it follows that the determinant of $pr_S \circ w$ is non-zero.

Conversely, if $\pi_S(W) \neq 0$, then $\det(pr_S(w)) \neq 0$ and therefore $pr_S(w)$ is invertible and bijective. Then $pr_S\colon W \to H_S$ is bijective as it is the composition $pr_S \circ w \circ w^{-1}$ of the two bijective operators $pr_S \circ w$ and w^{-1}. This implies $W \in U_S$.

(2) If $W \in \Sigma_S$, then $W \in U_S$ and $\pi_S(W) \neq 0$ by (1) . We know from Proposition 34 that

$$W \in U_S \Rightarrow S \geq S_W$$

and that the negation of this proposition is

$$S < S_W \Rightarrow W \notin U_S.$$

So we get that if $S' < S = S_W$, then $W \notin U_{S'}$ if and only if $\pi_{S'}(W) = 0$. This implies that if $S' < S = S_W$, then $W \notin \Sigma_{S'}$ and $\pi_{S'}(W) = 0$, as if $W \notin U_{S'}$, then $W \notin \Sigma_{S'}$.

Conversely, if $\pi_S(W) \neq 0$, then $W \in U_S$ implies $S \geq S_W$. We know that $W \in U_{S_W}$ and so $\pi_{S_W}(W) \neq 0$. Suppose $S > S_W$, then by assumption, $\pi_{S_W}(W) = 0$, which is a contradiction to $W \in U_{S_W}$. We conclude that $S = S_W$ and so $W \in \Sigma_S$.

(3) We use the same arguments as in item (2) and Proposition 36. We observe $W \in C_S \Rightarrow W \in U_S \Leftrightarrow \pi_{S_W}(W) \neq 0$. By negation of $W \in U_S \Rightarrow S \leq S^W$, which is $S > S^W \Rightarrow W \notin U_S$, we get that if $S' > S = S^W$, then $W \notin U_{S'}$ if and only if $\pi_{S'}(W) = 0$. This implies that if $S' > S = S^W$, then $W \notin C_{S'}$ and $\pi_{S'}(W) = 0$, as if $W \notin U_{S'}$, then $W \notin C_{S'}$.

Conversely, if $\pi_S(W) \neq 0$, then $W \in U_S$ and then $S^W \geq S$. We know that $W \in U_{S^W}$ implies $\pi_{S^W}(W) \neq 0$. Suppose $S^W > S$. It follows that $\pi_{S^W}(W) = 0$, which is a contradiction. We conclude that $S^W = S$ and so $W \in C_S$.

(4) If $W \in Gr_0(H)$, then there exist only finitely many $T_{pq} \neq 0$. Proposition 38 yields that $\pi_S(W)$ is the determinant of a finite $S \setminus S' \times S' \setminus S$ submatrix of T. It follows easily from combinatorial analysis that there exist only finitely many possibilities making the determinant of the submatrix different from zero.

Conversely, suppose $\pi_{S'}(W) = 0$ except for finitely many S'. Assume there exist infinitely many non-vanishing T_{pq}. We know that there exist infinitely many $S' \in \mathfrak{S}$ such that $\operatorname{card}(S' \setminus S) = \operatorname{card}(S \setminus S') = 1$. If we consider the (1×1)-submatrix of T defined as in the proof of Proposition 38, the determinant of this matrix is $T_{s'_k s_j}$, where $(S' \setminus S) = \{s'_k\}$ and $(S \setminus S') = \{s_j\}$. As there exist infinitely many non-vanishing T_{pq}, it follows that there exist infinitely many S' such that $\pi_{S'}(W) \neq 0$, which is a contradiction to our assumptions.

(5) Suppose $W \in Gr_\omega(H)$, i.e. $r^{p-q}T_{pq}$ is bounded for all $(p,q) \in \mathbb{Z} \setminus S \times S$ for some $0 < r < 1$. We remind that for $S' \in \mathfrak{S}$ with $\operatorname{virtcard}(S') = d$

$$\begin{aligned} l(S) - l(S') &= \sum_{k \geq -d} (k - s_k) - \sum_{k \geq -d} (k - s'_k) \\ &= \sum_{k \geq -d} (s'_k - k + k - s_k) = \sum_{s'_k \in A} s'_k - \sum_{s_k \in B} s_k, \end{aligned} \tag{19}$$

where $A = S' \setminus S$ and $B = S \setminus S'$. We know that $\operatorname{card}(A) = n = \operatorname{card}(B)$ and so

$$r^{a_1 - g(1)} T_{a_1 g(1)} \cdot \ldots \cdot r^{a_n - g(n)} T_{a_n g(n)} = r^{l(S) - l(S')} T_{a_1 g(1)} \cdot \ldots \cdot T_{a_n g(n)}, \tag{20}$$

where g is the bijective map from $\{1, ..., n\}$ to B and we write $A = \{a_1, ..., a_n\}$ for the set of indixes. The product (20) is bounded as $r^{a_i - g(i)} T_{a_i g(i)}$ is bounded for all $1 \leq i \leq n$ and the finite product of bounded elements is bounded.

We know that $\pi_{S'}(w) = \det(T_{A \times B})$ and by calculating the determinant of $T_{A \times B}$ we get that $r^{l(S)} r^{-l(S')} \pi_{S'}(w)$ is bounded as it is a finite sum of bounded elements. As $l(S) \geq 0$, $r^{l(S)}$ is different from zero and a bounded constant. We conclude that $r^{-l(S')} \pi_{S'}(w)$ is bounded for all $S' \in \mathfrak{S}$ with virtual cardinal d. For $S' \in \mathfrak{S}$ with virtual cardinal different from d, $\pi_{S'}(w)$ vanishes and so bounded.

Conversely, suppose $r^{-l(S')} \pi_{S'}(w)$ is bounded for some $0 < r < 1$ for all $S' \in \mathfrak{S}$. This implies that $r^{l(S) - l(S')} \pi_{S'}(w)$ is bounded as $l(S) \geq 0$ is constant. Suppose S' is such that $\operatorname{card}(S' \setminus S) = 1$. It follows that $\det(T_{A \times B}) = T_{a_1 b_1}$. This gives $r^{l(S) - l(S')} \pi_{S'}(w) = r^{a_1 - b_1} T_{a_1 b_1}$. Therefore, $r^{p-q} T_{pq}$ is bounded for some $0 < r < 1$ and for all $(p,q) \in \overline{S} \times S$, i.e. $W \in Gr_\omega(H)$.

(6) Suppose $l(S')^m \pi_{S'}(w)$ is bounded for each m and for all $S' \in \mathfrak{S}$. Suppose S' is such that $\text{card}(S' \setminus S) = 1$, $\text{virtcard}(S') = d$, and $S = S_W$. We know that

$$\pi_{S'}(w) \neq 0 \quad \Leftrightarrow \quad W \in U_{S'} \quad \Rightarrow \quad S' \geq S_W = S$$

and so $s_k \geq s'_k$ for all $k \geq -d$, that gives $0 \leq l(S) \leq l(S')$, i.e. $l(S') - l(S) \geq 0$. Suppose that $s_j \in S \setminus S'$ and $s_k \in S' \setminus S$ which implies that

$$l(S') \geq l(S') - l(S) = s_j - s'_k = \mid s_j - s'_k \mid \geq 0$$

and so $l(S')^m \geq \mid s_j - s'_k \mid^m$ for each m. We conclude that

$$\infty > l(S')^m \pi_{S'}(w) \geq \mid s_j - s'_k \mid^m \pi_{S'}(w) = \mid s_j - s'_k \mid^m T_{s'_k s_j}.$$

As we can repeat this for every $s_k \in \overline{S}$ and $s_j \in S$ it follows that $W \in Gr_\infty(H)$.

