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LOCAL WELL-POSEDNESS FOR THE PERIODIC KORTEWEG-DE VRIES EQUATION IN ANALYTIC GEVREY CLASSES

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Abstract. Motivated by the work of Grujić and Kalisch, [Z. Grujić and H. Kalisch, *Local well-posedness of the generalized Korteweg-de Vries equation in spaces of analytic functions*, Differential and Integral Equations **15** (2002) 1325–1334], we prove the local well-posedness for the periodic KdV equation in spaces of periodic functions analytic on a strip around the real axis without shrinking the width of the strip in time.

1 Introduction This paper studies the local well-posedness of the Cauchy problem for the generalized periodic Korteweg-deVries equation (GKdV)

$$\begin{cases} \partial_t u + \partial^3_{xxx} u + u^k \partial_x u = 0 & u:\ \mathbb{T}\times[0,T]\to\mathbb{R} \\ u(x,0) = u_0(x) & x\in\mathbb{T} \end{cases} \tag{1}$$

with initial data $u_0(x)$ in a class of periodic functions analytic in a symmetric strip around the real axis. The number k is taken to be a positive integer and $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ is the torus. For $\sigma>0$, $s\in\mathbb{R}$, denote Gevrey classes $G^{\sigma,s}$ to be the subset of $L^2(\mathbb{T})$ such that

$$\|u_0\|^2_{G^{\sigma,s}} = \sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} e^{2\sigma\langle n\rangle} |\widehat{u_0}(n)|^2 < \infty$$

where $\langle n\rangle := 1+|n|$ and $\widehat{u_0}(n)$ denotes the Fourier transform of u_0 on torus.

In [18], Kato and Masuda introduced a method of obtaining spatial analyticity of solution for a large class of semi-linear evolution equations, and the research on Gevrey regularity for the solution of the semi-linear equations goes back to the work of Foias and Temam [10]. Further results concerning periodic solutions of Navier-Stokes equations in Gevrey spaces have been obtained by Biswas [1]. We refer to [2, 12] for the study of Kuramoto-Sivashinsky equation. For a treatment of a more general case of nonlinear parabolic equations, we refer the reader to [9]. Also, a number of authors have obtained solutions in Gevrey spaces without strong regularizing effects. Here we mention the recent work of Kukavica and Vicol on the three-dimensional Euler equations [21], and a body of work concerning KdV-like equations (see, for example, Hayashi [14, 15], Bouard et al. [5], Grujić and Kalisch

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[13], Bona et al. [4]). As explained in [3, 16, 17], analyticity of solution of the KdV equation plays an essential role in the numerical study of the equation.

The example constructed in [11] shows that the solution of GKdV equation with an appropriate analytic data may not be analytic in the time variable t. So, we must restrict our attention to the spatial analyticity of the solution of GKdV. Grujić and Kalisch [13] proved local well-posedness of non-periodic GKdV for a strip without shrinking the width of the strip in time. It is of interest to know whether it is possible to establish the same result for the periodic case.

Kato's smoothing effect was shown to be useful in the proof of the main theorem in [13]. However, this technique cannot be used in dealing with GKdV with periodic boundary data. Our approach is in the spirit of [8, Theorem 1] and the proof relies on the Bourgain's bilinear estimate [6], multilinear estimate in [22] and linear estimates in [7, 8]. In addition, the proof reveals some new aspects in the estimation of the time-cutoff function which are essential in the proof of the main nonlinear estimate which is given in Lemma 3.2.

Denote by $C([0,T],G^{\sigma,s})$ the space of continuous functions from the time interval $[0,T]$ into $G^{\sigma,s}$. We will prove the following theorem.

Theorem 1.1. *Let $s \geq 1$ and $k \geq 1$. For initial data in $G^{\sigma,s}$, $\sigma > 0$, there exists a small positive time T, such that the initial-value problem (1) is well-posed in the space $C([0,T],G^{\sigma,s})$.*

The paper is organized as follows. In Section 2, we set up notations and terminologies and deal with linear estimates. Section 3 is devoted to the study of bilinear estimates, and Section 4 provides a proof of the multilinear estimate. In Section 5, Theorem 1.1 is proved via a contraction argument.

2 Some notations and linear estimates Throughout this paper, $A \lesssim B$ denotes the estimate $A \leq CB$, where the constant $C > 0$ possibly depending on s, k and *independent* of σ. We say that $A \approx B$, if $A \lesssim B$ and $B \lesssim A$. We also denote by $A \ll B$ the estimate $A \lesssim \frac{1}{K}B$ for a large constant $K > 0$. The Lebesgue classes on the integer set and real line are denoted by l^p and L^q respectively, while the following notation is used to denote the $l^p - L^q$ space-time norms: $\|f(n,\lambda)\|_{l^p_n L^q_\lambda} = \|\|f(n,\lambda)\|_{L^q_\lambda}\|_{l^p_n}$.

Let $u(x,t)$ be a function defined on the cylinder $\mathbb{T}\times\mathbb{R}$ and $s,\ b \in \mathbb{R}$. The space-time Fourier transform of $u(x,t)$ is defined by

$$\hat{u}(n,\lambda) = \int_{\mathbb{R}}\int_{\mathbb{T}} u(x,t)e^{-2\pi i\lambda t - 2\pi inx}dxdt,$$

where $n \in \mathbb{Z}$. We denote by $\mathscr{F}_t[u(x,t)]$ the partial Fourier transform of u in variable t and by $\mathscr{F}_x[u(x,t)]$ the partial Fourier transform in variable x. We define the $X^{s,b} = X^{s,b}_{\tau=\xi^3}(\mathbb{T}\times\mathbb{R})$ norm of $u(x,t)$ by

$$\|u\|_{X^{s,b}} = \left\|\langle\lambda - n^3\rangle^b\langle n\rangle^s\hat{u}(n,\lambda)\right\|_{l^2_n L^2_\lambda},$$

where $\langle\cdot\rangle := 1 + |\cdot|$. This norm was introduced by Bourgain [6] and the space-time symbol is adapted to the linear part of KdV equation.

The low-regularity study of (1) is usually considered in spaces $X^{s,\frac{1}{2}}$ (see [6, 8, 22]). In order to overcome difficulty in persistence property in this case, authors [8] and [22] introduced the function space $Y^{s,b}$ to be the subset of $X^{s,b}$ such that

$$\|u\|_{Y^{s,b}} = \|u\|_{X^{s,b}} + \|\langle n\rangle^s\hat{u}(n,\lambda)\|_{l^2_n L^1_\lambda} < \infty.$$

It is indicated in [13] that we have to introduce another family of function spaces which are adapted to the study of Gevrey regularity. For $\sigma \geq 0$, define $X^{\sigma,s,b}$ norm of $u(x,t)$ by

$$\|u\|_{X^{\sigma,s,b}} = \left\| \langle \lambda - n^3 \rangle^b \langle n \rangle^s e^{\sigma\langle n \rangle} \hat{u}(n,\lambda) \right\|_{l^2_n L^2_\lambda}.$$

We shall use the space $Y^{\sigma,s,b}$ which equipped with the norm

$$\|u\|_{Y^{\sigma,s,b}} = \|u\|_{X^{\sigma,s,b}} + \left\| e^{\sigma\langle n \rangle} \langle n \rangle^s \hat{u}(n,\lambda) \right\|_{l^2_n L^1_\lambda}.$$

