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**Basic Modern Algebraic
Geometry**

**Introduction to Grothendieck's Theory of
Schemes**

Basic Modern Algebraic Geometry

Introduction to Grothendieck's Theory of Schemes

by

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1 Preliminaries

1.1 Categories

1.1.1 Objects and morphisms

A *category* \mathcal{C} is defined by the following data:

1. A collection of *objects* denoted by $\text{Obj}(\mathcal{C})$
2. For any two objects $A, B \in \text{Obj}(\mathcal{C})$ there is a *set* denoted by $\text{Hom}_{\mathcal{C}}(A, B)$, and referred to as *the set of morphisms from A to B*.
3. For any three objects A, B and C there is a *rule of composition for morphisms*, that is to say, a mapping

$$\text{Hom}_{\mathcal{C}}(A, B) \times \text{Hom}_{\mathcal{C}}(B, C) \longrightarrow \text{Hom}_{\mathcal{C}}(A, C)$$

denoted as

$$(\varphi, \psi) \mapsto \psi \circ \varphi$$

In general the collection \mathcal{C} is not a set, in the technical sense of set theory. Indeed, the collection of *all possible sets*, which we denote by Set , form a category. For two sets A and B the set $\text{Hom}_{\text{Set}}(A, B)$ is the set of all mappings from A to B ,

$$\text{Hom}_{\text{Set}}(A, B) = \{\varphi \mid \varphi : A \longrightarrow B\}.$$

For the category Set the composition of morphisms is nothing but the usual composition of mappings.

For a general category we impose some conditions on the rule of composition of morphisms, which ensures that all properties of mappings of sets, which are expressible in terms of diagrams, are valid for the rule of composition of morphisms in any category.

Specifically, this is an immediate consequence of the following two conditions:

Condition 1.1.1.1 *There is a morphism $\text{id}_A \in \text{Hom}_{\mathcal{C}}(A, A)$, referred to as the identity morphism on A , such that for all $\varphi \in \text{Hom}_{\mathcal{C}}(A, B)$ we have $\varphi \circ \text{id}_A = \varphi$, and for all $\psi \in \text{Hom}_{\mathcal{C}}(C, A)$ we have $\text{id}_A \circ \psi = \psi$.*

and

Condition 1.1.1.2 *Composition of morphisms is associative, in the sense that whenever one side in the below equality is defined, so is the other and equality holds:*

$$(\varphi \circ \psi) \circ \xi = \varphi \circ (\psi \circ \xi)$$

1.1.2 Small categories

A category \mathcal{S} such that $\text{Obj}(\mathcal{S})$ is a set is called a *small category*. Such categories are important in certain general constructions which we will come to later.

1.1.3 Examples: Groups, rings, modules and topological spaces

We have seen one example already, namely the category *Set*. We list some others below.

Example 1.1.3.1 *The collection of all groups form a category, the morphisms being the group-homomorphisms. This category is denoted by Grp*

Example 1.1.3.2 *The collection of all Abelian groups form a category, the morphisms being the group-homomorphisms. This category is denoted by Ab .*

Example 1.1.3.3 *The collection of all rings form a category, the morphisms being the ring-homomorphisms. This category is denoted by Ring .*

Example 1.1.3.4 *The collection of all commutative rings with 1 form a category, the morphisms being the ring-homomorphisms which map 1 to 1. This category is denoted by Comm . Note the important condition of the unit element being mapped to the unit element!*

Example 1.1.3.5 *Let A be a commutative ring with 1. The collection of all A -modules form a category, the morphisms being the A -homomorphisms. This category is denoted by Mod_A .*

Example 1.1.3.6 *The class of all topological spaces, together with the continuous mappings, form a category which we denote by Top .*

1.1.4 The dual category

If \mathcal{C} is a category, then we get another category \mathcal{C}^* by keeping the objects, but putting

$$\text{Hom}_{\mathcal{C}^*}(A, B) = \text{Hom}_{\mathcal{C}}(B, A).$$

It is a trivial exercise to verify that \mathcal{C}^* is then a category. It is referred to as the *dual category of \mathcal{C}* .

Instead of writing $\varphi \in \text{Hom}_{\mathcal{C}}(A, B)$, we employ the notation

$$\varphi : A \longrightarrow B,$$

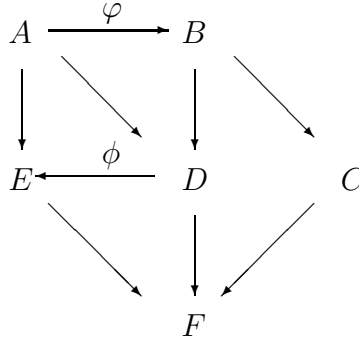
which is more in line with our usual thinking. If we have the situation

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ f \downarrow & & \downarrow g \\ C & \xrightarrow{\psi} & D \end{array}$$

and the two compositions are the same, then we say that *the diagram commutes*. This language is also used for diagrams of different shapes, such as triangular ones, with the obvious modification. A complex diagram consisting of several sub-diagrams is called commutative if all the subdiagrams commute, and this is so if all the subdiagrams commute in the diagram obtained by adding in some (or all) compositions: Thus for instance the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\varphi} & B & & \\ \downarrow & & \downarrow & \searrow & \\ E & \xleftarrow{\phi} & D & & C \\ & \searrow & & \swarrow & \\ & & F & & \end{array}$$

commutes if and only if all the subdiagrams in



commute.

If A is an object in the category \mathcal{C} , then we define the *fiber category over A* , denoted by \mathcal{C}_A , by taking as objects

$$\{(B, \varphi) \mid \varphi : B \longrightarrow A\},$$

and letting

$$\text{Hom}_{\mathcal{C}_A}((B, \varphi), (C, \psi)) = \{f \in \text{Hom}_{\mathcal{C}}(B, C) \mid \psi \circ f = \varphi\}$$

1.1.5 The topology on a topological space viewed as a category

Let X be a topological space. We define a category $\text{Top}(X)$ by letting the objects be the set of all open subsets of X , and for two open subsets U and V we let $\text{Hom}(U, V)$ be the set whose only element is the inclusion mapping if $U \subseteq V$, and \emptyset otherwise. This is a category, as is easily verified. If $U \subseteq X$ is an open subset, then the category $\mathcal{T}op(X)_U$ is nothing but the category $\mathcal{T}op(U)$.

1.1.6 Monomorphisms and epimorphisms

We frequently encounter two important classes of morphisms in a general category:

Definition 1.1.6.1 (Monomorphisms) *Let $f : Y \longrightarrow X$ be a morphism in the category \mathcal{C} . We say that f is a monomorphism if $f \circ \psi_1 = f \circ \psi_2$ implies that $\psi_1 = \psi_2$.*

In other words, the situation

$$\begin{array}{ccccc}
& & \psi_1 & & \\
Z & \xrightarrow{\quad} & Y & \xrightarrow{f} & X \\
& & \psi_2 & &
\end{array}$$

where

$$f \circ \psi_1 = f \circ \psi_2$$

implies that $\psi_1 = \psi_2$.

To say that $f : X \longrightarrow Y$ is a monomorphism is equivalent to asserting that for all Z the mapping

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(Z, Y) &\xrightarrow{\text{Hom}_{\mathcal{C}}(Z, f)} \text{Hom}_{\mathcal{C}}(Z, Z) \\ \psi &\mapsto f \circ \psi \end{aligned}$$

is an injective mapping of sets.

Proposition 1.1.6.1 *1. The composition of two monomorphisms is again a monomorphism*

2. If $f \circ g$ is a monomorphism then so is g .

The dual concept to a monomorphism is that of an *epimorphism*:

Definition 1.1.6.2 (Epimorphisms) *Let $f : X \longrightarrow Y$ be a morphism in the category \mathcal{C} . We say that f is an epimorphism if*

$$\psi_1 \circ f = \psi_2 \circ f \implies \psi_1 = \psi_2.$$

In other words, the situation

$$\begin{array}{ccc} & & \psi_1 \\ X & \xrightarrow{f} & Y & \xrightarrow{\quad} & Z \\ & & & \xrightarrow{\quad} & \\ & & & & \psi_2 \end{array}$$

where

$$\psi_1 \circ f = \psi_2 \circ f$$

implies that $\psi_1 = \psi_2$. To say that $f : X \longrightarrow Y$ is an epimorphism is equivalent to asserting that for all Z the mapping

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(Y, Z) &\xrightarrow{\text{Hom}_{\mathcal{C}}(f, Z)} \text{Hom}_{\mathcal{C}}(X, Z) \\ \psi &\mapsto \psi \circ f \end{aligned}$$

is an injective mapping of sets.

Proposition 1.1.6.2 1. *The composition of two epimorphisms is again an epimorphism*
 2. *If $f \circ g$ is an epimorphism then so is f .*

For some of the categories we most frequently encounter, monomorphisms are injective mappings, while the epimorphisms are surjective mappings. This is the case for $\mathcal{S}et$, as well as for the category $\mathcal{M}od_R$ of R -modules over a ring R . But for topological spaces a morphism (i.e. a continuous mapping) is an epimorphism if and only if the image of the source space is dense in the target space. Monomorphisms are the injective, continuous mappings, however.

At any rate, this phenomenon motivates the usual practice of referring to epimorphisms as *surjections* and monomorphisms as *injections*. A morphism which is both is said to be *bijective*. But this concept must not be confused with that of *an isomorphism*. The latter is always bijective, but the former need not always be an isomorphism.

1.1.7 Isomorphisms

A morphism

$$\varphi : A \longrightarrow B$$

is said, as in the examples cited above, to be an *isomorphism* if there is a morphism

$$\psi : B \longrightarrow A,$$

such that the two compositions are the two identity morphisms of A and B , respectively. In this case we say that A and B are *isomorphic* objects, and as is easily seen the relation of being isomorphic is an equivalence relation on the class $\text{Obj}(\mathcal{C})$. We write, as usual, $A \cong B$. A category such that the collection of isomorphism classes of objects is *a set*, is referred to as *an essentially small category*.

If $\varphi : A \longrightarrow B$ is an isomorphism, then the inverse $\psi : B \longrightarrow A$ is uniquely determined: Indeed, assume that

$$\psi \circ \varphi = \psi' \circ \varphi = \text{id}_A \text{ and } \varphi \circ \psi = \varphi \circ \psi' = \text{id}_B$$

then multiplying the first relation to the right with ψ' and using associativity we get $\psi = \psi'$. We put, as usual, $\varphi^{-1} = \psi$.

1.2 Functors

1.2.1 Definition of covariant and contravariant functors

Given two categories \mathcal{C} and \mathcal{D} . A *covariant functor* from \mathcal{C} to \mathcal{D} is a mapping

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

and for any two objects A and B in \mathcal{C} a mapping, by abuse of notation also denoted by F ,

$$F : \text{Hom}_{\mathcal{C}}(A, B) \longrightarrow \text{Hom}_{\mathcal{D}}(F(A), F(B)),$$

which maps identity morphisms to identity morphisms and is compatible with the composition, namely $F(\varphi \circ \psi) = F(\varphi) \circ F(\psi)$.

We shall refer to the category \mathcal{C} as the *source* category for the functor F , and to \mathcal{D} as the *target* category.

As is easily seen the composition of two covariant functors is again a covariant functor.

A *contravariant* functor is defined in the same way, except that it *reverses the morphisms*. Another way of expressing this is to define a contravariant functor

$$T : \mathcal{C} \longrightarrow \mathcal{D}$$

as a covariant functor

$$T : \mathcal{C} \longrightarrow \mathcal{D}^*,$$

or equivalently as a covariant functor

$$T : \mathcal{C}^* \longrightarrow \mathcal{D}.$$

In particular the *identity mapping* of objects and morphisms from \mathcal{C} to itself is a covariant functor, referred to as the *identity functor on \mathcal{C}* .

Example 1.2.1.1 *The assignment*

$$A - \text{mod} \longrightarrow B - \text{mod}$$

which to an A -module assigns a B -module, where B is an A -algebra by

$$T_B : M \mapsto M \otimes_A B$$

is a covariant functor.

Example 1.2.1.2 *The assignment*

$$A - \text{mod} \longrightarrow A - \text{mod}$$

$$h_N : M \mapsto \text{Hom}_A(M, N)$$

where N is a fixed A -module, is a contravariant functor.

Example 1.2.1.3 *The assignment*

$$A - \text{mod} \longrightarrow A - \text{mod}$$

$$h^N : M \mapsto \text{Hom}_A(N, M)$$

where N is a fixed A -module, is a covariant functor.

Example 1.2.1.4 *The assignment*

$$F : \mathcal{A}b \longrightarrow \mathcal{G}rp$$

which merely regards an Abelian group as a general group, is a covariant functor. This is an example of so-called forgetful functors, to be treated below.

1.2.2 Forgetful functors

The functor

$$T : \mathcal{A}b \longrightarrow \mathcal{S}et$$

which to an Abelian group assigns the underlying set, is called a *forgetful* functor. Similarly we have forgetful functors between many categories, where the effect of the functor merely is do disregard part of the structure of the objects in the source category. Thus for instance, we gave forgetful functors into the category $\mathcal{S}et$ from $\mathcal{T}op$, $\mathcal{C}omm$, etc, and from $\mathcal{M}od_A$ to $\mathcal{A}b$, and so on.

1.2.3 The category of functors $\mathcal{F}un(\mathcal{C}, \mathcal{D})$

The category of covariant functors $\mathcal{F}un(\mathcal{C}, \mathcal{D})$ from the category \mathcal{C} to the category \mathcal{D} is defined by letting the objects be the covariant functors from \mathcal{C} to \mathcal{D} , and for two such functors T and S we let

$$\text{Hom}_{\mathcal{F}un(\mathcal{C}, \mathcal{D})}(S, T)$$

be collections

$$\{\Psi_A\}_{A \in \text{Obj}(\mathcal{C})}$$

of morphisms

$$\Psi_A : S(A) \longrightarrow T(A),$$

such that whenever $\varphi : A \longrightarrow B$ is a morphism in \mathcal{C} , then the following diagram commutes:

$$\begin{array}{ccc} S(A) & \xrightarrow{\Psi_A} & T(A) \\ S(\varphi) \downarrow & & \downarrow T(\varphi) \\ S(B) & \xrightarrow{\Psi_B} & T(B) \end{array}$$

Morphisms of functors are often referred to as *natural transformations*. The commutative diagram above is then called *the naturality condition*.

1.2.4 Functors of several variables

We may also define a functor of n “variables”, i.e. an assignment T which to a tuple of objects (A_1, A_2, \dots, A_n) from categories \mathcal{C}_i , $i = 1, 2, \dots, n$ assigns an object $T(A_1, A_2, \dots, A_n)$ in a category \mathcal{D} , and which is covariant in some of the variables, contravariant in others, and such that the obvious generalization of the naturality condition holds. In particular we speak of *bifunctors* when there are two source categories. The details are left to the reader.

1.3 Isomorphic and equivalent categories

1.3.1 The collection of all categories regarded as a large category. Isomorphic categories

We may regard the categories themselves as a category, the objects then being the categories and the morphisms being the covariant functors. Strictly speaking this “category” violates the requirement that $\text{Hom}_{\mathcal{C}}(A, B)$ be a set, so the language “the categories of all categories” should be viewed as an informal way of speaking. We then, in particular, get the notion of *isomorphic categories*: Explicitly, two categories \mathcal{C} and \mathcal{D} are isomorphic if there are covariant functors

$$S : \mathcal{C} \longrightarrow \mathcal{D} \text{ and } T : \mathcal{D} \longrightarrow \mathcal{C}$$

such that

$$S \circ T = \text{id}_{\mathcal{D}} \text{ and } T \circ S = \text{id}_{\mathcal{C}}.$$

1.3.2 Equivalent categories

The requirement of having an equal sign in the relations above is so strong as to render the concept of limited usefulness. But bearing in mind that the functors from \mathcal{C} to \mathcal{D} do form a category, we may amend the definition by requiring only that the two composite functors above be *isomorphic* to the respective identity functors. We get the important notion of

Definition 1.3.2.1 (Equivalence of Categories) *Two categories \mathcal{C} and \mathcal{D} are equivalent if there are covariant functors*

$$S : \mathcal{C} \longrightarrow \mathcal{D} \text{ and } T : \mathcal{D} \longrightarrow \mathcal{C}$$

such that there are isomorphisms Ψ and Φ of covariant functors

$$\Psi : S \circ T \xrightarrow{\cong} \text{id}_{\mathcal{D}}$$

and

$$\Phi : T \circ S \xrightarrow{\cong} \text{id}_{\mathcal{C}},$$

such that

$$S \circ \Phi = \Psi \circ S$$

in the sense that

$$S(\Phi_A) = \Psi_{S(A)} \text{ for all objects } A \text{ in } \mathcal{C},$$

and moreover,

$$T \circ \Psi = \Phi \circ T$$

in the sense that

$$T(\Psi_B) = \Phi_{T(B)} \text{ for all objects } B \text{ in } \mathcal{D}.$$

The functors are then referred to as equivalences of categories, and the two categories are said to be equivalent.

We express the two compatibility conditions by saying that S and T commutes with Φ, Ψ .

Proposition 1.3.2.1 *A covariant functor*

$$S : \mathcal{C} \longrightarrow \mathcal{D}$$

is an equivalence of categories if and only if the following two conditions are satisfied:

1. For all $A_1, A_2 \in \text{Obj}(\mathcal{C})$, S induces a bijection

$$\text{Hom}_{\mathcal{C}}(A_1, A_2) \longrightarrow \text{Hom}_{\mathcal{D}}(S(A_1), S(A_2))$$

2. For all $B \in \text{Obj}(\mathcal{D})$ there exists $A \in \text{Obj}(\mathcal{C})$ such that $B \cong S(A)$.

Proof. We follow [BD], pages 26 - 30. Assume first that $S : \mathcal{C} \longrightarrow \mathcal{D}$ is an equivalence, and let T be the functor going the other way as in the definition. Then for all $B \in \text{Obj}(\mathcal{D})$ we have the isomorphism $\Phi_B : T(S(B)) \longrightarrow B$, hence the condition 2. is satisfied.

To prove 1., we construct an inverse t to the mapping s

$$\text{Hom}_{\mathcal{C}}(A_1, A_2) \longrightarrow \text{Hom}_{\mathcal{D}}(S(A_1), S(A_2))$$

$$f \mapsto S(f)$$

as follows: For all $g : \text{Hom}_{\mathcal{D}}(S(A_1), S(A_2))$ the morphism $t(g)$ is the unique morphism which makes the diagram below commutative:

$$\begin{array}{ccc} T(S(A_1)) & \xrightarrow{T(g)} & T(S(A_2)) \\ \Phi_{A_1} \downarrow \cong & & \cong \downarrow \Phi_{A_2} \\ A_1 & \xrightarrow[t(g)]{} & A_2 \end{array}$$

Then $t(S(f)) = f$ since the diagram below commutes:

$$\begin{array}{ccc} T(S(A_1)) & \xrightarrow{T(S(f))} & T(S(A_2)) \\ \Phi_{A_1} \downarrow \cong & & \cong \downarrow \Phi_{A_2} \\ A_1 & \xrightarrow[t(g)]{} & A_2 \end{array}$$

Conversely, let $g : \text{Hom}_{\mathcal{D}}(S(A_1), S(A_2))$. Then the diagram below commutes

$$\begin{array}{ccc} S(T(S(A_1))) & \xrightarrow{S(T(g))} & S(T(S(A_2))) \\ S(\Phi_{A_1}) \downarrow \cong & & \cong \downarrow S\Phi_{A_2} \\ S(A_1) & \xrightarrow{st(g)} & S(A_2) \end{array}$$

Thus

$$s(t(g)) = S(\Phi_{A_2}) \circ S(T(g)) \circ S(\Phi_{A_1})^{-1}$$

$$S(\Phi_{A_2}) \circ (\Psi_{S(A_2)}^{-1} \circ g \circ \Psi_{S(A_1)}) \circ S(\Phi_{A_1})$$

which is equal to g since S commutes with Φ, Ψ .

For the sufficiency, for each object $B \in \text{Obj}(\mathcal{D})$ we choose ¹ an object $A_B \in \text{Obj}(\mathcal{C})$ and an isomorphism $\beta_B : B \rightarrow S(A_B)$.

We now define a functor

$$T : \mathcal{D} \rightarrow \mathcal{C}$$

by

$$B \mapsto A_B$$

and

$$(\varphi : B_1 \rightarrow B_2) \mapsto (T(\varphi) : A_{B_1} = T(B_1) \rightarrow A_{B_2} = T(B_2)),$$

where $T(\varphi)$ is the unique morphism which corresponds to

$$(\beta_{B_2} \circ \varphi \circ (\beta_{B_1})^{-1} : S(A_{B_1}) \rightarrow S(A_{B_2})).$$

To complete the proof we have to define isomorphisms of functors, commuting with S and T ,

$$\Psi : S \circ T \xrightarrow{\cong} \text{id}_{\mathcal{D}}$$

and

$$\Phi : T \circ S \xrightarrow{\cong} \text{id}_{\mathcal{C}}.$$

¹... disregarding objections raised by the expert on axiomatic set theory...

This is left to the reader as an exercise, and may be found in [BD] on page 30. \square

Remark It will be noted that only the condition that S commutes with Φ, Ψ is used in proving the criterion for equivalence. Thus if S commutes with Φ, Ψ then it follows that T commutes with Φ, Ψ .

1.3.3 When are two functors isomorphic?

It is useful to be able to determine when a morphism of functors

$$S, T : \mathcal{C} \longrightarrow \mathcal{D},$$

is an isomorphism. If it is an isomorphism, then it follows that for all objects A in \mathcal{C} , $S(A) \cong T(A)$. But the existence of isomorphisms $S(A) \cong T(A)$ for all objects A does not imply that the functors S and T are isomorphic. Instead, we have the following result:

Proposition 1.3.3.1 *A morphism of functors $S, T : \mathcal{C} \longrightarrow \mathcal{D}$*

$$\Gamma : S \longrightarrow T$$

is an isomorphism if and only if all Γ_A are isomorphisms.

