A Pick function related to the sequence of volumes of the unit ball in $n$-space\

Christian Berg \( ^{†} \) and Henrik L. Pedersen \( ^‡ \)

\( ^† \) Institute of Mathematical Sciences, University of Copenhagen
Universitetsparken 5; DK-2100 København Ø, Denmark
E-mail berg@math.ku.dk

\( ^‡ \) Department of Basic Sciences and Environment
Faculty of Life Sciences, University of Copenhagen
Thorvaldsensvej 40, DK-1871 Frederiksberg C
E-mail henrikp@dina.kvl.dk

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Abstract

We show that

\[
F_a(x) = \frac{\ln \Gamma(x + 1)}{x \ln(ax)}
\]

is a Pick function for $a \geq 1$ and find its integral representation. We also consider the function

\[
f(x) = \left( \frac{\pi^{x/2}}{\Gamma(1 + x/2)} \right)^{1/(x \ln x)}
\]

and show that $\ln f(x + 1)$ is a Stieltjes function and that $f(x + 1)$ is completely monotonic on $[0, \infty]$. In particular $f(n) = \Omega_n^{1/(n \ln n)}$, $n \geq 2$ is a Hausdorff moment sequence. Here $\Omega_n$ is the volume of the unit ball in Euclidean $n$-space.

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1 Introduction and results

Since the appearance of the paper [3], monotonicity properties of the functions

\[ F_a(x) = \frac{\ln \Gamma(x + 1)}{x \ln(ax)}, \quad x > 0, a > 0 \]  

have attracted the attention of several authors in connection with monotonicity properties of the volume \( \Omega_n \) of the unit ball in Euclidean \( n \)-space. A recent paper about inequalities involving \( \Omega_n \) is [2].

Let us first consider the case \( a = 1 \). In [9] the authors proved that \( F_1 \) is a Bernstein function, which means that it is positive and has a completely monotonic derivative, i.e.,

\[ (-1)^{n-1} F_1^{(n)}(x) \geq 0, \quad x > 0, n \geq 1. \]  

This extended monotonicity and concavity proved in [4] and [12] respectively.

We actually proved a stronger statement than (2), namely that the reciprocal function \( x \ln x / \ln \Gamma(x + 1) \) is a Stieltjes transform, i.e. belongs to the Stieltjes cone \( S \) of functions of the form

\[ g(x) = c + \int_0^\infty \frac{d\mu(t)}{x + t}, \quad x > 0, \]  

where \( c \geq 0 \) and \( \mu \) is a non-negative measure on \([0, \infty[\) satisfying

\[ \int_0^\infty \frac{d\mu(t)}{1 + t} < \infty. \]

The result was obtained using the holomorphic extension of the function \( F_1 \) to the cut plane \( \mathcal{A} = \mathbb{C} \setminus (-\infty, 0] \), leading to an explicit formula for the measure \( \mu \) in (3). Our derivation used the fact that the holomorphic function \( \log \Gamma(z) \) only vanishes in \( \mathcal{A} \) at the points \( z = 1 \) and \( z = 2 \), a result interesting in itself and included as an appendix in [9]. A simpler proof of the non-vanishing of \( \log \Gamma(z) \) appeared in [10].

In a subsequent paper [10] we proved an almost equivalent result, namely that \( F_1 \) is a Pick function, and obtained the following representation formula

\[ F_1(z) = 1 - \int_0^\infty \frac{d_1(t)}{z + t} dt, \quad z \in \mathcal{A} \]  

where

\[ d_1(t) = \frac{\ln |\Gamma(1 - t)| + (k - 1) \ln t}{t((\ln t)^2 + \pi^2)} \quad \text{for} \quad t \in ]k - 1, k[, \quad k = 1, 2, \ldots \]  

and \( d_1(t) \) tends to infinity when \( t \) approaches 1, 2, \ldots. Since \( d_1(t) > 0 \) for \( t > 0 \), (2) is an immediate consequence of (4).
We recall that a Pick function is holomorphic function $\varphi$ in the upper half-plane $\mathbb{H} = \{ z = x + iy \in \mathbb{C} \mid y > 0 \}$ satisfying $\Im \varphi(z) \geq 0$ for $z \in \mathbb{H}$, cf. [11].

For $a = 2$ Anderson and Qiu proved in [4] that $F_2$ is strictly increasing on $[1, \infty[$, thereby proving a conjecture from [3]. Alzer proved in [2] that $F_2$ is concave on $[4, \infty[$. In [14] the concavity was extended to the optimal interval $\left[ \frac{1}{2}, \infty \right]$. 