By the Riemann-Lebesgue lemma, the Fourier transform of an L^1 function is continuous and bounded, and we have the embedding property

$$Y^{\sigma,s,b} \subset C([0,T], G^{\sigma,s}) \subset L^\infty([0,T], G^{\sigma,s}). \tag{2}$$

We will also need the space $Z^{\sigma,s,b}$ with the norm defined by

$$\|u\|_{Z^{\sigma,s,b}} = \|u\|_{X^{\sigma,s,-b}} + \left\| \frac{e^{\sigma\langle n \rangle}\langle n \rangle^s}{\langle \lambda - n^3 \rangle} \hat{u}(n,\lambda) \right\|_{l^2_n L^1_\lambda}.$$

Consider initial value problem of the Airy equation on $\mathbb{T}$:

$$\begin{cases} \partial_t w + \partial^3_{xxx} w = 0 \\ \qquad w(x,0) = w_0(x), \ x \in \mathbb{T}. \end{cases} \tag{3}$$

The explicit solution of the initial value problem (3) can be expressed in terms of the semigroup $S(t)$ via Fourier transform,

$$w(x,t) = S(t)w_0 = c \sum_{n\in\mathbb{Z}} e^{2\pi i(xn+tn^3)} \widehat{w_0}(n).$$

We shall establish linear estimates for the propagator $S(t)$. Let $\psi(t)$ be a bump function supported in $[-2,2]$ and equal to one on $[-1,1]$. Denote by $0 < \delta < 1$ a small constant which need to be determined later.

Lemma 2.1. *We have*

$$\|\psi(t/\delta)S(t)u_0\|_{Y^{\sigma,s,\frac{1}{2}}} \lesssim \|u_0\|_{G^{\sigma,s}}$$

for all $s \in \mathbb{R}$ and $\sigma \geq 0$.

Proof. Let us first write $\widehat{\psi(t/\delta)S(t)}u_0(n,\lambda) = \widehat{u_0}(n)\delta\hat{\psi}(\delta(\lambda - n^3))$. By the definition of $X^{\sigma,s,b}$,

$$\|\psi(t/\delta)S(t)u_0\|^2_{X^{\sigma,s,\frac{1}{2}}} = \sum_n e^{2\sigma\langle n \rangle}\langle n \rangle^{2s} |\widehat{u_0}(n)|^2 \int_{\mathbb{R}} \langle \lambda \rangle \delta^2 |\hat{\psi}(\delta\lambda)|^2 d\lambda.$$

Since $\int_{\mathbb{R}} \langle \lambda \rangle \delta^2 |\hat{\psi}(\delta\lambda)|^2 d\lambda \lesssim 1 + \delta$, we get $\|\psi(t/\delta)S(t)u_0\|_{X^{\sigma,s,\frac{1}{2}}} \lesssim \|u_0\|_{G^{\sigma,s}}$. On the other hand, we see at once that $\left\| e^{\sigma\langle n \rangle}\langle n \rangle^s \widehat{\psi(t/\delta)S(t)}u_0 \right\|^2_{l^2_n L^1_\lambda} \lesssim \|u_0\|^2_{G^{\sigma,s}}$, which completes the proof. □

Having established Lemma 2.1, we repeat the proof of [8, Lemma 3.1], and we get Lemma 2.2.

Lemma 2.2. *We have*

$$\left\| \psi(t/\delta) \int_0^t S(t-t')F(t')dt' \right\|_{Y^{\sigma,s,\frac{1}{2}}} \lesssim \|F\|_{Z^{\sigma,s,\frac{1}{2}}}$$

for all $s \in \mathbb{R}$, $\sigma \geq 0$ and test functions F on $\mathbb{T} \times \mathbb{R}$.

We also need to estimate the cutoff function $\psi(t/\delta)u$ in the space $X^{\sigma,s,\frac{1}{2}}$. We present a proof in a spirit of [20, Lemma 3.2].

Lemma 2.3. *Let $\sigma \geq 0$. We have*

$$\|\psi(t/\delta)u\|_{X^{\sigma,s,\frac{1}{2}}} \lesssim \|u\|_{Y^{\sigma,s,\frac{1}{2}}}$$

for all $s \in \mathbb{R}$ and $\sigma \geq 0$.

Proof. By the definition of $Y^{\sigma,s}$, the proof is reduced to showing that, if $a = n^3$ then

$$\int_{\mathbb{R}} |\hat{u} *_\lambda (\delta\hat{\psi}(\delta\lambda))(l)|^2 \langle l-a\rangle dl \lesssim \int_{\mathbb{R}} |\hat{u}(n,\lambda)|^2 \langle \lambda - a\rangle d\lambda + \|\hat{u}(n,\lambda)\|^2_{L^1_\lambda} \qquad (4)$$

where $*_\lambda$ is the convolution in variable λ.

According to the proof of [20, Lemma 3.2], we have

$$\begin{aligned}&\int_{\mathbb{R}} |\hat{u} *_\lambda (\delta\hat{\psi}(\delta\lambda))(l)|^2 \langle l-a\rangle dl \\ &\lesssim \int_{\mathbb{R}} |e^{2\pi iat}\mathscr{F}_x[u](n,t)\partial_t^{\frac{1}{2}}\psi(\delta^{-1}t)|^2 dt + \int_{\mathbb{R}} |\hat{u}(n,\lambda)|^2 |\lambda - a| d\lambda\end{aligned}$$

and

$$\int_{\mathbb{R}} |\hat{u} *_\lambda (\delta\hat{\psi}(\delta\lambda))(l)|^2 dl \lesssim \int_{\mathbb{R}} |\hat{u}(n,\lambda)|^2 d\lambda.$$

By the Plancherel theorem and the Young inequality,

$$\begin{aligned}\int_{\mathbb{R}} |e^{2\pi iat}\mathscr{F}_x[u](n,t)\partial_t^{\frac{1}{2}}\psi(\delta^{-1}t)|^2 dt &= \left\|\widehat{e^{2\pi in^3t}}u(n,\lambda) *_\lambda \widehat{\partial_t^{\frac{1}{2}}}\psi(\delta^{-1}t)(\lambda)\right\|^2_{L^2_\lambda} \\ &\leq \left\|\hat{u}(n,\lambda-n^3)\right\|^2_{L^1_\lambda} \left\|\lambda^{\frac{1}{2}}\delta\hat{\psi}(\delta\lambda)\right\|^2_{L^2_\lambda} \\ &\lesssim \|\hat{u}(n,\lambda)\|^2_{L^1_\lambda},\end{aligned}$$

which shows (4), and the proof of Lemma 2.3 is completed. □

3 Bilinear estimates

The bilinear estimate is a standard technique in dealing with nonlinear term in the equation. This kind of technique has been used and developed by many authors (See, for instance [6, 13, 19, 23]).

