STABILITY OF SOLITARY WAVES FOR A NONLINEARLY DISPERSIVE EQUATION

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(Communicated by Walter A. Strauss)

Abstract. Solitary-wave solutions of a nonlinearly dispersive equation are considered. It is found that solitary waves are peaked or smooth waves, depending on the wave speed. The stability of the smooth solitary waves also depends on the wave speed. Orbital stability is proved for some wave speeds, while instability is proved for others.

1. Introduction. Consideration is given to the stability of solitary-wave solutions of a nonlinearly dispersive model equation. The equation in view is

\[ u_t - u_{xxt} + \omega u_x + 3uu_x = \gamma(2u_xu_{xx} + uu_{xxx}), \]

(1.1)

where \( \gamma \in \mathbb{R} \) and \( \omega \geq 0 \). This equation has recently been investigated with regard to well-posedness and singularity formation by Yin [25]. He also showed existence of smooth solitary waves for certain values of the parameters \( \omega \) and \( \gamma \). For \( \gamma = 1 \), the equation is known as the Camassa-Holm equation. It was first found by Fokas and Fuchssteiner [18], and later recovered as a water-wave model by Camassa, Holm and Hyman [7, 8]. Recently, Johnson has given an account of the role of the Camassa-Holm equation as a long-wave model for water waves in a long uniform channel [21]. Besides being a model equation for water waves, one reason for the interest in the Camassa-Holm equation is its integrable bi-Hamiltonian structure. This property has led to many interesting developments, a sample of which can be found in [6, 7, 8, 9, 17, 18, 19], and the references contained therein. One aspect of the integrability of the equation in case \( \gamma = 1 \) is that the solitary-wave solutions are solitons [7, 14], similar to the solitary-wave solutions of the Korteweg-deVries equation. Yet another application of the Camassa-Holm equation arises in the context of differential geometry, where it can be seen as a re-expression for geodesic flow on an infinite-dimensional Lie group [10, 24].

For \( \omega = 0 \), equation (1.1) reduces to a model equation for mechanical vibrations in a compressible elastic rod, as derived by Dai and Huo [15]. The range of the parameter \( \gamma \) is roughly from \(-29.5\) to \(3.4\). In this case, the equation is not a long-wave model, but rather a narrow-banded spectrum approximation.

In this article, the dynamic stability of solitary-wave solutions of (1.1) will be considered. It will be shown that for all \( \gamma \), the equation admits smooth stable solitary waves. When \( \gamma < 1 \), all solitary waves are stable. On the other hand,
when $\gamma > 1$, the solitary waves are either stable or unstable, depending on the 
wave speed.

The stability of solitary-wave solutions of the Camassa-Holm equation was proved 
in [14] by showing that the equation fits into the general framework of the theory 
of Grillakis, Shatah and Strauss [20]. For $\omega = 0$, but $\gamma \neq 1$, stability of the solitary 
waves was proved in [13] by a similar method. Specializing further, for $\gamma = 1$ and 
$\omega = 0$, the equation has the so-called peakon solutions. These are solutions of the form 
\[ u(x, t) = ce^{-|x-ct|}, \]
where $c$ is the wave speed. The stability of these special solutions was proved 
by Constantin and Strauss [12] using an elementary method. One other special 
case occurs when $\gamma = 0$. The equation is then known as the regularized long-wave 
equation [2], and there is an explicit formula for the solitary waves [4]. In fact, in 
this case it is possible to prove asymptotic stability of the solitary waves [23].

If $\omega$ and $\gamma$ are fixed to some values other than the special combinations mentioned 
above, then the stability properties of the solitary waves depend on the speed $c$. If 
$\gamma < 1$, then all solitary waves are orbitally stable in $H^1$. For $\gamma > 1$, solitary waves 
with speed $c$ close to $\omega$ are stable, but solitary waves with speed close to but less 
than $\frac{\omega^2}{\gamma-1}$ are unstable. If $c = \frac{\omega^2}{\gamma-1}$, the solitary waves are peaked waves similar 
to (1.2). For the peaked solitary waves, orbital stability remains unknown. The 
possible combinations are summarized in Figure 1.