To show the inverse statement we assume that $W \in Gr_\infty(H)$, i.e. $\mid p - q \mid^m T_{pq}$ is bounded for all $(p,q) \in \overline{S} \times S$ and for each m. As the Plücker embedding is continuous, it's enough to prove it for a dense subset of U_d. We consider the dense subset U_S of U_d where $S = \{-d, -d+1, ...\}$. We know that there exists $c \in \mathbb{R} \setminus \{0\}$ such that $(c \prod_{i=1}^{n} \mid x_i \mid)^m = (\sum_{i=1}^{n} \mid x_i \mid)^m$ for $x_i \neq 0$. Then

$$\begin{aligned} c^m \prod_{j=1}^{n} |k_j - g(j)|^m T_{k_j g(j)} &= (\sum_{j=1}^{n} |k_j - g(j)|)^m \prod_{i=1}^{n} T_{k_i g(i)} \\ &\geq [\sum_{j=1}^{n} (k_j - g(j))]^m \prod_{i=1}^{n} T_{k_i g(i)} = l(S')^m \prod_{i=1}^{n} T_{k_i g(i)}, \end{aligned} \tag{21}$$

where $\{-d, -d+1, ...\} \setminus S' = \{k_1, ..., k_n\}$ and $g\colon \{1, ..., n\} \to S' \setminus \{-d, -d+1, ...\}$ is a bijective map. This is bounded since it is the finite product of bounded elements $\mid k_j - g(j) \mid^m T_{k_j g(j)}$. As

$$\begin{aligned} \sum_{g \in G} \text{sign}(g) c^m \prod_{j=1}^{n} \mid k_j - g(j) \mid^m T_{k_j g(j)} &\geq \sum_{g \in G} \text{sign}(g) l(S')^m \prod_{i=1}^{n} T_{k_i g(i)} \\ &= l(S')^m \sum_{g \in G} \text{sign}(g) \prod_{i=1}^{n} T_{k_i g(i)} \\ &= l(S')^m \pi_{S'}(w), \end{aligned} \tag{22}$$

where G is the set of all bijective maps from $\{1, ..., n\}$ to $S' \setminus \{-d, -d+1, ...\}$ and $\sum_{g \in G} \text{sign}(g) c^m \prod_{j=1}^{n} |k_j - g(j)|^m T_{k_j g(j)}$ is bounded, we get that $l(S')^m \pi_{S'}(w)$ is bounded for all $S' \in \mathfrak{S}$ and for each m. □

5.6. **The $\mathbb{C}^\times_{\leq 1}$-action.**

We will define the action of the circle group $\mathbb{T} := \{z \in \mathbb{C} \mid \|z\|_{\mathbb{C}} = 1\}$ on $H = L^2(S^1, \mathbb{C})$ by the rotation on the circle S^1.

Definition 50. *The* **rotation** *by $\mathbb{T}$ on H is the map $r_u \colon H \to H$ defined by $u := \exp(i\alpha) \in \mathbb{T}$, $\alpha \in \mathbb{R}$,*

$$z = \exp(ix) \mapsto z_u = \exp(-i\alpha)\exp(ix) = \exp(i(-\alpha + x)).$$

Proposition 41. *The rotation r_u preserves the polarization $H = H_+ \oplus H_-$ and its fixed points are the subspaces $H_S \in Gr(H)$, $S \in \mathfrak{S}$.*

Proof. Since $H_+ = \operatorname{span}\{z^k \mid k \geq 0\}$ and $H_- = \operatorname{span}\{z^k \mid k < 0\}$, we see that $r_u(z^k) = z_u^k = \exp(-ik\alpha)z^k$. Then

$$\operatorname{span}\{z_u^k \mid k \geq 0\} = \operatorname{span}\{z^k \mid k \geq 0\} = H_+$$

and

$$\operatorname{span}\{z_u^k \mid k < 0\} = \operatorname{span}\{z^k \mid k < 0\} = H_-$$

as $\exp(-ik\alpha) \in \mathbb{C}$. It follows easily that $H_S = \operatorname{span}\{z^k \mid k \in S\}$ is fixed by r_u according to $z^k \in H_+$ or $z^k \in H_-$ for all $k \in S$. □

Definition 51. *The map $R_u \colon Gr(H) \to Gr(H)$ is defined by*

$$R_u(W) = \operatorname{span}\{r_u(w_s) \mid s \geq -d\},$$

where $\{w_s\}_{s\geq -d}$ is a basis of W with $\operatorname{virtdim}(W) = d$.

By Definition 51 and Proposition 41 one sees that $R_u(W) \in Gr(H)$, since r_u preserves the decomposition $H = H_+ \oplus H_-$.

Corollary 14. *If $W = graph(T)$ is an element of $U_S \cong HS(H_S, H_S^\perp)$, then $R_u(W)$ can be identified with the result of the action $L \colon HS(H_S, H_S^\perp) \to HS(H_S, H_S^\perp)$ on $\mathbb{T}$ defined by*

$$T = (T_{pq})_{\overline{S}\times S} \mapsto T_u = (u^{q-p}T_{pq})_{\overline{S}\times S}.$$

Proof. Let $W = \operatorname{span}\{w_q = z^q + \sum_{p\in\overline{S}} T_{pq}z^p \mid q \in S\}$, where $\{w_q\}_{q\in S}$ is the canonical basis. Then

$$r_u(w_q) = u^{-q}z^q + \sum_{p\in\overline{S}} T_{pq}u^{-p}z^p = u^{-q}(z^q + \sum_{p\in\overline{S}} T_{pq}u^{q-p}z^p),$$

and $\operatorname{span}\{r_u(w_q) \mid q \in S\} = \operatorname{span}\{z^q + \sum_{p\in\overline{S}} T_{pq}u^{q-p}z^p \mid q \in S\}$. We conclude that $R_u(W)$ can be identified with the graph of the operator

$$L(T) = T_u = (u^{q-p}T_{pq})_{\overline{S}\times S}.$$

□

Corollary 15. *The map* $R\colon \mathbb{T} \times Gr(H) \to Gr(H)$ *defined by*

$$R(u, W) := R_u(W),$$

is continuous but not differentiable. More precisely, it is not differentiable in the first variable.

Proof. We identify $W \in Gr(H)$ with $T \in HS(H_S, H_S^{\perp})$ and study the map R defined by

$$(u, T) \mapsto T_u = (u^{q-p} T_{pq})_{\bar{S}\times S} \in HS(H_S, H_S^{\perp}).$$

Suppose that R is differentiable in the first variable u and that $u_h = \exp(i(\alpha + h))$. Then the differential quotient

$$\lim_{h\to 0} \| \frac{(u_h^{q-p} T_{pq})_{\bar{S}\times S} - (u^{q-p} T_{pq})_{\bar{S}\times S}}{h} \|_{HS} = \lim_{h\to 0} \| \frac{((u_h^{q-p} - u^{q-p}) T_{pq})_{\bar{S}\times S}}{h} \|_{HS}$$
$$= \lim_{h\to 0} \| (\frac{(u_h^{q-p} - u^{q-p})}{h} T_{pq})_{\bar{S}\times S} \|_{HS} = \| (i(q-p) u^{q-p} T_{pq})_{\bar{S}\times S} \|_{HS}$$

exists. The above limit has to converge with respect to the H-S norm, i.e.

$$\| (i(q-p) u^{q-p} T_{pq})_{\bar{S}\times S} \|_{HS}^2 = \sum_q (\sum_p \| (q-p) u^{q-p} T_{pq} \|_{\mathbb{C}})^2 < \infty.$$

Suppose $T = (T_{pq})_{(\mathbb{Z}\setminus\mathbb{N})\times\mathbb{N}}$ with $T_{-1q} = \frac{1}{1+q} \neq 0$ and $T_{pq} = 0$ for $p < -1$ with

$$\sum_q (\sum_p \| T_{pq} \|_{\mathbb{C}})^2 = \sum_q \| T_{-1q} \|^2 = \sum_{q\geq 1} \frac{1}{q^2} < \infty.$$

Then the series

$$\| (i(q+1) u^{q-p} T_{pq})_{(\mathbb{Z}\setminus\mathbb{N})\times\mathbb{N}} \|_{HS}^2 = \sum_{q\geq 1} \| u \|^{2q}$$

diverges. This is a contradiction to the existence of the differential quotient for all H-S operators and so we conclude that the map R is not differentiable in the first variable.