Lemma 3.1. *Let $s \geq 0$, $\sigma \geq 0$, and suppose the functions u, v satisfy $\int_{\mathbb{T}} udx = 0$ and $\int_{\mathbb{T}} vdx = 0$. Assume that $\|v\|_{Y^{\sigma,s,\frac{1}{2}}} < \infty$ and $\|\psi(t/\delta)u\|_{X^{\sigma,s,\frac{1}{2}}} < \infty$. Then*

$$\left\|\psi(t/\delta)^2\partial_x(uv)\right\|_{X^{\sigma,s,-\frac{1}{2}}} \lesssim \delta^{\frac{1}{12}}\|v\|_{Y^{\sigma,s,\frac{1}{2}}} \|\psi(t/\delta)u\|_{X^{\sigma,s,\frac{1}{2}}}.$$

Proof. The main idea of the proof is due to Bourgain [6, page 221].

Since $\int_{\mathbb{T}} u = 0$ and $\int_{\mathbb{T}} v = 0$, we write

$$\begin{aligned}f(n,\lambda) &= \langle\lambda - n^3\rangle^{\frac{1}{2}}|n|^s e^{\sigma\langle n\rangle}|\widehat{\psi(t/\delta)u}(n,\lambda)|, \\ g(n,\lambda) &= \langle\lambda - n^3\rangle^{\frac{1}{2}}|n|^s e^{\sigma\langle n\rangle}|\widehat{\psi(t/\delta)v}(n,\lambda)|.\end{aligned}$$

Let $h(n,\lambda) \in l_n^2 L_\lambda^2$ and $\|h\|_{l_n^2 L_\lambda^2} \le 1$, we introduce a trilinear form:

$$\Lambda(f,g,h) = \sum_{n\neq 0} \sum_{\substack{n_1\neq 0,\\ n_1\neq n}} \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{e^{\sigma\langle n\rangle} e^{-\sigma\langle n-n_1\rangle} e^{-\sigma\langle n_1\rangle} h(n,\lambda) f(n_1,\lambda_1)}{\langle \lambda - n^3\rangle^{\frac12} \langle \lambda_1 - n_1^3\rangle^{\frac12}} \times \frac{g(n-n_1,\lambda-\lambda_1)|n|^{s+1}|n_1|^{-s}|n-n_1|^{-s}}{\langle \lambda - \lambda_1 - (n-n_1)^3\rangle^{\frac12}} d\lambda d\lambda_1.$$

Thus we need only to estimate $\Lambda(f,g,h)$.

Since $|n| \lesssim |n_1||n-n_1|$ and $e^{\sigma|n|}e^{-\sigma|n-n_1|}e^{-\sigma|n_1|} \le 1$, we obtain

$$|\Lambda(f,g,h)| \lesssim \sum_{n\neq 0} \sum_{\substack{n_1\neq 0,\\ n_1\neq n}} \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(n_1,\lambda_1) g(n-n_1,\lambda-\lambda_1) h(n,\lambda)||n| d\lambda_1}{\langle \lambda - n^3\rangle^{\frac12} \langle \lambda_1 - n_1^3\rangle^{\frac12} \langle \lambda - \lambda_1 - (n-n_1)^3\rangle^{\frac12}}.$$

From resonance identity $n^3 = (n-n_1)^3 + n_1^3 + 3nn_1(n-n_1)$, we get

$$\max\left\{|\lambda-\lambda_1-(n-n_1)^3|,\ |\lambda_1-n_1^3|,\ |\lambda-n^3|\right\} \ge |n||n_1||n-n_1|. \tag{5}$$

As pointed out in [6, Theorem 7.41], we have

$$|\Lambda(f,g,h)| \lesssim \|FG\|_{L_x^2 L_t^2} \|h\|_{l_n^2 L_\lambda^2} \qquad \text{if} \quad |\lambda - n^3| \gtrsim n^2,$$
$$|\Lambda(f,g,h)| \lesssim \|G\|_{L_x^4 L_t^4} \|H\|_{L_x^4 L_t^4} \|f\|_{l_n^2 L_\lambda^2} \qquad \text{if} \quad |\lambda_1 - n_1^3| \gtrsim n^2,$$

where $\hat F(n,\lambda) = f(n,\lambda)\langle\lambda-n^3\rangle^{-\frac12}$, $\hat G(n,\lambda) = g(n,\lambda)\langle\lambda-n^3\rangle^{-\frac12}$ and $\hat H(n,\lambda) = h(n,\lambda)\langle\lambda-n^3\rangle^{-\frac12}$. Let us focus on the first of the above cases. Recalling that $\|h\|_{l_n^2 L_\lambda^2} \le 1$ by assumption, and using Cauchy-Schwarz, it appears that we have to estimate the terms $\|F\|_{L_x^4 L_t^4}$ and $\|G\|_{L_x^4 L_t^4}$. Recalling the Strichartz estimate [6, Proposition 7.15]

$$\|F\|_{L_x^4 L_t^4} \lesssim \|F\|_{X^{0,\frac13}} \tag{6}$$

it becomes plain that the terms $\|F\|_{X^{0,\frac13}}$ and $\|G\|_{X^{0,\frac13}}$ have to be controlled. To this end, define a a square-integrable function

$$\theta(x,t) = |\partial_x|^s e^{\sigma(I+|\partial_x|)} u(x,t) = \mathscr{F}_x^{-1}\left[|n|^s e^{\sigma\langle n\rangle} \mathscr{F}_x u\right](x,t)$$

where I denotes the identity operator. We also set

$$\hat\vartheta(n,\lambda) = |n|^s e^{\sigma\langle n\rangle} \widehat{\psi(t/\delta)} u(n,\lambda) = \mathscr{F}_t\left[\psi(t/\delta)(\mathscr{F}_x\theta)(n,t)\right](\lambda).$$

Using the Strichartz estimate (6) for the function $\psi(t/\delta)\theta$ yields

$$\begin{aligned} \sum_{n\neq 0}\int_{\mathbb{R}} |\hat\vartheta(n,\lambda)|^2 d\lambda &= \int_{\mathbb{R}}\int_{\mathbb{T}} \chi_{[-3,3]}(t/\delta)\left|\psi(t/\delta)\theta(x,t)\right|^2 dxdt \\ &\lesssim \delta^{\frac12}\|\psi(t/\delta)\theta\|^2_{L_x^4 L_t^4} \lesssim \delta^{\frac12}\|\psi(t/\delta)\theta\|^2_{X^{0,\frac13}} \\ &= \delta^{\frac12}\|\psi(t/\delta)u\|^2_{X^{\sigma,s,\frac13}}. \end{aligned} \tag{7}$$

By Hölder's inequality and (7), we get

$$\begin{aligned} \|F\|^2_{X^{0,\frac13}} &= \sum_{n\neq 0}\int_{\mathbb{R}} \langle\lambda-n^3\rangle^{-\frac13} f(n,\lambda)^2 d\lambda \\ &= \sum_{n\neq 0}\int_{\mathbb{R}} \left(|\widehat{\psi(t/\delta)}u(n,\lambda)|^2 |n|^{2s} e^{2\sigma\langle n\rangle}\right)^{\frac13} f(n,\lambda)^{\frac43} d\lambda \\ &\le \delta^{\frac16}\,\|\psi(t/\delta)u\|^2_{X^{\sigma,s,\frac12}}. \end{aligned} \tag{8}$$