Proof. One way is by definition. We need to show that if all Γ_A are isomorphisms in \mathcal{D} , then Γ is an isomorphism of functors. We let $\Delta_A = \Gamma_A^{-1}$. We have to show that this defines a morphism of functors

$$\Delta : T \longrightarrow S,$$

which is then automatically inverse to Γ . We have to show that the following diagram commutes, for all $\varphi : A \longrightarrow B$:

$$\begin{array}{ccc} T(A) & \xrightarrow{\Delta_A} & S(A) \\ T(\varphi) \downarrow & & \downarrow S(\varphi) \\ T(B) & \xrightarrow{\Delta_B} & S(B) \end{array}$$

In fact, we have the commutative diagram

$$\begin{array}{ccc} S(A) & \xrightarrow{\Gamma_A} & T(A) \\ S(\varphi) \downarrow & & \downarrow T(\varphi) \\ S(B) & \xrightarrow{\Gamma_B} & T(B) \end{array}$$

or

$$T(\varphi) \circ \Gamma_A = \Gamma_B \circ S(\varphi).$$

This implies that

$$\Delta_B \circ (T(\varphi) \circ \Gamma_A) \circ \Delta_A = \Delta_B \circ (\Gamma_B \circ S(\varphi)) \circ \Delta_A,$$

from which the claim follows by associativity of composition. \square

1.3.4 Left and right adjoint functors

Let \mathcal{A} and \mathcal{B} be two categories and let

$$F : \mathcal{A} \longrightarrow \mathcal{B} \text{ and } G : \mathcal{B} \longrightarrow \mathcal{A}$$

be covariant functors. Assume that for all objects

$$A \in \text{Obj}(\mathcal{A}) \text{ and } B \in \text{Obj}(\mathcal{B})$$

there are given bijections

$$\Phi_{A,B} : \text{Hom}_{\mathcal{B}}(F(A), B) \longrightarrow \text{Hom}_{\mathcal{A}}(A, G(B))$$

which are functorial in A and in B . Then call F left adjoint to G , and G right adjoint to F , or, less precisely, that F and G are adjoint functors.

For contravariant functors the definition is analogous, or as some prefer, reduced to the covariant case by passage to the dual category of \mathcal{A} or \mathcal{B} .

Example 1.3.4.1 *Let $\varphi : R \longrightarrow S$ be a homomorphism of commutative rings. Recall that if M is an S -module, then we define an R -module denoted by $M_{[\varphi]}$ by putting $rm = \varphi(r)m$ whenever $r \in R$ and $m \in M$. The covariant functor*

$$S - \text{modules} \longrightarrow R - \text{modules}$$

$$M \mapsto M_{[\varphi]}$$

is called the *Reduction of Structure-functor*. This functor has a left adjoint, called the *Extension of Structure-functor*

$$\begin{aligned} R - \text{modules} &\longrightarrow S - \text{modules} \\ N &\mapsto N \otimes_S R. \end{aligned}$$

In a more fancy language one may express the definition above by saying that

Definition 1.3.4.1 *The functor F is left adjoint to the functor G , or G is right adjoint to F , where*

$$\begin{array}{ccc} & F & \\ \mathcal{A} & \xrightarrow{\quad} & \mathcal{B} \\ & \xleftarrow{\quad} & \\ & G & \end{array}$$

provided that there is an isomorphism of bifunctors

$$\Phi : \text{Hom}_{\mathcal{B}}(F(),) \xrightarrow{\cong} \text{Hom}_{\mathcal{A}}(, G())$$

Whenever we have a *morphism* of bifunctors as above, i.e. functorial mappings

$$\Phi_{A,B} : \text{Hom}_{\mathcal{B}}(F(A), B) \longrightarrow \text{Hom}_{\mathcal{A}}(A, G(B)),$$

then the morphism $\text{id}_{F(A)}$ is mapped to a morphism $\varphi_A : A \longrightarrow G(F(A))$. We then obtain a morphism of functors

$$\varphi : \text{id}_{\mathcal{A}} \longrightarrow G \circ F.$$

We then have that $\Phi_{A,B}$ is given by

$$(F(A) \longrightarrow B) \mapsto (A \xrightarrow{\varphi_A} G(F(A)) \longrightarrow G(B))$$

Similarly, a morphism of bifunctors Ψ in the opposite direction yields a morphism of functors

$$\psi : F \circ G \longrightarrow \text{id}_{\mathcal{B}},$$

and $\Psi_{A,B}$ is then given by

$$(A \longrightarrow G(B)) \mapsto (F(A) \longrightarrow F(G(B)) \xrightarrow{\psi_B} B)$$

The assertion that Φ and Ψ are inverse to one another may then be expressed solely in terms of commutative diagrams, involving F , G , φ and ψ . We do not pursue this line of thought any further here.

1.4 Representable functors

1.4.1 The functor of points

We now turn to the very important and useful notion of a *representable functor*. Because we shall mainly use this in the contravariant case, we shall take that approach here, although of course the contravariant and the covariant cases are essentially equivalent by the usual trick of passing to the dual category.

So let \mathcal{C} be a category, and let $X \in \text{Obj}(\mathcal{C})$. We define a contravariant functor

$$h_X : \mathcal{C} \longrightarrow \text{Set}$$

putting

$$h_X(Y) = \text{Hom}_{\mathcal{C}}(Y, X)$$

for any object Y in \mathcal{C} , and for any morphism $\varphi : Y_1 \longrightarrow Y_2$ we let

$$h_X(\varphi) : \text{Hom}_{\mathcal{C}}(Y_2, X) \longrightarrow \text{Hom}_{\mathcal{C}}(Y_1, X)$$

be given by

$$\psi \mapsto \psi \circ \varphi.$$

It is easily verified that h_X so defined is a contravariant functor. We shall extend a notation from algebraic geometry, and refer to the functor h_X as *the functor of points of the object X* . We also shall refer to the set $h_X(Y) = \text{Hom}_{\mathcal{C}}(Y, X)$ as the set of Y -valued points of the object X in \mathcal{C} .

1.4.2 A functor represented by an object

We are now ready for the important

Definition 1.4.2.1 *A contravariant functor*

$$F : \mathcal{C} \longrightarrow \text{Set}$$

is said to be representable by the object X of \mathcal{C} if there is an isomorphism of functors

$$\Psi : h_X \longrightarrow F.$$

Remark 1.4.2.1 For the covariant case we define the covariant functor

$$h^X : \mathcal{C} \longrightarrow \mathcal{S}et$$

by

$$h^X(Y) = \text{Hom}_{\mathcal{C}}(X, Y),$$

which is a covariant functor. We then similarly get the notion of a representable covariant functor $\mathcal{C} \longrightarrow \mathcal{S}et$. The details are left to the reader. Of course this amounts to applying the contravariant case to the category \mathcal{C}^* , as pointed out above.

1.4.3 Representable functors and Universal Properties: Yoneda's Lemma

A vast number of constructions in mathematics are best understood as representing an appropriate functor. The key to a unified understanding of this lies in the theorem below.

Given a contravariant functor

$$F : \mathcal{C} \longrightarrow \mathcal{S}et.$$

Let X be an object in \mathcal{C} , and let $\xi \in F(X)$. For all objects Y of \mathcal{C} we then define a mapping as follows:

$$\Phi_Y : h_X(Y) \longrightarrow F(Y)$$

$$\varphi \mapsto F(\varphi)(\xi).$$

It is an easy exercise to verify that this is a *morphism of contravariant functors*,

$$\Phi : h_X \longrightarrow F.$$

We now have the

Theorem 1.4.3.1 (Yoneda's Lemma) *The functor F is representable by the object X if and only if there exists an element $\xi \in F(X)$ such that the corresponding Φ is an isomorphism of contravariant functors. This is the case if and only if all Φ_Y are bijective.*

Proof. By Proposition 1.3.3.1 Φ is an isomorphism if and only if all Φ_Y are bijective. Thus, if all Φ_Y are bijective then F is representable via the isomorphism Φ .

On the other hand, if F is representable, then there is an isomorphism of functors

$$\Psi : h_X \xrightarrow{\cong} F.$$

Put $\xi = \Psi_X(\text{id}_X)$ and let $\varphi \in h_X(Y)$. For any object Y we get the commutative diagram

$$\begin{array}{ccc} h_X(X) & \xrightarrow{\Psi_X} & F(X) \\ \varphi \circ (\cdot) \downarrow & & \downarrow F(\varphi) \\ h_X(Y) & \xrightarrow{\Psi_Y} & F(Y) \end{array}$$

noting what happens to id_X in this commutative diagram, we find the relation

$$F(\varphi)(\xi) = \Psi_Y(\varphi),$$

thus $\Psi_Y = \Phi_Y$ which is therefore bijective. \square

We say that the object X represents the functor F and that the element $\xi \in F(X)$ is the *universal element*. This language is tied to the following *Universal Mapping Property* satisfied by the pair (X, ξ) :

The **Universal Mapping Property** of the pair (X, ξ) representing the contravariant functor F is formulated as follows:

For all elements $\eta \in F(Y)$ there exists a unique morphism

$$\varphi : Y \longrightarrow X$$

such that

$$F(\varphi)(\xi) = \eta$$

.

In fact, this is nothing but a direct translation of the assertion that Φ_Y be bijective.

Another remark to be made here, is that two objects representing a representable functor are isomorphic by a unique isomorphism. The universal elements correspond under the mapping induced by this isomorphism. The proof of this observation is left to the reader.

1.5 Some constructions in the light of representable functors

1.5.1 Products and coproducts

Let \mathcal{C} be a category. Let $B_i, i = 1, 2$ be two objects in \mathcal{C} . Define a functor

$$F : \mathcal{C} \longrightarrow \mathcal{S}et$$

by

$$B \mapsto \{(\psi_1, \psi_2) \mid \psi_i \in \text{Hom}_{\mathcal{C}}(B, B_i), i = 1, 2\}$$

If this functor is representable, then the representing object, unique up to a unique isomorphism, is denoted by $B_1 \times B_2$, and referred to as the *product* of B_1 and B_2 . The universal element (p_1, p_2) is of course a pair of two morphisms, from $B_1 \times B_2$ to B_1 , respectively B_2 :

$$\begin{array}{ccc} B_1 \times B_2 & \xrightarrow{p_2} & B_2 \\ p_1 \downarrow & & \\ B_1 & & \end{array}$$

The morphism p_1 and p_2 are called the first and second projection, respectively. The product $B_1 \times B_2$ and the projections solve the following so called *universal problem*: For all morphisms f_1 and f_2 as below, there exists a unique morphism h such that the triangular diagrams commute:

$$\begin{array}{ccccc} A & & & & \\ & \searrow^{f_2} & & & \\ & & \exists! h & & \\ & \searrow^{f_1} & \downarrow p_1 & \longrightarrow & B_2 \\ & & B_1 & & \\ & & & & \end{array}$$

We obtain, as always, a dual notion by applying the above to the category \mathcal{C}^* . Specifically, we consider the functor

$$G : \mathcal{C} \longrightarrow \mathit{Set}$$

by

$$B \mapsto \{(\ell_1, \ell_2) \mid \ell_i \in \mathit{Hom}_{\mathcal{C}}(B_i, B), i = 1, 2\}$$

Whenever this functor is representable, the representing object is denoted by $B_1 \coprod B_2$ and referred to as the *coproduct* of B_1 and B_2 . The morphisms η_i , $i = 1, 2$ are called the canonical injections:

$$\begin{array}{ccc} B_1 & & \\ \downarrow \eta_1 & & \\ B_1 \coprod B_2 & \longleftarrow & B_2 \\ & \eta_2 & \end{array}$$

We similarly define products and coproducts of sets of objects, in particular infinite sets of objects. For a set of morphisms

$$\varphi_i : B \longrightarrow B_i, i \in I,$$

we get

$$(\varphi_i \mid i \in I) : B \longrightarrow \prod_{i \in I} B_i.$$

such that all the appropriate diagrams commute.

Further, if $\psi_i : A_i \longrightarrow B_i$ are morphisms for all $i \in I$, then we get a morphism

$$\psi = \prod_{i \in I} \psi_i : \prod_{i \in I} A_i \longrightarrow \prod_{i \in I} B_i$$

uniquely determined by making all of the following diagrams commutative:

$$\begin{array}{ccc} \prod_{i \in I} A_i & \xrightarrow{\psi} & \prod_{i \in I} B_i \\ \text{pr}_i \downarrow & & \downarrow \text{pr}_i \\ A_i & \xrightarrow{\psi_i} & B_i \end{array}$$

1.5.2 Products and coproducts in $\mathcal{S}et$.

In the category $\mathcal{S}et$ products exist, and are nothing but the usual set-theoretic product:

$$\prod_{i \in I} A_i = \{(a_i | i \in I) \mid a_i \in A_i\}.$$

The coproduct is the *disjoint union* of all the sets:

$$\coprod_{i \in I} A_i = \{(a_i, i) \mid i \in I, a_i \in A_i\}.$$

Adding the index as a second coordinate only serves to make the union disjoint.

1.5.3 Fibered products and coproducts

When we apply the above concepts to the categories \mathcal{C}_A , respectively \mathcal{C}^A , then we get the notions of *fibered* products and coproducts, respectively. We go over this version in detail, as it is important in algebraic geometry.

Let A be an object in the category \mathcal{C} . Let (B_i, φ_i) , $i = 1, 2$ be two objects in \mathcal{C}_A . Define a functor

$$F : \mathcal{C}_A \longrightarrow \mathcal{S}et$$

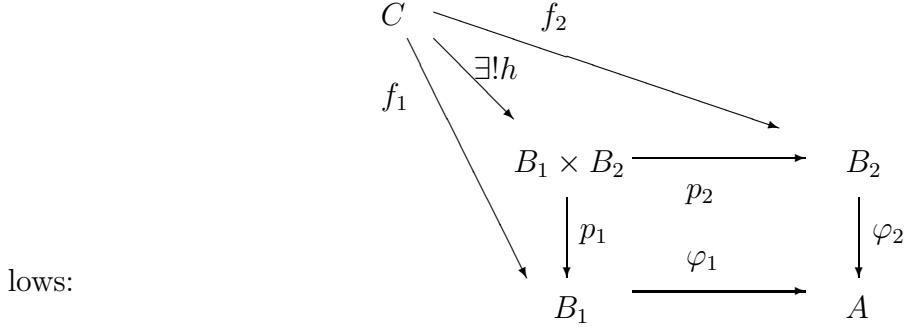
by

$$(B, \varphi) \mapsto \{(\psi_1, \psi_2) \mid \psi_i \in \text{Hom}_{\mathcal{C}_A}((B, \varphi), (B_i, \varphi_i)), i = 1, 2\}$$

If this functor is representable, then the representing object, unique up to a unique isomorphism, is denoted by $B_1 \times_A B_2$, and referred to as the *fibered product* of B_1 and B_2 over A . The universal element (p_1, p_2) is of course a pair of two morphisms, from $B_1 \times_A B_2$ to B_1 , respectively B_2 such that the following diagram commutes:

$$\begin{array}{ccc} B_1 \times_A B_2 & \xrightarrow{p_2} & B_2 \\ p_1 \downarrow & & \downarrow \varphi_2 \\ B_1 & \xrightarrow{\varphi_1} & A \end{array}$$

The morphisms p_1 and p_2 are called the first and second projection, respectively. We may illustrate the universal property of the fibered product as fol-



where all the triangular diagrams are commutative.

We obtain, as always, a dual notion by applying the above to the category \mathcal{C}^* . Specifically, we consider the functor

$$G : \mathcal{C}^A \longrightarrow \mathcal{S}et$$

by

$$(\tau, B) \mapsto \{(\ell_1, \ell_2) \mid \ell_i \in \text{Hom}_{\mathcal{C}^A}((\tau, B), (\tau_i, B_i)), i = 1, 2\}$$

Whenever this functor is representable, the representing object is denoted by $B_1 \coprod_A B_2$ and referred to as the *fibred coproduct* of B_1 and B_2 . The morphisms ℓ_i , $i = 1, 2$ are called the canonical injections, and the following diagram commutes:

$$\begin{array}{ccc}
A & \xrightarrow{\tau_1} & B_1 \\
\tau_2 \downarrow & & \downarrow \ell_1 \\
B_2 & \xrightarrow{\ell_2} & B_1 \coprod_A B_2
\end{array}$$

1.5.4 Abelian categories

In the category of Abelian groups, or more generally the category of modules over a commutative ring A , the product and the coproduct of two objects always exist. And moreover, they are isomorphic. In fact, it is easily seen that the direct sum $M_1 \oplus M_2$ of two A -modules satisfies the universal properties of both $M_1 \times M_2$ and $M_1 \coprod M_2$. The universal elements of the appropriate functors are given by the A -homomorphisms

$$p_1(m_1, m_2) = m_1, p_2(m_1, m_2) = m_2,$$

$$\ell_1(m_1) = (m_1, 0), \ell_2(m_2) = (0, m_2).$$

This also applies to finite products and coproducts in $\mathcal{A}b$: They exist, and are equal (canonically isomorphic.)²

Another characteristic feature of this category is that $\text{Hom}_{\mathcal{C}}(M, N)$ is an Abelian group.

The category of A -modules is an example of an *Abelian category*. The two properties noted above are part of the defining properties of this concept. Another important part of the definition of an Abelian category, is the existence of a *zero object*. For the category of A -modules this is the A -module consisting of the element 0 alone. It is denoted by $\mathbf{0}$, and has the property that for any object X there is a unique morphism from it to X , and a unique morphism from X to $\mathbf{0}$. We say that $\mathbf{0}$ is both *final and cofinal* in \mathcal{C} .

1.5.5 Product and coproduct in the category Comm

An important category from commutative algebra is *not Abelian*, however. Namely, the category Comm . In Comm , the product of two objects, of two commutative rings with 1 A and B , is the ring $A \times B$ consisting of all pairs (a, b) with $a \in A$ and $b \in B$. The *coproduct*, however, is the tensor product $A \otimes B$. In the category Comm_R of R -algebras the coproduct is $A \otimes_R B$.

1.5.6 Localization as representing a functor

Let A be a commutative ring with 1, and $S \subset A$ a multiplicatively closed subset. For any element $a \in A$ and A -module N , we say that a is invertible on N provided that the A -homomorphism

$$\mu_a : N \longrightarrow N \text{ given by } \mu_a(n) = an,$$

is invertible, i.e. is a bijective mapping on N . The subset S of A is said to be invertible on N if all elements in S are.

We now let \mathcal{C} be the subcategory of Mod_A of modules where S is invertible, and let M be an A -module. Define $F : \mathcal{C} \longrightarrow \text{Set}$ by

$$F(N) = \text{Hom}_A(M, N).$$

²But although infinite products and coproducts do exist, they are not equal: The direct sum of a family of Abelian groups $\{A_i\}_{i \in \mathbb{N}}$ is the subset of the direct product $A_1 \times A_2 \times \cdots \times A_n \times \cdots$ consisting of all tuples such that only a finite number of coordinates are different from the zero element in the respective A_i 's.

In introductory courses in commutative algebra we construct the localization of M in S , the A -module $S^{-1}M$, with the canonical A -homomorphism $\tau : M \longrightarrow S^{-1}M$. It is a simple exercise to show that the pair $(S^{-1}M, \tau)$ represents the functor F .

1.5.7 Kernel and Cokernel of two morphisms

Let

$$\begin{array}{ccc} & \varphi_1 & \\ & \longrightarrow & \\ A & \longrightarrow & B \\ & \longrightarrow & \\ & \varphi_2 & \end{array}$$

be two morphisms in the category \mathcal{C} .

We consider the functor

$$F : \mathcal{C} \longrightarrow \mathcal{S}et$$

given by

$$F(X) = \{\varphi \in \text{Hom}_{\mathcal{C}}(X, A) \mid \varphi_1 \circ \varphi = \varphi_2 \circ \varphi.\}$$

This functor is contravariant. If it is representable, then we refer to the representing object as the *kernel* of the pair φ_1, φ_2 .

Thus an object N together with a morphism $\iota : N \longrightarrow A$ is called a *kernel* for φ_1 and φ_2 if the following two conditions are satisfied:

1. We have $\varphi_1 \circ \iota = \varphi_2 \circ \iota$,
2. If $\varphi : X \longrightarrow A$ is a morphism such that $\varphi_1 \circ \varphi = \varphi_2 \circ \varphi$, then there exists a unique morphism $\psi : X \longrightarrow N$ such that $\iota \circ \psi = \varphi$

The dual concept is that of a *cokernel*. $\tau : B \longrightarrow M$ is a cokernel for the morphisms $\varphi_1, \varphi_2 : A \longrightarrow B$ if the following universal property holds:

1. We have $\tau \circ \varphi_1 = \tau \circ \varphi_2$,
2. If $\varphi : B \longrightarrow X$ is a morphism such that $\varphi \circ \varphi_1 = \varphi \circ \varphi_2$, then there exists a unique morphism $\psi : M \longrightarrow X$ such that $\psi \circ \tau = \varphi$

We write

$$\ker(f, g) \text{ and } \text{coker}(f, g)$$

for the kernel, respectively the cokernel, of the pair (f, g) .

1.5.8 Kernels and cokernels in some of the usual categories

It is easily verified that kernels and cokernels exist in the categories $\mathcal{A}b$, $\mathcal{M}od_R$, $\mathcal{T}op$ and $\mathcal{S}et$.

Two mappings of sets,

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B$$

have a kernel and a cokernel: The kernel is defined as

$$K = \{a \in A \mid f(a) = g(a)\}$$

and the map η is the obvious inclusion. The cokernel is defined as $C = B / \sim$, where \sim is the equivalence relation on B generated by the relation ρ given below: ³

$$\begin{array}{c} b_1 \rho b_2 \\ \Downarrow \\ \exists a \in A \text{ such that } f(a) = b_1, g(a) = b_2. \end{array}$$

As is immediately seen, this is a cokernel for (f, g) .

For the category $\mathcal{T}op$, these sets carry a natural topology: Namely the induced topology from the space $A \times B$ in the former case, the quotient topology in the latter. We get a kernel and a cokernel in $\mathcal{T}op$ with this choice of topology on the set-theoretic versions.