We want to prove that $R(u, T)$ is continuous. Suppose that the sequence $\{u_n, T^n\} \subset \mathbb{T} \times HS(H_S, H_S^{\perp})$ converges, i.e. for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that the inequality $n \geq N$ implies

$$\| u_n - u \|_{\mathbb{C}} < \varepsilon \qquad \text{and} \qquad \| T^n - T \|_{HS} < \varepsilon.$$

Furthermore, we know that the existence of the norm $\|T\|_{HS}$ implies existence of $M \in \mathbb{N}$ such that

$$\sum_{q,p: |q-p| > M} \| T_{pq} \|^2 < \varepsilon.$$

We also know that for any $|q-p| \leq M$ we can find $N_{|q-p|} \in \mathbb{N}$ such that for all $n \geq N_{|q-p|}$ we also have $\| u_n^{q-p} - u^{q-p} \| < \varepsilon$. Since $N_0 := \max_{|q-p|\leq M} \{N_{|q-p|}\}$

exists due to $\{N_{|q-p|} \mid\mid q-p \mid\leq M\}$ is finite, we get that for any $n \geq N_0$ the inequality

$$\|u_n^j - u^j\| < \varepsilon \quad \text{for all} \quad 0 \leq j \leq M$$

holds.
We define $N_2 := \max\{N_0, N_1\}$, where N_1 is such that for all $n \geq N_1$ the inequality $\|T^n - T\|_{HS} < \varepsilon$ holds. Then for all $n \geq N_2$ we have

$$\begin{aligned}
&\|R(u_n, T^n) - R(u,T)\|_{HS} = \|R(u_n, T^n) - R(u_n, T) + R(u_n, T) - R(u,T)\|_{HS} \\
&\leq \|R(u_n, T^n) - R(u_n, T)\|_{HS} + \|R(u_n, T) - R(u,T)\|_{HS} \\
&= \Big[\sum_{q,p} \|u_n^{q-p}\|^2 \|T_{pq}^n - T_{pq}\|^2\Big]^{\frac{1}{2}} + \Big[\sum_{q,p} \|(u_n^{q-p} - u^{q-p})T_{pq}\|^2\Big]^{\frac{1}{2}} \\
&= \Big[\sum_{q,p} \|T_{pq}^n - T_{pq}\|^2\Big]^{\frac{1}{2}} \\
&+ \Big[\sum_{q,p:|q-p|>M} \|(u_n^{q-p} - u^{q-p})T_{pq}\|^2 + \sum_{q,p:|q-p|\leq M} \|(u_n^{q-p} - u^{q-p})T_{pq}\|^2\Big]^{\frac{1}{2}} \\
&\leq \|T^n - T\|_{HS} + [2 \sum_{q,p:|q-p|>M} \|T_{pq}\|^2 + \varepsilon^2 \sum_{q,p:|q-p|\leq M} \|T_{pq}\|^2]^{\frac{1}{2}} \\
&\leq \varepsilon + [2\varepsilon + \varepsilon^2 \|T\|_{HS}^2]^{\frac{1}{2}}.
\end{aligned}$$

This implies $\|R(u_n, T^n) - R(u,T)\|_{HS} \to 0$ as $n \to \infty$, that shows the continuity of $R(u,T)$. □

Definition 52. *The* **$\mathbb{T}$-orbit** *of a point $W \in Gr(H)$ is the map $o_W \colon \mathbb{T} \to Gr(H)$ defined by $u \mapsto R_u(W)$.*

Proposition 42.

1. *The $\mathbb{T}$-orbit of W is smooth if and only if $W \in Gr_\infty(H)$.*
2. *The $\mathbb{T}$-orbit of W is real-analytic if and only if $W \in Gr_\omega(H)$.*
3. *$\mathbb{T}$ acts smoothly on the manifold $Gr_\infty(H)$ endowed with its C^∞-topology.*

Proof. (1) Suppose that o_W is smooth. Then the differential operator exists and all its derivatives are continuous. It follows that

$$\|(i^m(q-p)^m u^{q-p} T_{pq})_{\bar{S}\times S}\|_{HS} < \infty.$$

As the H-S-norm is the sum of positive summands

$$\infty > \mid q-p \mid^m \|u\|_{\mathbb{C}}^{q-p} \|T_{pq}\|_{\mathbb{C}} = \mid q-p \mid^m \|T_{pq}\|_{\mathbb{C}},$$

we conclude that every summand is bounded and $\mid q-p \mid^m T_{pq}$ is bounded for all $(p,q) \in \bar{S} \times S$ and for each m. Therefore, $graph(T) = W \in Gr_\infty(H)$.

Conversely, suppose $graph(T) = W \in Gr_\infty(H)$, i.e. $\mid q-p \mid^m T_{pq}$ is bounded for all $(p,q) \in \bar{S}\times S$ and for each m. Then $\|(|q-p|^m\|T_{pq}\|_{\mathbb{C}})_{\bar{S}\times S}\|_{HS}$ is bounded by arguments of Proposition 29. As

$$|q-p|^m\|T_{pq}\|_{\mathbb{C}} = \|i\|_{\mathbb{C}}^m|q-p|^m\|u\|_{\mathbb{C}}^{q-p}\|T_{pq}\ _{\mathbb{C}}$$

it follows that

$$\|(i^m(q-p)^m u^{q-p}T_{pq})_{\bar{S}\times S}\|_{HS} \leq \|(|q-p|^m\|T_{pq}\|_{\mathbb{C}})_{\bar{S}\times S}\|_{HS} < \infty.$$

This implies that the differential quotient exists for each m.

Now we want to show that every derivative is continuous. Suppose that sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{T}$ converges to $u \in \mathbb{T}$, i.e. $\|u_n - u\|_{\mathbb{C}} \to 0$. Remind the third binomial equation $a^n - b^n = (a-b)a^{n-1}\sum_{k=0}^{n-1}(\frac{b}{a})^k$. The group $\mathbb{T}$ is multiplicative, i.e. $\frac{a}{b} \in \mathbb{T}$ and $ab \in \mathbb{T}$ that implies $|\frac{a}{b}| = |ab| = 1$. From these two facts we conclude that

$$\begin{aligned}\|u_n^{q-p} - u^{q-p}\|_{\mathbb{C}} &= \|(u_n - u)u_n^{q-p-1}\sum_{k=0}^{q-p-1}(\frac{u}{u_n})^k\|_{\mathbb{C}} \\ &\leq \|u_n - u\|_{\mathbb{C}}\|u_n^{q-p-1}\|_{\mathbb{C}}\sum_{k=0}^{q-p-1}\|\frac{u}{u_n}\|_{\mathbb{C}}^k \leq \|u_n - u\|_{\mathbb{C}} \mid q-p \mid .\end{aligned}$$

Thus

$$\begin{aligned}&\|(\mid q-p\mid^m u_n^{q-p}T_{pq})_{\bar{S}\times S} - (\mid q-p\mid^m u^{q-p}T_{pq})_{\bar{S}\times S}\|_{HS} \\ =&\|(\mid q-p\mid^m T_{pq}(u_n^{q-p} - u^{q-p}))_{\bar{S}\times S}\|_{HS} \\ \leq&\|(\mid q-p\mid^m T_{pq}(u_n - u)(q-p))_{\bar{S}\times S}\|_{HS} \\ \leq&\|u_n - u\|_{\mathbb{C}}\|(\mid q-p\mid^{m+1} T_{pq})_{\bar{S}\times S}\|_{HS} \to 0\end{aligned}$$

as $n \to \infty$ since $\|(\mid q-p\mid^{m+1} T_{pq})_{\bar{S}\times S}\|_{HS}$ is bounded. We conclude that every derivative exists and continuous, which is equivalent to the statement that the $\mathbb{T}$-orbit of W is smooth.

(2) For a given $graph(T) = W \in Gr_\omega(H)$ we write $o_W(u) = \sum_{k=-\infty}^{\infty} o_k u^k$, where $o_k := B_k T$ with B_k is defined as a linear operator of the matrix form $\bar{S}\times S$ by

$$B_k := \begin{cases} b_{i,j} = 1 & \text{if } j = i-k \\ b_{i,j} = 0 & \text{if } j \neq i-k. \end{cases}$$

Denote $T^k := o_k = B_k T$. We claim that $\|r^{-|k|}T^k\|_{HS}$ is bounded. Since $\|r^{-|k|}T^k_{pq}\|_{\mathbb{C}}$ and $\|T^k\|_{HS}$ are bounded it follows that

$$\|r^{-|k|}T^k\|_{HS} = r^{-|k|}\|T^k\|_{HS} < \infty.$$

We conclude that o_W is real-analytic.

Conversely, if we suppose that o_W is real-analytic, i.e. $r^{-|k|}T^k < \infty$, then $r^{-|k|}T^k_{pq} < \infty$ for all k and $(p,q) \in \bar{S}\times S$. Thus $W = graph(T) \in Gr_\omega(H)$.