Making use of the argument above, we deduce

$$\|G\|_{X^{0,\frac{1}{3}}} \lesssim \delta^{\frac{1}{12}} \|\psi(t/\delta)v\|_{X^{\sigma,s,\frac{1}{2}}} \lesssim \delta^{\frac{1}{12}} \|v\|_{Y^{\sigma,s,\frac{1}{2}}}$$

from Lemma 2.3. Thus the estimate in the case $|\lambda - n^3| \gtrsim n^2$ may be continued as follows.

$$\|FG\|_{L_x^2 L_t^2} \leq \|F\|_{L_x^4 L_t^4} \|G\|_{L_x^4 L_t^4} \lesssim \delta^{\frac{1}{12}} \|v\|_{Y^{\sigma,s,\frac{1}{2}}} \|\psi(t/\delta)u\|_{X^{\sigma,s,\frac{1}{2}}} . \tag{9}$$

For the case when $|\lambda_1 - n_1^3| \gtrsim n^2$, we use the Strichartz estimate (6) to find $\|H\|_{L_x^4 L_t^4} \lesssim \|h\|_{l_n^2 L_\lambda^2} \leq 1$. Recalling the definition of $f(n,\lambda)$, a similar argument yields as in the previous case yields

$$\|G\|_{L_x^4 L_t^4} \|f\|_{l_n^2 L_\lambda^2} \lesssim \delta^{\frac{1}{12}} \|\psi(t/\delta)u\|_{X^{\sigma,s,\frac{1}{2}}} \|v\|_{Y^{\sigma,s,\frac{1}{2}}} \tag{10}$$

Finally, interchanging f and g, we obtain

$$|\Lambda(f,g,h)| \lesssim \delta^{\frac{1}{12}} \|\psi(t/\delta)u\|_{X^{\sigma,s,\frac{1}{2}}} \|v\|_{Y^{\sigma,s,\frac{1}{2}}} \tag{11}$$

for the case $|\lambda - \lambda_1 - (n-n_1)^3| \gtrsim n^2$ by symmetry. Now based on (9)-(11), we have

$$\begin{aligned}\left\|\partial_x \left(\psi(t/\delta)^2 uv\right)\right\|_{X^{\sigma,s,-\frac{1}{2}}} &= \sup_{\|h\|_{l_n^2 L_\lambda^2} \leq 1} |\Lambda(f,g,h)| \\ &\lesssim \delta^{\frac{1}{12}} \|v\|_{Y^{\sigma,s,\frac{1}{2}}} \|\psi(t/\delta)u\|_{X^{\sigma,s,\frac{1}{2}}} .\end{aligned}$$

□

Remark 1. Note we have actually proved that

$$\|\partial_x(uv)\|_{X^{\sigma,s,-\frac{1}{2}}} \lesssim \|u\|_{X^{\sigma,s,\frac{1}{2}}} \|v\|_{X^{\sigma,s,\frac{1}{2}}} \tag{12}$$

for $s \geq 0$ and $\sigma \geq 0$.

The bilinear estimate for periodic KdV equation in Sobolev spaces with negative indices has been studied by Kenig, Ponce and Vega [19]. As the counterexample shows in [19, Theorem 1.4], the boundedness of the quadratic term fails for Sobolev indices below $-\frac{1}{2}$.

Corollary 1. *For functions u, v satisfying $\int_{\mathbb{T}} u = 0$, $\int_{\mathbb{T}} v = 0$, the estimate (12) holds for $s \geq -\frac{1}{2}$.*

Proof. According to the above remark, we only need to consider the case $-\frac{1}{2} \leq s \leq 0$. Let $\rho = -s \geq 0$, we follow the definition of multiplier bounds which was introduced by Tao [23]. It remains to show that

$$\left\| \frac{e^{\sigma\langle n\rangle} e^{-\sigma\langle n-n_1\rangle} e^{-\sigma\langle n_1\rangle} |n|^{1-\rho} |n_1|^\rho |n-n_1|^\rho}{\langle \lambda - n^3\rangle^{\frac{1}{2}} \langle \lambda_1 - n_1^3\rangle^{\frac{1}{2}} \langle \lambda - \lambda_1 - (n-n_1)^3\rangle^{\frac{1}{2}}} \right\|_{[3,\mathbb{Z}\times\mathbb{R}]} \lesssim 1.$$

Since $e^{\sigma|n|} e^{-\sigma|n-n_1|} e^{-\sigma|n_1|} \leq 1$, the comparison principle [23, Lemma 3.1] reduce this estimate to

$$\left\| \frac{|n|^{1-\rho} |n_1|^\rho |n-n_1|^\rho}{\langle \lambda - n^3\rangle^{\frac{1}{2}} \langle \lambda_1 - n_1^3\rangle^{\frac{1}{2}} \langle \lambda - \lambda_1 - (n-n_1)^3\rangle^{\frac{1}{2}}} \right\|_{[3,\mathbb{Z}\times\mathbb{R}]} \lesssim 1,$$

which has been proved by Kenig, Ponce and Vega [19, Theorem 1.2]. □

In order to estimate the bilinear term in space of $Z^{\sigma,s,\frac{1}{2}}$, it will necessary to analyze the proof of [8, Proposition 1]. We will prove the following result in analogy with discussions in [8, Proposition 1].

Lemma 3.2. *Let* $s \geq \frac{1}{2}$, $\sigma \geq 0$, $\int_{\mathbb{T}} udx = 0$, $\int_{\mathbb{T}} vdx = 0$ *and* $0 \leq \kappa \ll 1$. *Assume that* $\int_{\mathbb{T}} uvdx = 0$, $\|v\|_{Y^{\sigma,s-1,\frac{1}{2}}} < \infty$ *and* $\|\psi(t/\delta)u\|_{X^{\sigma,s-1,\frac{1}{2}}} < \infty$. *Then*

$$\left\| \frac{\langle n\rangle^s e^{\sigma\langle n\rangle} \widehat{\psi(t/\delta)^2 uv}(n,\lambda)}{\langle \lambda - n^3\rangle^{1-\kappa}} \right\|_{l^2_n L^1_\lambda} \lesssim \delta^{\frac{1}{200}} \|\psi(t/\delta)u\|_{X^{\sigma,s-1,\frac{1}{2}}} \|v\|_{Y^{\sigma,s-1,\frac{1}{2}}}.$$

Proof. Since $\int_{\mathbb{T}} uv = 0$, the quantity $\langle n\rangle^s$ can be replaced with $|n|^s$ in the left hand side of the estimate. Let square-integrable functions u_1 and u_2 be defined by

$$\widehat{u_1}(n,\lambda) = \langle \lambda - n^3\rangle^{\frac{1}{2}} |n|^{s-1} e^{\sigma\langle n\rangle} \widehat{\psi(t/\delta)v}(n,\lambda)$$
$$\widehat{u_2}(n,\lambda) = \langle \lambda - n^3\rangle^{\frac{1}{2}} |n|^{s-1} e^{\sigma\langle n\rangle} \widehat{\psi(t/\delta)u}(n,\lambda).$$