If f and g are morphisms in $\mathcal{M}od_R$, then the set theoretic kernel is automatically an R -module. The same is true for the cokernel, since the relation \sim is actually a *congruence relation* for the operations, that is to say, it is compatible with the operations. ⁴

³A relation ρ on a set B generates an equivalence relation \sim by putting $a \sim b$ if either $a = b$, or there is a sequence $a \sim a_1 \sim a_2 \sim \dots \sim a_n = b$, or a sequence $b = b_1 \sim b_2 \sim \dots \sim b_m = a$. As is immediately verified \sim so defined is an equivalence relation on the set A .

⁴Hence addition and multiplication with an element in R may be defined on the set of equivalence classes by performing the operations on elements representing the classes and taking the resulting classes. The details are left to the reader.

1.5.9 Exactness

The diagrams

$$X \begin{array}{c} \xrightarrow{\quad} \\ \twoheadrightarrow \end{array} Y \longrightarrow Z$$

or

$$X \longrightarrow Y \begin{array}{c} \xrightarrow{\quad} \\ \twoheadrightarrow \end{array} Z$$

are said to be *exact* if the former is a cokernel-diagram or the latter a kernel-diagram, respectively.

1.5.10 Kernels and cokernels in Abelian categories

In an Abelian category we have the usual concept of kernel and cokernel of a single morphism. The link between this and the case of a pair of morphisms are the definitions

$$\ker(f) = \ker(f, 0) \text{ and } \operatorname{coker}(f) = \operatorname{coker}(f, 0),$$

where 0 denotes the zero morphism.

1.5.11 The inductive and projective limits of a covariant or a contravariant functor

Let \mathcal{J} and \mathcal{C} be two categories, where \mathcal{J} is small, and

$$F : \mathcal{J} \longrightarrow \mathcal{C}$$

be a covariant functor. We define a functor

$$L : \mathcal{C} \longrightarrow \mathcal{S}et$$

by

$$L(A) = \left\{ \left\{ v_X \right\}_{X \in \operatorname{Obj}(\mathcal{J})} \left| \begin{array}{l} v_X : A \longrightarrow F(X) \\ \text{such that for all morphisms} \\ \alpha : X \longrightarrow Y \text{ we have} \\ F(\alpha) \circ v_X = v_Y \end{array} \right. \right\}$$

When this functor is representable, we call the representing object *the projective limit of the functor F* .

Similarly we define the *inductive limit* of a covariant functor: We define a functor

$$S : \mathcal{C} \longrightarrow \mathcal{S}et$$

by

$$S(A) = \left\{ \left\{ v_X \right\}_{X \in \text{Obj}(\mathcal{J})} \left| \begin{array}{l} v_X : F(X) \longrightarrow A \\ \text{such that for all morphisms} \\ \alpha : X \longrightarrow Y \text{ we have} \\ F(\alpha) \circ v_Y = v_X \end{array} \right. \right\}$$

When this functor is representable, we call the representing object *the inductive limit of the functor F* .

We have the following universal properties for the two limits introduced above, the inductive limit which is denoted by $\lim_{\longrightarrow X \in \text{Obj}(\mathcal{J})} (F)$ and the projective limit which is denoted by $\lim_{\longleftarrow X \in \text{Obj}(\mathcal{J})} (F)$:

Universal property of the inductive limit: For all objects A of \mathcal{C} and objects X of \mathcal{J} with morphisms $v_X : F(X) \longrightarrow A$ compatible with morphisms in \mathcal{J} , there exists a unique morphism $\lim_{\longrightarrow} (F) \longrightarrow A$, compatible with the v_X 's.

Universal property of the projective limit: For all objects A of \mathcal{C} and objects X of \mathcal{J} with morphisms $v_X : A \longrightarrow F(X)$ compatible with morphisms in \mathcal{J} , there exists a unique morphism $A \longrightarrow \lim_{\longleftarrow} (F)$, compatible with the v_X 's.

The subscript $X \in \text{Obj}(\mathcal{J})$ is deleted when no ambiguity is possible.

We define the inductive and the projective limit for contravariant functors similarly, or rely on the definition for the covariant case by regarding a contravariant functor

$$F : \mathcal{J} \longrightarrow \mathcal{C}$$

as a covariant functor

$$G : \mathcal{J}^* \longrightarrow \mathcal{C}$$

Then $\lim_{\longrightarrow} (F) = \lim_{\longrightarrow} (G)$ and $\lim_{\longleftarrow} (F) = \lim_{\longleftarrow} (G)$.⁵ Another name for the

⁵Note that if we turn the contravariant functor into a covariant one as $H : \mathcal{J} \longrightarrow \mathcal{C}^*$, then the two kinds of limits are interchanged.

inductive limit is *direct limit*, while the projective limit is called *inverse limit*.

To sum up, for a covariant functor we have for all morphisms $\varphi : X \longrightarrow Y$ in \mathcal{J} :

$$\varinjlim(F) \longrightarrow F(X) \xrightarrow{F(\varphi)} F(Y) \longrightarrow \varinjlim(F),$$

and for a contravariant functor we have for all morphisms $\varphi : X \longrightarrow Y$ in \mathcal{J} :

$$\varprojlim(F) \longrightarrow F(Y) \xrightarrow{F(\varphi)} F(X) \longrightarrow \varprojlim(F).$$

1.5.12 Projective and inductive systems and their limits

Let I be a partially ordered set, that is to say a set I where there is given an ordering-relation \leq such that

1. $i \leq i$
2. $i \leq j$ and $j \leq k \implies i \leq k$

This is a rather general definition. Frequently the first condition is strengthened to the assertion that $i \leq j$ and $j \leq i \iff i = j$. Also, a related concept is that of a *directed* set, which is a partially ordered set where any two elements have an “upper bound”.

We now generalize the definition of the category $\mathcal{T}op(X)$ as follows: We define the category $\mathcal{J}nd(I)$ by letting the objects be the elements of I , and putting

$$\text{Hom}_{\mathcal{J}nd(I)}(i, j) = \begin{cases} \emptyset & \text{if } i \not\leq j, \\ \{\iota_{i,j}\} & \text{if } i \leq j. \end{cases} \quad (1)$$

As we see, the category $\mathcal{T}op(X)$ is the result of applying this to the partially ordered set of open subsets in the topological space X .

An *inductive system* in a category \mathcal{C} over I is by definition a covariant functor

$$F : \mathcal{J}nd(I) \longrightarrow \mathcal{C},$$

and a *projective system* in \mathcal{C} over I is contravariant functor between the same categories, regarded as a covariant functor

$$F : \mathcal{J}nd(I)^* \longrightarrow \mathcal{C}.$$

Usually we write F_i instead of $F(i)$ in the above situations, and refer to $\{F_i\}_{i \in I}$ as an inductive, respectively projective, system.

Note that if we give I the partial ordering \preceq by letting $i \preceq j \Leftrightarrow j \leq i$ and denote the resulting partially ordered set by I^* , then $\mathcal{I}nd(I)^* = \mathcal{I}nd(I^*)$.

We let $\{F_i\}_{i \in I}$ be an inductive system, and define a functor

$$L : \mathcal{C} \longrightarrow \mathcal{S}et$$

as follows:

$$L(X) = \left\{ \{\varphi_i\}_{i \in I} \mid \varphi_i \in \text{Hom}_{\mathcal{C}}(F_i, X) \text{ and if } i \leq j \text{ then } \varphi_i = \varphi_j \circ \iota_{i,j} \right\}$$

If this functor happens to be representable, then we denote the representing object by

$$\lim_{\longrightarrow i \in I} F_i,$$

and note that the universal element is a collection of morphisms, compatible with the inductive structure,

$$F_i \longrightarrow \lim_{\longrightarrow i \in I} F_i.$$

The universal property amounts to that whenever we have such a set of compatible morphisms,

$$F_i \longrightarrow Y,$$

then they factor uniquely through a morphism

$$\lim_{\longrightarrow i \in I} F_i \longrightarrow Y.$$

Similarly we define the *projective limit*, denoted by $\lim_{\longleftarrow i \in I} F_i$.

As in the general case we may sum this up as follows, for an inductive system $\{F_i\}_{i \in I}$ over a partially ordered set I , where $i \leq j$:

$$\lim_{\longleftarrow i \in I} F_i \longrightarrow F_i \longrightarrow F_j \longrightarrow \lim_{\longrightarrow i \in I} F_i.$$

1.5.13 On the existence of projective and inductive limits

There are several results on the existence of inductive and projective limits. The most general theorem is the following, which we prove following [BD]:

Theorem 1.5.13.1 *Let \mathcal{C} be a category. Then every covariant functor from a small category \mathcal{A}*

$$F : \mathcal{A} \longrightarrow \mathcal{C}$$

has an inductive limit if and only if \mathcal{C} has infinite coproducts and cokernels always exist in \mathcal{C} , and every functor as above has a projective limit if and only if \mathcal{C} has infinite products and kernels always exist in \mathcal{C} .

Proof. We prove the assertion for projective limits, noting that the inductive case follows by replacing the target category by its dual.

To show that the condition is necessary, we note that the product of a family of objects in the category \mathcal{C} may be viewed as a projective limit: Indeed, let $\{C_i\}_{i \in I}$ denote any set of objects from \mathcal{C} . Let \mathcal{A} denote the category defined by $\text{Obj}(\mathcal{A}) = I$ and $\text{Hom}_{\mathcal{A}}(i, j) = \{\text{id}_i\}$ if $i = j$, empty otherwise. Define a functor

$$\begin{aligned} F : \mathcal{A} &\longrightarrow \mathcal{C} \\ i &\mapsto C_i. \end{aligned}$$

As is immediately seen, the assertion that $\lim_{\longleftarrow F} C_i$ exists is equivalent to the assertion that $\prod_{i \in I} C_i$ exists. To show the necessity of the last part of the condition, let

$$f_1, f_2 : C_1 \longrightarrow C_2$$

be two morphisms in \mathcal{C} . Let \mathcal{A} be the category consisting of two objects denoted by 1 and 2, and such that $\text{Hom}_{\mathcal{A}}(1, 2) = \{\varphi_1, \varphi_2\}$. Apart from the identity morphisms, these are the only morphisms in \mathcal{A} . Define the functor F by

$$\begin{aligned} F : \mathcal{A} &\longrightarrow \mathcal{C} \\ i &\mapsto C_i \text{ for } i = 1, 2, \\ f_i &\mapsto \varphi_i \text{ for } i = 1, 2. \end{aligned}$$

The kernel of f_1 and f_2 is the projective limit $\lim_{\leftarrow F} C_i$.

To prove the sufficiency, put

$$\Pi = \prod_{X \in \text{Obj}(\mathcal{A})} F(X),$$

and let

$$\text{pr}_X : \Pi \longrightarrow F(X)$$

denote the projections. Further, let

$$\Upsilon = \prod_{Y \in \text{Obj}(\mathcal{A}), \alpha \in \text{Hom}_{\mathcal{A}}(X, Y)} F(Y)_\alpha,$$

where $F(Y)_\alpha = F(Y)$ for all α and let

$$\text{pr}_{Y, \alpha} : \Upsilon \longrightarrow F(Y)$$

denote the projections.

Now each morphism $\alpha : X \longrightarrow Y$ yields a morphism

$$F(\alpha) \circ \text{pr}_X : \Pi \longrightarrow F(Y),$$

hence by the universal property of the product Υ there is a unique morphism v which makes the following diagrams commutative:

We also have a morphism $\text{pr}_Y : \Pi \longrightarrow F(Y)$, for any given Y and morphism $X_\alpha : \longrightarrow Y$. This yields a morphism w such that the following diagrams are commutative:

$$\begin{array}{ccc} \Pi & \xrightarrow{w} & \Upsilon \\ \text{pr}_Y \downarrow & & \downarrow \text{pr}_{Y, \alpha} \\ F(Y) & \xrightarrow{=} & F(Y) \end{array}$$

Let

$$\ell : L \longrightarrow \Pi$$

be the kernel of (v, w) , so in particular $v \circ \ell = w \circ \ell$. Let $\ell_X = \text{pr}_X \circ \ell$, and consider the system

$$\ell_X : L \longrightarrow F(X).$$

We claim that this is the projective limit of the functor F . First of all, we have to show that the compositions behave right, namely that whenever

$$\alpha : X \longrightarrow Y$$

is a morphism in \mathcal{A} , then

$$\ell_Y : L \xrightarrow{\ell_X} F(X) \xrightarrow{F(\alpha)} F(Y).$$

Indeed, we have

$$F(\alpha) \circ \ell_X = F(\alpha) \circ \text{pr}_X \circ \ell = \text{pr}_{Y,\alpha} \circ v \circ \ell = \text{pr}_{Y,\alpha} \circ w \circ \ell = \text{pr}_Y \circ \ell = \ell_Y.$$

We finally show that the universal property of the projective limit is satisfied. So let the family of morphisms

$$s_X : S \longrightarrow F(X), \quad X \in \text{Obj}(\mathcal{A})$$

be such that whenever $\alpha : X \longrightarrow Y$ is a morphism in \mathcal{A} , then $F(\alpha) \circ s_X = s_Y$. In particular we obtain a unique morphism $\sigma : S \longrightarrow \Pi$, such that $s_X = \text{pr}_X \circ \sigma$. We now have

$$\text{pr}_{Y,\alpha} \circ v \circ \sigma = F(\alpha) \circ \text{pr}_X \circ \sigma = F(\alpha) \circ s_X = s_Y$$

and

$$\text{pr}_{Y,\alpha} \circ w \circ \sigma = \text{pr}_Y \circ \sigma = s_Y.$$

Thus by the universal property of the product Υ it follows that

$$v \circ \sigma = w \circ \sigma,$$

and hence σ factors uniquely through the kernel L : There is a unique $S \xrightarrow{s} L$ such that $\sigma = \ell \circ s$. Thus $\ell_X \circ s = \text{pr}_X \circ \ell \circ s = \text{pr}_X \circ \sigma = s_X$, and we are done. \square

We note that we are now guaranteed the existence of inductive and projective limits, in a rather general setting, in the categories Set , Top , and Mod_A . It is, however, useful in our practical work to have a good description of these limits. This is particularly important in the case of some inductive limits we encounter in *sheaf theory*. This is the subject of the next section.

1.5.14 An example: The stalk of a presheaf on a topological space

Let X be a topological space, and let $x \in X$ be a point. Let \mathcal{B} be a set consisting of open subsets of X containing x , such that the following condition holds:

For any two open subsets U and V containing x there is an open subset $W \in \mathcal{B}$ contained in $U \cap V$.

We say that \mathcal{B} is a basis for the system of open neighborhoods around $x \in X$. Let \mathcal{C} be one of the categories $\mathcal{S}et$ or $\mathcal{M}od_A$, and let

$$\mathcal{F} : \mathcal{T}op_X \longrightarrow \mathcal{C}$$

be a contravariant functor. We refer to \mathcal{F} as a *presheaf* of \mathcal{C} on the topological space X . If $\iota_{U,V} : U \hookrightarrow V$ is the inclusion mapping of U into V , then $\mathcal{F}(\iota_{U,V}) : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$ is denoted by $\rho_{V,U}^{\mathcal{F}}$ and referred to as the *restriction morphism from V to U* . \mathcal{F} is deleted from the notation when no ambiguity is possible. The image of an element f under the restriction morphism $\rho_{V,U}$ is referred to as the restriction of f from V to U .

Then we get the inductive limit $\lim_{\longrightarrow V \in \mathcal{B}} \mathcal{F}(V)$ as follows: We form the disjoint union of all $\mathcal{F}(V)$:

$$M(x) = \coprod_{V \in \mathcal{B}} \mathcal{F}(V).$$

We define an equivalence relation in $M(x)$ by putting $f \sim g$ for $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$ provided that they have the same restriction to a smaller open subset in \mathcal{B} . We then have

$$\lim_{\longrightarrow V \in \mathcal{B}} \mathcal{F}(V) = M(x) / \sim.$$

In fact, for the category $\mathcal{M}od_A$ we find well defined addition and scalar multiplication which makes this set into an A -module by putting

$$[f_U] + [g_V] = [\rho_{U,W}(f_U) + \rho_{V,W}(g_V)]$$

where f_U and g_V are elements in $\mathcal{F}(U)$ and $\mathcal{F}(V)$, respectively. We also let

$$a[g_V] = [ag_V].$$

It is easy to see that there are canonical isomorphisms between $\lim_{\longrightarrow V \in \mathcal{B}} \mathcal{F}(V)$ and $\lim_{\longrightarrow V \in \mathcal{D}} \mathcal{F}(V)$ when \mathcal{B} and \mathcal{D} are two bases for the neighborhood system

at x . In particular we may take all the open subsets containing x : Indeed, we consider first the case when \mathcal{B} is arbitrary and \mathcal{D} is the set of all open subsets containing x . Then $\mathcal{D} \supset \mathcal{B}$ induces a morphism

$$\lim_{\rightarrow V \in \mathcal{B}} \mathcal{F}(V) \longrightarrow \lim_{\rightarrow V \in \mathcal{D}} \mathcal{F}(V),$$

which is an isomorphism since whenever U is an open subset containing x , there is an open subset V in \mathcal{B} contained in U . Since we may do this for all bases for the neighborhood system around x , the claim follows.

The inductive limit defined above is denoted by \mathcal{F}_x , and referred to as *the stalk* of the presheaf \mathcal{F} at the point $x \in X$. For an open subset V containing the point x we have, in particular, a mapping of sets or an A -homomorphism

$$\iota_{V,x} : \mathcal{F}(V) \longrightarrow \mathcal{F}_x$$

of $\mathcal{F}(V)$ into the stalk at the point x . Frequently we write f_x instead of $\iota_{V,x}(f)$.

1.6 Grothendieck Topologies, sheaves and presheaves

1.6.1 Grothendieck topologies

A Grothendieck topology \mathcal{G} consists of a category $\text{Cat}(\mathcal{G})$ and a set $\text{Cov}(\mathcal{G})$, called *coverings*, of families of morphisms

$$\{\varphi_i : U_i \longrightarrow U\}_{i \in I}$$

in $\text{Cat}(\mathcal{G})$ such that

1. All sets consisting of one isomorphism are coverings
2. If $\{\varphi_i : U_i \longrightarrow U\}_{i \in I}$ and $\{\varphi_{i,j} : U_{i,j} \longrightarrow U_i\}_{j \in I_i}$ are coverings, then so is the set consisting of all the compositions

$$\{\varphi_{i,j} : U_{i,j} \longrightarrow U\}_{i \in I, j \in I_i}$$

3. If $\{\varphi_i : U_i \longrightarrow U\}_{i \in I}$ is a covering, and $V \longrightarrow U$ is a morphism in $\text{Cat}(\mathcal{G})$, then the products $U_i \times_U V$ exist for all $i \in I$ and the projections $\{U_i \times_U V \longrightarrow V\}_{i \in I}$ is a covering.

At this point we offer one example only, namely the following: Let X be a topological space, and let $\text{Cat}(\mathcal{G})$ be $\mathcal{T}op(X)$. Whenever U is an open subset, a covering is given as the set of all open injections of the open subsets in an open covering in the usual sense:

$$\{\varphi_i : U_i \longrightarrow U\}_{i \in I} \in \text{Cov}(\mathcal{G}) \iff U = \bigcup_{i \in I} U_i.$$

The verification that this is a Grothendieck topology is simple, perhaps modulo the following hint: If V and W are open subsets of the open set U , then $V \times_U W = V \cap W$ in the topology $\mathcal{T}op(X)$.

1.6.2 Presheaves and sheaves on Grothendieck topologies

Let \mathcal{G} be a Grothendieck topology and \mathcal{C} be a category with infinite products. A presheaf on \mathcal{G} of \mathcal{C} is a contravariant functor

$$\mathcal{F} : \text{Cat}(\mathcal{G}) \longrightarrow \mathcal{C}.$$

If $\varphi : U \longrightarrow V$ is a morphism in $\text{Cat}(\mathcal{G})$, then we say, as for a topological space, that $\mathcal{F}(\varphi)$ is the restriction morphism from $\mathcal{F}(V)$ to $\mathcal{F}(U)$, and when the objects of \mathcal{C} have an underlying set, then we refer to the image of individual elements $s \in \mathcal{F}(V)$ as the restriction of s from V to U .

\mathcal{F} is said to be a *sheaf* if it satisfies the following condition:

Condition 1.6.2.1 (Sheaf Condition) *If $\{\varphi_i : U_i \longrightarrow U\}_{i \in I}$ is a covering, then the diagram below is exact:*

$$\mathcal{F}(U) \xrightarrow{\alpha} \prod_{i \in I} \mathcal{F}(U_i) \begin{array}{c} \xrightarrow{\beta} \\ \xrightarrow{\gamma} \end{array} \prod_{i,j \in I} \mathcal{F}(U_i \times_U U_j)$$

Here $\alpha = (\mathcal{F}(\varphi_i)|_{i \in I})$ is the canonical morphism which is determined by the universal property of the product, and β, γ also come from the universal property of the product by means of the two sets of morphisms:

$$\begin{array}{l} \mathcal{F}(U_i) \xrightarrow{\mathcal{F}(pr_1)=\beta_{i,j}} \mathcal{F}(U_i \times_U U_j) \\ \mathcal{F}(U_j) \xrightarrow{\mathcal{F}(pr_2)=\gamma_{j,i}} \mathcal{F}(U_i \times_U U_j) \end{array}$$

so

$$\beta = (\beta_j|_{j \in I}) \text{ where } \beta_j = \prod_{i \in I} \beta_{i,j}$$

and

$$\gamma = (\gamma_j | j \in I) \text{ where } \gamma_j = \prod_{i \in I} \gamma_{j,i}.$$

1.6.3 Sheaves of *Set* and sheaves of Mod_A

We have a simple but clarifying result on sheaves of *Set* and sheaves of Mod_A :

Proposition 1.6.3.1 *Let*

$$\mathcal{F} : \text{Cat}(\mathcal{G}) \longrightarrow \text{Mod}_A$$

be a presheaf on the Grothendieck topology \mathcal{G} , and

$$T : \text{Mod}_A \longrightarrow \text{Set}$$

be the forgetful functor. Then \mathcal{F} is a sheaf if and only if $T \circ \mathcal{F}$ is a sheaf.