To close the introduction, let us mention that equation (1.1) has three invariant 
integrals. The first is $\int_{-\infty}^{\infty} u \, dx$, and we do not make use of it. The other two are 
\[ E(u) = \int_{-\infty}^{\infty} (u^2 + u_x^2) \, dx, \] 
and 
\[ F(u) = \int_{-\infty}^{\infty} (u^3 + \gamma uu_x^2 + \omega u^2) \, dx. \]
Simple calculations show that $E(u)$ is invariant with respect to the time $t$. To see 
that $F$ is invariant, note that the operator $J = \frac{1}{2} \partial_x (1 - \partial_x^2)^{-1}$ is Hamiltonian, and 
that (1.1) can be written as a Hamiltonian system in the form 
\[ u_t + JF'(u) = 0. \] 

2. **Solitary waves.** As was mentioned in the Introduction, equation (1.1) features 
both smooth and peaked solitary waves. To be specific, for fixed $\omega$ and $\gamma$, there is 
one special speed $c = \frac{\omega^2}{\gamma-1}$, for which the solitary-wave solution is a peaked wave, 
similar to (1.2).

Following the usual method of obtaining an equation for solitary waves, suppose 
there are solutions of the form 
\[ u(x, t) = \phi(x - ct), \]
where $\phi(\xi)$ is a function which decays at infinity, and has a positive maximum. The 
independent variable is $\xi = x - ct$, and $c > \omega$. Inserting this form into the equation 
(1.1), there appears the ordinary differential equation 
\[ -c\phi_x + c\phi_{\xi\xi\xi} + \omega\phi_x + 3\phi\phi_x = \gamma (2\phi\phi_x + \phi\phi_{\xi\xi}). \]
Figure 1. Solitary waves exist for $c > \omega$, and $c \leq \frac{\omega}{\gamma - 1}$. In this figure, $\omega = 5$. The horizontally shaded area denotes the region of stable smooth solitary waves. Peak solitary waves exist on the curve $c = \frac{\omega}{\gamma - 1}$. Solitary waves with speeds $c$ close to the curve $c = \frac{\omega}{\gamma - 1}$ are unstable.

Since $\phi(\xi)$ approaches zero as $\xi \to \pm \infty$, this equation can be integrated to

$$(\omega - c)\phi + c\phi_{\xi\xi} + \frac{3}{2}\phi^2 = \gamma\phi_x^2 + \gamma\int_{-\infty}^{\xi} \phi\phi_{\xi\xi\xi\xi}.\tag{2.1}$$

The second term on the right can be integrated by parts to yield

$$\gamma\phi\phi_{\xi\xi} - \gamma\int_{-\infty}^{\xi} \phi_x\phi_{\xi\xi\xi} = \gamma\phi\phi_{\xi\xi} - \frac{\gamma}{2}\phi_x^2.\tag{2.2}$$

Next, we multiply by $\phi_x$ to obtain

$$(\omega - c)\phi_{\xi} + c\phi_{\xi\xi}\phi_x + \frac{3}{2}\phi^2\phi_x = \frac{\gamma}{2}\phi_x^2 + \gamma\phi\phi_{\xi\xi\xi},$$

and integrating once more yields

$$(\omega - c)\phi^2 + c\phi_x^2 + \phi^3 = \gamma\phi\phi_x^2.\tag{2.3}$$

Rearranging the equation, this appears as

$$(c - \omega)\left[\frac{c}{c - \omega}\phi_x^2 - \phi^2\right] = \phi(\gamma\phi_x^2 - \phi^2).\tag{2.4}$$
It is apparent that when $\gamma = \frac{c}{c - \omega}$, the solutions of (2.3) are peaked solitary waves given by the formula
\[ \phi(\xi) = (c - \omega)e^{-\sqrt{c-\omega}\xi}, \] (2.4)
where the wave speed is
\[ c = \frac{\omega \gamma}{\gamma - 1}. \] (2.5)
If $c$ does not have the special form (2.5), an explicit formula for the solutions has not been found, but phase plane analysis of equation (2.2) shows that since $c > \omega$, $\phi$ is a positive smooth function of maximal height $c - \omega$, symmetric around and monotonically decreasing from its crest. Moreover, $\phi$, $\phi_{\xi}$ and $\phi_{\xi\xi}$ are all exponentially decaying at infinity.