Since $e^{\sigma|n|} \leq e^{\sigma|n_1|} e^{\sigma|n-n_1|}$ and $|n|^{s-\frac{1}{2}} \leq |n-n_1|^{s-\frac{1}{2}} |n_1|^{s-\frac{1}{2}}$, we obtain

$$\begin{aligned}
&\frac{|n|^s e^{\sigma\langle n\rangle} |\widehat{\psi(t/\delta)^2 uv}(n,\lambda)|}{\langle \lambda - n^3\rangle^{1-\kappa}} \\
\leq &\sum_{\substack{n_1\neq 0,\\ n_1\neq n}} \int_{\mathbb{R}} \frac{|\widehat{u_1}(n-n_1,\lambda-\lambda_1)\widehat{u_2}(n_1,\lambda_1)| d\lambda_1}{\langle \lambda - n^3\rangle^{1-\kappa} \langle \lambda_1 - n_1^3\rangle^{\frac{1}{2}} \langle \lambda - \lambda_1 - (n-n_1)^3\rangle^{\frac{1}{2}}} \\
&\qquad \times \frac{e^{\sigma\langle n\rangle}|n|^s}{e^{\sigma\langle n_1\rangle} e^{\sigma\langle n-n_1\rangle} |n_1|^{s-1} |n-n_1|^{s-1}} \\
\leq &\sum_{\substack{n_1\neq 0,\\ n_1\neq n}} \int_{\mathbb{R}} \frac{|\widehat{u_1}(n-n_1,\lambda-\lambda_1)\widehat{u_2}(n_1,\lambda_1)| d\lambda_1}{\langle \lambda - n^3\rangle^{1-\kappa} \langle \lambda_1 - n_1^3\rangle^{\frac{1}{2}} \langle \lambda - \lambda_1 - (n-n_1)^3\rangle^{\frac{1}{2}}} \\
&\qquad \times |n|^{\frac{1}{2}} |n_1|^{\frac{1}{2}} |n-n_1|^{\frac{1}{2}} \\
:= &S(n,\lambda).
\end{aligned}$$

To estimate $\|S(n,\lambda)\|_{l^2_n L^1_\lambda}$ we note that the resonance relation (5) enables us to distinguish three cases once again.

If $|\lambda - \lambda_1 - (n-n_1)^3| \geq |n||n_1||n-n_1|$, $S(n,\lambda)$ can be dominated by

$$S(n,\lambda) \leq \sum_{\substack{n_1\neq 0,\\ n_1\neq n}} \int_{\mathbb{R}} \frac{|\widehat{u_1}(n-n_1,\lambda-\lambda_1)\widehat{u_2}(n_1,\lambda_1)| d\lambda_1}{\langle \lambda - n^3\rangle^{\frac{2}{3}-\kappa} \langle \lambda - n^3\rangle^{\frac{1}{3}} \langle \lambda_1 - n_1^3\rangle^{\frac{1}{2}}}.$$

Taking first the L^1_λ-norm, using the Cauchy-Schwarz inequality, and recognizing that $\int_{\mathbb{R}} |\langle \lambda - n^3\rangle^{-\frac{2}{3}+\kappa}|^2 d\lambda$ is finite, it follows from duality that

$$\begin{aligned}
\|S(n,\lambda)\|_{l^2_n L^1_\lambda} \lesssim \sup_{\|\widehat{u_3}\|_{l^2_n L^2_\lambda} \leq 1} \sum_{n,n_1} \int_{\mathbb{R}^2} &\widehat{u_1}(n-n_1,\lambda-\lambda_1)\widehat{u_2}(n_1,\lambda_1)\langle \lambda_1 - n_1^3\rangle^{-\frac{1}{2}} \\
&\times \widehat{u_3}(n,\lambda)\langle \lambda - n^3\rangle^{-\frac{1}{3}} d\lambda_1 d\lambda.
\end{aligned} \tag{13}$$

Now define $\widehat{u_2'}(n_1,\lambda_1) = \widehat{u_2}(n_1,\lambda_1)\langle \lambda_1 - n_1^3\rangle^{-\frac{1}{2}}$ and $\widehat{u_3'}(n,\lambda) = \overline{\widehat{u_3}(n,\lambda)\langle \lambda - n^3\rangle^{-\frac{1}{3}}}$. Note that from (6) and (8), we gain the estimates

$$\|u_2'\|_{L^4_x L^4_t} \lesssim \|u_2'\|_{X^{0,\frac{1}{3}}} \lesssim \delta^{\frac{1}{12}} \|\psi(t/\delta)u\|_{X^{\sigma,s-1,\frac{1}{2}}} \tag{14}$$

and

$$\|u_3'\|_{L^4_x L^4_t} \lesssim \|u_3'\|_{X^{0,\frac{1}{3}}} = \|\widehat{u_3}\|_{l^2_n L^2_\lambda}. \tag{15}$$

Thus, using Parseval's relation, (14)-(15) and Lemma 2.3, the estimate takes the form

$$\begin{aligned}\|S(n,\lambda)\|_{l_n^2 L_\lambda^1} &\lesssim \sup_{\|\widehat{u_3}\|_{l_n^2 L_\lambda^2}\le 1} \int_{\mathbb{T}\times\mathbb{R}} u_1 u_2' u_3' dtdx \\ &\lesssim \sup_{\|\widehat{u_3}\|_{l_n^2 L_\lambda^2}\le 1} \|u_1\|_{L_x^2 L_t^2} \|u_2'\|_{L_x^4 L_t^4} \|u_3'\|_{L_x^4 L_t^4} \\ &\lesssim \delta^{\frac{1}{12}} \|\psi(t/\delta)u\|_{X^{\sigma,s-1,\frac{1}{2}}} \|v\|_{Y^{\sigma,s-1,\frac{1}{2}}}.\end{aligned} \tag{16}$$

By symmetry, we also have

$$\|S(n,\lambda)\|_{l_n^2 L_\lambda^1} \lesssim \delta^{\frac{1}{12}} \|v\|_{Y^{\sigma,s-1,\frac{1}{2}}} \|\psi(t/\delta)u\|_{X^{\sigma,s-1,\frac{1}{2}}} \tag{17}$$

for the case $|\lambda_1 - n_1^3| \ge |n||n_1||n-n_1|$.

We now turn to the remaining case $|\lambda - n^3| \ge |n||n_1||n-n_1|$. This will be split into three subcases. Suppose first that we also have

$$|\lambda - \lambda_1 - (n-n_1)^3| \gtrsim (\delta|n||n-n_1||n_1|)^{\frac{1}{100}}.$$