We will now describe the main results of the present paper.

We also denote by $F_a$ the holomorphic extension of (1) to $\mathcal{A}$ with an isolated singularity at $z = 1/a$, which is a simple pole with residue $\ln \Gamma(1 + 1/a)$ assuming $a \neq 1$, while $z = 1$ is a removable singularity for $F_1$. For details about this extension see the beginning of section 2. Using the residue theorem we obtain:

**Theorem 1.1** For $a > 0$ the function $F_a$ has the integral representation

$$F_a(z) = 1 + \ln \Gamma(1 + 1/a) - \int_0^\infty \frac{d_a(t)}{z + t} \, dt, \quad z \in \mathcal{A} \setminus \{ 1/a \},$$

where

$$d_a(t) = \frac{\ln |\Gamma(1-t)| + (k-1) \ln (at)}{t((\ln(at))^2 + \pi^2)} \quad \text{for} \quad t \in [k-1,k[, \quad k = 1,2,\ldots, \quad (7)$$

and $d_a(0) = 0, d_a(k) = \infty, k = 1,2,\ldots$. We have $d_a(t) \geq 0$ for $t \geq 0, a \geq 1/2$ and $F_a$ is a Pick function for $a \geq 1$ but not for $0 < a < 1$.

From this follows the monotonicity property conjectured in [14]:

**Corollary 1.2** Assume $a \geq 1$. Then

$$(-1)^{n-1} F_a^{(n)}(x) > 0, \quad x > 1/a, n = 1,2,\ldots, \quad (8)$$

In particular, $F_a$ is strictly increasing and strictly concave on the interval $[1/a, \infty[$.

The function

$$f(x) = \left( \frac{\pi^{x/2}}{\Gamma(1 + x/2)} \right)^{1/(x \ln x)}$$

has been studied because the volume $\Omega_n$ of the unit ball in $\mathbb{R}^n$ is

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(1 + n/2)}, n = 1,2,\ldots.$$

We prove the following integral representation of the extension of $\ln f(x + 1)$ to the cut plane $\mathcal{A}$.

\footnote{This is slightly improved in Remark 2.6 below.}
Theorem 1.3 For $z \in \mathcal{A}$ we have

$$\log f(z+1) = -\frac{1}{2} + \frac{\ln(2/\sqrt{\pi})}{z} + \ln(\sqrt{\pi}) + \frac{1}{2} \int_{1}^{\infty} \frac{d_2((t-1)/2)}{z+t} \, dt. \quad (10)$$

In particular $1/2 + \log f(x+1)$ is a Stieltjes function and $f(x+1)$ is completely monotonic.

We recall that completely monotonic functions $\varphi : ]0, \infty[ \to \mathbb{R}$ are characterized by Bernstein’s theorem as

$$\varphi(x) = \int_{0}^{\infty} e^{-xt} \, d\mu(t), \quad (11)$$

where $\mu$ is a positive measure on $]0, \infty[$ such that the integrals above make sense for all $x > 0$.

We also recall that a sequence $\{a_n\}_{n \geq 0}$ of positive numbers is a Hausdorff moment sequence if it has the form

$$a_n = \int_{0}^{1} x^n \, d\sigma(x), \quad n \geq 0, \quad (12)$$

where $\sigma$ is a positive measure on the unit interval. Note that $\lim_{n \to \infty} a_n = \sigma\{1\}$.

For a discussion of these concepts see [7] or [17]. It is clear that if $\varphi$ is completely monotonic with the integral representation (11), then $a_n = \varphi(n+1), n \geq 0$ is a Hausdorff moment sequence, because

$$a_n = \int_{0}^{\infty} e^{-(n+1)t} \, d\mu(t) = \int_{0}^{1} x^n \, d\sigma(x),$$

where $\sigma$ is the image measure of $e^{-t} \, d\mu(t)$ under $e^{-t}$. Since $\lim_{x \to \infty} f(x+1) = e^{-1/2}$ we get

**Corollary 1.4** The sequence

$$f(n+2) = \Omega_{n+2}^{1/(n+2)\ln(n+2))}, n = 0, 1, \ldots \quad (13)$$

is a Hausdorff moment sequence tending to $e^{-1/2}$.