(3) The fact that $\mathbb{T}$ acts smoothly on the manifold $Gr_\infty(H)$ means that the map $R_{Gr_\infty(H)}\colon \mathbb{T} \to Gr(H)$, defined by $u \mapsto (u^{q-p}T_{pq})$ where $W_T \in Gr_\infty(H)$ and $T \in HS(H_S, H_S^\perp)$, is smooth. We get

$$\begin{aligned}\lim_{h\to 0} \|\frac{(u_h^{q-p}T_{pq})_{\overline{S}\times S} - (u^{q-p}T_{pq})_{\overline{S}\times S}}{h}\|_{HS} &= \|((q-p)u^{q-p}T_{pq})_{\overline{S}\times S}\|_{HS} \\ &= \|((q-p)T_{pq})_{\overline{S}\times S}\|_{HS} < \infty. \qquad (23)\end{aligned}$$

The operator $((q-p)^m T_{pq})_{\overline{S}\times S}$ is a H-S operator for each m since $W_T \in Gr_\infty(H)$. The inequality (23) holds for all derivatives, which guarantees the smoothness of R in the C^∞-topology. □

We are able to extend the function R to a function

$$R_{\leq 1}\colon \mathbb{C}^\times_{\leq 1} \times Gr(H) \to Gr(H) \text{ by } R_{\leq 1}(u, W) := R_u(W).$$

This function has interesting properties which are stated in the following proposition.

Proposition 43.

1. *The map $R_{<1}$ is holomorphic on the open set $\mathbb{C}^\times_{<1} \times Gr(H)$.*
2. *The map $R_{<1}$ maps $W \in Gr(H)$ to $R_{<1}(W) \in Gr_\omega(H)$.*
3. *The map $R_{\mathbb{C}^\times}|_{Gr_0(H)}\colon \mathbb{C}^\times \times Gr_0(H) \to Gr_0(H)$ is holomorphic.*

Proof. (1) Let $A \in \mathbb{C}^{\bar{S}\times S}$ be a H-S operator and u, $u_h \in \mathbb{C}$. Then

$$\begin{aligned}&\lim_{h\to 0}\left\|\frac{(u_h^{q-p}(T_{pq} + hA_{pq}))_{\bar{S}\times S} - (u^{q-p}T_{pq})_{\bar{S}\times S}}{h}\right\|_{HS} \\ =&\lim_{h\to 0}\left\|\frac{(u_h^{q-p}T_{pq})_{\bar{S}\times S} + ((u_h)^{q-p}hA_{pq})_{\bar{S}\times S} - (u^{q-p}T_{pq})_{\bar{S}\times S}}{h}\right\|_{HS} \\ =&\lim_{h\to 0}\|(u_h^{q-p}(q-p)T_{pq})_{\bar{S}\times S} + (u_h^{q-p}A_{pq})_{\bar{S}\times S}\|_{HS}.\end{aligned}$$

Since $\|u_h\|_{\mathbb{C}} < 1$ and the series $\sum_{k=0}^\infty a^k k$ converges for $|a| < 1$, it follows that $\sum_{k=0}^\infty (a^k k)^2$ converges. This implies that $(a^{q-p}(q-p))_{\bar{S}\times S}$ is a H-S operator and that the H-S norm of $(u_h^{q-p}(q-p)T_{pq})_{\bar{S}\times S}$ is bounded. It is obvious that the H-S norm of $(u_h^{q-p}A_{pq})_{\bar{S}\times S}$ is finite. We conclude that the above limits exist and so $R_{<1}$ is holomorphic.

(2) The map $R_{<1}$ maps $T = (T_{pq}) \mapsto (u^{q-p}T_{pq})$.

If $q-p > 0$, then $\|u^{q-p}T_{pq}\|_{\mathbb{C}} = \|u\|_{\mathbb{C}}^{q-p}\|T_{pq}\|_{\mathbb{C}} < \|T_{pq}\|_{\mathbb{C}} < \infty$ as $\|u\|_{\mathbb{C}} < 1$.

If $q-p < 0$, then $\mid q-p \mid < \infty$ and so $\|u^{q-p}T_{pq}\|_{\mathbb{C}} < \infty$. We conclude that $(u^{q-p}T_{pq})_{\bar{S}\times S} \in Gr_\omega(H)$.

(3) Consider the differential quotient

$$\lim_{h\to 0}\|[(u_h^{q-p}(q-p)T_{pq})_{\bar{S}\times S} + (u_h^{q-p}A_{pq})_{\bar{S}\times S}]\|_{HS}$$

and observe that $q-p$ is finite as $W \in Gr_0(H)$ and so T has only finitely many non-vanishing entries. Furthermore, u^{q-p} is bounded for $u \in \mathbb{C}^\times$ and thus the H-S norms of $(u_h^{q-p}(q-p)T_{pq})_{\bar{S}\times S}$ and $(u_h^{q-p}A_{pq})_{\bar{S}\times S}$ are defined and bounded. Hence it is holomorphic on $\mathbb{C}^\times \times Gr_0(H)$. □

The stratification of $Gr(H)$ can be described by the action of the semigroup $\mathbb{C}^\times_{\leq 1}$.

Proposition 44.

(1) *The stratum Σ_S of $Gr(H)$ corresponding to S consists precisely of the points $W \in Gr(H)$ such that $R_u(W) \to H_S$ as $u \to 0$.*
(2) *The Schubert cell C_S of $Gr(H)$ corresponding to S consists precisely of the points $W \in Gr_0(H)$ such that $R_u(W) \to H_S$ as $u \to \infty$.*

Proof. (1) Let $W \in \Sigma_S$. We know from Proposition 34 that if $graph(T) = W \in \Sigma_S$, then all matrix entries T_{pq} vanish if $p > q$ with $T \in HS(H_S, H_S^\perp)$. From this it follows that for every non-vanishing matrix entry the inequality $q - p > 0$ holds. So we have $R_u(W) = R_u(T) = (u^{q-p}T_{pq})_{\bar{S}\times S}$ with positive powers of u. If u goes to 0, then u^{q-p} goes to 0 since $q-p$ positive. So every matrix entry of T_u vanishes, so the $graph(T_u) = (H_S, 0) = H_S$.

Conversely, we know that $graph(T) = W$ and $(u^{q-p}T_{pq}) \to (0)$ for $u \to 0$. We conclude that $T_{pq} = 0$ for $p > q$. From this it follows that Tz^s is of finite order smaller than s for all $s \in S$, which implies that $W \in \Sigma_S$.

(2) Let $W \in C_S$, then the proof is almost reverse to the first part. All matrix entries T_{pq} vanish if $p < q$ with $T \in HS(H_S, H_S^\perp)$. From this it follows that for every non-vanishing matrix entry we have $q - p < 0$. So since $R_u(W) = R_u(T) = (u^{q-p}T_{pq})_{\bar{S}\times S}$ have only negative powers of u and if u goes to infinity, then u^{q-p} goes to 0 for $q-p$ negative. So every matrix entry of T_u vanishes, that gives $graph(T_u) = (H_S, 0) = H_S$.

Inverse, suppose $graph(T) = W \in Gr_0(H)$ and $R_u(W) \to H_S$ which is equivalent to $(u^{q-p}T_{pq}) \to (0)$ for $u \to \infty$. This implies that $T_{pq} = 0$ for $p < q$. From this we conclude that $graph(T) = W \in C_S$. □

Proposition 45. *The Plücker embedding*

$$\pi \colon Gr(H) \to P(\mathbb{H})$$

is equivariant with respect to $\mathbb{C}^\times_{\leq 1}$ if the map $R_u \colon \mathbb{H} \to \mathbb{H}$ is defined by

$$(R_u(h))_S := u^{l(S)} h_S.$$

The statement can be written as $\pi_S(R_u(W)) = \lambda u^{l(S)}\pi_S(W)$, where $\lambda \in \mathbb{C}$ is a non-zero factor which can be identified as the determinant factor related two admissible basis.