Proof. Kernels are the same in the two categories. \square

For presheaves of *Set* and Mod_A the sheaf-condition takes on a concrete form. We have the

Proposition 1.6.3.2 *A presheaf \mathcal{F} on the Grothendieck topology \mathcal{G} is a sheaf if and only if the following two conditions are satisfied:*

Sheaf Condition 1: If $\{\varphi_i : U_i \longrightarrow U\}$ is a covering, and if s' and $s'' \in \mathcal{F}(U)$ have the same restrictions to U_i for all $i \in I$, then they are equal.

Sheaf Condition 2: If $\{\varphi_i : U_i \longrightarrow U\}$ is a covering, and if there is given $s_i \in \mathcal{F}(U_i)$ for each $i \in I$ such that s_i and s_j have the same restrictions to $U_i \cap U_j$, then there exists $s \in \mathcal{F}(U)$ such that the restriction of s to U_i is equal to s_i .

Proof. Immediate from the description of kernels in the categories *Set* and Mod_R . \square

The following example contains much of the geometric intuition behind the concept of a sheaf:

Example 1.6.3.1 (Continuous mappings) Let X be a topological space, and let $\mathcal{C}^0(U)$ denote the set of all continuous functions from the open subset U to the field of real numbers \mathbb{R} , with the usual topology given by the metric $d(r, s) = |s - r|$. Then \mathcal{C}^0 is a sheaf of Ab on $\mathcal{T}op_X$.

If X is an open subset of \mathbb{R}^N for some N , then we let $\mathcal{C}^m(U)$ denote the set of all functions on U which are m times differentiable. This is also a sheaf of Ab . We use this notation for $m = \infty$ as well.

We may replace \mathbb{R} by the field of complex numbers, also with the usual topology.

1.6.4 The category of presheaves of \mathcal{C} and the full subcategory of sheaves.

The category of presheaves of \mathcal{C} on the Grothendieck topology \mathcal{G} is the category of contravariant functors from $\mathcal{C}at(\mathcal{G})$ to \mathcal{C} , hence in particular it is a category. The *sheaves* form a *subcategory*, where we keep the morphisms but subject the objects to the additional sheaf-condition. We say that the sheaves form a *full subcategory* of the category of presheaves.

1.6.5 The sheaf associated to a presheaf.

We have the following general fact, valid for presheaves on any Grothendieck topology \mathcal{G} :

Proposition 1.6.5.1 Let \mathcal{F} be a presheaf of Set or Mod_R on a Grothendieck topology \mathcal{G} . Letting $Sheaves_{\mathcal{G}}$ denote the category of sheaves of Set or Mod_R , as the case may be, the following functor is representable:

$$\begin{aligned} Sheaves_{\mathcal{G}} &\longrightarrow Set \\ \mathcal{H} &\longmapsto \text{Hom}(\mathcal{F}, \mathcal{H}). \end{aligned}$$

Remark-Definition In other words, the presheaf \mathcal{F} determines uniquely a sheaf $[\mathcal{F}]$ and a morphism $\tau_{\mathcal{F}}$

$$\mathcal{F} \xrightarrow{\tau_{\mathcal{F}}} [\mathcal{F}]$$

such that whenever $\mathcal{F} \longrightarrow \mathcal{H}$ is a morphism from the presheaf \mathcal{F} to the sheaf \mathcal{H} , then there exists a unique morphism

$$[\mathcal{F}] \longrightarrow \mathcal{H},$$

such that the appropriate diagram commutes. The sheaf $[\mathcal{F}]$ is referred to as the sheaf associated to the presheaf \mathcal{F} .

Proof of the proposition. For simplicity we consider the case when \mathcal{G} is the usual topology on a topological space. We also only treat the case of a presheaf of $\mathcal{M}od_R$, as the case of $\mathcal{S}et$ is an obvious modification. We make the following important definition:

$$[\mathcal{F}](U) = \left\{ (\xi_x)_{x \in U} \in \prod_{x \in U} \mathcal{F}_x \left| \begin{array}{l} \forall x \in U \exists V \subset U \text{ such} \\ \text{that } \exists \eta_V \in \mathcal{F}(V) \text{ with} \\ \iota_{V,y}(\eta_V) = (\eta_V)_y = \xi_y \text{ for all } y \in V \end{array} \right. \right\}$$

The definition of the restriction map from U to some $W \subset U$ is obvious, it is denoted by $\rho_{U,W}^{[\mathcal{F}]}$. Likewise, it is an immediate exercise to check that this is a sheaf of $\mathcal{M}od_A$ on the topological space X . Also, the definition of $\tau_{\mathcal{F}}$ is obvious:

$$\begin{aligned} \tau_{\mathcal{F}} : \mathcal{F}(U) &\longrightarrow [\mathcal{F}](U) \\ f &\longmapsto (f_x | x \in U). \end{aligned}$$

Clearly $\tau_{\mathcal{F},x}$ is an isomorphism for all $x \in X$.

To verify the universal property, let

$$\varphi : \mathcal{F} \longrightarrow \mathcal{H}$$

be a morphism from \mathcal{F} to a sheaf \mathcal{H} . We have to define a morphism

$$[\varphi] : [\mathcal{F}] \longrightarrow \mathcal{H}$$

which makes the appropriate diagram commutative. Now φ yields, for all $x \in X$,

$$\varphi_x : \mathcal{F}_x \longrightarrow \mathcal{H}_x.$$

Thus we also have

$$\psi(x) : [\mathcal{F}]_x \longrightarrow \mathcal{H}_x,$$

and to show is that there is a morphism of sheaves

$$\psi : [\mathcal{F}] \longrightarrow \mathcal{H}$$

such that $\psi(x) = \psi_x$. Let U be an open subset of X , and let $\xi_U = (\xi_x)_{x \in U} \in [\mathcal{F}](U)$, where of course the ξ_x 's satisfy the condition in the definition of $[\mathcal{F}](U)$. In particular there is an open covering of U by open subsets V where there are $\eta_V \in \mathcal{F}(V)$ such that for all $y \in V$ we have $\xi_y = (\eta_V)_y$. We put $\varphi_V(\eta_V) = \zeta_V \in \mathcal{H}(V)$. Then it is easy to see that by the sheaf condition of \mathcal{H} these elements ζ_V may be glued to an element $\zeta_U \in \mathcal{H}(U)$. As is easily verified, putting $\psi_U(\xi_U) = \zeta_U$ gives a morphism of sheaves $\psi : [\mathcal{F}] \rightarrow \mathcal{H}$, and we put $\psi = [\varphi]$.

Uniqueness is a consequence of the following lemma, the proof of which is left to the reader:

Lemma 1.6.5.2 *Given two morphisms of sheaves on the topological space X ,*

$$\psi, \phi : \mathcal{A} \rightarrow \mathcal{B}$$

such that for all $x \in X$

$$\psi_x = \phi_x : \mathcal{A}_x \rightarrow \mathcal{B}_x.$$

Then $\psi = \phi$.

The verification that the appropriate diagram commutes is also straightforward. \square

We finally note that the assignment

$$\mathcal{F} \rightarrow [\mathcal{F}]$$

defines a covariant functor

$$\text{Presheaves}_{\mathfrak{G}} \rightarrow \text{Seaves}_{\mathfrak{G}}$$

1.6.6 The category of Abelian sheaves

We conclude this introductory section by summarizing the basic properties of the category of Abelian groups on a topological space X . This category is denoted by $\mathcal{A}b_X$. It is commonly referred to as the *category of Abelian sheaves on X* . All of this is valid in more general settings, say for modules over commutative rings, etc.

The sum of two Abelian sheaves \mathcal{A} and \mathcal{B} is defined by

$$(\mathcal{A} \oplus \mathcal{B})(U) = \mathcal{A}(U) \oplus \mathcal{B}(U),$$

which does indeed define an Abelian sheaf on X . For a morphism of Abelian sheaves,

$$\varphi : \mathcal{A} \longrightarrow \mathcal{B},$$

we define the Abelian sheaf $\ker(\varphi)$ by

$$\ker(\varphi)(U) = \ker(\varphi_U),$$

and let the restriction homomorphisms be the restrictions of the corresponding ones for the sheaf \mathcal{A} . It is a simple exercise to verify that $\ker(\varphi)$ so defined is an Abelian sheaf. For the definition of $\operatorname{coker}(\varphi)$, however, the situation is different: In this case we only get a *presheaf* by

$$U \mapsto \operatorname{coker}(\varphi_U).$$

It is important to reflect on the significance of this difference. We define $\operatorname{coker}(\varphi)$ by taking the associated sheaf to the above presheaf. Similarly we have to define the Abelian sheaf $\operatorname{im}(\varphi)$, by first defining the obvious presheaf, then taking the associated sheaf.

Proposition 1.6.6.1 (Monomorphisms and epimorphisms) *φ is a monomorphism if and only if $\ker(\varphi)$ is the zero sheaf, $\mathbf{0}$. Moreover, $\operatorname{coker}(\varphi) = \mathbf{0}$ if and only if f is an epimorphism.*

Remark The terms injective, respectively surjective, are also used.

The *proof* of the proposition a is simple routine exercise, and is left to the reader.

We also have the following simple result, the proof of which is likewise left to the reader as an exercise:

Proposition 1.6.6.2 *Let*

$$\varphi : \mathcal{A} \longrightarrow \mathcal{B}$$

be a morphism of Abelian sheaves. The following are equivalent:

1. φ is an isomorphism
2. All φ_U are bijective

3. All φ_x are bijective

4. φ is a monomorphism and an epimorphism

Remark Thus we have another example where “isomorphism” and “bijection” is the same thing. As we know, this is not always the case in general categories.

If $\iota : \mathcal{S} \hookrightarrow \mathcal{F}$ is the inclusion of a *subsheaf* (obvious definition) into the Abelian sheaf \mathcal{F} , then $\text{coker}(\iota)$ is denoted by \mathcal{F}/\mathcal{S} .

A sequence of Abelian sheaves

$$\dots \longrightarrow \mathcal{A}_{i-1} \xrightarrow{\varphi_{i-1}} \mathcal{A}_i \xrightarrow{\varphi_i} \mathcal{A}_{i+1} \longrightarrow \dots$$

is said to be *exact* at \mathcal{A}_i if $\text{im}(\varphi_{i-1}) = \text{ker}(\varphi_i)$.

In the category of Abelian sheaves $\text{Hom}(\mathcal{A}, \mathcal{B})$ is always an Abelian group with the obvious definition of addition. The category is, in fact, an Abelian category. Functors compatible with the additive structure on the Hom-sets are called additive functors. Here are two examples:

1.6.6.1 The direct image f_* Let $f : X \longrightarrow Y$ be a continuous mapping of topological spaces. We define a functor referred to as *the direct image under f* ,

$$f_* : \mathcal{A}b_X \longrightarrow \mathcal{A}b_Y$$

by putting

$$f_*(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)).$$

As is easily seen, this defines an Abelian sheaf on Y , and moreover, $f_*(\)$ is a covariant additive functor from $\mathcal{A}b_X$ to $\mathcal{A}b_Y$.

The fiber $f_*(\mathcal{F})_{f(x)}$ is related to \mathcal{F}_x in the following manner: By definition

$$f_*(\mathcal{F})_{f(x)} = \lim_{\longrightarrow \{V \subset Y \mid f(x) \in V\}} \mathcal{F}(f^{-1}(V)) \xrightarrow{\text{canonical}} \mathcal{F}_x$$

where the homomorphism labeled “canonical” is the one coming from forming \lim_{\longrightarrow} over an inductive system and over a subsystem.

1.6.6.2 The inverse image f^* We also define an “inverse image functor” for any continuous mapping $f \rightarrow Y$

$$f^* : \mathcal{A}b_Y \longrightarrow \mathcal{A}b_X$$

by first defining a presheaf

$$f^{-1}(\mathcal{G})(U) = \lim_{\longleftarrow V|V \supset f(U)} \mathcal{G}(V),$$

and then taking the associated sheaf. This is also a covariant, additive functor.

Remark. The notation f^* is used in a variety of different situations. Here f is a continuous mapping and the categories are categories of sheaves of \mathcal{A} on topological spaces. When f is a morphism of schemes, as encountered in algebraic geometry later in this book, and the categories are categories of Modules on these schemes, then f^* will have a different meaning.

We have

$$f^*(\mathcal{G})_x = \mathcal{G}_{f(x)}.$$

Indeed, we show that $f^{-1}(\mathcal{G})_x = \mathcal{G}_{f(x)}$. We have by the definition

$$f^*(\mathcal{G})_x = \lim_{\longrightarrow \{V \subset Y | x \in V\}} \left(\lim_{\longrightarrow \{U \subset Y | f(V) \subset U\}} \mathcal{G}(U) \right) \xrightarrow{\text{canonical}} \lim_{\longrightarrow \{U | f(x) \in U\}} \mathcal{G}(U) = \mathcal{G}_{f(x)}.$$

where again the homomorphism labeled “canonical” is the one coming from forming \lim_{\longrightarrow} over an inductive system and over a subsystem. In this case this homomorphism is an isomorphism, however, since the “subsystem” in question is actually the whole system: In fact, take $f(x) \in U \subset Y$, and put $V = f^{-1}(U)$. Then $x \in V$ and $f(V) \subset U$.

Whenever X is a subspace of Y and f is the natural injection, we write $\mathcal{F}|_X$ instead of $f^*(\mathcal{F})$. If f is an open embedding, this is nothing but the obvious restriction to open subsets contained in U .

1.6.6.3 Sheaf Hom $\mathcal{H}om$ For two Abelian sheaves \mathcal{A} and \mathcal{B} we define the sheaf $\mathcal{H}om(\mathcal{A}, \mathcal{B})$, referred to as *the Sheaf Hom* of \mathcal{A} and \mathcal{B} , as the associated sheaf of the presheaf

$$U \mapsto \text{Hom}(\mathcal{A}|_U, \mathcal{B}|_U)$$

The category $\mathcal{A}b_X$ plays an important role in algebraic geometry, and we will return to it as we need more specialized or advanced features.

1.6.7 Direct and inverse image of Abelian sheaves

We use the material from Section 1.3.4 to study the pair functors

$$\begin{array}{ccc} & f_* & \\ \mathcal{A}b_X & \xrightarrow{\quad} & \mathcal{A}b_Y \\ & \xleftarrow{\quad} & \\ & f^* & \end{array}$$

Indeed, they are adjoint, as we shall now explain.

For all Abelian sheaves \mathcal{G} on Y we define the functorial morphism of Abelian sheaves

$$\rho_{\mathcal{G}} : \mathcal{G} \longrightarrow f_*(f^*(\mathcal{G}))$$

by letting

$$\rho_{\mathcal{G},V} : \mathcal{G}(V) \xrightarrow{\text{canonical}} f^{-1}(\mathcal{G})(f^{-1}(V)) \xrightarrow{\tau_{f^{-1}(V)}} f_*(f^{-1}(\mathcal{G}))(V)$$

where the last homomorphism is the one coming from the morphism of a presheaf to its associated sheaf. ⁶

We next define functorial homomorphisms

$$\sigma_{\mathcal{F},V} : f^*(f_*(\mathcal{F}))(V) \longrightarrow \mathcal{F}(V)$$

as follows:

$$\begin{aligned} f^{-1}(f_*(\mathcal{F}))(V) &= \lim_{\longrightarrow \{U \subset Y \mid f(V) \subset U\}} f_*(\mathcal{F})(U) \\ &= \lim_{\longrightarrow \{U \subset Y \mid f(V) \subset U\}} \mathcal{F}(f^{-1}(U)) \longrightarrow \mathcal{F}(V), \end{aligned}$$

where the last homomorphism comes from the restrictions from $f^{-1}(U)$ to V . We now obtain a morphism of presheaves $f^{-1}(f_*(\mathcal{F})) \longrightarrow \mathcal{F}$ and hence a morphism of sheaves $f^*(f_*(\mathcal{F})) \longrightarrow \mathcal{F}$ as claimed.

We now have the following result:

Proposition 1.6.7.1 *The morphism of functors defined above $\rho : \text{id}_{\mathcal{A}b_Y} \longrightarrow f_* \circ f^*$ defines an isomorphism of bifunctors*

$$\Phi : \text{Hom}_X(f^*(\quad), \quad) \longrightarrow \text{Hom}_Y(\quad, f_*(\quad)),$$

thus f_* is right adjoint to f^* . The inverse functor Ψ of Φ is given by the morphism $\sigma : f^* \circ f_* \longrightarrow \text{id}_{\mathcal{A}b_X}$ defined above.

⁶One readily verifies that direct image f_* of a presheaf commutes with forming the associated sheaf.

Proof. See [EGA] I page from 30 onwards.

We introduce the following notation: The image of $\mu : f^*(\mathcal{G}) \longrightarrow \mathcal{F}$ under $\Phi_{\mathcal{G},\mathcal{F}}$ is denoted by $\mu^b : \mathcal{G} \longrightarrow f_*(\mathcal{F})$, whereas the preimage of $\nu : \mathcal{G} \longrightarrow f_*(\mathcal{F})$ is denoted by $\nu^\sharp : f^*(\mathcal{G}) \longrightarrow \mathcal{F}$

2 Schemes: Definition and basic properties

2.1 The affine spectrum of a commutative ring

2.1.1 The Zariski topology on the set of prime ideals

Let A be a commutative ring with 1. We consider the set of all prime ideals in A , that is to say all ideals $\mathfrak{p} \neq A$ such that

$$ab \in \mathfrak{p} \text{ and } a \notin \mathfrak{p} \Rightarrow b \in \mathfrak{p}.$$

We denote the set of all prime ideals in A by $\text{Spec}(A)$. For $a \in A$ we define the subset $D(a) \subseteq \text{Spec}(A)$ by

$$D(a) = \{\mathfrak{p} \in \text{Spec}(A) \mid a \notin \mathfrak{p}\}$$

and we put

$$V(a) = \{\mathfrak{p} \in \text{Spec}(A) \mid a \in \mathfrak{p}\}$$

As is easily seen,

$$D(a) \cap D(b) = D(ab),$$

hence all the subsets $D(a)$ as $a \in A$ constitute a *basis for a topology* on $\text{Spec}(A)$.

Definition 2.1.1.1 (The Zariski Topology) *The topology referred to above is called the Zariski topology on $\text{Spec}(A)$.*

It is easily seen that the closed subsets in this topology are given as

$$\mathcal{F} = \{F \mid F = V(S)\}$$

where $S \subset A$ and

$$V(S) = \{\mathfrak{p} \mid \mathfrak{p} \supset S\}.$$

Evidently $V(S) = V((S)A)$, thus the closed subsets of $\text{Spec}(A)$ are described by the ideals in A in this manner. Note that $V(A) = \emptyset$.

We similarly have that all the open subsets of $\text{Spec}(A)$ are described as

$$U = D(S)$$

where

$$D(S) = \{\mathfrak{p} \mid \mathfrak{p} \not\supseteq S\}.$$

We note that $D(S) = D((S)A)$.

This establishes an important relation between the closed subsets of the topological space $\text{Spec}(A)$ and the ideals in the ring A . We summarize this as follows:

Proposition 2.1.1.1 1. Let \mathfrak{a} and \mathfrak{b} be two ideals in A . Then

$$V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b}).$$

2. Let $\{\mathfrak{a}_i\}_{i \in I}$ be any family of ideals in A . Then

$$V\left(\sum_{i \in I} \mathfrak{a}_i\right) = \bigcap_{i \in I} V(\mathfrak{a}_i).$$

3. We have for all ideals \mathfrak{a} that $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$

4. V establishes a bijective correspondence between the radical ideals in A and the closed subsets of $\text{Spec}(A)$.

Proof. 1. is a direct consequence of the well known fact from commutative algebra, that if \mathfrak{p} is a prime ideal then for any ideals \mathfrak{a} and \mathfrak{b}

$$\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p} \text{ and } \mathfrak{b} \not\subseteq \mathfrak{p} \Rightarrow \mathfrak{a} \subseteq \mathfrak{p}.$$

2. For any ideal \mathfrak{J} , in particular for a prime, it is true that it contains all the \mathfrak{a}_i 's if and only if it contains their sum.

3. If $\mathfrak{p} \supseteq \sqrt{\mathfrak{a}}$, then in particular $\mathfrak{p} \supseteq \mathfrak{a}$. On the other hand if $\mathfrak{p} \supseteq \mathfrak{a}$, and if $a \in \sqrt{\mathfrak{a}}$, then for some integer N we have $a^N \in \mathfrak{a}$, thus $a^N \in \mathfrak{p}$, thus $a \in \mathfrak{p}$. Hence $\mathfrak{p} \supseteq \sqrt{\mathfrak{a}}$.

4. This assertion follows already from the previous ones, but we note the inverse mapping to V : Namely, letting

$$I(F) = \bigcap_{\mathfrak{p} \in F} \mathfrak{p},$$

we get a radical ideal such that $V(I(F)) = F$. The details of this simple verification is left to the reader. \square

Example 2.1.1.1 *If k is a field, then $\text{Spec}(k)$ consists of a single point.*

Example 2.1.1.2 *Let \mathbb{Z} be the ring of integers. Then $\text{Spec}(\mathbb{Z})$ is the set*

$$\{0, 2, 3, 5, 7, \dots\}$$

consisting of the set of all prime numbers and the number 0. The closure of the set consisting of 0 alone is all of $\text{Spec}(\mathbb{Z})$, while the closure of any other point is the point itself. The point 0 is referred to as the generic point of $\text{Spec}(\mathbb{Z})$, while the others are closed points.

Recall that if Δ is a multiplicatively closed subset of A , then there is a bijective correspondence between the prime ideals in A which do not intersect Δ , and the prime ideals in $\Delta^{-1}A$ given by

$$\mathfrak{p} \mapsto \mathfrak{P} = (\mathfrak{p})\Delta^{-1}A.$$

In particular, if $\mathfrak{P} = (\mathfrak{p})A_a$, then \mathfrak{P} is a prime in A_a , and all primes of A_a are obtained in this manner.