Even for smooth solitary waves, there are some conditions on the parameters $\omega$ and $\gamma$, and the speed $c$. In (2.3) it was assumed that $c \neq \omega$, but this can be seen directly from (2.2). Rearranging (2.2) as
\[ \phi_{\xi}^2(c - \gamma \phi) = \phi^2(c - \omega - \phi), \] (2.6)
it is clear that if $c = \omega$, then $\phi$ must be identically zero, because $\phi = 0$ at all points where $\phi_{\xi} = 0$. Since by assumption the solitary wave has a positive maximum, it also follows from (2.6) that $\omega < c$. Moreover, it can be seen that $\phi_{\xi}$ has only a single zero, located at the crest of the wave. Equation (2.6) also shows that in order for a unique homoclinic orbit to exist, the condition $c(\gamma - 1) < \omega \gamma$ is necessary. When $\gamma < 1$, this condition is always satisfied. On the other hand, when $\gamma > 1$ the peaked solitary waves arise as the limiting case of this relation.

3. Stability of smooth solitary waves. In this section, the dynamic stability of smooth solitary-wave solutions of (1.1) is proved. The proper notion of stability in the present context is orbital stability, where the invariance of the equation under a group action is taken into account. For (1.1), the group action is translation of the independent spatial variable $x$, and the following definition of stability is adopted from [14].

**Definition** The solitary wave $\phi$ of (1.1) is stable if for every $\epsilon > 0$ there is $\delta > 0$ such that if $u \in C([0,T);H^1(\mathbb{R}))$ for some $0 < T \leq \infty$ is a solution to (1.1) with $\|u(\cdot,0) - \phi\|_{H^1} \leq \delta$, then for every $t \in [0,T)$
\[ \inf_{s \in \mathbb{R}} \|u(\cdot,t) - \phi(\cdot - s)\|_{H^1} \leq \epsilon. \]

Let us briefly explain why it is essential to consider the infimum over all translations. Phase plane analysis of (2.2) shows that solitary waves of larger amplitude travel at a higher speed. So in particular, two solitary waves which may differ ever so slightly in height will drift apart as time passes, even though their crests may have been perfectly aligned initially. As a consequence, the usual notion of Lyapunov stability is not appropriate for the problem at hand. Instead, as was noted by Benjamin and Bona [1, 3], the proper framework to study the stability of solitary waves is the stability in shape. Taking the infimum over all translations effectively measures the difference in shape of two wave profiles. With this notion of stability and the appropriate definition, the following theorem can be stated.

**Theorem 3.1.** For $\gamma < 1$, all smooth solitary-wave solutions of (1.1) are stable.
In order to prove this general theorem, the assumption $\gamma < 1$ must be made. In the next section it will be seen that indeed for $\gamma > 1$, there exist unstable solitary waves.

Yin [25] has shown local well-posedness of equation (1.1) in $H^s$, when $s > \frac{3}{2}$. As was also shown by Yin, global well-posedness is prohibited by the existence of smooth solutions that develop an infinite slope in a finite time. Since local well-posedness in $H^1(\mathbb{R})$ has not been proved yet, it is worth note that in Theorem 1 can be rephrased as follows. If an initial disturbance $u(\cdot, 0)$ is close enough in $H^1$ to a solitary wave, and there is an associated solution $u(\cdot, t)$, then as long as the corresponding solution exists in $H^1$, it will stay close to a translate of the solitary wave.

To prove the orbital stability of the smooth solitary waves, use is made of the general theory of Grillakis, Shatah and Strauss [20]. Note that the present situation is slightly simpler than the case of the Korteweg-deVries equation [5], because the symplectic operator $J = \frac{1}{2} \partial_y (1 - \partial_x^2)^{-1}$ is surjective. First, observe that the equation for the solitary waves can be written in variational form as

$$cE'(\phi) = F'(\phi).$$  \tag{3.1}

Linearizing (3.1) about a solitary wave $\phi$ of speed $c$ yields the operator $$\mathcal{L}_c = (2\gamma \phi - 2c) \partial_x^2 + 2\gamma \phi' \partial_x - 6\phi + 2\gamma \phi'' + 2(c - \omega)$$ $$= -\partial_x ((2c - 2\gamma \phi) \partial_x) - 6\phi + 2\gamma \phi'' + 2(c - \omega).$$