Proof. Suppose $W \in Gr(H)$ and $w \colon z^{-d}H_+ \to W$ is an admissible basis for W. We construct an admissible basis of $R_u(W)$ from w. The orthogonal projection of w on $z^{-d}H_+$ is an operator with determinant, i.e. $pr_{z^{-d}H_+} \circ w = 1 + t$, where $t\colon z^{-d}H_+ \to z^{-d}H_+$ is a trace class operator. Observe the operator R_u is invertible but has no determinant and

$$pr_{z^{-d}H_+} \circ R_u = R_u\,|_{z^{-d}H_+} \circ pr_{z^{-d}H_+}$$

as R_u preserves the decomposition of $H = H_+ \oplus H_-$. Unfortunately $R_u \circ w$ is not an admissible basis because

$$pr_{z^{-d}H_+} \circ R_u \circ w = R_u\,|_{z^{-d}H_+} \circ pr_{z^{-d}H_+} \circ w = R_u\,|_{z^{-d}H_+} + R_u\,|_{z^{-d}H_+} \circ t$$

has no determinant. Nevertheless, $R_u \circ w \circ R_u^{-1}$ gives the admissible basis since

$$\begin{aligned} pr_{z^{-d}H_+} \circ R_u \circ w \circ R_u^{-1} &= R_u\,|_{z^{-d}H_+} \circ pr_{z^{-d}H_+} \circ w \circ R_u^{-1} \\ &= R_u\,|_{z^{-d}H_+} (R_u^{-1} + tR_u^{-1}) \\ &= R_u\,|_{z^{-d}H_+} R_u^{-1} + R_u\,|_{z^{-d}H_+} tR_u^{-1} \\ &= 1 + R_u\,|_{z^{-d}H_+} tR_u^{-1}, \end{aligned}$$

or, in other words, $pr_{z^{-d}H_+} \circ R_u \circ w \circ R_u^{-1}$ has a determinant if $R_u\,|_{z^{-d}H_+} tR_u^{-1}$ is of trace class. This is true as R_u is bounded and the space of trace class operators is two sided ideal in the space of bounded operators. The map $R_u \circ w \circ R_u^{-1}$ is invertible as all three operators are invertible and so we conclude that $R_u \circ w \circ R_u^{-1}$ is an admissible basis of $R_u(W)$.

By the continuity of the Plücker embedding it is enough to prove the proposition for a dense subset of $Gr(H)$. We know that Plücker coordinates of W with respect to S is zero if the virtual cardinal of S is different from the virtual dimension of W. So it is enough to show it for a dense subset of U_d. We choose the dense subset U_S with $S := \{-d, -d+1, -d+2, ...\}$ and the following basis $w_q = z^q + \sum_p T_{pq} z^p$ with $q \in S$. Remind that the virtual cardinal of S coincides with the virtual dimension of W and that the Plücker coordinates of W with respect to S' vanish if S' is of virtual cardinal different from d. We define two sets

$$A := S' \setminus S = \{a_1, ..., a_k\}, \qquad B := S \setminus S' = \{b_1, ..., b_k\}.$$

We claim that $l(S') = \sum_{i=1}^k (b_i - a_i)$. Indeed,

$$l(S') = \sum_{i \ge -d} (i - s_i') = \sum_{i \ge -d} (s_i - s_i') = \sum_{i=1}^{k} (b_i - a_i).$$

Proposition 38 shows that $\pi_{S'}(W)$ is the determinant of the submatrix of T formed from the rows A and columns B. We denote this submatrix by $T^{A\times B}$. Now we need to calculate the determinant of $R_u \circ T^{A\times B} \circ R_u^{-1}$. In order to do this, we would like to understand the form of entries of $T^{A\times B}$.

For z^{b_n}, $n \in \{1, ..., k\}$,

$$R_u(T(R_u^{-1}(z^{b_n}))) = R_u(T(u^{b_n} z^{b_n})) = R_u(\sum_{i=1}^{k} T_{a_i b_n}^{A\times B} u^{b_n} z^{a_i}) =$$

$$= \sum_{i=1}^{k} u^{-a_i} T_{a_i b_n}^{A\times B} u^{b_n} z^{a_i} = \sum_{i=1}^{k} u^{b_n - a_i} T_{a_i b_n}^{A\times B} z^{a_i}.$$

From this it follows that the matrix entries of $R_u \circ T^{A\times B} \circ R_u^{-1}$ are $u^{b_n - a_i} T_{a_i b_n}^{A\times B}$. If we now take the determinant of this matrix it is obvious that every summand of the determinant has the factor $\sum_{i=1}^{k} (b_i - a_i) = l(S^{'})$. We conclude that $\det(R_u \circ T^{A\times B} \circ R_u^{-1}) = u^{l(S)} \det(T^{A\times B})$. The equality $\det(T^{A\times B}) = \pi_{S'}(W)$ implies

$$\pi_{S'}(R_u(W)) = \det(R_u \circ T^{A\times B} \circ R_u^{-1}) = u^{l(S)} \det(T^{A\times B}) = u^{l(S^{'})} \pi_{S'}(W).$$

□

5.7. **The determinant bundle.**

In this section we shall construct a holomorphic line bundle Det on the Grassmannian $Gr(H)$.

Definition 53. *For $W \in Gr(H)$ of virtual dimension d we define the fibre* $\mathrm{Det}(W)$ *by* $\mathrm{Det}(W) := \{\lambda\Lambda \mid \lambda \in \mathbb{C}; \Lambda = w_{-d} \wedge w_{-d+1} \wedge ...,$ *where* $w = \{w_k\}_{k \geq -d}$ *is an admissible basis of* $W\}$. *We define an element of* $\mathrm{Det}(W)$ *by* $[\lambda, w]$, *where w is an admissible basis of W and $\lambda \in \mathbb{C}$.*

Furthermore, we define the determinant bundle Det *of $Gr(H)$ by*

$$\mathrm{Det} := \bigcup_{W \in Gr(H)} \mathrm{Det}(W).$$

Proposition 46. *If $w^{'}$ and w are admissible basis of W, then*

$$[\lambda, w] = [\lambda \det(t), w^{'}]$$

where $t = (t_{ij})$ is the relating matrix between w and $w^{'}$ such that $w_i = \sum_j t_{ij} w_j^{'}$.

Proof. We know that $w_i = \sum_j t_{ij} w_j^{'}$. So

$$[\lambda, w] = \lambda w_{-d} \wedge w_{-d+1} \wedge ... = \lambda(\sum_j t_{(-d)j} w_j^{'}) \wedge (\sum_j t_{(-d+1)j} w_j^{'}) \wedge ...$$

$$= \lambda(\sum_{\sigma \in \phi} \prod_{i \geq -d} t_{i\sigma(i)}) w_{-d}^{'} \wedge w_{-d+1}^{'} \wedge ...$$

$$= \lambda \det(t) w_{-d}^{'} \wedge w_{-d+1}^{'} \wedge ... = [\lambda \det(t), w^{'}],$$

where ϕ is the set of all permutations of the set $\{-d, -d+1, -d+2, ...\}$.

□

Proposition 47. *The fibre* Det(W) *of the line bundle* Det *is one dimensional complex vector space.*

Proof. We take any admissible basis w of Det(W). Then any element $[\lambda, w^{'}]$ of Det(W) can be written as $\lambda \det(t)[1, w]$, where t is the matrix relating $w^{'}$ and w. We get

$$[\lambda, w^{'}] = [\lambda \det(t), w] = \lambda \det(t)[1, w]$$

and $\lambda \det(t) \in \mathbb{C}$. We conclude that $[1, w]$ is a basis of the complex vector space Det(W). □

Proposition 48. *The line bundle* Det *is a complex manifold.*

Proof. We have to show that the transition between two open sets are holomorphic. For each indexing set $S \in \mathfrak{S}$ we have the open set $U_S \subset Gr(H)$, identified with the graphs of H-S operators $T\colon H_S \to H_S^{\perp}$. For every graph W_T of T we take the canonical basis, which is also an admissible basis, $w_i = z^q + \sum_{p \in \mathbb{Z} \setminus S} T_{pq} z^p$ with $q = s_i \in S = \{s_{-d}, s_{-d+1}, ...\}$. Define the function $\psi_S\colon (\mathbb{C} \times U_S) \to \text{Det}$ by $(\lambda, W_T) \mapsto [\lambda, w]$, where w is the canonical basis defined above.

We claim that it is bijective to its image. If $(\lambda_1, W_{T_1}) \neq (\lambda_2, W_{T_2})$, then $w_1 \neq w_2$ and/or $\lambda_1 \neq \lambda_2$. From this it follows that $[\lambda_1, w_1] \neq [\lambda_2, w_2]$, such that ψ_S is injective. The surjectivity is obvious. We can now identify the elements of Det above U_S with the elements of the image of ψ_S of $(\mathbb{C} \times U_S)$.