A Hausdorff moment sequence is clearly decreasing and convex and by the Cauchy-Schwarz inequality is even logarithmically convex, meaning that $a_n^2 \leq a_{n-1} a_{n+1}, n \geq 1$. The latter property was obtained in [14] in a different way.
2 Properties of the function $F_a$

In this section we will study the holomorphic extension of the function $F_a$ defined in (1). First a few words about notation. We use $\ln$ for the natural logarithm but only applied to positive numbers. The holomorphic extension of $\ln$ from the open half-line $]0, \infty[$ to the cut plane $\mathcal{A} = \mathbb{C} \setminus [-\infty, 0]$ is denoted $\log z = \ln |z| + i \arg z$, where $-\pi < \arg z < \pi$ is the principal argument. The holomorphic branch of the logarithm of $\Gamma(z)$ for $z$ in the simply connected domain $\mathcal{A}$ which equals $\ln \Gamma(x)$ for $x > 0$ is denoted $\log \Gamma(z)$. The imaginary part of $\log \Gamma(z)$ is a continuous branch of argument of $\Gamma(z)$ which we denote $\arg \Gamma(z)$, i.e.,

$$\log \Gamma(z) = \ln |\Gamma(z)| + i \arg \Gamma(z), \quad z \in \mathcal{A}.$$ 

We shall use the following property of $\log \Gamma(z)$, cf. [9, Lemma 2.1]

**Lemma 2.1** We have, for any $k \geq 1$,

$$\lim_{z \to t, 3z > 0} \log \Gamma(z) = \ln |\Gamma(t)| - i\pi k$$

for $t \in ]-k, -k+1[$ and

$$\lim_{z \to t, 3z > 0} |\log \Gamma(z)| = \infty$$

for $t = 0, -1, -2, \ldots$.

The expression

$$F_a(z) = \frac{\log (z + 1)}{z \log(az)}$$

clearly defines a holomorphic function in $\mathcal{A} \setminus \{1/a\}$, and $z = 1/a$ is a simple pole unless $a = 1$, where the residue $\ln \Gamma(1 + 1/a)$ vanishes.

**Lemma 2.2** For $a > 0$ and $t \leq 0$ we have

$$\lim_{y \to 0^+} \Im F_a(t + iy) = \pi d_a(-t), \quad (14)$$

where $d_a$ is given by (7).

**Proof.** For $-1 < t < 0$ we get

$$\lim_{y \to 0^+} F_a(t + iy) = \frac{\ln \Gamma(1 + t)}{t(\ln|a|t| + i\pi)},$$

hence $\lim_{y \to 0^+} \Im F_a(t + iy) = \pi d_a(-t)$. For $-k < t < -k+1$, $k = 2, 3, \ldots$ we find using Lemma 2.1

$$\lim_{y \to 0^+} F_a(t + iy) = \frac{\ln |\Gamma(1 + t)| - i(k - 1)\pi}{t(\ln|a|t| + i\pi)},$$
hence \( \lim_{y \to 0^+} \Im F_a(t + iy) = \pi d_a(-t) \) also in this case.

For \( t = -k, \ k = 1, 2, \ldots \) we have

\[
|F_a(-k + iy)| \geq \frac{|\ln|\Gamma(-k + 1 + iy)||}{|k + iy||\log(a(-k + iy))|} \to \infty
\]

for \( y \to 0^+ \) because \( \Gamma(z) \) has poles at \( z = 0, -1, \ldots \). Finally, for \( t = 0 \) we get (14) from the next Lemma.

\[ \Box \]

**Lemma 2.3** For \( a > 0 \) we have

\[
\lim_{z \to 0, z \in A} |F_a(z)| = 0.
\]

**Proof.** Since \( \log \Gamma(z + 1)/z \) has a removable singularity for \( z = 0 \) the result follows because \( |\log(az)| \geq |\ln|a||z|| \to \infty \) for \( |z| \to 0, z \in A \). \[ \Box \]

**Lemma 2.4** For \( a > 0 \) we have the radial behaviour

\[
\lim_{r \to \infty} F_a(re^{i\theta}) = 1 \text{ for } -\pi < \theta < \pi,
\]

and there exists a constant \( C_a > 0 \) such that for \( k = 1, 2, \ldots \) and \( -\pi < \theta < \pi \)

\[
|F_a((k + \frac{1}{2})e^{i\theta})| \leq C_a.
\]

**Proof.** We first note that

\[
F_a(z) = F_1(z) \frac{\log(z)}{\log(az)},
\]

and since

\[
\lim_{|z| \to \infty, z \in A} \frac{\log(z)}{\log(az)} = 1
\]

it is enough to prove the results for \( a = 1 \). We do this by using a method introduced in [9, Prop. 2.4].