Example 2.1.1.3 *Let A be a commutative ring, and let $a \in A$. Then $\text{Spec}(A/(a)A)$ is homeomorphic as a topological space with the subspace $V(a)$ of $\text{Spec}(A)$. Letting, as usual, A_a denote the localization of A in the multiplicatively closed set $S = \{1, a, a^2, a^3, \dots\}$ of all powers of a , we get $\text{Spec}(A_a)$ homeomorphic to $D(a)$. The observant reader may feel uneasy about the case when a is nilpotent, since in this case A_a is not defined as a commutative ring with 1. We should have made an exception ruling this case out.⁷*

2.1.2 The structure sheaf on $\text{Spec}(A)$

The complement of any set-theoretic union of prime ideals in a commutative ring with 1 is a multiplicatively closed subset. Indeed, let $\{\mathfrak{p}_i\}_{i \in I}$ be a set of prime ideals, and let Δ be the complement in A of the set $\cup_{i \in I} \mathfrak{p}_i$. Then if $a, b \in \Delta$ we have $a, b \notin \mathfrak{p}_i \forall i \in I$, thus $ab \notin \mathfrak{p}_i \forall i \in I$, thus $ab \in \Delta$.

⁷Some authors prefer to set the definitions up so that for nilpotent a , A_a is the zero ring (in which $1 = 0$). The zero ring, if allowed to be counted among the commutative rings with 1, will have an empty Spec , in any case, as A itself is by definition never a prime ideal. And of course in this case $D(a) = \emptyset$ as well.

Now for all open $U \subseteq \text{Spec}(A)$ let $\Delta(U)$ denote the multiplicatively closed subset of A given by the complement of the union of all primes $\mathfrak{p} \in U$. Note that for two open subsets U and V of $\text{Spec}(A)$ we have

$$U \subseteq V \Rightarrow \Delta(U) \supseteq \Delta(V).$$

We define a presheaf of $\mathcal{C}omm$, \mathcal{O}' on the topological space $\text{Spec}(A)$ by

$$\mathcal{O}'(U) = \Delta(U)^{-1}A$$

and for $U \subset V$ open subsets, we define the restriction map by

$$\begin{aligned} \rho_{V,U}^{\mathcal{O}'} : \Delta(V)^{-1}A &\longrightarrow \Delta(U)^{-1}A \\ \frac{a}{s} &\longmapsto \frac{a}{s}, \end{aligned}$$

which makes sense as $\Delta(U) \supseteq \Delta(V)$.⁸

Definition 2.1.2.1 *We denote the associated sheaf of the presheaf \mathcal{O}' by $\mathcal{O}_{\text{Spec}(A)}$, or just \mathcal{O} when no ambiguity is possible. We refer to it as the structure sheaf of the pair $(\text{Spec}(A), \mathcal{O})$. The pair itself is called the affine spectrum associated to the commutative ring A , or also the spectrum of the ring A . From now on $\text{Spec}(A)$ will denote this pair, rather than just the underlying topological space. The commutative ring $\mathcal{O}(U)$ is also denoted by $\Gamma(U, \mathcal{O})$.*

Let $\mathfrak{U}(x)$ be the set of all open subsets in $\text{Spec}(A)$ containing the point $x \in \text{Spec}(A)$, corresponding to the prime ideal $\mathfrak{p}_x \subset A$. Then for all $U \in \mathfrak{U}(x)$,

$$\Delta(U) \subset \Delta(x) = \{s \in A \mid s \notin \mathfrak{p}_x\}$$

This inclusion induces a homomorphism in $\mathcal{C}omm$,

$$\varphi_{U,x} : \mathcal{O}'(U) \longrightarrow A_{\mathfrak{p}_x},$$

and as these homomorphisms are compatible with the restriction homomorphisms of \mathcal{O}' , we obtain a homomorphism of commutative rings with 1,

$$\varphi_x : \mathcal{O}'_x \longrightarrow A_{\mathfrak{p}_x}.$$

We have the following

⁸Since we adhere to the requirement that in an object of $\mathcal{C}omm$ $1 \neq 0$, the category on which this presheaf is defined is strictly speaking not $\mathcal{T}op_{\text{Spec}(A)}$, but rather the category obtained from it by deleting the empty set from the objects.

Lemma 2.1.2.1 φ is an isomorphism.

Proof. To show is that φ_x is bijective.

1. φ_x is surjective: Let $\alpha \in A_{\mathfrak{p}_x}$. Then $\alpha = \frac{a}{s}$, where $s \notin \mathfrak{p}_x$. Thus $\alpha = \varphi_{D(s),x}(\frac{a}{s})$, the latter fraction now to be understood as an element in the ring $\Delta(x)^{-1}A$. Then the image of this element in the inductive limit \mathcal{O}'_x is mapped to α by f_x . Thus f_x is onto.

2. φ_x is injective: It suffices to show that $\ker(\varphi_x) = 0$. Suppose that $f_x(\beta) = 0$. We wish to show that $\beta = 0$. There is an open subset $U \ni x$ and $s \in \Delta(U)$ and an element $b \in A$ such that $\beta = [\frac{b}{s}]$, in the notation we used describing the stalks. It suffices to show that the restriction of $\frac{b}{s}$ to some smaller open neighborhood containing x is zero. Now $\varphi_{D(s),x}(\frac{b}{s}) = \varphi_x(\beta) = 0$. Hence there exists $t \in \Delta(x)$ such that $tb = 0$. But then the restriction of $\frac{b}{s}$ to $U \cap D(t)$ is zero. \square

For all non empty open subsets $U \subset \text{Spec}(A)$ we have the homomorphism of $\mathcal{C}omm$, compatible with restriction to a smaller open subset,

$$\tau_U : \Delta(U)^{-1}A \longrightarrow \mathcal{O}_{\text{Spec}(A)}(U).$$

Moreover, for a not nilpotent and $U = D(a)$ we have

$$\{1, a, a^2, a^3 \dots\} \subset \Delta(D(a)),$$

which defines a homomorphism

$$\varsigma_a : A_a \longrightarrow \Delta(D(a))^{-1}A,$$

by

$$\frac{b}{a^n} \mapsto \frac{b}{a^n}.$$

Now we have the following:

Proposition 2.1.2.2 1. For all a not nilpotent ς_a is an isomorphism.

2. For all a not nilpotent $\tau_{D(a)}$ is an isomorphism.

Remark. In particular

$$\Gamma(\text{Spec}(A), \mathcal{O}) \cong A.$$

Proof. To show 1, we prove that ζ_a is injective and surjective. So suppose that $\zeta_a(\frac{b}{a^n}) = 0$. Then there is an element $c \in \Delta(D(a))$ such that $cb = 0$. But since $c \in \Delta(D(a))$ we have $D(c) \supseteq D(a)$, so $V(c) \subseteq V(a)$, hence $\sqrt{(c)A} \supseteq \sqrt{(a)A}$. In particular $a \in \sqrt{(c)A}$, thus a suitable power of a , say a^m is in $(c)A$, $a^m = rc$. But then we also have $a^m b = 0$, hence $\frac{b}{a^n} = 0$. Next, let $\frac{b}{c} \in \Delta(D(a))^{-1}A$. As above we find $m \in \mathbb{N}$ and $r \in A$ such that $a^m = rc$. Thus $\frac{b}{c} = \frac{rb}{a^m}$, which is in the image of ζ_a . Thus 1 is proven.

In the course of the proof above we have established the essential step in proving the following useful

Lemma 2.1.2.3 *We have the biimplication*

$$c \in \Delta(D(a)) \iff \exists r \in \Delta(D(a)) \text{ such that } rc = a^N \text{ for some } N$$

Proof of 2. By virtue of 1 it suffices to show that the composition

$$\eta_a : A_a \xrightarrow{\tau_a} \Delta(D(a))^{-1}A \longrightarrow \mathcal{O}_{\text{Spec}(A)}(D(a))$$

is an isomorphism. We write, for the canonical homomorphism from A_a to $A_{\mathfrak{p}}$ where $a \notin \mathfrak{p}$,

$$\begin{aligned} A_a &\longrightarrow A_{\mathfrak{p}} \\ \frac{b}{a^n} &\mapsto \left(\frac{b}{a^n} \right)_{\mathfrak{p}}. \end{aligned}$$

Then

$$\eta_a \left(\frac{b}{a^n} \right) = \left(\left(\frac{b}{a^n} \right)_{\mathfrak{p}} \mid \mathfrak{p} \in D(a) \right).$$

By virtue of part 1, the homomorphism η_a for $a \in A$ is the same as the homomorphism η_1 for $1 \in A_a$. Thus it suffices to show that in general $\eta_1 = \eta$ is bijective:

$$\begin{aligned} \eta : A &\longrightarrow \mathcal{O}(\text{Spec}(A)) \\ b &\mapsto (b_{\mathfrak{p}} \mid \mathfrak{p} \in \text{Spec}(A)). \end{aligned}$$

η is injective: Suppose that $\eta(b) = 0$. Then for all prime ideals \mathfrak{p} of A there is $p \notin \mathfrak{p}$ such that $sb = 0$. Hence $\mathfrak{a} = \text{Ann}(b)$ is contained in no prime ideal, thus $1 \in \text{Ann}(b)$, so $b = 0$.

η is surjective: Recall that

$$\mathcal{O}(\mathrm{Spec}(A)) = \left\{ (s_{\mathfrak{p}} | \mathfrak{p} \in \mathrm{Spec}(A)) \left| \begin{array}{l} \forall \mathfrak{p} \in \mathrm{Spec}(A) \exists V \subset \mathrm{Spec}(A) \text{ such} \\ \text{that } \exists s_V \in \Delta(V)^{-1}A \text{ with} \\ (s_V)_{\mathfrak{q}} = s_{\mathfrak{q}} \text{ for all } \mathfrak{q} \in \mathrm{Spec}(A) \end{array} \right. \right\}$$

Clearly we may assume that all the open subsets V are of the form $D(a_i)$ as i runs through some indexing set I . We then have

$$\mathrm{Spec}(A) = \bigcup_{i \in I} D(a_i),$$

thus

$$\bigcap_{i \in I} V(a_i) = V((a_i | i \in I)A) = \emptyset.$$

Hence

$$(a_i | i \in I)A = A,$$

in particular we have for some indices i_1, i_2, \dots, i_r

$$c_{i_1} a_{i_1} + \dots + c_{i_r} a_{i_r} = 1,$$

and we may assume that $I = \{1, 2, \dots, r\}$.

Now $s_{D(a_i)} \in \Delta(D(a_i))^{-1}A$, thus by the lemma

$$s_{D(a_i)} = \frac{b_i}{a_i^{n_i}}.$$

However, since $D(a_i) = D(a_i^{n_i})$ for all i , and the localizations are the same as well, we may assume that all $n_i = 1$. Thus

$$s_{D(a_i)} = \frac{b_i}{a_i}.$$

To compare this for different values of i , consider the canonical homomorphisms

$$A_{a_i} \xrightarrow{\varphi_i} A_{a_i a_j} \xleftarrow{\varphi_j} A_{a_j}.$$

We show that

$$\varphi_i\left(\frac{b_i}{a_i}\right) = \varphi_j\left(\frac{b_j}{a_j}\right):$$

Indeed, letting $\varphi_i(\frac{b_i}{a_i}) - \varphi_j(\frac{b_j}{a_j}) = b_{i,j}$, the image of $\beta_{i,j}$ in $(A_{a_i a_j})_{\mathfrak{P}}$ is zero for all prime ideals \mathfrak{P} of $(A_{a_i a_j})$. Thus as above, $\beta_{i,j} = 0$.

Hence we have the identity

$$\frac{b_i a_j}{a_i a_j} = \frac{b_j a_i}{a_j a_i}$$

in $(A_{a_i a_j})$, and thus there are non negative integers $m_{i,j}$ such that

$$(a_i a_j)^{m_{i,j}} (b_i a_j - b_j a_i) = 0,$$

and being finite in number, we may assume that these integers are equal, say to M , and get the relation

$$a_i^M a_j^{M+1} b_i = a_j^M a_i^{M+1} b_j.$$

As

$$\frac{b_i}{a_i} = \frac{a_i^M b_i}{a_i^{M+1}},$$

we may replace b_i by $a_i^M b_i$ and a_i by a_i^{M+1} , and finally obtain the simple relation⁹

$$a_i b_j = a_j b_i.$$

Using the c_1, \dots, c_N which we found above with the property that

$$1 = c_1 a_1 + \dots + c_N a_N,$$

we let

$$b = b_1 c_1 + \dots + b_N c_N.$$

We claim that in A_{a_i} ,

$$\frac{b}{1} = \frac{b_i}{a_i}.$$

Indeed,

$$b a_i = \sum_{j=1}^N c_j b_j a_i = \sum_{j=1}^N c_j b_i a_j = b_i.$$

Thus η is surjective and the proof is complete. \square

⁹The argument would be much simpler if A were an integral domain. However, an important aspect of scheme-theory is to have a theory which is valid in the presence of zero-divisors and even nilpotent elements.

2.1.3 Examples of affine spectra

2.1.3.1 Spec of a field. The simplest possible cases are the affine spectra of fields: if k is a field, then $\text{Spec}(k)$ has an underlying topological space consisting of one point, $X = \{s\}$ where s corresponds to the zero ideal of k . The structure sheaf is simply given by $\mathcal{O}(s) = k$.

2.1.3.2 Spec of the ring of integers. $\text{Spec}(\mathbb{Z})$ has as underlying topological space the set

$$\{0, 2, 3, 5, \dots, p, \dots\},$$

the set of 0 and all prime numbers. The topology is given by the open sets being the whole space as well as the empty set and the complements of all finite sets of prime numbers. The structure sheaf has \mathbb{Q} as stalk in the point 0, called the generic point, and at a prime number the stalk is \mathbb{Z} localized at that prime.

2.1.3.3 The scheme-theoretic affine n -space over a field k . We consider $\text{Spec}(k[X_1, X_2, \dots, X_n])$, the affine spectrum of the polynomial ring in n variables over the field k . It is referred to as the *scheme theoretic* affine n -space over the field k . It is denoted by \mathbb{A}_k^n . Note that we distinguish between this and k^n , which is identified with a special set of closed points in \mathbb{A}_k^n , namely those corresponding to maximal ideals of the type

$$\mathfrak{m} = (X_1 - a_1, X_2 - a_2, \dots, X_n - a_n).$$

If k is not algebraically closed, there are of course other closed points than these: Namely, all maximal ideals are closed points, and to capture these as points of the above type we have to *extend the base* to the algebraic closure K of k . Note that it is definitely *not true* that $\mathbb{A}_k^n \subset \mathbb{A}_K^n$. The reader should take a few moments to contemplate this phenomenon.

2.1.3.4 Affine spectra of finite type over a field. Let \mathfrak{a} be an ideal in $\text{Spec}(k[X_1, X_2, \dots, X_n])$, the polynomial ring in n variables. Let $B = \text{Spec}(k[X_1, X_2, \dots, X_n])/\mathfrak{a}$. Then $\text{Spec}(B)$ has as underlying topological space a closed subset of the affine n -space over k . An affine spectrum of this kind is called *an affine spectrum of finite type over k* . They constitute the class of closed subschemes of \mathbb{A}_k^n . We return to this later.

2.1.4 The sheaf of modules \widetilde{M} on $\text{Spec}(A)$

The construction of $\mathcal{O}_{\text{Spec}(A)}$ has an important generalization:

Definition 2.1.4.1 *Let M be an A -module. Then the sheaf \widetilde{M} on $\text{Spec}(A)$ is the sheaf associated to the presheaf \mathcal{M} defined by $\mathcal{M}(U) = \Delta(U)^{-1}M$, with the restriction maps being the canonical ones induced from localization:*

$$U \subset V \Rightarrow \Delta(V)^{-1}M \longrightarrow \Delta(U)^{-1}M, \frac{m}{s} \mapsto \frac{m}{s}.$$

We immediately observe that for all open subsets $U \subset \text{Spec}(A)$, $\widetilde{M}(U)$ is a module over the ring $\mathcal{O}_{\text{Spec}(A)}(U)$. Moreover, if $V \supset U$ then the restriction map

$$\rho_{V,U}^{\widetilde{M}} : \widetilde{M}(V) \longrightarrow \widetilde{M}(U)$$

is an $\mathcal{O}_{\text{Spec}(A)}(V) - \mathcal{O}_{\text{Spec}(A)}(U)$ homomorphism.

Definition 2.1.4.2 *If M and N are modules over A and B , respectively, and if $\varphi : A \longrightarrow B$ is a homomorphism of rings, then a mapping $f : M \longrightarrow N$ is called an A, B -homomorphism if it is additive and $f(am) = \varphi(a)f(m)$.*

The following observations are proved in exactly the same fashion as the corresponding ones for the sheaf $\mathcal{O}_{\text{Spec}(A)}$:

Proposition 2.1.4.1 *1. The canonical homomorphism for $\mathfrak{p} \in U$*

$$\begin{aligned} \Delta(U)^{-1}M &\longrightarrow M_{\mathfrak{p}} \\ \frac{m}{s} &\mapsto \frac{m}{s} \end{aligned}$$

induces an isomorphism

$$\varphi_x : \mathcal{M}_x \xrightarrow{\cong} M_{\mathfrak{p}_x}$$

where x corresponds to (is equal to) the prime ideal \mathfrak{p}_x .

2. The canonical homomorphism

$$\begin{aligned} \Delta(D(a))^{-1}M &\longrightarrow M_a \\ \frac{m}{s} &\mapsto \frac{m}{s} \end{aligned}$$

is an isomorphism

3. The morphism which maps a presheaf to its associated sheaf induces a homomorphism over the basis open sets $D(a)$

$$\tau_{D(a)} : \Delta(D(a))^{-1}M \longrightarrow \widetilde{M}(D(a))$$

which is an isomorphism.

We observe that for all non-empty open subsets $U \subset \text{Spec}(A)$, $\Delta(U)^{-1}M$ is a module over $\Delta(U)^{-1}A$, and that the restriction mappings are bi-homomorphisms as defined in Definition 2.1.4.2. Thus we have the same situation for the associated sheaves:

$\widetilde{M}(U)$ is an $\mathcal{O}_{\text{Spec}(A)}(U)$ -module, and restrictions of \widetilde{M} are bi-homomorphisms for the corresponding restrictions of $\mathcal{O}_{\text{Spec}(A)}$.

Definition 2.1.4.3 A sheaf \mathcal{M} of modules satisfying the above is called an \mathcal{O}_X -Module on $X = \text{Spec}(A)$. A morphism $f : \mathcal{M} \longrightarrow \mathcal{N}$ of sheaves between two \mathcal{O}_X -Module on X is called an \mathcal{O}_X -homomorphism if all f_U are $\mathcal{O}_X(U)$ -homomorphisms.

If $f : M \longrightarrow N$ is a homomorphism of A -modules, then we have a \mathcal{O}_X -homomorphism $\widetilde{f} : \widetilde{M} \longrightarrow \widetilde{N}$. Thus $M \mapsto \widetilde{M}$ is a covariant functor from the category of A -modules to the category of \mathcal{O}_X -Modules on $X = \text{Spec}(A)$.

2.2 The category of Schemes

2.2.1 First approximation: The category of Ringed Spaces

A *ringed space* is a pair (X, \mathcal{O}_X) consisting of a topological space X and a sheaf \mathcal{O}_X of $\mathcal{C}omm$ on X , defined for all non empty open subsets of X . By abuse of notation the pair (X, \mathcal{O}_X) is also denoted by X . The topological space is referred to as *the underlying topological space*, while the sheaf \mathcal{O}_X is called the *structure sheaf* of X .

A morphism from the ringed space (X, \mathcal{O}_X) to the ringed space (Y, \mathcal{O}_Y)

$$(f, \theta) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

is a pair consisting of a continuous mapping $f : X \longrightarrow Y$ and a homomorphism of sheaves of $\mathcal{C}omm$,

$$\theta : \mathcal{O}_Y \longrightarrow f_*(\mathcal{O}_X).$$

The ringed spaces thus form a category, denoted by $\mathcal{R}s$.

Note that whenever (X, \mathcal{O}_X) is a ringed space, and $f : X \rightarrow Y$ is a continuous mapping, then $Y = (Y, f_*(\mathcal{O}_X))$ is a ringed space and the pair (f, id) is a morphism from X to Y .

2.2.1.1 Function sheaves The most common ringed spaces are topological spaces X with various kinds of function sheaves, which usually take their values in a field K . Frequently the field is either \mathbb{R} or \mathbb{C} . The sheaf \mathcal{O}_X may be the sheaf of all continuous functions on the respective open subsets, or when X looks locally like an open subset of \mathbb{R}^n or \mathbb{C}^n we may consider functions which are n times differentiable, or algebraic functions when X is an algebraic variety over the field K , and so on.

If (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) are the ringed spaces obtained by taking the sheaves of continuous functions (say to \mathbb{R} or to \mathbb{C}) on the two topological spaces X and Y , and if $f : X \rightarrow Y$ is any continuous mapping, then composition with the restriction of f yields a morphism of sheaves

$$\theta : \mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X),$$

where as asserted,

$$\begin{aligned} \theta_U : \mathcal{O}_Y(U) &\rightarrow f_*(\mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}(U)) \\ (U \xrightarrow{\varphi} K) &\mapsto (\theta_U(\varphi) : f^{-1}(U) \xrightarrow{\varphi|_U} U \rightarrow K), \end{aligned}$$

where K is \mathbb{R} \mathbb{C} or for that matter, any topological ring (commutative with 1).

Similarly, if the topological spaces have more structure, like being differentiable manifolds, algebraic varieties etc., then this also works if we use morphisms in the category to which X and Y belong, instead of just continuous mappings. The details of these considerations are left to the reader.

2.2.1.2 The constant sheaf Another type of ringed spaces is obtained by taking any topological space X and letting \mathcal{O}_X be the *sheaf associated* to the presheaf defined by

$$\mathcal{O}'(U) = A,$$

where A is a fixed ring. Of course \mathcal{O}' is not a sheaf (why?), and the sheaf \mathcal{O}_X so defined is referred to as the *constant sheaf of A on X* .

Any topological space can be made into a ringed space by adding to it the constant sheaf of any ring whatsoever.