Let $\widehat{u_1'}(n-n_1,\lambda-\lambda_1) = \widehat{u_1}(n-n_1,\lambda-\lambda_1)\langle\lambda-\lambda_1-(n-n_1)^3\rangle^{-\frac{1}{3}}$, and let $\widehat{u_2'}(n_1,\lambda_1) = \widehat{u_2}(n_1,\lambda_1)\langle\lambda_1 - n_1^3\rangle^{-\frac{1}{2}}$ as before. Then we deduce that

$$\begin{aligned}S(n,\lambda) &\le \sum_{\substack{n_1\ne 0,\\ n_1\ne n}} \int_{\mathbb{R}} \frac{|n|^{\frac{1}{2}}|n_1|^{\frac{1}{2}}|n-n_1|^{\frac{1}{2}}}{\langle\lambda-n^3\rangle^{1-\kappa}} \frac{\widehat{u_1'}(n-n_1,\lambda-\lambda_1)\widehat{u_2'}(n_1,\lambda_1)}{\langle\lambda-\lambda_1-(n-n_1)^3\rangle^{\frac{1}{6}}} d\lambda_1 \\ &\le \sum_{\substack{n_1\ne 0,\\ n_1\ne n}} \int_{\mathbb{R}} \frac{|n|^{\frac{1}{2}}|n_1|^{\frac{1}{2}}|n-n_1|^{\frac{1}{2}}}{\langle\lambda-n^3\rangle^{1-\kappa}} \frac{\widehat{u_1'}(n-n_1,\lambda-\lambda_1)\widehat{u_2'}(n_1,\lambda_1)}{(\delta|n|\ |n_1|\ |n-n_1|)^{\frac{1}{600}}} d\lambda_1,\end{aligned}$$

and the estimate continues as

$$\begin{aligned}\|S(n,\lambda)\|_{l_n^2 L_\lambda^1} &\lesssim \delta^{-\frac{1}{600}} \left\| \langle\lambda-n^3\rangle^{-\frac{1}{2}-\frac{1}{600}+\kappa} \sum_{n_1} \int_{\mathbb{R}} \widehat{u_1'}(n-n_1,\lambda-\lambda_1)\widehat{u_2'}(n_1,\lambda_1)d\lambda_1 \right\|_{l_n^2 L_\lambda^1} \\ &\lesssim \delta^{-\frac{1}{600}} \|u_1'u_2'\|_{L_x^2 L_t^2} \lesssim \delta^{-\frac{1}{600}} \|u_1'\|_{L_x^4 L_t^4} \|u_2'\|_{L_x^4 L_t^4}\end{aligned}$$

by using the Cauchy-Schwarz inequality, and the Plancherel theorem in the same way as in the previous case. It follows from (14) and (15) that

$$\|S(n,\lambda)\|_{l_n^2 L_\lambda^1} \lesssim \delta^{\frac{1}{12}-\frac{1}{600}} \|v\|_{Y^{\sigma,s-1,\frac{1}{2}}} \|\psi(t/\delta)u\|_{X^{\sigma,s-1,\frac{1}{2}}}. \tag{18}$$

Similarly, for the second subcase $|\lambda_1 - n_1^3| \gtrsim (\delta|n||n-n_1||n_1|)^{\frac{1}{100}}$, the argument above can be repeated, and (18) holds, as well.

We proceed to consider the third subcase where

$$\max\left\{|\lambda-\lambda_1-(n-n_1)^3|,\ |\lambda_1-n_1^3|\right\} \ll (\delta|n||n_1||n-n_1|)^{\frac{1}{100}}.$$

Since δ is taken to be a small number, we have $|\lambda - n^3| \approx |n||n_1||n-n_1|$. Therefore, it is plain that $\|S(n,\lambda)\|_{L_\lambda^1}$ can be majorized by

$$\sum_{\substack{n_1\ne 0,\\ n_1\ne n}} \int_{\mathcal{A}_2}\int_{\mathcal{A}_1} (|n||n_1||n-n_1|)^{\kappa-\frac{1}{2}} \widehat{u_1}(n-n_1,\lambda-\lambda_1)\widehat{u_2}(n_1,\lambda_1)d\lambda_1 d\lambda,$$

where the domain of integration is given by

$$\mathcal{A}_1(n,n_1,\lambda) = \{\lambda_1\in\mathbb{R}\ :\ |\lambda-\lambda_1-(n-n_1)^3| \le (\delta|n||n_1||n-n_1|)^{\frac{1}{100}}\}$$

and

$$\mathcal{A}_2(n,n_1)=\{\lambda_1\in\mathbb{R}\ :\ |\lambda_1-n_1^3|\le(\delta|n||n_1||n-n_1|)^{\frac{1}{100}}\}$$

Using the Cauchy-Schwarz inequality in each integral, the last expression is dominated by

$$\delta^{\frac{1}{200}+\frac{1}{200}}\sum_{\substack{n_1\neq 0,\\ n_1\neq n}}(|n||n_1||n-n_1|)^{\kappa-\frac{1}{2}+\frac{1}{200}+\frac{1}{200}}\|\widehat{u_1}(n-n_1,\lambda)\|_{L^2_\lambda}\|\widehat{u_2}(n_1,\lambda)\|_{L^2_\lambda}.$$

Now since $|n||n_1||n-n_1|$ takes only nonzero integer values, we may write

$$\begin{aligned}&\|S(n,\lambda)\|_{l^2_nL^1_\lambda}\\ &\lesssim\delta^{\frac{1}{100}}\left\|\sum_{n_1}\langle nn_1n-n_1\rangle^{\kappa-\frac{1}{2}+\frac{1}{100}}\|\widehat{u_1}(n-n_1,\lambda)\|_{L^2_\lambda}\|\widehat{u_2}(n_1,\lambda)\|_{L^2_\lambda}\right\|_{l^2_n}\\ &\lesssim\delta^{\frac{1}{100}}\|\widehat{u_1}(n,\lambda)\|_{l^2_nL^2_\lambda}\|\widehat{u_2}(n,\lambda)\|_{l^2_nL^2_\lambda}.\end{aligned}\tag{19}$$

Now recalling the definition of $\widehat{u_1}$ and $\widehat{u_2}$, it becomes clear that the estimated can be concluded in the same way as the previous cases. For more details of the last step we refer the reader to [8, page 200]. Combining estimates (16)-(19), we finish the proof of the lemma. □

4 A multilinear estimate

We shall use the multilinear estimate in a variant of [22, Lemma 4.2].

Lemma 4.1. *If $k\ge 1$, $s\ge 1$ and $\sigma\ge 0$, then*

$$\left\|\psi(t/\delta)\prod_{i=1}^k u_i\right\|_{X^{\sigma,s-1,\frac{1}{2}}}\lesssim\prod_{i=1}^k\|u_i\|_{Y^{\sigma,s,\frac{1}{2}}}.$$

Proof. Denote $h(n,\lambda)\in l^2_nL^2_\lambda$ and $\|h\|_{l^2_nL^2_\lambda}\le 1$. We let $\epsilon>0$ be a sufficiently small number, it follows that

$$\left\|\lambda^{\frac{1}{2}-\epsilon}\delta\hat{\psi}(\delta\lambda)\right\|_{L^2_\lambda}\lesssim\delta^\epsilon\lesssim 1,\quad\left\|\lambda^{-\epsilon}\delta\hat{\psi}(\delta\lambda)\right\|_{L^1_\lambda}\lesssim\delta^\epsilon\lesssim 1.\tag{20}$$

Since $s\ge 1>\frac{1}{2}$, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\left\|e^{\sigma\langle n\rangle}\hat{u}(n,\lambda)\right\|_{l^1_nL^1_\lambda}&=\sum_n\langle n\rangle^{-s}\langle n\rangle^s e^{\sigma\langle n\rangle}\int_{\mathbb{R}}|\hat{u}(n,\lambda)|d\lambda\\ &\lesssim\left\|e^{\sigma\langle n\rangle}\langle n\rangle^s\hat{u}(n,\lambda)\right\|_{l^2_nL^1_\lambda}\le\|u\|_{Y^{\sigma,s,\frac{1}{2}}}.\end{aligned}\tag{21}$$