Define

\[
R_k = \{ z = x + iy \in \mathbb{C} \mid -k \leq x < -k + 1, 0 < y \leq 1 \} \text{ for } k \in \mathbb{Z}
\]

and

\[
R = \bigcup_{k=0}^{\infty} R_k, \quad S = \{ z = x + iy \in \mathbb{C} \mid x \leq 1, |y| \leq 1 \}.
\]

By Lemma 2.1 it is clear that

\[
M_k = \sup_{|\theta| < \pi} |F_1((k + \frac{1}{2})e^{i\theta})| < \infty
\]

for each \( k = 1, 2, \ldots \), so it is enough to prove that \( M_k \) is bounded for \( k \to \infty \).
Stieltjes ([16, formula 20]) found the following formula for \(\log \Gamma(z)\) for \(z\) in the cut plane \(\mathcal{A}\)

\[
\log \Gamma(z + 1) = \ln \sqrt{2\pi} + (z + 1/2) \log z - z + \mu(z). \tag{19}
\]

Here

\[
\mu(z) = \sum_{n=0}^{\infty} h(z + n) = \int_{0}^{\infty} \frac{P(t)}{z + t} dt,
\]

where \(h(z) = (z + 1/2) \log(1 + 1/z) - 1\) and \(P\) is periodic with period 1 and \(P(t) = 1/2 - t\) for \(t \in [0, 1]\). A derivation of these formulas can also be found in [5]. The integral above is improper, and integration by parts yields

\[
\mu(z) = \frac{1}{2} \int_{0}^{\infty} \frac{Q(t)}{(z + t)^2} dt, \tag{20}
\]

where \(Q\) is periodic with period 1 and \(Q(t) = t - t^2\) for \(t \in [0, 1]\). Note that by (20) \(\mu\) is a completely monotonic function. For further properties of Binet’s function \(\mu\) see [13].

We claim that

\[|\mu(z)| \leq \frac{\pi}{8} \text{ for } z \in \mathcal{A} \setminus S.\]

In fact, since \(0 \leq Q(t) \leq 1/4\), we get for \(z = x + iy \in \mathcal{A}\)

\[|\mu(z)| \leq \frac{1}{8} \int_{0}^{\infty} \frac{dt}{(t + x)^2 + y^2}.\]

For \(x > 1\) we have

\[\int_{0}^{\infty} \frac{dt}{(t + x)^2 + y^2} \leq \int_{0}^{\infty} \frac{dt}{(t + 1)^2} = 1,
\]

and for \(x \leq 1, |y| \geq 1\) we have

\[\int_{0}^{\infty} \frac{dt}{(t + x)^2 + y^2} = \int_{1}^{\infty} \frac{dt}{t^2 + y^2} < \int_{1}^{\infty} \frac{dt}{t^2 + 1} = \pi.
\]

Since

\[F_1(z) = 1 + \frac{\ln \sqrt{2\pi} + 1/2 \log z - z + \mu(z)}{z \log z},\]

for \(z \in \mathcal{A}\), we immediately get (15) and

\[|F_1(z)| \leq 2 \tag{21}\]

for all \(z \in \mathcal{A} \setminus S\) for which \(|z|\) is sufficiently large. In particular, there exists \(N_0 \in \mathbb{N}\) such that

\[|F_1((k + \frac{1}{2})e^{i\theta})| \leq 2 \text{ for } k \geq N_0, \ (k + \frac{1}{2})e^{i\theta} \in \mathcal{A} \setminus S. \tag{22}\]
By continuity the quantity
\[ c = \sup \{ |\log \Gamma(z)| \mid z = x + iy, \frac{1}{2} \leq x \leq 1, 0 \leq y \leq 1 \} \] (23)
is finite.

We will now estimate the quantity \(|F_1((k + \frac{1}{2})e^{i\theta})|\) when \((k + \frac{1}{2})e^{i\theta} \in S\), and since \(F_1(z) = F_1(z)\), it is enough to consider the case when \((k + \frac{1}{2})e^{i\theta} \in R_{k+1}\). To do this we use the relation

\[ \log \Gamma(z + 1) = \log \Gamma(z + k + 1) - \sum_{l=1}^{k} \log(z + l) \] (24)

for \(z \in A\) and \(k \in \mathbb{N}\). Equation (24) follows from the fact that the functions on both sides of the equality sign are holomorphic functions in \(A\), and they agree on the positive half-line by repeated applications of the functional equation for the Gamma function.