2.2.1.3 The affine spectrum of a commutative ring Clearly $\text{Spec}(A)$ which we have defined above is a ringed space. Moreover, if $\varphi : A \longrightarrow B$ is a homomorphism of $\mathcal{C}omm$, then we obtain a morphism of ringed spaces

$$\text{Spec}(\varphi) : \text{Spec}(B) \longrightarrow \text{Spec}(A)$$

as follows: The mapping of topological spaces $f : \text{Spec}(B) \longrightarrow \text{Spec}(A)$ is given by

$$\mathfrak{q} \mapsto \varphi^{-1}(\mathfrak{q}).$$

As is easily seen, we then have

$$f^{-1}(D(\mathfrak{a})) = D((\varphi(\mathfrak{a})B),$$

hence f is a continuous mapping.

Recall the notation of Example 1.3.4.1. We then have the

Proposition 2.2.1.1 *There is an isomorphism, functorial in M :*

$$\varrho_M : f_*(\widetilde{M}) \longrightarrow \widetilde{M}_{[\varphi]}.$$

Proof. The assertion of the proposition is immediate from the following general and useful lemma, when applied to the basis consisting of the open subsets of the form $D(a)$:

Lemma 2.2.1.2 *Let X be a topological space, and let \mathcal{B} be a basis for the topology on X . Let \mathcal{F} and \mathcal{G} be two sheaves of $\mathcal{A}b$ on X , such that for all $W \in \mathcal{B}$ there is an isomorphism*

$$\varphi_W : \mathcal{F}(W) \longrightarrow \mathcal{G}(W),$$

which is compatible with the restriction homomorphisms in the sense that all the diagrams

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\rho_{V,W}^{\mathcal{F}}} & \mathcal{F}(W) \\ \varphi_V \downarrow & & \downarrow \varphi_W \\ \mathcal{G}(V) & \xrightarrow{\rho_{V,W}^{\mathcal{G}}} & \mathcal{G}(W) \end{array}$$

are commutative. Then \mathcal{F} and \mathcal{G} are isomorphic.

Proof of the lemma. We have to define isomorphisms φ_U for all open subsets $U \subset X$, not just the basis open subsets. This is a simple application of the definition of sheaves: Let V be any open subset, and let $f \in \mathcal{F}(V)$. We have $V = \cup_{i \in I} W_i$, a covering by open subsets from \mathcal{B} . Let $g_i = \varphi_{W_i}(f|_{W_i})$, the image by φ_{W_i} of the restriction of f to W_i . For any basis open set $W \subset W_i \cap W_j$ we then have $g_i|_W = g_j|_W$, since the two diagrams

$$\begin{array}{ccc} \mathcal{F}(W_i) & \xrightarrow{\rho_{W_i, W}^{\mathcal{F}}} & \mathcal{F}(W) & \mathcal{F}(W_j) & \xrightarrow{\rho_{W_j, W}^{\mathcal{F}}} & \mathcal{F}(W) \\ \varphi_{W_i} \downarrow & & \downarrow \varphi_W & \varphi_{W_j} \downarrow & & \downarrow \varphi_W \\ \mathcal{G}(W_i) & \xrightarrow{\rho_{W_i, W}^{\mathcal{G}}} & \mathcal{G}(W) & \mathcal{G}(W_j) & \xrightarrow{\rho_{W_j, W}^{\mathcal{G}}} & \mathcal{G}(W) \end{array}$$

commute. Thus the g_i s glue to a unique $g \in \mathcal{G}(V)$, we put $\varphi_V(f) = g$. We now have to show that φ_V so defined is in fact an isomorphism of Abelian groups, and that it is compatible with restriction. This is straightforward and is left to the reader. \square (of the lemma)

To complete the proof of the proposition, we only need to apply the lemma to the basis for the topology on $\text{Spec}(A)$ consisting of the open subsets $D(a)$. \square

To proceed with the definition of $\text{Spec}(\varphi)$, we note that the homomorphism φ gives a homomorphism of A -modules denoted by the same letter, $\varphi : A \longrightarrow B_{[\varphi]}$, hence a morphism of $\mathcal{O}_{\text{Spec}(A)}$ -Modules

$$\theta = \tilde{\varphi} : \tilde{A} = \mathcal{O}_{\text{Spec}(A)} \longrightarrow \widetilde{B_{[\varphi]}} = f_*(\mathcal{O}_{\text{Spec}(B)}),$$

Remark 2.2.1.3 We follow [EGA] and identify $\widetilde{B_{[\varphi]}}$ with $f_*(\mathcal{O}_{\text{Spec}(B)})$ via the canonical isomorphism ρ_B .

We make the definition

$$\text{Spec}(\varphi) = (f, \theta) = (\varphi^{-1}(\quad), \tilde{\varphi}).$$

From now on we adopt the notation of [EGA] and write ${}^a\varphi$ for the mapping $\varphi^{-1}(\quad)$.

It is easily seen that Spec of a composition is the composition of the Spec 's (in reverse order), and that the Spec of the identity on A is the identity on $\text{Spec}(A)$. We may sum our findings up as follows:

Proposition 2.2.1.4 *Spec is a contravariant functor*

$$\text{Spec} : \mathcal{C}omm \longrightarrow \mathcal{R}s.$$

2.2.2 Second approximation: Local Ringed Spaces

Some of the ringed spaces X we have seen so far have the important property that for all points $x \in X$ the fiber $\mathcal{O}_{X,x}$ of the structure sheaf \mathcal{O}_X at x is a *local ring*. This is certainly so for $\text{Spec}(A)$, and also for the *function spaces* where the functions take their values in a field. Thus for instance, let (X, \mathcal{O}_X) be the topological space X together with the sheaf \mathcal{O}_X of continuous real valued functions on the open subsets. Then the ring $\mathcal{O}_{X,x}$ is the ring of *germs of continuous functions* at x : It is the ring of equivalence classes of function elements (f, U) where U is an open subset containing x and f is a continuous real valued function defined on U . We have the evaluation homomorphism

$$\begin{aligned} \varphi_x : \mathcal{O}_{X,x} &\longrightarrow \mathbb{R}, \\ [(f, U)] &\mapsto f(x). \end{aligned}$$

Clearly this is well defined, it is a ring-homomorphism and it is surjective as the constant functions are continuous.

Let $\mathfrak{m}_{X,x} = \ker(\varphi_x)$. This is a maximal ideal since $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \cong \mathbb{R}$. We show that $\mathfrak{m}_{X,x}$ is the only maximal ideal in $\mathcal{O}_{X,x}$. It suffices to show that if f is a continuous function on $U \ni x$ such that $f(x) \neq 0$, then $[(f, U)]$ is invertible in $\mathcal{O}_{X,x}$. Indeed, as f is continuous $f^{-1}(0)$ is a closed subset of U , not containing x . Thus if $V = U - f^{-1}(0)$, then $[f|_V, V]$ is invertible. Since this element is equal to $[(f, U)]$, we are done.

Definition 2.2.2.1 *A ringed space (X, \mathcal{O}_X) is called a local ringed space provided that all the fibers $\mathcal{O}_{X,x}$ of the structure sheaf are local rings. A morphism of ringed spaces between two local ringed spaces*

$$f = (f, \theta) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

is said to be a morphism of local ringed spaces provided that the morphism of sheaves

$$\theta : \mathcal{O}_Y \longrightarrow f_*(\mathcal{O}_X)$$

has the following property:

Whenever $f(x) = y$, the homomorphism θ_x^\sharp which is the composition

$$\theta_x^\sharp : f^*(\mathcal{O}_Y)_x = \mathcal{O}_{Y,y} \xrightarrow{\theta_y} f_*(\mathcal{O}_X)_y \xrightarrow{\text{canonical}} \mathcal{O}_{X,x}$$

is a local homomorphism in the sense that the maximal ideal of $\mathcal{O}_{Y,y}$ is mapped into the maximal ideal of $\mathcal{O}_{X,x}$.¹⁰

The category thus obtained is denoted by \mathcal{Lrs} . We note that $\text{Spec}(\varphi)$ is a morphism of local ringed spaces, and also that the morphism between two function spaces obtained from a continuous mapping by composition is a morphism of local ringed spaces.

For all points x of a local ringed space (X, \mathcal{O}_X) we have a field $\mathfrak{k}(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$, which plays a key role in the theory. For $X = \text{Spec}(A)$ $\mathfrak{k}(\mathfrak{p})$ is the quotient field of the integral domain A/\mathfrak{p} , thus this field varies from point to point in general. However, for local ringed spaces where the structure sheaf is a sheaf of functions with values in a fixed field, the *local fields* $\mathfrak{k}(x)$ are all equal to this fixed field.

If $U \subset X$ is an open subset and $f \in \mathcal{O}_X(U)$, $f(x)$ denotes the image of f under the composition

$$\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,x} \longrightarrow \mathfrak{k}(x).$$

If $f \in \mathcal{O}_X(X)$, then we put

$$X_f = \{x \in X \mid f(x) \neq 0\}$$

Then

Lemma 2.2.2.1 X_f is an open subset of X .

Proof. The assertion $f(x) \neq 0$ is equivalent to the assertion that the image of f in $\mathcal{O}_{X,x}$ be a unit. Thus if $x \in X_f$, then there exists an open subset U containing x and an element $g \in \mathcal{O}_X(U)$ such that $f|_U g = 1$. Hence $U \subset X_f$. \square

We have the following important result, which shows that the definition of morphisms between local ringed spaces made above is exactly right for our purposes:

¹⁰It is easily seen that this condition is equivalent to the assertion that the inverse image of the maximal ideal of the target local ring be the maximal ideal of the source local ring.

Proposition 2.2.2.2 *Let A and B be two commutative rings, and let*

$$(f, \theta) : \text{Spec}(B) = X \longrightarrow \text{Spec}(A) = S$$

be a morphism of ringed spaces. Then $(f, \theta) = \text{Spec}(\varphi)$ for

$$\varphi : A \xrightarrow{\tau_A} \mathcal{O}_S(S) \xrightarrow{\theta_S} \mathcal{O}_X(X) \xrightarrow{\tau_B^{-1}} B$$

if and only if it is a morphism of local ringed spaces.

Remark 2.2.2.3 We shall use the convention that $\tau_A : A \longrightarrow \mathcal{O}_S(S)$ denotes the canonical isomorphism τ_1 for A , similar for τ_B . To avoid unwieldy notation, we adhere from now on to the convention of [EGA] of identifying the rings A and $\mathcal{O}_S(S)$ via the canonical isomorphism τ_A , when there is no danger of misunderstandings.

Striking as this result may be, it is only the starting point of several generalizations. We here present the ultimate version, due to *John Tate*. See [EGA] II, Errata et addenda on page 217.

Theorem 2.2.2.4 *Let $(S, \mathcal{O}_S) \cong \text{Spec}(A)$ and let (X, \mathcal{O}_X) be any local ringed space. Then the mapping*

$$\begin{aligned} \rho = \rho_{X,S} : \text{Hom}_{\mathcal{L}rs}((X, \mathcal{O}_X), (S, \mathcal{O}_S)) &\longrightarrow \text{Hom}_{\text{Comm}}(\mathcal{O}_S(S), \mathcal{O}_X(X)) \\ (f, \theta) &\mapsto \theta_S \end{aligned}$$

is bijective.

We first note that the theorem implies the proposition. Indeed, the “only if” part is trivial as $\text{Spec}(\varphi)$ is a morphism of $\mathcal{L}rs$. The “if” part follows since $\rho((f, \theta)) = \rho(\text{Spec}(\varphi)) = \theta_S$.¹¹

Proof of the theorem. We assume first that $S = \text{Spec}(A)$. We prove bijectivity of ρ by constructing an inverse. We make the canonical identification of A_f with $\mathcal{O}_S(D(f))$ for all $f \in A$. For any homomorphism $\varphi : \mathcal{O}_S(S) = A \longrightarrow \mathcal{O}_X(X)$, we define a mapping of topological spaces

$${}^a\varphi : X \longrightarrow S$$

¹¹We have $\rho(\text{Spec}(\varphi)) = \tilde{\varphi}_S \in \text{Hom}_{\text{Comm}}(\mathcal{O}_S(S), \mathcal{O}_X(X))$ by the identification of Remark 2.2.1.3.

by letting ${}^a\varphi(x) = \mathfrak{p}_x$ where

$$\mathfrak{p}_x = \{f \in A \mid \varphi(f)(x) = 0.\}$$

\mathfrak{p}_x is a prime ideal, being the kernel of a homomorphism into a field. Note that this definition generalizes the previous definition of ${}^a\varphi$, made in the case when X is affine.

As is easily checked ${}^a\varphi^{-1}(D(f)) = X_{\varphi(f)}$, and hence ${}^a\varphi$ is a continuous mapping. We next define a morphism of \mathcal{O}_X -Modules on S

$$\tilde{\varphi} : \mathcal{O}_S \longrightarrow {}^a\varphi_*(\mathcal{O}_X)$$

by first defining

$$\begin{aligned} \tilde{\varphi}_{D(f)} : A_f &\longrightarrow \mathcal{O}_X(X_{\varphi(f)}) \\ \frac{s}{f^n} &\mapsto (\varphi(s)_{X_{\varphi(f)}})((\varphi(f)|_{X_{\varphi(f)}})^{-1})^n \end{aligned}$$

It is easily seen that the following diagram commutes,

$$\begin{array}{ccc} A_f & \xrightarrow{\tilde{\varphi}_{D(f)}} & \mathcal{O}_X(X_{\varphi(f)}) \\ \mathcal{O}_S \downarrow \rho_{D(f), D(fg)} & & \downarrow \rho_{X_{\varphi(f)}, X_{\varphi(fg)}} \mathcal{O}_X \\ A_{fg} & \xrightarrow{\tilde{\varphi}_{D(fg)}} & \mathcal{O}_X(X_{\varphi(fg)}) \end{array}$$

and hence we may extend the set of homomorphisms $\tilde{\varphi}_{D(f)}$ to a morphism of \mathcal{O}_X -Modules on S as asserted above.

We thus have defined a morphism of $\mathcal{R}s$:

$$\sigma(\varphi) : (X, \mathcal{O}_X) \longrightarrow (S, \mathcal{O}_S).$$

This is actually a morphism of $\mathcal{L}rs$. Indeed, the homomorphism

$$\mathcal{O}_{S, {}^a\varphi(x)} = A_{\mathfrak{p}_x} \longrightarrow \mathcal{O}_{X, x}$$

maps the element $\frac{s}{f}$, where $f \notin \mathfrak{p}_x$, to the element $(\varphi(s)_{X_{\varphi(f)}})(\varphi(f)|_{X_{\varphi(f)}})^{-1}$. If $s \in \mathfrak{p}_x$ then $(\varphi(s)_{X_{\varphi(f)}})(\varphi(f)|_{X_{\varphi(f)}})^{-1} \in \mathfrak{m}_{X, x}$ by the definition of ${}^a\varphi(x)$, and $\tilde{\varphi}_x^\sharp$ is a local homomorphism.

It remains to show that ρ and σ are inverse to one another.

First of all, with the identifications we have made,

$$\tilde{\varphi}_S = \varphi.$$

Hence $\rho \circ \sigma$ is the identity on $\text{Hom}_{\text{eomm}}(\mathcal{O}_S(S), \mathcal{O}_X(X))$. To show that $\sigma \circ \rho$ is the identity, start with a morphism of local ringed spaces

$$(\psi, \theta) : (X, \mathcal{O}_X) \longrightarrow (S, \mathcal{O}_S)$$

and let $\varphi = \theta_S$. Since

$$\theta_x^\sharp : \mathcal{O}_{S, \psi(x)} \longrightarrow \mathcal{O}_{X, x}$$

is a local homomorphism it induces an embedding of fields

$$\theta^x : \mathfrak{k}(\psi(x)) \hookrightarrow \mathfrak{k}(x).$$

such that for all $f \in A$ we have $\theta^x(f(\psi(x))) = \varphi(f)(x)$. Then

$$f(\psi(x)) = 0 \iff \varphi(f)(x) = 0,$$

thus $\psi =^a \varphi$. It remains to show that

$$\tilde{f} = \theta : \mathcal{O}_S \longrightarrow \psi_*(\mathcal{O}_X) (=^a \varphi_*(\mathcal{O}_X)),$$

To prove this we note first that the following two diagrams are commutative:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & \mathcal{O}_X(X) & & A & \xrightarrow{\varphi} & \mathcal{O}_X(X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A_{\mathfrak{p}_{\psi(x)}} & \xrightarrow{\tilde{\varphi}_x^\sharp} & \mathcal{O}_{X, x} & & A_{\mathfrak{p}_{\psi(x)}} & \xrightarrow{\theta_x^\sharp} & \mathcal{O}_{X, x} \end{array}$$

The diagonal mapping $\alpha : A \longrightarrow \mathcal{O}_{X, x}$ is a homomorphism which maps the multiplicatively close subset $\Delta = A - \mathfrak{p}_{\psi(x)}$ into the group of units of the local ring $\mathcal{O}_{X, x}$, since the inverse image of its maximal ideal is $\mathfrak{p}_{\psi(x)}$. Thus by the universal property of localization α factors uniquely through $A_{\mathfrak{p}_{\psi(x)}}$, and so the two bottom homomorphisms are equal.

This implies that $\theta^\sharp = \tilde{\varphi}^\sharp$ and hence that $\theta = \tilde{\varphi}$. Thus the proof is complete in the case when $S = \text{Spec}(A)$.

We finally treat the general case, when we have an isomorphism $\ell : S \xrightarrow{\cong} \text{Spec}(A) = T$. In particular there is an isomorphism $\theta_\ell : \mathcal{O}_T \longrightarrow \ell_*(\mathcal{O}_S)$.

We have a commutative diagram¹²

$$\begin{array}{ccc} \text{Hom}_{\mathcal{L}_{rs}}(X, S) & \xrightarrow{\rho_{X, S}} & \text{Hom}_{\text{eomm}}(\mathcal{O}_S(S), \mathcal{O}_X(X)) \\ \text{Hom}_{\mathcal{L}_{rs}}(X, \ell) \downarrow & & \downarrow \text{Hom}_{\text{eomm}}(\alpha_T, \mathcal{O}_X(X)) \\ \text{Hom}_{\mathcal{L}_{rs}}(X, T) & \xrightarrow{\rho_{X, T}} & \text{Hom}_{\text{eomm}}(\mathcal{O}_T(T), \mathcal{O}_X(X)) \end{array}$$

¹²From now on we write X instead of (X, \mathcal{O}_X) .

Here the two vertical and the bottom horizontal maps are bijective, hence so is top horizontal mapping. This completes the proof. \square

Definition 2.2.2.2 *A local ringed space which is isomorphic to $\text{Spec}(A)$ for some commutative ring A is called an affine scheme.*

Corollary 2.2.2.5 *A local ringed space Y is an affine scheme if and only if $\rho_{X,Y}$ is bijective for all local ringed spaces X .*

Proof. The “if” part is the theorem. Assume that all $\rho_{X,Y}$ are bijective, and put $A = \mathcal{O}_Y(Y)$. We then have isomorphisms of functors

$$\text{Hom}_{\mathcal{L}rs}(\cdot, Y) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}omm}(A, \mathcal{O}_{(\cdot)}(\cdot)) \xleftarrow{\cong} \text{Hom}_{\mathcal{L}rs}(\cdot, \text{Spec}(A))$$

by hypothesis and the theorem. Thus the functors h_Y and $h_{\text{Spec}(A)}$ are isomorphic, thus $Y \cong \text{Spec}(A)$. \square

For a general local ringed space Z we put

$$S(Z) = \text{Spec}(\mathcal{O}_Z(Z)).$$

We then have functorial mappings

$$\text{Hom}_{\mathcal{L}rs}(X, Z) \xrightarrow{\rho_{X,Z}} \text{Hom}_{\mathcal{C}omm}(\mathcal{O}_Z(Z), \mathcal{O}_X(X)) \xleftarrow{\rho_{X,S(Z)}} \text{Hom}_{\mathcal{L}rs}(X, S(Z)),$$

which yield a morphism of contravariant functors

$$h_Z \longrightarrow h_{S(Z)},$$

thus a morphism of local ringed spaces

$$\epsilon_Z : Z \longrightarrow S(Z).$$

We obtain the further

Corollary 2.2.2.6 *Z is an affine scheme if and only if ϵ_Z is an isomorphism.*

Proof. By the previous corollary Z is an affine scheme if and only if all $\rho_{X,Z}$ are bijective. The claim follows from this. \square

Remark In the literature, textbooks and other, we frequently encounter assertions of the following type: “Let X be an affine scheme. Then $X = \text{Spec}(A)$...” A statement like this is justified when we identify X with $S(X)$ by ϵ_X , and this identification will be made throughout this book without further comments.

We note a final, important corollary:

Corollary 2.2.2.7 *The category Comm^* is equivalent to the category of affine schemes, AffSch . More generally, if $S \cong \text{Spec}(A)$ then AffSch_S is equivalent to the category of commutative A -algebras.*

Proof. This is immediate by Proposition 1.3.2.1 and the last corollary. A proof using only the definition of equivalent categories runs as follows: Let

$$F : \text{Comm} \longrightarrow \text{AffSch}$$

be the functor Spec , and let

$$G : \text{AffSch} \longrightarrow \text{Comm}$$

be the functor $X \mapsto \mathcal{O}_X(X)$. Then the canonical isomorphism $\tau_A : A \longrightarrow \mathcal{O}_{\text{Spec}(A)}(\text{Spec}(A))$ yields an isomorphism

$$\text{id}_{\text{Comm}} \longrightarrow G \circ F,$$

and the isomorphism

$$\epsilon_X : X \longrightarrow \text{Spec}(\mathcal{O}_X(X))$$

yields an isomorphism

$$\text{id}_{\text{AffSch}} \longrightarrow F \circ G.$$

This completes the proof. \square

2.2.3 Definition of the category of Schemes

The most important object under study in modern algebraic geometry is that of a *scheme*. A scheme is a geometric object which also embodies a vast generalization of the concept of a *commutative ring*:

Definition 2.2.3.1 *A scheme is a local ringed space X with the following property:*

$\forall x \in X \exists U \ni x$, an open subset of X , such that $(U, \mathcal{O}_X|_U)$ is an affine scheme, i.e., the morphism $\epsilon_U : (U, \mathcal{O}_X|_U) \longrightarrow \text{Spec}(\mathcal{O}_X(U))$ is an isomorphism.

A morphism $f : X \longrightarrow Y$ from one scheme to another is a morphism between them when viewed as local ringed spaces.