Note that $2c - 2\gamma \phi$ is bounded below by a positive constant. Since $\phi$, $\phi'$ and $\phi''$ are exponentially decaying, the spectral equation $\mathcal{L}_c v = \lambda v$ can be transformed by the Liouville transformation

$$z = \int_0^x \frac{1}{2c - 2\gamma \phi(y)} dy,$$

and

$$\psi(z) = (2c - 2\gamma \phi(x))^{\frac{3}{2}} v(x),$$

into

$$\mathcal{L}_c \psi(z) = (-\partial_z^2 + q(z) + 2(c - \omega))\psi(z)$$ $$= \lambda \psi(z),$$

where

$$q_c(z) = -6\phi(x) + \frac{3\gamma}{2} \phi''(x) - \frac{\gamma^2 [\phi'(x)]^2}{4(c - \gamma \phi(x))}.$$ 

Since $q_c$ has exponential decay, it can be shown that the operator $\mathcal{L}$ has continuous spectrum $[2(c - \omega), \infty)$, and there are finitely many eigenvalues below $2(c - \omega)$. Moreover, the $n$-th eigenvalue in increasing order from the left has an associated eigenfunction with exactly $(n - 1)$ zeroes (cf. Dunford and Schwartz [16]). Note that (2.1) shows that $\mathcal{L}_c(\phi') = 0$, and we know that $\phi'$ has exactly one zero. Therefore 0 is the second eigenvalue from the left. Thus, it appears that there is only one negative eigenvalue for the operator $\mathcal{L}_c$. This is exactly the situation in which the theory in [20] can be applied. The crucial part in applying this result is now to show that the function $d(c) = cE(\phi) - F(\phi)$, is convex. This will be established by the following lemma.

**Lemma 3.2.** The function $d(c) = cE(\phi) - F(\phi)$, is convex.
Proof: Consider the first derivative
\[ d'(c) = \langle cE'(\phi) - F'(\phi), \frac{\partial \phi}{\partial c} \rangle + E(\phi) = E(\phi). \]

It will be shown that this is an increasing function of \( c \). Using (2.6), and the fact that \( \phi \) is even, one may write
\[ d'(c) = \int_{-\infty}^{\infty} \phi^2 + \phi^2 \frac{c - \omega - \phi}{c - \gamma \phi} d\xi. \]

Next, using (2.6) and the fact that \( \phi \) is positive and \( \phi_\xi \) is negative on \((0, \infty)\), we obtain
\[ d'(c) = 2 \int_0^\infty \frac{\phi}{\sqrt{c - \gamma \phi}} \phi \frac{2c - \omega - (\gamma + 1)\phi}{\sqrt{c - \gamma \phi}} d\xi. \]

Now the substitution \( y = c - \phi \) yields
\[ d'(c) = 2 \int_0^c (c - y) \frac{2c - \omega - (\gamma + 1)(c - y)}{\sqrt{c - \omega - (c - y)(c - y)}} dy. \]

This integral will be an increasing function of \( c \) if the integrand is. To check this, differentiate the integrand with respect to \( c \). The result is
\[ \frac{x}{c - \gamma x} \left[ \frac{\omega + (1 - \gamma)x}{2 \sqrt{c - \gamma x \sqrt{c - \omega - x}}} \right], \]
which is clearly positive for the values of \( x \) considered.
\[ \square \]
4. **Instability of smooth solitary waves.** In the previous section, the stability of solitary waves for $\gamma < 1$ was proved. For the case when $\gamma = 1$, the stability of solitary waves was proved by Constantin and Strauss in [14]. It turns out that as soon as $\gamma > 1$, there exist both stable and unstable solitary waves for equation (1.1). Note that in this case, the speed $c$ of the solitary waves is restricted by the relation $c(\gamma - 1) < \omega \gamma$.

**Theorem 4.1.** For $\gamma > 1$, there exist stable and unstable smooth solitary-wave solutions of (1.1).

According to the theory of Grillakis, Shatah and Strauss [20], and the discussion given in Section 3, to prove instability of a solitary wave, it is enough to show that the function $d'(c)$ is concave in a neighborhood of a solitary wave with speed $c_0$.

Thus the proof of Theorem 2 will follow from the following two lemmas.