For \(z = (k + \frac{1}{2})e^{i\theta} \in R_{k+1}\) we get \(|\log \Gamma(z + 1)| \leq c\) by (23), and hence by (24)

\[ |\log \Gamma(z + 1)| \leq c + \sum_{l=1}^{k} |\log(z + l)| \leq c + k\pi + \sum_{l=1}^{k} |\ln |z + l||. \]

For \(l = 1, \ldots, k - 1\) we have \(k - l < |z + l| < k + 2 - l\), hence \(0 < \ln |z + l| < \ln(k+2-l)\). Furthermore, \(1/2 \leq |z+k| \leq \sqrt{2}\), hence \(-\ln 2 < \ln |z+k| \leq (\ln 2)/2\). Inserting this we get

\[ |\log \Gamma(z + 1)| \leq c + k\pi + \sum_{j=2}^{k+1} \ln j < c + k\pi + k \ln(k + 1). \]

From this we get for \(z = (k + \frac{1}{2})e^{i\theta} \in R_{k+1}\)

\[ |F_1(z)| \leq \frac{c + k\pi + k \ln(k + 1)}{(k + \frac{1}{2}) \ln(k + \frac{1}{2})} \] (25)

which tends to 1 for \(k \to \infty\). Combined with (22) we see that there exists \(N_1 \in \mathbb{N}\) such that

\[ |F_1((k + \frac{1}{2})e^{i\theta})| \leq 2 \text{ for } k \geq N_1, -\pi < \theta < \pi, \]

which shows that \(M_k\) from (18) is a bounded sequence. \(\square\)

**Lemma 2.5** Let \(a > 0\). For \(k = 1, 2, \ldots\) there exists an integrable function \(f_{k,a} : [-k, -k + 1[ \to [0, \infty]\) such that

\[ |F_a(x + iy)| \leq f_{k,a}(x) \text{ for } -k < x < -k + 1, 0 < y \leq 1. \] (26)
Proof. For $z = x + iy$ as above we get using (24)

$$|\log \Gamma(z + 1)| \leq |\log \Gamma(z + k + 1)| + \sum_{l=1}^{k} |\log(z + l)| \leq L + k\pi + \sum_{l=1}^{k} |\ln|z + l||,$$

where $L$ is the maximum of $|\log \Gamma(z)|$ for $z \in \mathbb{R}$. We only treat the case $k \geq 2$ because the case $k = 1$ is a simple modification combined with Lemma 2.3.

For $l = 1, \ldots, k - 2$ we have $1 < |z + l| < 1 + k - l$, and for $l = k - 1, k$

$$|\log \Gamma(z + 1)| \leq (L + k\pi + \sum_{l=1}^{k-1} |\ln|z + l|| + |\ln|x + k||, \tag{27}$$

so as $f_{k,1}$ we can use the right-hand side of (27) divided by $(k - 1) \ln(k - 1)$.

Using (17) we next define

$$f_{k,a}(x) = f_{k,1}(x) \max_{z \in \mathbb{R}_k} |\log z| |\log(az)|.$$

Proof of Theorem 1.1

For fixed $w \in \mathcal{A} \setminus \{1/a\}$ we choose $\varepsilon > 0, k \in \mathbb{N}$ such that $\varepsilon < |w|, 1/a < k + \frac{1}{2}$ and consider the positively oriented contour $\gamma(k, \varepsilon)$ in $\mathcal{A}$ consisting of the half-circle $z = \varepsilon e^{i\theta}, \theta \in [-\pi, \pi]$ and the half-lines $z = x \pm i\varepsilon, x \leq 0$ until they cut the circle $|z| = k + \frac{1}{2}$, which closes the contour. By the residue theorem we find

$$\frac{1}{2\pi i} \int_{\gamma(k, \varepsilon)} \frac{F_a(z)}{z - w} \, dz = F_a(w) + \frac{\ln \Gamma(1 + 1/a)}{1/a - w}.$$

We now let $\varepsilon \to 0$ in the contour integration. By Lemma 2.3 the contribution from the half-circle with radius $\varepsilon$ will tend to zero, and by Lemma 2.2 and Lemma 2.5 we get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{F_a((k + \frac{1}{2})e^{i\theta})}{(k + \frac{1}{2})e^{i\theta} - w} (k + \frac{1}{2})e^{i\theta} \, d\theta + \int_{-k-\frac{1}{2}}^{0} \frac{d_a(-t)}{t - w} \, dt = F_a(w) + \frac{\ln \Gamma(1 + 1/a)}{1/a - w}.$$