We will only prove the Lemma 4.1 for $k\ge 3$, since the situation will be simpler when we deal with the case $k=1$ and $k=2$. We let $v_3=\prod_{i=3}^k u_i$. Since $e^{\sigma|n|}\le e^{\sigma|n-n_3|}e^{\sigma|n_3-n_4|}\cdots e^{\sigma|n_{k-1}|}$, it follows from (21) and the Young inequality,

$$\begin{aligned}&\left\|e^{\sigma\langle n\rangle}\widehat{v_3}\right\|_{l^1_nL^1_\lambda}\\ &=\sum_{n,n_3,\dots,n_{k-1}}\int_{\mathbb{R}}\int_{\mathbb{R}^{k-3}}e^{\sigma\langle n\rangle}|\widehat{u_3}(n-n_3,\lambda-\lambda_3)||\widehat{u_4}(n_3-n_4,\lambda_3-\lambda_4)|\times\cdots\\ &\quad\times|\widehat{u_{k-1}}(n_{k-2}-n_{k-1},\lambda_{k-2}-\lambda_{k-1})||\widehat{u_k}(n_{k-1},\lambda_{k-1})|d\lambda_3\cdots d\lambda_{k-1}d\lambda\end{aligned}$$

$$\leq \left\| e^{\sigma\langle n\rangle}\widehat{u_3} * \cdots * e^{\sigma\langle n\rangle}\widehat{u_k} \right\|_{l_n^1 L_\lambda^1}$$
$$\leq \prod_{i=3}^{k} \left\| e^{\sigma\langle n\rangle}\widehat{u_i} \right\|_{l_n^1 L_\lambda^1} \leq \prod_{i=3}^{k} \|u_i\|_{Y^{\sigma,s,\frac{1}{2}}}. \tag{22}$$

The multilinear form $\Lambda(h, u_1, u_2, v_3)$ is defined by

$$\Lambda(h, u_1, u_2, v_3) = \sum_{n,n_1,n_2} \int_{\mathbb{R}^4} e^{\sigma\langle n\rangle} \langle \lambda - n^3\rangle^{\frac{1}{2}-\epsilon} \langle n\rangle^{s-1+2\epsilon} |h(n,\lambda)|$$
$$\times |\widehat{u_1}(n-n_1, \lambda-\lambda_1)||\widehat{u_2}(n_1-n_2, \lambda_1-\lambda_2)|$$
$$\times |\widehat{v_3}(n_2, \lambda_2-\lambda_3)||\delta\hat{\psi}(\delta\lambda_3)| d\lambda_1 d\lambda_2 d\lambda_3 d\lambda$$

and, consequently,

$$\left\| \psi(t/\delta) \prod_{i=1}^{k} u_i \right\|_{X^{\sigma,s-1+2\epsilon,\frac{1}{2}-\epsilon}} = \sup_{\|h(n,\lambda)\|_{l_n^2 L_\lambda^2} \leq 1} \Lambda(h, u_1, u_2, v_3).$$

Let u_1', u_2' and v_3' be square integrable functions such that

$$\widehat{u_1'} = e^{\sigma\langle n\rangle}\widehat{u_1}, \quad \widehat{u_2'} = e^{\sigma\langle n\rangle}\widehat{u_2}, \quad \text{and} \quad \widehat{v_3'} = e^{\sigma\langle n\rangle}\widehat{v_3}.$$

Since $e^{\sigma|n|} \leq e^{\sigma|n-n_1|}e^{\sigma|n_1-n_2|}e^{\sigma|n_2|}$, we have

$$\Lambda(h, u_1, u_2, v_3)$$
$$\leq \sum_{n,n_1,n_2} \int_{\mathbb{R}^4} \langle \lambda - n^3\rangle^{\frac{1}{2}-\epsilon} \langle n\rangle^{s-1+2\epsilon} |h(n,\lambda)||\widehat{u_1'}(n-n_1, \lambda-\lambda_1)| \tag{23}$$
$$\times |\widehat{u_2'}(n_1-n_2, \lambda_1-\lambda_2)||\widehat{v_3'}(n_2, \lambda_2-\lambda_3)||\delta\hat{\psi}(\delta\lambda_3)| d\lambda_1 d\lambda_2 d\lambda_3 d\lambda$$

We denote by $\Lambda'(h, u_1', u_2', v_3')$ the right hand side of (23). As in the proof of [22, Lemma 4.2], estimate (20) gives

$$\Lambda'(h, u_1', u_2', v_3') \lesssim \|u_1'\|_{Y^{s,\frac{1}{2}}} \|u_2'\|_{Y^{s,\frac{1}{2}}} \|v_3'\|_{l_n^1 L_\lambda^1}.$$

Combining this estimate with (22) and (23), we get

$$\left\| \psi(t/\delta) \prod_{i=1}^{k} u_i \right\|_{X^{\sigma,s-1,\frac{1}{2}-\epsilon}} \lesssim \prod_{i=1}^{k} \|u_i\|_{Y^{\sigma,s,\frac{1}{2}}}.$$

The Lemma 4.1 follows for $k \geq 3$ by letting $\epsilon \to 0$ and the Fatou lemma. □

5 Proof of Theorem 1.1 It is indicated in [22] and [8] that up to a gauge transform, we can rewrite (1) as follows:

$$\begin{cases} \partial_t u + \partial_{xxx}^3 u + \mathbf{P}(\mathbf{P}(u^k)\partial_x u) = 0 \\ \qquad u(x,0) = u_0(x), \qquad x \in \mathbb{T}, \end{cases} \tag{24}$$

where $\mathbf{P}$ is the projection operator defined by $\mathbf{P}(u) = u - \int_{\mathbb{T}} u(x,t)dx$. The well-posedness problem of (1) is reduced to consider the initial value problem (24).

Since we have the embedding property (2), it is necessary to use the contraction principle on function space $Y^{\sigma,s,\frac{1}{2}}$. Let $r = \|u_0\|_{G^{\sigma,s}} < \infty$. By Lemma 2.1, there exists a constant $c_1 > 0$ such that

$$\|\psi(t/\delta)S(t)u_0\|_{Y^{\sigma,s,\frac{1}{2}}} \leq c_1 \|u_0\|_{G^{\sigma,s}}.$$

We aim to show that the integral operator

$$\Gamma(u) = \psi(t/\delta)S(t)u_0 - \psi(t/\delta)\int_0^t S(t-t')\psi^2(t'/\delta)\mathbf{P}\left(\mathbf{P}(u^k)\partial_x u\right)dt'$$

is a contraction map on the set $\mathfrak{B} = \{\|u\|_{Y^{\sigma,s,\frac{1}{2}}} \leq 2c_1 r\}$.