Now we consider the change of coordinates on this manifold. Suppose that $W_T \in U_S \cap U_{S'}$ and $W_T = W_{T'}$, where $T^{'}: H_{S'} \to H_{S'}^{\perp}$. We know from Subsection 5.1 that $T^{'} = (c + dT)(a + bT)^{-1}$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is the matrix of the permutation relating S to $S^{'}$, i.e. $A\colon H_S \oplus H_S^{\perp} \to H_{S'} \oplus H_{S'}^{\perp}$ and for all $x \in H : A(x) = x$. Note that the submatrix $a\colon H_S \to H_{S'}$ is a matrix which differs from the identity matrix of H_S by an operator of finite rank. It is also known that $b\colon H_S^{\perp} \to H_{S'}$ is a H-S operator. Then bT is an operator of trace class, since T is a H-S operator. Suppose $h\colon H_S \to H_{S'}$ is an operator of trace class, more precisely an operator of finite rank defined by $h := a - \text{Id}\mid_{H_S}$. We get an operator of trace class

$$a + bT - \text{Id}\mid_{H_S} = h + bT.$$

We conclude that $a+bT$ has a determinant. Notice that bT takes the form

$$\left(\begin{array}{ccc|c} 1 & \cdots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & 1 & \\ \hline & T_{A\times S\cap S'} & & T_{A\times B} \end{array}\right),$$

where the identity matrix is a $S^{'}\cap S\times S\cap S^{'}$ matrix.

We claim that $a+bT$ is the matrix $t=(t_{ij})$ relating the canonical basis w of W_T with the canonical basis $w^{'}$ of $W_{T'}$. To prove this we observe that in every basis element $w_j^{'}$ of the canonical basis $w^{'}$ there is exactly one z^p such that $p\in S^{'}$. We conclude that t_{ij} is determind by w_i and $z^{s_j^{'}}$. So it is enough to check that $t_{ij}z^{s_j^{'}}$ is exactly a part of w_i, i.e.

$$w_i=t_{ij}z^{s_j^{'}}+\sum_{k\in\mathbb{Z}\setminus\{s_j^{'}\}}a_kz^k.$$

We calculate t_{ij} from $w_i=\sum_j t_{ij}w_j^{'}$. For this we have to make a case-by-case analysis for $s_j^{'}\in S^{'}$. If $s_j^{'}\in S$, then

$$t_{ij}=\begin{cases}0 \text{ if } s_i\neq s_j^{'} \\ 1 \text{ if } s_i=s_j^{'}\end{cases}.$$

If $s_j^{'}\notin S$, then $t_{ij}=T_{s_j^{'}s_i}$. This coincides with the definition of $a+bT$ and so we get that $t=a+bT$. If we define $\lambda^{'}:=\lambda\det(a+bT)$, then

$$[\lambda,w]=[\lambda\det(a+bT),w^{'}]=[\lambda^{'},w^{'}].$$

Now we define the transition $\chi\colon\mathbb{C}\times U_S\to\mathbb{C}\times U_{S'}$ by

$$\chi:=\psi_{S'}^{-1}\circ\psi_S,\qquad(\lambda,W_T)\mapsto(\lambda^{'},W_{T'}).$$

Since the graph W_T of T is determined by T, we can identify the second coordinate function with the holomorphic function

$$\Xi\colon I_{01}\to HS(H_{S'},H_{S'}^{\perp})$$

where I_{01} is the notation from the proof of Proposition 25, defined by

$$T\mapsto T^{'}=(c+dT)(a+bT)^{-1}.$$

Also the first coordinate function $\varphi\colon\mathbb{C}\times HS(H_S,H_S^{\perp})\to\mathbb{C}$ defined by $(\lambda,T)\mapsto\lambda\det(a+bT)$ is holomorphic. Furthermore, we get that χ is holomorphic and so we get that Det is a complex manifold. □

Theorem 4. *The action of* $GL_{res}(H)$ *on* $Gr(H)$ *is covered by an action of* $GL^{\sim}_{res}(H)$ *on the line bundle* Det*, i.e. for every* $A \in GL_{res}(H)$ *there exists an extension* $K \in GL^{\sim}_{res}(H)$ *of* A *such that* K *acts on the line bundle* Det*.*

Proof. The idea of the proof is to extend an action of $GL_{res,0}(H)$ on $Gr(H)$ to an action of $GL^{\sim}_{res,0}(H)$ on the line bundle Det and in the second step making use of the central extensions over both to get an action of $GL^{\sim}_{res}(H)$ which covers an action of $GL_{res}(H)$.

We start by considering

$$Gr^0(H) := \{W \in Gr(H) \mid \text{virtdim}(W) = 0\}.$$

This is a connected component by Proposition 27 of $Gr(H)$ of equal virtual dimension 0. We take an admissible basis $w\colon H_+ \to W$ of W. By definition this is an isomorphism and we can write it as a $\mathbb{Z} \times \mathbb{N}$ matrix

$$w = \begin{pmatrix} w_+ \\ w_- \end{pmatrix},$$

where $w_+ := pr_+ \circ w\colon H_+ \to H_+$, $w_- := pr_- \circ w\colon H_+ \to H_-$ and $pr_\pm\colon W \to H_\pm$ are orthogonal projections of W to $H_\pm$. By definition of the admissible basis w_+ has a determinant as $\text{virtdim}\, W = d = 0$ and so $z^{-d}H_+ = H_+$.

Throughout the proof we identify $A \in GL(H)$ with the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a and d are Fredholm operators and b and c are H-S operators. We define $GL_{res,0}(H)$ by

$$GL_{res,0}(H) := \{A \in GL_{res}(H) \mid \text{ind}(a) = 0\}.$$

We define the subgroup $\mathfrak{E}$ of $GL_{res,0}(H) \times GL(H_+)$ by

$$\mathfrak{E} := \{(A, q) \in GL_{res,0}(H) \times GL(H_+) \mid aq^{-1} \text{ has a determinant } \}.$$

Now we define an action of $\mathfrak{E}$ mapping the set of admissible basis of $W \in Gr^0(H)$ to the set of admissible basis of $A(W) \in Gr^0(H)$ by

$$(A, q).w := Awq^{-1}.$$

We state that it is well defined if and only if Awq^{-1} is an admissible basis of $A(W) \in Gr^0(H)$.

Proof. The map $Awq^{-1}\colon H_+ \to A(W)$ is linear, continuous and isomorphic, that maps z^k to v^k, where v is an admissible basis of $A(W) \in Gr^0(H)$ and A, w, q^{-1} are continuous, linear and isomorphic. We have

$$\text{virtdim}(A(W)) = \text{virtdim}(W) + \text{ind}(a) = \text{virtdim}(W)$$

since $A \in GL_{res,0}(H)$. So we conclude that A maps $W \in Gr^0(H)$ to the element $A(W) \in Gr^0(H)$.

Now we prove that the composition of the orthogonal projection $pr_+ \colon A(W) \to H_+$ and the map Awq^{-1} has a determinant. We have

$$Awq^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_+q^{-1} \\ w_-q^{-1} \end{pmatrix} = \begin{pmatrix} aw_+q^{-1} + bw_-q^{-1} \\ cw_+q^{-1} + dw_-q^{-1} \end{pmatrix}$$

with

$$pr_+ \circ Awq^{-1} = aw_+q^{-1} + bw_-q^{-1}.$$

We will show that aw_+q^{-1} has a determinant and that bw_-q^{-1} is an operator of trace class. We know that w_+ has a determinant, therefore $t := w_+ - \mathrm{Id}$ is of trace class. If we multiply this equation from the left by the bounded operator a and from the right by the bounded operator q^{-1}, we get

$$aw_+q^{-1} - aq^{-1} = atq^{-1} \quad \text{and} \quad aw_+q^{-1} = aq^{-1} + atq^{-1}.$$

Trace class operators form a two sided ideal in the class of bounded operators, therefore atq^{-1} is an operator of trace class. The operator aq^{-1} has a determinant and so $aq^{-1} - \mathrm{Id}$ is of trace class. But then $aq^{-1} - \mathrm{Id} + atq^{-1}$ is of trace class and so is $aw_+q^{-1} - \mathrm{Id}$. We can conclude that aw_+q^{-1} has a determinant.