For $k \to \infty$ the integrand in the first integral converges to 1 for each $\theta \in [-\pi, \pi]$ and by Lemma 2.4 Lebesgue’s theorem on dominated convergence can be applied, so we finally get

$$F_a(w) = 1 + \frac{\ln \Gamma(1 + 1/a)}{w - 1/a} - \int_{0}^{\infty} \frac{d_a(t)}{t + w} \, dt.$$

The last integral above appears as an improper integral, but we shall see that the integrand is Lebesgue integrable. We show below that $d_a(t) \geq 0$ when
$a \geq 1/2$ and for these values of $a$ the integrability is obvious. The function $d_a$ tends to 0 for $t \to 0$ and has a logarithmic singularity at $t = 1$ so $d_a$ is integrable over $[0,1]$. For $k - 1 < t < k$, $k \geq 2$ we have

$$d_a(t) = \frac{(\ln(t))^2 + \pi^2}{(\ln(at))^2 + \pi^2} d_1(t) + \frac{(k - 1) \ln a}{t ((\ln(at))^2 + \pi^2)},$$

and the factor in front of $d_1(t)$ is a bounded continuous function with limit 1 at 0 and at infinity. Therefore

$$\int_1^\infty \frac{|d_a(t)|}{t} dt < \infty$$

follows from the finiteness of the corresponding integral for $a = 1$ provided that we establish

$$S := \sum_{k=2}^\infty (k - 1) \int_{k-1}^k \frac{dt}{t^2 ((\ln(at))^2 + \pi^2)} < \infty.$$  

Choosing $N \in \mathbb{N}$ such that $aN > 1$, we can estimate

$$S < \sum_{k=1}^\infty \int_{ka}^{(k+1)a} \frac{dt}{t(\ln^2(t) + \pi^2)} < \int_a^N \frac{dt}{t(\ln^2(t) + \pi^2)} + \sum_{k=N+1}^\infty \int_{ka}^{(k+1)a} \frac{dt}{t \ln^2(t)}$$

$$= \int_a^N \frac{dt}{t(\ln^2(t) + \pi^2)} + \frac{1}{\ln(aN)} < \infty.$$

We next examine positivity of $d_a$.

For $0 < t < 1$ we have

$$d_a(t) = \frac{\ln |\Gamma(1-t)|}{t((\ln(at))^2 + \pi^2)} > 0$$

because $\Gamma(s) > 1$ for $0 < s < 1$.

For $k \geq 2$ and $t \in ]k-1,k[$ the numerator $N_a$ in $d_a$ can be written

$$N_a(t) = \ln \Gamma(k-t) + \sum_{l=1}^{k-1} \ln \frac{ta}{t-l},$$

where we have used the functional equation for $\Gamma$, hence

$$N_a(t) \geq \sum_{l=1}^{k-1} \ln \frac{k}{k-l} + (k - 1) \ln a = (k - 1) \ln k - \ln \Gamma(k) + (k - 1) \ln a,$$

because $\Gamma(k-t) > 1$ and $t/(t-l)$ is decreasing for $k - 1 < t < k$. From (19) we get

$$\ln \Gamma(k) = \ln \sqrt{2\pi} + (k - 1/2) \ln k + \mu(k)$$

(29)
and in particular for $k = 2$

$$\mu(2) = 2 - \frac{3}{2} \ln 2 - \ln \sqrt{2\pi}.$$  

Using (29) we find

$$N_a(t) \geq k - \frac{1}{2} \ln k - \ln \sqrt{2\pi} - \mu(k) + (k - 1) \ln a \geq k - \frac{1}{2} \ln k - 2 + \frac{3}{2} \ln 2 + (k - 1) \ln a,$$

because $\mu$ is decreasing on $]0, \infty[$ as shown by (20).

For $a \geq 1/2$ and $k - 1 < t < k$ with $k \geq 2$ we then get

$$N_a(t) \geq k(1 - \ln 2) - \frac{1}{2} \ln k + \frac{5}{2} \ln 2 - 2 \geq 0,$$

because the sequence $c_k, k \geq 2$ on the right-hand side is increasing with $c_2 = 0$.