It is easy to check that $\partial_x u = \mathbf{P}(\partial_x u)$, $\mathbf{P}\partial_x = \partial_x \mathbf{P}$ and $\|\partial_x v\|_{Y^{\sigma,s-1,\frac{1}{2}}} \approx \|v\|_{Y^{\sigma,s,\frac{1}{2}}}$ for $v \in Y^{\sigma,s,\frac{1}{2}}$ and $\int_{\mathbb{T}} v(x,t)dx = 0$. It follows from Lemma 3.1 and Lemma 4.1 that

$$\begin{aligned}\left\|\psi(t/\delta)^2\mathbf{P}\left[\mathbf{P}(u^k)\partial_x u\right]\right\|_{X^{\sigma,s,-\frac{1}{2}}} &\approx \left\|\psi(t/\delta)^2\partial_x\left[\mathbf{P}(u^k)\mathbf{P}(\partial_x u)\right]\right\|_{X^{\sigma,s-1,-\frac{1}{2}}} \\ &\lesssim \delta^{\frac{1}{200}}\|u\|_{Y^{\sigma,s,\frac{1}{2}}}\left\|\psi(t/\delta)u^k\right\|_{X^{\sigma,s-1,\frac{1}{2}}} \\ &\lesssim \delta^{\frac{1}{200}}\|u\|^{k+1}_{Y^{\sigma,s,\frac{1}{2}}}.\end{aligned}$$

On the other hand, by Lemma 2.3, Lemma 3.2 with $\kappa = 0$, and Lemma 4.1,

$$\begin{aligned}&\left\|\frac{\langle n\rangle^s e^{\sigma\langle n\rangle}\widehat{\mathbf{P}(\psi(t/\delta)u^k)\mathbf{P}(\psi}(t/\delta)\partial_x u)(n,\lambda)}{\langle\lambda - n^3\rangle}\right\|_{l^2_n L^1_\lambda} \\ &\lesssim \delta^{\frac{1}{200}}\|\partial_x u\|_{Y^{\sigma,s-1,\frac{1}{2}}}\left\|\psi(t/\delta)u^k\right\|_{X^{\sigma,s-1,\frac{1}{2}}} \\ &\lesssim \delta^{\frac{1}{200}}\|u\|^{k+1}_{Y^{\sigma,s,\frac{1}{2}}}.\end{aligned}$$

Therefore, we have

$$\left\|\psi(t/\delta)^2\mathbf{P}\left(\mathbf{P}(u^k)\partial_x u\right)\right\|_{Z^{\sigma,s,\frac{1}{2}}} \lesssim \delta^{\frac{1}{200}}\|u\|^{k+1}_{Y^{\sigma,s,\frac{1}{2}}}.$$

Combining this estimate with Lemma 2.2, we deduce that there exists a constant $c_2 > 0$ such that

$$\|\Gamma(u)\|_{Y^{\sigma,s,\frac{1}{2}}} \leq c_1\|u_0\|_{G^{\sigma,s}} + c_2\delta^{\frac{1}{200}}\|u\|^{k+1}_{Y^{\sigma,s,\frac{1}{2}}}.$$

If we take

$$T < \delta < \left(\frac{1}{2^{k+1}c_2(c_1 r)^k}\right)^{200}$$

then $\Gamma(\mathfrak{B}) \subset \mathfrak{B}$.

We are now in a position to verify that Γ is a contraction. By a similar argument as above, it is not hard to show that

$$\|\Gamma(u) - \Gamma(v)\|_{Y^{\sigma,s,\frac{1}{2}}} \lesssim \delta^{\frac{1}{200}}\sum_{k-1\leq l\leq k}\|\psi(t/\delta)P_l(u,v)\|_{X^{\sigma,s-1,\frac{1}{2}}}\|u-v\|_{Y^{\sigma,s,\frac{1}{2}}},$$

where $P_l(u,v)$ is a homogeneous polynomial of degree l. Since $u, v \in \mathfrak{B}$, there exists a constant $c_3 > 0$ by Lemma 4.1, such that

$$\|\Gamma(u) - \Gamma(v)\|_{Y^{\sigma,s,\frac{1}{2}}} \leq c_3\delta^{\frac{1}{200}}r^k\|u-v\|_{Y^{\sigma,s,\frac{1}{2}}}.$$

If we set

$$T < \delta < \min\left\{\left(\frac{1}{2^{k+1}c_2(c_1 r)^k}\right)^{200}, \left(\frac{1}{2r^k c_3}\right)^{200}\right\},$$

then Γ is a contraction on $\mathfrak{B}$. It follows that Γ has a unique fixed point u in $\mathfrak{B}$ and u solves the initial value problem (1).

To prove continuous dependence on the initial data, suppose u and $\bar{u}$ are solutions corresponding to initial data u_0 and $\bar{u}_0$. Following the argument above, we arrive at

$$\|u-\bar{u}\|_{Y^{\sigma,s,\frac{1}{2}}} \le c\|u_0-\bar{u}_0\|_{G^{\sigma,s}} + \frac{1}{2}\|u-\bar{u}\|_{Y^{\sigma,s,\frac{1}{2}}}.$$

Combining this inequality with (2), continuous dependence in $C([0,T],G^{\sigma,s})$ of the solution on the initial data in $G^{\sigma,s}$ is immediate, as shown by the estimate

$$\|u-\bar{u}\|_{L^\infty([0,T],G^{\sigma,s})} \le c\|u-\bar{u}\|_{Y^{\sigma,s,\frac{1}{2}}} \le c\|u_0-\bar{u}_0\|_{G^{\sigma,s}}.$$

Remark 2. If we consider the integral operator

$$\Phi(u) = \psi(t)S(t)u_0 - \psi(t)\int_0^t S(t-t')\psi^2(t')\mathbf{P}\left[\mathbf{P}(u^k)\partial_x u\right]dt',$$

from a similar contraction argument and Corollary 1, it is a simple matter to establish the following corollary.

Corollary 2. *Let $s \ge \frac{1}{2}$ when $k=1$ and $s \ge 1$ when $k \ge 2$. The initial-value problem (1) is well-posed in the space $C([0,1],G^{\sigma,s})$ if initial data in $G^{\sigma,s}$, $\sigma>0$ is sufficiently small.*

Remark 3. Similarly as in the proof of [13, Lemma 6], we can prove the uniqueness of the solution (1) in $C([0,T],G^{\sigma,s})$ when $s>\frac{3}{2}$.

In fact, if $s>\frac{3}{2}$, from Hölder inequality,

$$\begin{aligned}\|\partial_x u\|_{L_x^\infty L_t^\infty} &= \sup_{0\le t\le T}\|\partial_x u\|_{L_x^\infty}\\ &\le \sup_{0\le t\le T}\left\|ne^{\sigma\langle n\rangle}\mathscr{F}_x u(n,t)\right\|_{l_n^1} \lesssim \sup_{0\le t\le T}\|u(\cdot,t)\|_{G^{\sigma,s}}<\infty.\end{aligned}\tag{25}$$

Suppose u and v are solutions to (1) in $C([0,T],G^{\sigma,s})$ with $u(x,0)=v(x,0)$. Let $e=u-v$. Using the fact $ee_{xxx}=\partial_x(ee_{xx})-\frac{1}{2}\partial_x(e_x^2)$, we get the estimate

$$\frac{d}{dt}\|e(\cdot,t)\|^2_{L^2(\mathbb{T})} \le cP(u,u_x,v,v_x)\|e(\cdot,t)\|^2_{L^2(\mathbb{T})}$$

where $P(u,u_x,v,v_x)$ is a polynomial with respect to u, u_x, v and v_x. From (25) and Gronwall's inequality, we know that $e\equiv 0$.

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