We know that the orthogonal projection $pr_- \colon W \to H_-$ is a H-S operator and the composition of a bounded operator and a H-S operator is also a H-S operator. Since w is linear and bounded, we deduce that $pr_- \circ w = w_-$ is a H-S operator. We also know that b is a H-S operator and as the product of two H-S operators is an operator of trace class, the operators bw_- and bw_-q^{-1} are of trace class by the boundedness of q^{-1}. It follows that

$$pr_+ \circ Awq^{-1} - \mathrm{Id} = aw_+q^{-1} - \mathrm{Id} + bw_-q^{-1}$$

is of trace class as $aw_+q^{-1} - \mathrm{Id}$ and bw_-q^{-1} are of trace class. This yields that $pr_+ \circ Awq^{-1}$ has a determinant. □

We define an action of $\mathfrak{E}$ on Det by

$$(A, q).[\lambda, w] := [\lambda, (A, q).w] = [\lambda, Awq^{-1}].$$

We define the subgroup τ_1 of $\mathfrak{E}$ by $\tau_1 := \{(1, q) \in \mathfrak{E} \mid \det(q) = 1\}$. For $(1, q) \in \tau_1$ and $[\lambda, w] \in \mathrm{Det}$ we get

$$(1, q).[\lambda, w] = [\lambda, (1, q).w] = [\lambda, wq^{-1}].$$

As q is the matrix relating w and wq^{-1} since $wq^{-1}q = w$, we get

$$[\lambda, wq^{-1}] = [\lambda \det(q), wq^{-1}] = [\lambda, w]$$

and so $(1, q).[\lambda, w] = [\lambda, w]$. We conclude that τ_1 acts trivially on Det and so we obtain that $\mathfrak{E}/\tau_1$ acts on Det. We know that $\mathfrak{E}/\tau_1 = GL^{\sim}_{res,0}(H)$, which is the identity component of $GL^{\sim}_{res}(H)$, i.e. we defined an action of $GL^{\sim}_{res,0}(H)$ on Det which covers an action of $GL_{res,0}(H)$ on $Gr^0(H)$.

Now we construct an action of $GL_{res,0}(H)$ on the part of Det over $Gr^d(H)$. Let $\sigma\colon H \to H$ be a shift operator

$$x \mapsto zx \quad \text{for } x \in H_+ \quad , \quad x \mapsto x \quad \text{for } x \in H_-.$$

It is a Fredholm operator with index -1 and the index of the adjoint operator σ^* is $+1$, see Proposition 17. Recall that we defined an automorphism

$$\tilde{\sigma}\colon \mathfrak{E}/\tau_1 = GL^{\sim}_{res,0}(H) \to \mathfrak{E}/\tau_1 = GL^{\sim}_{res,0}(H)$$

by

$$(A,q) \mapsto (\sigma A \sigma^{-1}, q_\sigma) = \begin{cases} (\sigma A\sigma^{-1}, \sigma q \sigma^{-1}) & \text{on} \quad \sigma(H_+) \\ (\sigma A \sigma^{-1}, 1) & \text{on} \quad H_+ \ominus \sigma(H_+). \end{cases}$$

Furthermore, we define the action $\sigma_{\mathrm{Det}}\colon$ Det $\to$ Det by

$$[\lambda, w] \mapsto \sigma_{\mathrm{Det}}.[\lambda, w] := [\lambda, \sigma w].$$

Now we define the action of $A \in GL_{res,0}(H)$ on Det $|_{Gr^d(H)}$ to Det $|_{Gr^d(H)}$ as the action

$$\sigma_{\mathrm{Det}}^{-d} \circ \tilde{\sigma}^d(A) \circ \sigma_{\mathrm{Det}}^d$$

which maps $[\lambda, w] \mapsto [\lambda, Aw(q_\sigma^{-1})^d]$. Here Det $|_{Gr^d(H)}$ denotes Det over the component $Gr^d(H)$. It is clear that $Aw(q_\sigma^{-1})^d$ is linear, continuous and is an isomorphism from $z^{-d}H_+$ to $A(W)$. Furthermore, we see that

$$\mathrm{virtdim}(A(W)) = \mathrm{ind}(a) + \mathrm{virtdim}(W) = 0 + d = d.$$

By repeating the argument of the first part of this proof on page 83 we see that the composition of the orthogonal projection $pr\colon A(W) \to z^{-d}H_+$ and $Aw(q_\sigma^{-1})^d$, $pr \circ Aw(q_\sigma^{-1})^d$ has a determinant.

Remind that $GL^{\sim}_{res}(H)$ is a semidirect product of $GL^{\sim}_{res,0}(H)$ by the cyclic subgroup generated by σ, see Section 3. This implies that the action above defines an action of $GL^{\sim}_{res}(H)$ on Det which covers the action of $GL_{res}(H)$ on $Gr(H)$. □

It is a correct moment to define the Plücker embedding for Det. First we need the definition of the Plücker coordinates on the fibre Det(W).

Definition 54. *The Plücker coordinate $\pi_S\colon$ Det $\to \mathbb{C}$ is defined by*

$$[\lambda, w] \mapsto \lambda \pi_S(w).$$

We see, as π_S is a holomorphic function, that each Plücker coordinate can be regarded as a holomorphic section of the dual Det* of Det, which is linear on each fibre.

Definition 55. *The map R_u with $u \in \mathbb{C}^\times_{\leq 1}$ acting on $\{\xi_S\} \in \mathbb{H} = l^2((S))$ is defined by*

$$(R_u \xi)_S := u^{l^*(S)} \xi_S,$$

where $u^{l^*(S)} := l(S) + \frac{1}{2}d(d+1)$ *and* $d := \operatorname{virtcard}(S)$. *The map* R_u *with* $u \in \mathbb{C}^\times_{\leq 1}$ *acting on* $[\lambda, w] \in \operatorname{Det}$ *is defined by*

$$R_u[\lambda, w] := [\lambda u^{\frac{1}{2}d(d+1)}, R_u w R_u^{-1}].$$

The map $\pi\colon \operatorname{Det} \to \mathbb{H}$ *is defined by* $[\lambda, w] \mapsto \lambda\pi(w)$.

Proposition 49. *The map* $\pi\colon \operatorname{Det} \to \mathbb{H}$ *is equivariant with respect to* $\mathbb{C}^\times_{\leq 1}$, *i.e.* $\pi(R_u[\lambda, w]) = R_u\pi([\lambda, w])$.

Proof. The definition of the action $(R_u\xi)_S := u^{l^*(S)}\xi_S$ implies

$$R_u\pi([\lambda, w]) = R_u(\lambda\pi(w)) = \lambda R_u\pi(w) = \lambda u^{l^*(S)}\pi(w).$$

Since $\pi(R_u W) = u^{l(S)}\pi(W)$ and $R_u w R_u^{-1}$ is an admissible basis of $R_u W$, we conclude that

$$\begin{aligned}\pi(R_u[\lambda, w]) &= \pi([\lambda u^{\frac{1}{2}d(d+1)}, R_u w R_u^{-1}]) = \lambda u^{\frac{1}{2}d(d+1)}\pi(R_u w R_u^{-1}) \\ &= \lambda u^{\frac{1}{2}d(d+1)}\pi(R_u W) = \lambda u^{\frac{1}{2}d(d+1)} u^{l(S)}\pi(W) \\ &= \lambda u^{l^*(S)}\pi(w) = R_u\pi([\lambda, w]).\end{aligned}$$

So we get that $\pi\colon \operatorname{Det} \to \mathbb{H}$ is equivariant with respect to $\mathbb{C}^\times_{\leq 1}$.

□

5.8. $Gr(H)$ as the Kähler and symplectic manifold.

This subsection presents a brief idea why the Grassmannian is usefull in physics. We want to show that $Gr(H)$ is a Kähler manifold and it can be done in two different ways.

Proposition 50. $Gr(H)$ *is the Kähler manifold.*

Proof. (1) The Grassmannian $Gr(H)$ is a complex manifold by Corollary 25. If we could introduce a Hermitian metric on the tangent bundle of $Gr(H)$, then we would get a Kähler metric. The tangent bundle of $Gr(H)$ is $\bigcup_{W \in Gr(H)} HS(W, W^\perp)$ equipped with the manifold structure. As U_{res} acts transitively on $Gr(H)$ it is enough to define a Hermitian form on its tangent space at the point H_+, which is $HS(H_+, H_-)$. The space H_+ is invariant under left-composition of $U(H_+)$ and under right-composition of $U(H_-)$. We define the unique invariant inner product on the Hilbert space $HS(H_+, H_-)$ by

$$\begin{gathered} h\colon HS(H_+, H_-) \times HS(H_+, H_-) \to \mathbb{C} \\ (X, Y) \mapsto 2\operatorname{trace}(X^*Y). \end{gathered}$$

This inner product defines the Kähler structure on $Gr(H)$. Notice that the imaginary part of h is $\omega(X, Y) = -i\operatorname{trace}(X^*Y - Y^*X)$.