We also see that $d_a(t)$ tends to infinity for $t$ approaching the end points of the interval $]k - 1, k[$. For $z = 1/a + iy, y > 0$ we get from (6)

$$\Im F_a(1/a + iy) = -\frac{\ln \Gamma(1 + 1/a)}{y} + \int_0^\infty \frac{yd_a(t)}{(1/a + t)^2 + y^2} dt.$$

The last term tends to 0 for $y \to 0$ while the first term tends to $-\infty$ when $0 < a < 1$. This shows that $F_a$ is not a Pick function for these values of $a$. □

Remark 2.6 We proved in Theorem 1.1 that $d_a(t)$ is non-negative on $[0, \infty[$ for $a \geq 1/2$. This is not best possible, and we shall explain that the smallest value of $a$ for which $d_a(t)$ is non-negative is $a_0 = 0.3681154742...$

Replacing $k$ by $k + 1$ in the numerator $N_a$ for $d_a$ given by (7), we see that

$$N_a(t) = \ln |\Gamma(1 - t)| + k\ln(at) \text{ for } t \in ]k, k + 1[, \ k = 1, 2, \ldots$$

is non-negative if and only if

$$\ln(1/a) \leq \ln(k + s) + \frac{1}{k} \ln |\Gamma(1 - k - s)| \text{ for } s \in ]0, 1[, \ k = 1, 2, \ldots,$$

and using the reflection formula for $\Gamma$ this is equivalent to $\ln(1/a) \leq \rho(k, s)$ for all $0 < s < 1$ and all $k = 1, 2, \ldots$, where

$$\rho(k, s) = \ln(k + s) - \frac{1}{k} \ln \left( \Gamma(k + s) \frac{\sin(\pi s)}{\pi} \right).$$  (30)

Using Stieltjes’ formula (19), we find that

$$\rho(k, s) = 1 + \frac{\ln(\pi/2)}{2k} - (1/k) [(s - 1/2) \ln(s + k) + \ln \sin(\pi s) - s + \mu(s + k)]$$  (31)
for all $s \in ]0,1[$ and $k = 1, 2, \ldots$. For fixed $s \in ]0,1[$ we see that $\rho(k,s) \to 1$ as $k \to \infty$, so $\ln(1/a) \leq 1$ is a necessary condition for non-negativity of $d_a(t)$. This condition is not sufficient, because for $\ln(1/a) = 1$ the inequality $1 \leq \rho(k,s)$ is equivalent to

$$0 \geq \frac{1}{2} \ln(2/\pi) + (s - 1/2) \ln(s + k) + \ln \sin(\pi s) - s + \mu(s + k)$$

which does not hold when $k$ is sufficiently large and $1/2 < s < 1$.

For each $k = 1, 2, \ldots$ it is easy to verify that the function $\rho_k(s) = \rho(k,s)$ has a unique minimum $m_k$ over $]0,1[$, and clearly

$$\ln(1/a_0) = \inf \{m_k, k \geq 1\} \quad (32)$$

determines the smallest value of $a$ for which $d_a(t)$ is non-negative. Using Maple one obtains that $m_k$ is decreasing for $k = 1, \ldots, 510$ and increasing for $k \geq 510$ with limit 1. Therefore $m_{510} = \inf m_k = 0.9993586013...$ corresponding to $a_0 = 0.3681154742...$ We add that $m_1 = 1.6477352344.., m_{178} = 1.0000028637.., m_{179} = 0.9999936630...$

### 3 Properties of the function $f$

**Proof of Theorem 1.3** The function

$$\ln f(x) = \frac{(x/2) \ln \pi - \ln \Gamma(1 + x/2)}{x \ln x}$$

clearly has a meromorphic extension to $\mathcal{A} \setminus 1$ with a simple pole at $z = 1$ with residue $\ln 2$. We denote this meromorphic extension $\log f(z)$ and have

$$\log f(z + 1) = \frac{\ln \sqrt{\pi}}{\Log(z+1)} - \frac{1}{2} F_2 \left( \frac{z + 1}{2} \right).$$

Using the representation (6), we immediately get (10). It is well-known that $1/\Log(z+1)$ is a Stieltjes function, cf. [8, p.130], and the integral representation is

$$\frac{1}{\Log(z+1)} = \int_1^\infty \frac{dt}{(z + t)((\ln(t - 1))^2 + \pi^2)}. \quad (33)$$

It follows that $\ln(\sqrt{\pi}f(x + 1))$ is a Stieltjes function, in particular completely monotonic, showing that $\sqrt{\pi}f(x + 1)$ belongs to the class $\mathcal{L}$ of logarithmically completely monotonic functions studied in [15] and in [6]. Therefore also $f(x + 1)$ is completely monotonic. □
4 Representation of $1/F_a$

For $a > 0$ we consider the function

$$G_a(z) = 1/F_a(z) = \frac{z \log(a)}{\log \Gamma(z + 1)}$$ (34)

which is holomorphic in $\mathcal{A}$ with an isolated singularity at $z = 1$, which is a simple pole with residue $\ln a/\Psi(2) = \ln a/(1 - \gamma)$ if $a \neq 1$, while it is a removable singularity when $a = 1$. Here $\Psi(z) = \Gamma'(z)/\Gamma(z)$ and $\gamma$ is Euler’s constant.