The category of schemes is denoted by Sch . Let S be any scheme. The category Sch_S is referred to as the category of S -schemes. Recall that an S -scheme is then a pair (X, φ_X) , where $\varphi_X : X \longrightarrow S$ is a morphism, which we, by abuse of language, refer to as the structure sheaf of the S -scheme X . A morphism of S -schemes $f : X \longrightarrow Y$ is a morphism of schemes such that $\varphi_Y \circ f = \varphi_X$.

The first important task is to carry out the construction of finite products in the category of S -schemes. We prove the following:

Theorem 2.2.3.1 *Finite products exist in the category Sch_S .*

Proof. It suffices to construct the product $X_1 \times_S X_2$ for any two S -schemes X_1 and X_2 . This is done in several steps. First of all, we know by Corollary 2.2.2.7 that if $S = \text{Spec}(A)$, and $X_i = \text{Spec}(B_i)$, where the B_i are A -algebras, then $\text{Spec}(B_1 \otimes_A B_2)$ is the product of X_1 and X_2 in the category of affine schemes over S . But by Theorem 2.2.2.4 it follows that this is the product in the larger category $\mathcal{L}rs_S$, in particular in Sch_S : Indeed, for any local ringed space Z we have to show that there is an isomorphism, functorial in Z ,

$$\text{Hom}_{\mathcal{L}rs_S}(Z, \text{Spec}(B_1 \otimes_A B_2)) \xrightarrow{\cong} \text{Hom}_{\mathcal{L}rs_S}(Z, \text{Spec}(B_1)) \times \text{Hom}_{\mathcal{L}rs_S}(Z, \text{Spec}(B_2))$$

This follows by the theorem quoted since it provides functorial isomorphisms

$$\mathrm{Hom}_{\mathcal{L}_{rsS}}(Z, \mathrm{Spec}(B_1 \otimes_A B_2)) \xrightarrow{\cong} \mathrm{Hom}_A(B_1 \otimes_A B_2, \mathcal{O}_Z(Z))$$

and

$$\mathrm{Hom}_{\mathcal{L}_{rsS}}(Z, \mathrm{Spec}(B_i)) \xrightarrow{\cong} \mathrm{Hom}_A(B_i, \mathcal{O}_Z(Z))$$

for $i = 1, 2$ and moreover,

$$\mathrm{Hom}_A(B_1 \otimes_A B_2, \mathcal{O}_Z(Z)) \xrightarrow{\cong} \mathrm{Hom}_A(B_1, \mathcal{O}_Z(Z)) \times \mathrm{Hom}_A(B_2, \mathcal{O}_Z(Z))$$

by the universal property of \otimes_A .

To construct the product $X_1 \times_S X_2$ we first reduce to the case when S is an affine scheme. For this we employ the following general

Lemma 2.2.3.2 *Let $f : S' \rightarrow S$ be a morphism of schemes which is a monomorphism. Assume that the S -schemes X_1 and X_2 are such that the structure morphisms $\varphi_i : X_i \rightarrow S$ factor through S' , i.e., that there are morphisms $\psi_i : X_i \rightarrow S'$ such that the following diagrams commute:*

$$\begin{array}{ccc} X_i & \xrightarrow{\varphi_i} & S \\ & \searrow & \nearrow \\ \psi_i & & f \\ & S' & \end{array}$$

Then

$$X_1 \times_{S'} X_2 = X_1 \times_S X_2,$$

in the sense that if one of the products is defined, then so is the other and they are canonically isomorphic.

Proof of the lemma. If Z is an S -scheme and $f_i : Z \rightarrow X_i$ two S -morphisms, then $\varphi_Z = \varphi_1 \circ f_1 = \varphi_2 \circ f_2$, thus

$$f \circ \psi_1 \circ f_1 = f \circ \psi_2 \circ f_2,$$

so as f is a monomorphism,

$$\psi_1 \circ f_1 = \psi_2 \circ f_2 = \varphi',$$

and we may consider Z as an S' -scheme by φ' , and f_1, f_2 as S' -morphisms. This establishes a bijection between pairs of S -morphisms $f_i : Z \rightarrow X_i$ and pairs of S' -morphisms $f_i : Z \rightarrow X_i$, and the claim follows. \square (of the lemma).

Assume that U is an open, non empty subset of the scheme S such that $(U, \mathcal{O}_S|_U)$ is an affine scheme. We then say that U is an open, affine subscheme (or just subset by abuse of language) of S .

The lemma implies the following

Proposition 2.2.3.3 *Let X_i be two S -schemes with structure morphisms φ_i , and let U be an open affine subset of S such that $\varphi_i(X_i) \subseteq U$ for $i = 1, 2$. Then*

$$X_1 \times_S X_2 = X_1 \times_U X_2,$$

in the sense that if one of the products is defined, then so is the other and they are canonically isomorphic.

Proof of the proposition. Immediate as the inclusion $U \hookrightarrow S$ obviously is a monomorphism, \square (of proposition).

We need one more general observation, namely that being a product is a local property.

Proposition 2.2.3.4 *Let Z be an S -scheme and let $p_i : Z \rightarrow X_i$ be two S -morphisms.*

1. *Let U and V be open subschemes of X_1 and X_2 , respectively. Let*

$$W = p_1^{-1}(U) \cap p_2^{-1}(V).$$

Then if Z is a product of X_1 and X_2 , W is a product of U and V .

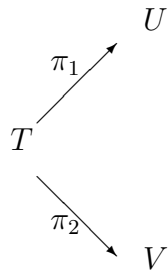
2. *Assume that*

$$X_1 = \cup_{\alpha \in I} X_{1,\alpha} \text{ and } X_2 = \cup_{\beta \in J} X_{2,\beta}$$

For all $(\alpha, \beta) \in I \times J$ put

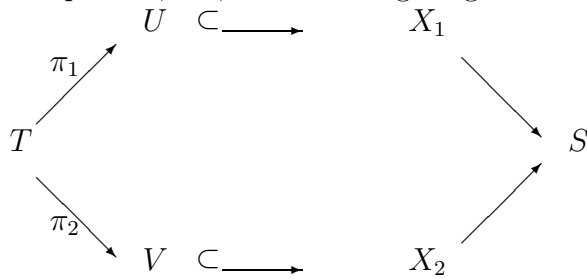
$$Z_{\alpha,\beta} = p_1^{-1}(X_{1,\alpha}) \cap p_2^{-1}(X_{2,\beta}),$$

and let $p_{1,\alpha,\beta}$ and $p_{2,\alpha,\beta}$ be the restrictions of p_1 and p_2 , respectively. Assume that $Z_{\alpha,\beta}$ is the product of $X_{1,\alpha}$ and $X_{2,\beta}$ with these morphisms as the projections. Then Z is the product of X_1 and X_2 with p_1 and p_2 as the projections.

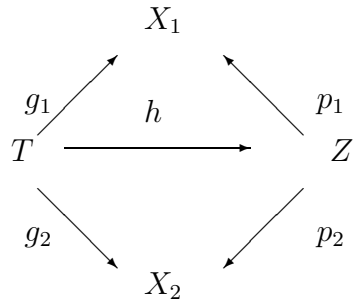


Proof. 1. Let

be S -morphisms, i.e., the following diagram commutes:

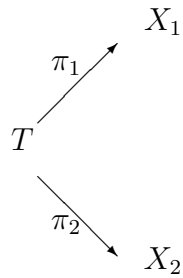


As $Z = X_1 \times_S X_2$ there is a unique $h : T \rightarrow Z$ such that the diagrams

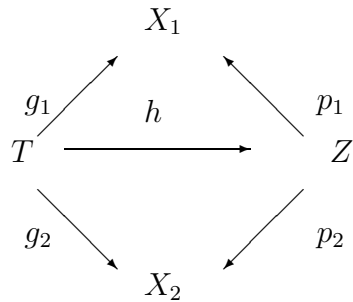


where q_i is the composition of π_i and the inclusion, commute. But this shows that h factors through W , and the claim follows.

2. Let



be S -morphisms. To show is that there is a unique S -morphism h such that the diagrams

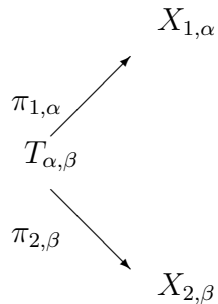


commute.

Uniqueness of h : Put

$$T_{\alpha,\beta} = \pi_1^{-1}(X_{1,\alpha}) \cap \pi_2^{-1}(X_{2,\beta}),$$

this yields an open covering of T . We then have the diagram



The restriction of h to $T_{\alpha,\beta}$ will then be a morphism

$$T_{\alpha,\beta} \longrightarrow Z_{\alpha,\beta}$$

which corresponds to the universal property of the product $Z_{\alpha,\beta}$ of $X_{1,\alpha}$ and $X_{2,\beta}$. Thus these restrictions are unique, hence so is h itself.

To show existence, define $T_{\alpha,\beta}$, $\pi_{1,\alpha}$ and $\pi_{2,\beta}$ as in the proof of uniqueness above. We get unique morphisms

$$h_{\alpha,\beta} : T_{\alpha,\beta} \longrightarrow Z_{\alpha,\beta}$$

$$\begin{array}{ccc}
 & X_{1,\alpha} & \\
 \pi_{1,\alpha} \nearrow & & \nwarrow p_{1,\alpha} \\
 T_{\alpha,\beta} & \xrightarrow{h_{\alpha,\beta}} & Z_{\alpha,\beta} \\
 \pi_{2,\beta} \searrow & & \swarrow p_{2,\beta} \\
 & X_{2,\beta} &
 \end{array}$$

such that the diagrams commute. It suffices to show that these $h_{\alpha,\beta}$ may be glued to a morphism $h : T \longrightarrow Z$. Thus we have to show that for all $\alpha, \gamma \in I$ and $\beta, \delta \in J$

$$h_{\alpha,\beta}|_{T_{\alpha,\beta} \cap T_{\gamma,\delta}} = h_{\gamma,\delta}|_{T_{\alpha,\beta} \cap T_{\gamma,\delta}}.$$

but by part 1. we have

$$Z_{\alpha,\beta} \cap Z_{\gamma,\delta} = (X_{1,\alpha} \cap X_{1,\gamma}) \times_S (X_{2,\beta} \cap X_{2,\delta})$$

and moreover

$$T_{\alpha,\beta} \cap T_{\gamma,\delta} = \pi_1^{-1}(X_{1,\alpha} \cap X_{1,\gamma}) \cap \pi_2^{-1}(X_{2,\beta} \cap X_{2,\delta})$$

and thus $h_{\alpha,\beta}|_{T_{\alpha,\beta} \cap T_{\gamma,\delta}}$ is the unique morphism coming from the universal property of the product $(X_{1,\alpha} \cap X_{1,\gamma}) \times_S (X_{2,\beta} \cap X_{2,\delta})$, hence it is equal to $h_{\gamma,\delta}|_{T_{\alpha,\beta} \cap T_{\gamma,\delta}}$ as claimed. \square

We are now ready to prove the key result which establishes the existence of finite fibered products in the category $\mathcal{S}ch$:

Proposition 2.2.3.5 *Let X_1 and X_2 be S -schemes, and let*

$$X_1 = \cup_{\alpha \in I} X_{1,\alpha} \text{ and } X_2 = \cup_{\beta \in J} X_{2,\beta}$$

be open coverings. Assume that all the products $X_{1,\alpha} \times_S X_{2,\beta}$ exist. Then $X_1 \times_S X_2$ also exists.

Proof. Let $i = (\alpha, \beta) \in I \times J = \mathcal{J}$, and put

$$Z'_i = X_{1,\alpha} \times_S X_{2,\beta}.$$

Let $j = (\gamma, \delta) \in \mathcal{J}$, and define the open subscheme $Z'_{i,j}$ of Z_i by

$$Z'_{i,j} = \text{pr}_{X_{1,\alpha}}^{-1}(X_{1,\alpha} \cap X_{1,\gamma}) \cap \text{pr}_{X_{2,\beta}}^{-1}(X_{2,\beta} \cap X_{2,\delta}).$$

Since $Z'_{i,j}$ is the product of the two intersections, there are unique isomorphisms $h_{i,j}$ and $h_{j,i}$ which yield isomorphisms $f_{i,j}$ by

$$f_{i,j} : Z'_{i,j} \xrightarrow{h_{i,j}} (X_{1,\alpha} \cap X_{1,\gamma}) \times_S (X_{2,\beta} \cap X_{2,\delta}) \xleftarrow{h_{j,i}} Z'_{j,i}.$$

Now for all $k = (\epsilon, \zeta) \in \mathcal{J}$ we have

$$(X_{1,\alpha} \cap X_{1,\gamma} \cap X_{1,\epsilon}) \times_S (X_{2,\beta} \cap X_{2,\delta} \cap X_{2,\zeta}) = Z'_{k,i} \cap Z'_{k,j},$$

from which it follows that

$$f_{i,k} = f_{i,j} \circ f_{j,k} \text{ on } Z'_{k,i} \cap Z'_{k,j} :$$

This important condition is referred to as the *Cocycle Condition*. We may visualize the situation as follows:

We now use the following general

Lemma 2.2.3.6 (Gluing-Lemma for ringed spaces) *Given a collection of ringed spaces $\{Z'_i\}_{i \in \mathcal{J}}$ with open sub-ringded spaces $Z'_{i,j}$ and isomorphisms $f_{i,j}$ as above, satisfying the Cocycle Condition. Then there exists a ringed space Z , with an open covering*

$$Z = \cup_{i \in \mathcal{J}} Z_i,$$

and isomorphisms $\varphi_i : Z'_i \longrightarrow Z_i$ such that $Z_{i,j}$ is mapped to $Z_i \cap Z_j$. If the Z'_i are local ringed spaces, respectively schemes, then so is Z .

Proof. The last assertion is of course obvious. To perform the gluing, we first put $Z'_{i,i} = Z'_i$, and let $f_{i,i}$ be the identity. We first glue the underlying topological spaces by introducing a relation \sim in the disjoint union of the sets Z_i as follows:

$$x \sim y \iff x \in Z'_i \text{ and } y \in Z'_j \text{ and } f_{i,j}(x) = y.$$

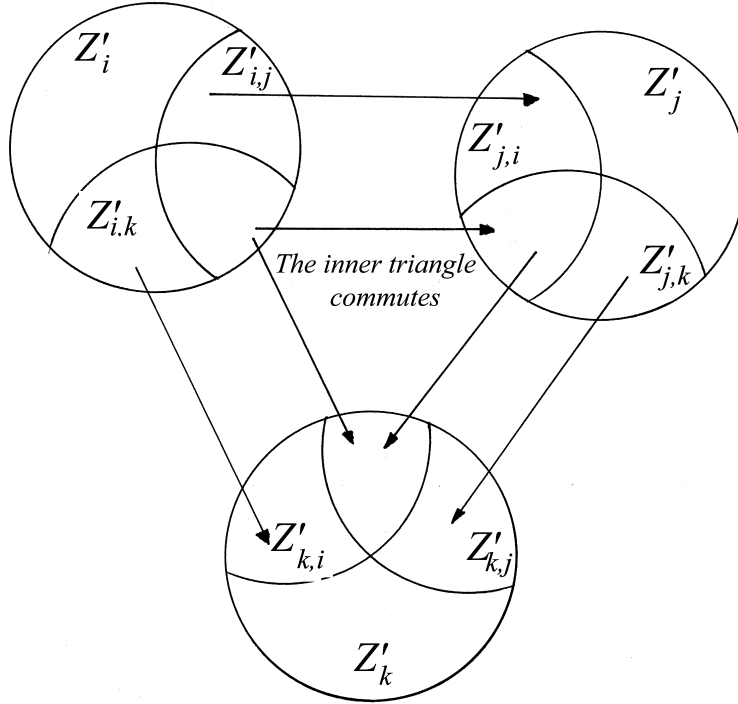


Figure 1: The Cocycle Condition

It follows in an obvious manner that this is an equivalence relation, transitivity uses the Cocycle Condition. As a set we define Z as the set of equivalence classes of this relation \sim . We get injective mappings

$$\varphi_i : Z'_i \hookrightarrow Z,$$

and clearly the images $\varphi_i(Z'_i) = Z_i$ do have the property that $Z_i \cap Z_j = \varphi_i(Z'_{i,j})$. Letting \mathcal{B} be the set of all images under φ_i of the open subsets of Z'_i , for all $i \in J$, we get a basis for a topology on Z , where $Z = \cup_{i \in J} Z_i$ is an open covering. Thus we are done gluing the topological spaces.

We now need to glue the structure sheaves as well. For this we have the following

Lemma 2.2.3.7 *Let Z be a topological space, with an open covering $Z = \cup_{\lambda \in L} Z_\lambda$. Assume that for all $\lambda \in L$ Z_λ has a sheaf \mathcal{F}_λ of $\mathcal{A}b$, and that for all*

$\lambda, \mu \in L$ we have isomorphisms

$$\varphi_{\lambda, \mu} : \mathcal{F}_\lambda|_{Z_\lambda \cap X_\mu} \xrightarrow{\cong} \mathcal{F}_\mu|_{Z_\mu \cap X_\lambda}$$

such that the cocycle condition holds on $Z_\lambda \cup X_\mu \cup Z_\nu$ for all λ, μ and ν in L . Then there exists a sheaf \mathcal{F} on Z with isomorphisms $\psi_\lambda : \mathcal{F}|_{Z_\lambda} \xrightarrow{\cong} \mathcal{F}_\lambda$ such that

$$\begin{array}{ccc} \mathcal{F}_\lambda|_{Z_\lambda \cap Z_\mu} & \xrightarrow{\varphi_{\lambda, \mu}} & \mathcal{F}_\mu|_{Z_\lambda \cap Z_\mu} \\ & \searrow \psi_\lambda|_{Z_\lambda \cap Z_\mu} & \swarrow \psi_\mu|_{Z_\lambda \cap Z_\mu} \\ & \mathcal{F}|_{Z_\lambda \cap Z_\mu} & \end{array}$$

commutes.

Proof. Let \mathcal{B} be a basis for the topology on Z consisting of the open subsets contained in Z_λ as λ runs through L . It is then enough to define $\mathcal{F}(V)$ for $V \in \mathcal{B}$: Indeed, we then define

$$\mathcal{F}_x = \lim_{\rightarrow_{x \in V \in \mathcal{B}}} \mathcal{F}(V),$$

and then define, for a general open subset U ,

$$\mathcal{F}(U) = \left\{ (\xi_x)_{x \in U} \subset \prod_{x \in U} \mathcal{F}_x \mid \begin{array}{l} \forall x \in U \exists V \in \mathcal{B} \text{ containing } x \text{ and} \\ f \in \mathcal{F}(V) \text{ such that} \\ \forall y \in V \text{ we have } \xi_y = f_y \end{array} \right\}$$

For all $V \in \mathcal{B}$ we now chose once and for all a $\lambda(V) \in L$ such that $V \subset U_{\lambda(V)}$. We define

$$\mathcal{F}(V) = \mathcal{F}_{\lambda(V)}(V),$$

and for $U \supset W$ we define $\rho_{U,W}^{\mathcal{F}}$ by

$$\begin{array}{ccc}
\mathcal{F}(U) = \mathcal{F}_{\lambda(U)}(U) & \xrightarrow{\rho_{U,W}^{\mathcal{F}}} & \mathcal{F}_{\lambda(U)}(W) \\
& \searrow \rho_{U,W}^{\mathcal{F}} & \swarrow (\varphi_{\lambda(U),\lambda(W)})_W \\
& & \mathcal{F}(W) = \mathcal{F}_{\lambda(W)}(W)
\end{array}$$

We have to verify that this definition of the restriction is transitive, and that follows from the cocycle condition. This completes the proof of the final lemma, and hence of the proposition. \square

The proposition has the following

Corollary 2.2.3.8 *Let $\varphi_i : X_i \longrightarrow S$, $i = 1, 2$ be morphisms of schemes, and let $S = \cup_{j \in J} S_j$ be an open covering. Let $X_{i,j} = \varphi_i^{-1}(S_j)$ for $i = 1, 2$ and $j \in J$. Then, if all $X_{1,j} \times_{S_j} X_{2,j}$ exist, $X_1 \times_S X_2$ exists.*

Proof. Immediate form the proposition by letting $Z'_i = X_{1,i} \times_S X_{2,i} = X_{1,j} \times_{S_j} X_{2,j}$, for all $i \in J$, and $Z'_{i,j} = p_{X_{1,i}}^{-1}(X_{1,i} \cap X_{1,j}) \cap p_{X_{2,i}}^{-1}(X_{2,i} \cap X_{2,j})$. $Z'_{i,j}$ is isomorphic with $(X_{1,i} \cap X_{1,j}) \times_S (X_{2,i} \cap X_{2,j})$, we get isomorphisms $\varphi_{i,j} : Z_{i,j} \longrightarrow Z'_{j,i}$, and any three of these do satisfy the cocycle condition. \square

We may now complete the proof of Theorem 2.2.3.1. It suffices to construct the product $X_1 \times_S X_2$ in the case when $S = \text{Spec}(A)$. For this we take affine open coverings $X_i = \cup_{j \in J_i} X_{i,j}$ for $i = 1, 2$, with $X_{i,j} = \text{Spec}(B_{i,j})$. For $\alpha \in J_1$, $\beta \in J_2$ we then have $Z_{\alpha,\beta} = X_{1,\alpha} \times_S X_{2,\beta} = \text{Spec}(B_{1,\alpha} \otimes_A B_{2,\beta})$. We are then done by Proposition 2.2.3.5. \square of the theorem.

As for coproducts in the category $\mathcal{S}ch_S$, the situation is much simpler: Indeed, the disjoint union of any family of S -schemes is again an S -scheme, and this disjoint union is, as one easily verifies, the coproduct in the category $\mathcal{S}ch_S$.

2.2.4 Formal properties of products

Finite products of S -schemes have a collection of formal properties, all of which are easy to prove and actually hold for products in any category:

They are consequences of the universal property which defines the product. We give a brief summary below.