**Lemma 4.2.** Suppose $\gamma > 1$. There is a value $c_1 > \omega$, such that $d''(c)$ is convex for $c < c_1$.

**Proof:** Remember that in the case at hand the derivative of $d$ is given by

$$d'(c) = 2 \int_\omega^c (c - y) \frac{(y - \omega) + (1 - \gamma)c + \gamma y}{\sqrt{y - \omega}(1 - \gamma)c + \gamma y} \, dy.$$

This function is defined for $\omega < c < \frac{\omega}{1 - \gamma}$. Taking the second derivative, there appears

$$d''(c) = 2 \int_\omega^c \frac{y - \omega + (1 - \gamma)c + \gamma y}{\sqrt{y - \omega}(1 - \gamma)c + \gamma y} \, dy + 2 \int_\omega^c (c - y) \frac{\sqrt{y - \omega}(1 - \gamma)c + \gamma y}{(y - \omega)(1 - \gamma)c + \gamma y} (1 - \gamma) \, dy$$

$$- 2 \int_\omega^c (c - y) \frac{[(y - \omega) + (1 - \gamma)c + \gamma y] \sqrt{y - \omega}(1 - \gamma)c + \gamma y}{(y - \omega)(1 - \gamma)c + \gamma y} \, dy.$$

Notice that combining the first two terms and rearranging the third yields

$$d''(c) = 2 \int_\omega^c \frac{-\omega + 2c + 2\gamma(y - c)}{\sqrt{y - \omega}(1 - \gamma)c + \gamma y} \, dy + (\gamma - 1) \int_\omega^c (c - y) \frac{y - \omega + (1 - \gamma)c + \gamma y}{\sqrt{y - \omega}(1 - \gamma)c + \gamma y} \, dy.$$

Now it appears that since $(1 - \gamma)c + \gamma y > 0$, the second term is positive for $\gamma > 1$. The first term is positive if the numerator is positive throughout the domain of integration. To show that this is possible, notice that the numerator is a linear function of $y$, and is always positive for $y = c$.

$$-\omega + 2c + 2\gamma(y - c) \bigg|_{y = c} = -\omega + 2c$$

$$> c,$$

and $c > 0$ by assumption. Thus the numerator is positive if it is also positive at the left endpoint of the interval. Evaluating the numerator at $y = \omega$ yields

$$-\omega + 2c + 2\gamma(y - c) \bigg|_{y = \omega} = (2\gamma - 1)\omega + 2\gamma(1 - \gamma).$$
This is positive for small enough $c$. To be precise, the last expression is positive as long as

$$c < \frac{(2\gamma - 1)\omega}{2(\gamma - 1)}.$$

\[\square\]

It follows that solitary waves with speeds close to $\omega$ are stable. On the other hand, the next lemma shows that solitary waves with speeds close to $\frac{\omega}{\gamma - 1}$ are unstable.

**Lemma 4.3.** Suppose $\gamma > 1$. There is a value $c_2 < \frac{\omega}{\gamma - 1}$, such that $d(c)$ is concave for $c > c_2$.

**Proof:** A lengthy computation shows that $d''(c)$, evaluated at the endpoint $\gamma = \frac{\omega}{\gamma - 1}$ is equal to negative infinity when $\gamma > 1$. In particular, we have the following result.

$$d'' \left( \frac{\omega\gamma}{\gamma - 1} \right) = K - \left( \frac{-2\omega\gamma + (\gamma + 1)\omega}{\gamma^2} \right) \int_\omega^{\frac{\omega\gamma}{\gamma - 1}} \frac{dy}{y - \omega} = K - \frac{\omega(1 - \gamma)}{\gamma^2} \int_\omega^{\frac{\omega\gamma}{\gamma - 1}} \frac{dy}{y - \omega},$$

where $K$ is a constant depending on $\omega$ and $\gamma$. Since $d'' \left( \frac{\omega}{\gamma - 1} \right) = -\infty$ for $\gamma > 1$, it appears that $d''(c)$ will be negative for values of $c$ close to $\frac{\omega}{\gamma - 1}$. Thus $d(c)$ is concave for those values.

\[\square\]

In light of these facts, it appears that for $\gamma > 1$, solitary waves with speed $c$ close to $\omega$ are stable, while solitary waves with speed $c$ close to $\frac{\omega}{\gamma - 1}$ are unstable. Since in the limit $c = \frac{\omega}{\gamma - 1}$, the theory fails to prove stability or instability, a different approach must be taken.

**Acknowledgments.** This work was performed while the author was supported by a research fellowship at Lund University. The author is pleased to acknowledge the support and encouragement of Professor Adrian Constantin as this work has developed.

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Received December 2002; revised May 2003.

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