(2) The form ω represents the Chern class of the line bundle Det on $Gr(H)$. This is equivalent to the fact that the Kähler structure of $Gr(H)$ is induced

from the standard structure on the projective space $P(\mathbb{H})$ by the Plücker embedding. The proof can be found in [9]. $\square$

Corollary 16. *The imaginary part ω of h coincides with the Lie algebra cocycle corresponding to the extension $GL^{\sim}_{res}(H)$ given by $t\colon \mathfrak{u}_{res} \times \mathfrak{u}_{res} \to \mathbb{C}$*

$$(A_1, A_2) \mapsto \frac{1}{4}\operatorname{trace}(J[J, A_1][J, A_2]),$$

where $\mathfrak{u}_{res} := \{A \in B(H) \mid A^* = -A \quad \text{and} \quad [J, A] \in HS(H)\}$.

Proposition 51. *The Hamiltonian function $F\colon Gr(H) \to \mathbb{R}$, which defines the flow on $Gr(H)$ corresponding to $\zeta \in \mathfrak{u}_{res}$, is given by*

$$F(W) = -i\operatorname{trace}(\zeta(J_W - J)),$$

where $J_W := gJg^{-1}$ with $g \in U_{res}$ and $W = g(H_+)$.

Proof. F is well defined since $[J, g] = Jg - gJ$ is a H-S operator and so $-(Jg - gJ)g^{-1} = gJg^{-1} - J$. This implies that $\zeta(gJg^{-1} - J)$ is a trace class operator.

The gradient of F at W along the tangent vector corresponding to $\eta \in \mathfrak{u}_{res}$ is $dF(W, \eta) = -i\operatorname{trace}(\zeta[\eta, J_W])$. The value of the invariant form ω at W on the tangent vectors defined by ζ, $\eta \in \mathfrak{u}_{res}$ is

$$\begin{aligned}\omega(W, \zeta, \eta) &= \omega(g^{-1}\zeta g, g^{-1}\eta g) = -i\operatorname{trace}(g^{-1}\zeta g[g^{-1}\eta g, J]) \\ &= -i\operatorname{trace}(\zeta[\eta, J_W]) = dF(W, \eta).\end{aligned}$$

$\square$

This proposition can not be applied directly to the rotation group action $\mathbb{T}$ on $Gr(H)$, since we saw that the smooth action is defined only on the submanifold $Gr_\infty(H)$.

Definition 56. *We define the* **energy** $E\colon Gr_\infty(H) \to \mathbb{R}$ *by*

$$E(W) = \operatorname{trace}((i\frac{d}{d\theta})(J_W - J)).$$

Proposition 52. *We can write E more generally as*

$$E(W) = \sum_{S \in \mathfrak{S}} l^*(S) \mid \pi_S(W) \mid^2 = \langle \Omega_W, i\frac{d}{d\theta} \cdot \Omega_W \rangle,$$

where $\{\pi_S(W)\}_{S \in \mathfrak{S}}$ are the Plücker coordinates of W, normalized such that $\sum_{S \in \mathfrak{S}} \mid \pi_S(W) \mid^2 = 1$, and Ω_W is the corresponding unit vector in $\mathbb{H}$.

We mentioned in the introduction that the Grassmannians can be used in some physical applications. One application can be found in quantum mechanics, where $Gr(H)$ is interpreted as the space of states of a classical system and $P(\mathbb{H})$ as the corresponding quantum state space. In this case

Ω_W represents the quantum state corresponding to W. Furthermore, Proposition 52 asserts that the classical energy $E(W)$ is the expected value of the quantum energy operator $i(\frac{d}{d\theta})$ in the state Ω_W.

6. Appendix

For convenience, we collect some the main theorems and definitions employed in this thesis.

Definition 57. *The operator norm for all $A \in B(H)$ is defined by*

$$\|A\|_{op} := \sup_{x \in H \setminus \{0\}} \frac{\|Ax\|_H}{\|x\|_H} = \sup_{x \in H, \|x\|_H = 1} \|Kx\|_H.$$

Definition 58. *Let H be a separable Hilbert-space and $\{e_i\}_{i \in N}$ an orthonormal family of H. We define $pr \colon H \to H$ as an orthogonal projection on* $\operatorname{span}\{e_i\}_{i \in N}$ *if it is of the form*

$$pr(x) := \sum_{i \in N} \langle e_i, x \rangle e_i.$$

Corollary 17. *If pr is an orthogonal projection, then*

$$\|pr\|_{op} \leq 1 \qquad \textit{and} \qquad \|1 - pr\|_{op} \leq 1.$$

Theorem 5. *If A, $B \in B(H)$, then*

$$\|ABx\|_H \leq \|A\|_{op} \|Bx\|_H.$$

Corollary 18. *Let pr be an orthogonal projection and $A \in B(H)$, then*

$$\|prAx\|_H \leq \|Ax\|_H, \qquad \|(1 - pr)Ax\|_H \leq \|Ax\|_H.$$

Theorem 6 (Parseval's identity)**.** *Let H be a separable Hilbert-space and $\{e_i\}_{i \in \mathbb{N}}$ be an orthonormal basis of H. Then*

$$\sum_{i \in \mathbb{N}} | \langle x, e_i \rangle |^2 = \|x\|_H^2$$

for every $x \in H$.

Theorem 7. *If $\{f_k\}_{k \in A}$ is an orthogonal subset of the Hilbert space H , then $\sum_{k \in A} f_k$ converges in H if and only if $\sum_{k \in A} \|f_k\|^2 < \infty$, and in this case*

$$\| \sum_{k \in A} f_k \|^2 = \sum_{k \in A} \|f_k\|^2.$$

Theorem 8 (Fubini's theorem)**.** *Suppose U, V are complete measure spaces. Suppose $f(x, y)$ is $U \times V$ measurable. If*

$$\int_{U \times V} | f(x, y) | \, \mathrm{d}(x, y) < \infty,$$

then

$$\int_U (\int_V f(x, y) \, \mathrm{d}y) \, \mathrm{d}x = \int_V (\int_U f(x, y) \, \mathrm{d}x) \, \mathrm{d}y = \int_{U \times V} | f(x, y) | \, \mathrm{d}(x, y).$$

Corollary 19. *If* $\sum_{n,k\in\mathbb{N}} a_{n,k}$ *with* $a_{n,k} \geq 0$ *exists, then*

$$\sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} a_{n,k} = \sum_{k\in\mathbb{N}}\sum_{n\in\mathbb{N}} a_{n,k}.$$

Theorem 9 (Bounded-inverse theorem)**.** *If* $T \in B(H)$ *bijective, then* $T^{-1} \in B(H)$*.*

Theorem 10 (Closed-graph theorem)**.** *Suppose* X *and* Y *are Banach spaces and* $T\colon X \to Y$ *is a linear operator. Then the graph of* T *is closed in* $X \times Y$ *if and only if* T *is continuous.*

Definition 59. *A map* $f\colon U \to E$*, where* U *is an open subset of the vector space* E*, is continuously differentiable if the limit*

$$Df(u,v) = \lim_{\mathbb{R}\ni t\to 0} t^{-1}(f(u+tv) - f(u))$$

exists for all $u \in U$ *and* $v \in E$*, and is continuous as a map* $Df\colon U \times E \to E$*. To say that* f *is holomorphic means that* f *is smooth, i.e. infinitely differentiable, and that* Df *is complex linear in the second variable.*

Definition 60. *If* E *is a complex vector space and the transition functions are holomorphic, then we have a complex manifold.*

The manifolds we consider will be paracompact topological spaces X modelled on some topological vector space E, in the sense that X is covered by an atlas of open sets $\{U_\alpha\}$ each of which is homeomorphic to an open set E_α of E by a given homeomorphism $\phi_\alpha\colon U_\alpha \to E_\alpha$. The vector space E will always be locally convex and complete. The transition functions between charts

$$\phi_\alpha(U_\alpha \cap U_\beta) \xrightarrow{\phi_\alpha^{-1}} U_\alpha \cap U_\beta \xrightarrow{\phi_\beta} \phi_\beta(U_\alpha \cap U_\beta)$$

are assumed to be smooth, i.e. infinitely differentiable.

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THE GRASSMANNIAN OF AN INFINITE DIMENSIONAL SEPARABLE HILBERT SPACE

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