**Theorem 4.1** For $a > 0$ the function $G_a$ has the integral representation

$$G_a(z) = 1 + \frac{\ln a}{(1 - \gamma)(z - 1)} + \int_0^\infty \frac{\rho_a(t)}{z + t} \, dt, \quad z \in \mathcal{A} \setminus \{1\},$$ (35)

where

$$\rho_a(t) = t \frac{\ln |\Gamma(1 - t)| + (k - 1) \ln(at)}{(\ln |\Gamma(1 - t)|)^2 + ((k - 1)\pi)^2} \quad \text{for} \quad t \in ]k - 1, k[, \quad k = 1, 2, \ldots, \quad (36)$$

and $\rho_a(0) = 1/\gamma, \rho_a(k) = 0, \ k = 1, 2, \ldots$, which makes $\rho_a$ continuous on $[0, \infty[$. We have $\rho_a(t) \geq 0$ for $t \geq 0$, $a \geq a_0 = 0.3681154742\ldots$, cf. Remark 2.6, and $G_a(x + 1)$ is a Stieltjes function for $a \geq 1$ but not for $0 < a < 1$.

**Proof.** We notice that for $-k < t < -k + 1, k = 1, 2, \ldots$ we get using Lemma 2.1

$$\lim_{y \to 0^+} G_a(t + iy) = \frac{t(\ln(|t|) + i\pi)}{\ln |\Gamma(1 + t)| - i(k - 1)\pi},$$

and for $t = -k, k = 1, 2, \ldots$ we get

$$\lim_{y \to 0^+} |G_a(-k + iy)| = 0$$

because of the poles of $\Gamma$, hence $\lim_{y \to 0^+} \Re G_a(t + iy) = -\pi \rho_a(-t)$ for $t < 0$.

For fixed $w \in \mathcal{A} \setminus \{1\}$ we choose $\varepsilon > 0, k \in \mathbb{N}$ such that $\varepsilon < |w|, 1 < k + \frac{1}{2}$ and consider the positively oriented contour $\gamma(k, \varepsilon)$ in $\mathcal{A}$ which was used in the proof of Theorem 1.1.

By the residue theorem we find

$$\frac{1}{2\pi i} \int_{\gamma(k, \varepsilon)} G_a(z) \, dz = G_a(w) + \frac{\ln a}{(1 - \gamma)(1 - w)}.$$

We now let $\varepsilon \to 0$ in the contour integration. The contribution from the $\varepsilon$-half circle tends to 0 and we get

$$\frac{1}{2\pi} \int_0^\pi G_a((k + \frac{1}{2})e^{i\theta}) \, d\theta - \int_{-k}^0 \rho_a(-t) \, dt = G_a(w) + \frac{\ln a}{(1 - \gamma)(1 - w)}.$$
Finally, letting $k \to \infty$ we get (35), leaving the details to the reader. Clearly, $\rho_a \geq 0$ if and only if $d_\alpha$ defined in (7) is non-negative. It follows that $G_a(x + 1)$ is a Stieltjes function for $a \geq 1$ but not for $0 < a < 1$, since in the latter case $\Re G_a(1 + iy) > 0$ for $y > 0$ sufficiently small. \hfill \Box

**Remark 4.2** The integral representation in Theorem 4.1 was established in [9, (6)] in the case of $a = 1$. Since

$$G_a(z) = G_1(z) + \ln(a)\frac{z}{\log \Gamma(z + 1)},$$

the formula for $G_a$ can be deduced from the formula for $G_1$ and the following formula

$$\frac{z}{\log \Gamma(z + 1)} = \frac{1}{(1 - \gamma)(z - 1)} + \int_0^\infty \frac{\tau(t) dt}{z + t}, \quad z \in \mathcal{A} \setminus \{1\}, \quad (37)$$

where

$$\tau(t) = \frac{(k - 1)t}{(\ln |\Gamma(1 - t)|)^2 + ((k - 1)\pi)^2} \quad \text{for} \quad t \in ]k - 1, k[ \setminus k \in \mathbb{Z}, \quad k = 1, 2, \ldots \quad (38)$$

**References**