Proposition 2.2.4.1 1. Let X_i be S -schemes, for $i = 1, 2$. Then

$$X_1 \times_S X_2 = X_2 \times_S X_1.$$

2. Let X_i be S -schemes, for $i = 1, 2, 3$. Then

$$(X_1 \times_S X_2) \times_S X_3 = X_1 \times_S (X_2 \times_S X_3),$$

and all the similar relations of associativity hold for any finite number of S -schemes.

Proof. 1. By the universal property.

2. The last assertion is a consequence of the formula given, by repeated application. The formula is immediate from the universal property. \square

Remark We say that products are commutative and associative.

We also have that

Proposition 2.2.4.2 (Triviality-Rule) For any S -scheme X , $X \times_S S = X$.

We have some basic constructions of morphisms. First of all, if $f_i : Z \rightarrow X_i$, $i = 1, 2$, are two S -morphisms then the unique S -morphism given by the universal property of the product is denoted by $(f_1, f_2)_S : Z \rightarrow X_1 \times_S X_2$. When no confusion is possible we write simply (f_1, f_2) . When $g_i : Z_i \rightarrow X_i$, $i = 1, 2$ are two S -morphisms, then composing with the first and the second projection yield two morphisms

$$f_i : Z_1 \times_S Z_2 \xrightarrow{\text{pr}_{Z_i}} Z_i \xrightarrow{g_i} X_i$$

$i = 1, 2$. We then put

$$g_1 \times_S g_2 = (f_1, f_2)_S,$$

in other words,

$$g_1 \times_S g_2 = (g_1 \circ \text{pr}_{Z_1}, g_2 \circ \text{pr}_{Z_2})_S.$$

Whenever we have an S -morphism $f : X \longrightarrow Y$, then we have the *graph* of f , which is defined as the morphism

$$\Gamma_f = (id_X, f) : X \longrightarrow X \times_S Y.$$

A special case is the diagonal of $X \times_S X$ for an S -scheme X , which is defined as

$$\Delta_{X/S} = \Gamma_{id_X} : X \longrightarrow X \times_S X.$$

If $\varphi : S' \longrightarrow S$ is a morphism of schemes and $f : X \longrightarrow S$ is a morphism, so X is an S -scheme, then we frequently denote the projection to S' by

$$f_{S'} : X_{S'} \longrightarrow S',$$

referring to the morphism and the scheme with the subscript S' as the *extension to S'* of the morphism f or the scheme X , respectively. Bearing in mind that $S \times_S S' = S'$, we have more generally for any morphism $f : X \longrightarrow Y$ the notation $f_{S'} = f \times id_{S'} : X_{S'} \longrightarrow Y_{S'}$.

This general concept of base extension is transitive in the following sense:

Proposition 2.2.4.3 *For two morphisms $S'' \longrightarrow S' \longrightarrow S$ we have $(X_{S'})_{S''} = X_{S''}$, and the similar relation for morphisms.*

Proof. The claim follows by the universal property. Indeed, letting $\varphi : S' \longrightarrow S$ be the structure morphism, then for any S' -scheme Z the mapping

$$\mathrm{Hom}_{S'}(Z, X_{S'}) \longrightarrow \mathrm{Hom}_S(Z, X)$$

$$f \mapsto \mathrm{pr}_X \circ f$$

is bijective, since any S -morphism $g : Z \longrightarrow X$ yields a unique

$$f = (g, \varphi) : Z \longrightarrow X_{S'}$$

such that $g = \mathrm{pr}_X \circ f$. Repeated application implies, in the situation of the proposition, that

$$\mathrm{Hom}_{S''}(Z, (X_{S'})_{S''}) = \mathrm{Hom}_S(Z, X) = \mathrm{Hom}_{S''}(Z, X_{S''}),$$

and the claim follows. \square

Along the same lines we have the

Proposition 2.2.4.4 1. *The following formula holds*

$$X_{S'} \times_{S'} Y_{S'} = (X \times_S Y)_{S'}.$$

2. *Let Y be an S -scheme, $f : X \rightarrow Y$ and $S' \rightarrow S$ morphisms. Then*

$$X_{S'} = X \times_Y Y_{S'},$$

and under this identification the second projection corresponds to $f_{S'}$.

Proof. 1. As in the proof of Proposition 2.2.4.3 we find that

$$\begin{aligned} \mathrm{Hom}_{S'}(Z, X_{S'}) \times \mathrm{Hom}_{S'}(Z, Y_{S'}) &= \mathrm{Hom}_S(Z, X) \times \mathrm{Hom}_S(Z, Y) \\ &= \mathrm{Hom}_S(Z, X \times_S Y) = \mathrm{Hom}_{S'}(Z, (X \times_S Y)_{S'}), \end{aligned}$$

and the claim follows. \square

2. We apply Proposition 2.2.4.3 to the situation

$$X \rightarrow Y \rightarrow S,$$

and the claim follows. \square

As an application of these ideas, we prove the following:

Proposition 2.2.4.5 *If the S -morphisms $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are monomorphisms, then so is $f \times_S g : X \times_S Y \rightarrow X' \times_S Y'$. In particular, the property of being a monomorphism is preserved by base extension.*

Proof. The latter assertion follows from the former, the identities being monomorphisms. If $h_i : Z \rightarrow X \times_S Y$, $i = 1, 2$ are two morphisms such that

$$(f \times_S g) \circ h_1 = (f \times_S g) \circ h_2,$$

then the compositions

$$Z \xrightarrow{\mathrm{pr}_X \circ h_i} X \xrightarrow{f} X'$$

are the same, and so are

$$Z \xrightarrow{\mathrm{pr}_Y \circ h_i} Y \xrightarrow{g} Y'.$$

Thus since f and g are monomorphisms,

$$\text{pr}_X \circ h_1 = \text{pr}_X \circ h_2 \text{ and } \text{pr}_Y \circ h_1 = \text{pr}_Y \circ h_2.$$

Hence $h_1 = h_2$ by the universal property of the product $X \times_S Y$. \square

An even simpler fact is the

Proposition 2.2.4.6 *For any S -morphism $f : X \longrightarrow Y$ the graph $\Gamma_f : X \longrightarrow X \times_S Y$ is a monomorphism*

Proof. Suppose that the two compositions

$$\begin{array}{ccc} & h_1 & \\ Z & \xrightarrow{\quad} & X \\ & \xrightarrow{\quad} & \\ & h_2 & \end{array} \quad \begin{array}{c} \Gamma_f \\ \longrightarrow \\ X \times_S Y \end{array}$$

are the same. Composing with pr_X we then get $h_1 = h_2$. \square

3 Properties of morphisms of schemes

3.1 Modules and Algebras on schemes

3.1.1 Quasi-coherent \mathcal{O}_X -Modules, Ideals and Algebras on a scheme X

Definition 3.1.1.1 *An \mathcal{O}_X -Module on the scheme X is a sheaf \mathcal{F} of $\mathcal{A}b$ on X , such that for all open $U \subset X$ $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$ -module and all restrictions $\rho_{U,V}^{\mathcal{F}} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ are $\mathcal{O}_X(U) - \mathcal{O}_X(V)$ -homomorphisms.*

We have seen one example, namely the sheaf \widetilde{M} on $\text{Spec}(A)$, for any A -module M . An \mathcal{O}_X -Module which is locally of this type is called *quasi-coherent*:

Definition 3.1.1.2 *An \mathcal{O}_X -Module \mathcal{F} on the scheme X is said to be quasi-coherent if for all $x \in X$ there exists an open affine $U = \text{Spec}(A)$ such that $\mathcal{F}|_U = \widetilde{M}$ for some A -module M .*

It is an important fact that a quasi-coherent \mathcal{O}_X -Module has a stronger property, namely:

Proposition 3.1.1.1 *An \mathcal{O}_X -Module \mathcal{F} on the scheme X is quasi-coherent if and only if for all open affine subschemes of X , $U = \text{Spec}(A)$, we have that $\mathcal{F}|_U = \widetilde{\mathcal{F}(U)}$.*

Proof. Will be provided in the final edition of the notes.

Definition 3.1.1.3 *A homomorphism of \mathcal{O}_X -modules on X is a morphism of sheaves of $\mathcal{A}b$,*

$$\varphi : \mathcal{F} \longrightarrow \mathcal{G},$$

such that all φ_U are $\mathcal{O}_X(U)$ -homomorphisms. φ is called injective, respectively surjective, if it is so as a morphism of sheaves.

The kernel, denoted $\ker(\varphi)$ is defined as the *sheaf*

$$\mathcal{K}(U) = \ker(\varphi_U),$$

and the cokernel $\text{coker}(\varphi)$ is the *associated sheaf* of the presheaf

$$\mathcal{C}(U) = \text{coker}(\varphi_U).$$

The latter is an \mathcal{O}_X -Module on X as is easily seen.

With these notions available we define *exact sequences* in the standard way, and note that the functor

$$\mathcal{M}od_A \longrightarrow \mathcal{O}_X - \text{Modules on } X = \text{Spec}(A)$$

$$M \mapsto \widetilde{M}$$

is an exact functor.

Kernels and cokernels of homomorphisms of quasi-coherent \mathcal{O}_X -Modules on X are again quasi-coherent, as one immediately verifies from the local structure as an \widetilde{M} .

An *Ideal* on X is defined as a quasi-coherent subsheaf \mathcal{I} of \mathcal{O}_X . A quasi-coherent \mathcal{O}_X -algebra in X , \mathcal{A} , is a \mathcal{O}_X -Module such that all $\mathcal{A}(U)$ are $\mathcal{O}_X(U)$ -algebras, and the restriction homomorphisms are homomorphisms of algebras as well. We define Ideals in a quasi-coherent Algebra on X as we did for

Ideals on X : Quasi-coherent submodules with the usual multiplicative ideal-property over all open subsets.

An important example of an Ideal on X is \mathcal{N}_X , the Ideal of nilpotent elements. For all open subsets U in X we let $\mathcal{N}_X(U)$ be the ideal of nilpotent elements in $\mathcal{O}_X(U)$, the nilpotent radical of that ring. We then obtain a quasi-coherent subsheaf of \mathcal{O}_X , so \mathcal{N}_X is an Ideal on X .

The quotient of a quasi-coherent \mathcal{O}_X -algebra on X by a quasi-coherent Ideal, is again a quasi-coherent \mathcal{O}_X -algebra on X . The usual algebraic operations of sum, intersection, radical etc. also carry over to this general situation.

3.1.2 Spec of an \mathcal{O}_X -Algebra on a scheme X

Let \mathcal{A} be a quasi-coherent \mathcal{O}_X -Algebra on a scheme X . For all open affine subschemes U of X we then have $\mathcal{A}|_U = \widetilde{\mathcal{A}(U)}$. Let $Z_{(U)} = \text{Spec}(\mathcal{A}(U))$. We then have morphisms $\pi_U : Z_U \rightarrow U$, and if $U \supset V$ are two open affine subschemes, then we have the obvious commutative diagram.

Proposition 3.1.2.1 *The $\pi_U : Z_U \rightarrow U$ may be glued to $\pi : Z \rightarrow X$, in such a way that $Z_{(U)}$ is identified with the open subset $\pi^{-1}(U) \subset Z$, and the open subsets*

$$\pi_U^{-1}(U \cap V) \text{ and } \pi_V^{-1}(U \cap V)$$

are identified.

Proof. Makes essential use of the quasi-coherent property, and proceeds along similar lines to the construction of the product of S -schemes. Will be provided in the final edition of the notes.

Definition 3.1.2.1 *The scheme Z of Proposition 3.1.2.1 is denoted by $\text{Spec}(\mathcal{A})$.*

We note the following general fact:

Proposition 3.1.2.2 *Let $f : X \rightarrow Y$ be a morphism and let \mathcal{A} be an Algebra on X . Then $f_*(\mathcal{A})$ is an Algebra on Y via $\theta : \mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$, and*

$$\text{Spec}(f_*(\mathcal{A})) \rightarrow Y$$

is the composition

$$\text{Spec}(\mathcal{A}) \rightarrow X \rightarrow Y.$$

Proof. It suffices to check this locally on Y , so we may assume that $Y = \text{Spec}(A)$. Then, we may assume that $X = \text{Spec}(B)$, since the equality of two given morphisms is a local question on the source scheme. But in the affine case the claim is obvious. \square

If \mathcal{J} is any Ideal on X , then the morphism

$$i = \pi : \text{Spec}(\mathcal{O}_X/\mathcal{J}) \longrightarrow \text{Spec}(\mathcal{O}_X) = X$$

is called a *canonical closed embedding*. A composition of an isomorphism and a canonical closed embedding is referred to as a *closed embedding*. An *open embedding* is just the inclusion of an open subscheme, and we shall not be too concerned with the distinction between closed embeddings and canonical closed embeddings: For all practical purposes all closed embeddings may be assumed to be canonical ones.

We immediately note that as a mapping of topological spaces, a closed embedding $i : Z \longrightarrow X$ identifies the source space with a closed subset of the target space. The corresponding $\theta : \mathcal{O}_X \longrightarrow i_*(\mathcal{O}_Z)$ is surjective as a morphism of sheaves.

We may define the *polynomial Algebra in X_1, \dots, X_N* over a scheme X , denoted by

$$\mathcal{A} = \mathcal{O}_X[X_1, \text{dots}, X_N]$$

by putting $\mathcal{A}(U) = \mathcal{O}_X(U)[X_1, \text{dots}, X_N]$ for all open subschemes $U \subset X$. This is an \mathcal{O}_X -Algebra on X , as is immediately verified. We put

$$\mathbb{A}_X^N = \text{Spec}(\mathcal{O}_X[X_1, \dots, X_N]),$$

referring to this scheme as the affine N -space over X . When $N = 1$ we speak of the affine line over X , etc.

3.1.3 Reduced schemes and the reduced subscheme X_{red} of X

An important example of a closed embedding is the case when $\mathcal{J} = \mathcal{N}_X$. In that case the source scheme is denoted by X_{red} , and the closed embedding is a homeomorphism as a mapping of topological spaces.

Since forming the nilpotent radical is compatible with localization, it follows that X_{red} is *reduced* in the following sense: ¹³

¹³A direct proof that all the local rings of X_{red} are without nilpotent elements runs as

Definition 3.1.3.1 A scheme X is said to be reduced if all its local rings are without nilpotent elements.

We have the following:

Proposition 3.1.3.1 The assignment

$$X \mapsto X_{\text{red}}$$

is a covariant functor from the category of schemes to itself.

Proof. We verify that a morphism $f : X \rightarrow Y$ gives rise to a morphism f_{red} which makes the following diagram commutative:

$$\begin{array}{ccc} X_{\text{red}} & \xrightarrow{f_{\text{red}}} & Y_{\text{red}} \\ i \downarrow & & \downarrow j \\ X & \xrightarrow{f} & Y \end{array}$$

where i and j are the closed embeddings. This is easily reduced to the fact that whenever $\varphi : A \rightarrow B$ is a homomorphism of commutative rings, then the nilpotent radical \mathfrak{N}_A of A is mapped into the nilpotent radical \mathfrak{N}_B of B , and thus there is a ring homomorphism φ_{red} which makes the diagram below commutative:

$$\begin{array}{ccc} A/\mathfrak{N}_A & \xrightarrow{\varphi_{\text{red}}} & B/\mathfrak{N}_B \\ \tau_A \uparrow & & \uparrow \tau_B \\ A & \xrightarrow{\varphi} & B \end{array}$$

Instead of piecing this together to obtain the globally defined morphism f_{red} , perfectly feasible as this may be, we now proceed by observing that the diagram above holds with \mathcal{A} and \mathcal{B} instead of A and B , i.e. for quasi-coherent Algebras on X , and Spec on such algebras is a contravariant functor. \square

follows: We may assume that $X = \text{Spec}(A)$, where A is without nilpotent elements. Let \mathfrak{p} correspond to the point $x \in X$. Suppose that $(\frac{a}{b})^n = 0$, where $a, b \in A$, $b \notin \mathfrak{p}$. Then $\exists c \notin \mathfrak{p}$ such that $ca^n = 0$. Thus $(ca)^n = 0$, hence $ca = 0$ as A has no nilpotents, thus $(\frac{a}{b}) = 0$. \square

3.1.4 Reduced and irreducible schemes and the “field of functions”

A scheme X is said to be *irreducible* if it is not the union of two proper closed subsets:

Definition 3.1.4.1 *The scheme X is said to be irreducible if*

$$X = X_1 \cup X_2 \text{ where } X_1 \text{ and } X_2 \text{ are closed in } X \implies X_1 = X \text{ or } X_2 = X.$$

The concept of an irreducible scheme is particularly powerful when the scheme is also reduced and locally Noetherian. We have the following:

Proposition 3.1.4.1 *Let X be a reduced and irreducible, locally Noetherian, scheme. Then there exists a unique point $x_{\text{=}} \in X$ such that $\overline{\{x_{\text{=}}\}} = X$. Moreover, the local ring $\mathcal{O}_{X,x_{\text{=}}}$, which we denote by $K(X)$, is a field, and as $x_0 \in U$ for all non empty open subsets of X there are canonical homomorphisms*

$$\rho_U : \mathcal{O}_X(U) \longrightarrow K(X),$$

which identify these rings as well as the local rings at all points of X with subrings of $K(X)$, in such a way that the restriction homomorphisms from the ring of an open subset to the ring of a smaller open subset are identified with the inclusion mappings.

Proof. Let $U = \text{Spec}(A)$ be an open affine subscheme, where A is Noetherian, and let x_0 be the point which corresponds to the prime ideal $(0) \subset A$ of the integral domain A . Indeed, A is necessarily an integral domain as the existence of more than one minimal prime ideals would yield a decomposition of X as a union of a finite number of proper closed subsets, namely the complement of U and the closures of the points corresponding to the minimal primes of A . Since the local ring at x_0 is without nilpotent elements, and has only one prime ideal, it is a field. The rest of the assertion of the proposition is immediate. \square

3.1.5 Irreducible components of Noetherian schemes

Let X be a Noetherian scheme. It then follows easily, by imitating the corresponding fact for the ideals in a Noetherian ring, that

Proposition 3.1.5.1 1. The set of closed subsets of X satisfy the descending chain condition.

2. Any collection of closed subsets of X has a minimal element.

3. All closed subsets of X may be written uniquely as the union of irreducible¹⁴ closed subsets.

3.2 Separated morphisms

3.2.1 Embeddings, graphs and the diagonal

We now know open and closed embeddings. We have the

Definition 3.2.1.1 A composition

$$Z \xrightarrow{i} U \xrightarrow{j} X$$

where j is an open embedding and i is a closed embedding is referred to as an embedding.

We shall derive several properties of embeddings. We start out with the

Proposition 3.2.1.1 Let

$$f : X \longrightarrow Y \text{ and } f' : X' \longrightarrow Y'$$

be two S -morphisms which are embeddings. Then so is

$$f \times_S f' : X \times_S X' \longrightarrow Y \times_S Y',$$

and if the two embeddings are open, respectively closed, then so is the product.

Proof. Whenever we have S -morphisms

$$X \xrightarrow{f_1} Y_1 \xrightarrow{f_2} Y$$

$$X' \xrightarrow{f'_1} Y'_1 \xrightarrow{f'_2} Y'$$

then

$$(f_2 \circ f_1) \times_S (f'_2 \circ f'_1) = (f_2 \times_S f'_2) \circ (f_1 \times_S f'_1),$$

¹⁴An irreducible subset is of course defined as for schemes, namely one which is not the union of proper closed subsets.

since they both solve the same universal problem. Hence it suffices to prove the assertions for open and closed embeddings. For open embeddings the claim is obvious, as $U \subset Y$ and $U' \subset Y'$ being two open subschemes yield the open subscheme $U \times_S U' \subset X \times X'$.

For closed embeddings, we may assume that $S = \text{Spec}(A)$, essentially by the same argument used to reduce the existence of $X \times_S X'$ to the case of S being affine. Since the question of being a closed embedding is local on the target space, we may assume that $Y = \text{Spec}(B)$ and $Y' = \text{Spec}(B')$, B and B' being A -algebras. Then we must have $X = \text{Spec}(B/\mathfrak{b})$ and $X' = \text{Spec}(B'/\mathfrak{b}')$, and hence $X \times_S X' = \text{Spec}((B \otimes_A B')/(\mathfrak{b}, \mathfrak{b}'))$. Thus the claim follows. \square

In particular it follows from the proposition that being an embedding, open or closed, is preserved by any base extension.

Moreover, we have the

Proposition 3.2.1.2 *All embeddings are monomorphisms.*

Proof. This is immediate for open embeddings. For closed embeddings we may assume that the target scheme is affine. Then so is the source scheme. In the situation

$$X \begin{array}{c} \longrightarrow \\ \twoheadrightarrow \end{array} \text{Spec}(B) \hookrightarrow \text{Spec}(A)$$

the two morphisms to the left coincide if they do so on some open covering of X , hence we may assume that $X = \text{Spec}(C)$. We then have the situation

$$C \begin{array}{c} \longleftarrow \\ \longleftarrow \end{array} B \xleftarrow{\varphi} A$$

where φ is surjective, hence an epimorphism in $\mathcal{C}omm$. Thus the claim follows. \square

We have seen that the diagonal of an S -scheme, and more generally the graph of any morphism, is a monomorphism. We have a stronger result:

Proposition 3.2.1.3 *The diagonal of any S -scheme*

$$\Delta_{X/S} : X \longrightarrow X \times_S X$$

is an embedding.

Proof. We may assume that $S = \text{Spec}(A)$. Now cover X by open affine subsets,

$$X = \bigcup_{i \in I} U_i, \text{ where } U_i = \text{Spec}(B_i).$$

Then if $V = \bigcup_{i \in I} U_i \times_S U_i$, the diagonal factors

$$\Delta_{X/S} = \Gamma_{\text{id}_X} : X \longrightarrow V \hookrightarrow X \times_S X,$$

as is easily seen. We show that the leftmost morphism is a closed embedding. It suffices to show that $U_i \longrightarrow U_i \times_{\text{Spec}(A)} U_i$ is a closed embedding for all $i \in I$. But this is clear, as the morphism

$$\text{Spec}(B) \longrightarrow \text{Spec}(B) \times_{\text{Spec}(A)} \text{Spec}(B) = \text{Spec}(B \otimes_A B)$$

corresponds to the multiplication map

$$B \otimes_A B \longrightarrow B,$$

which is surjective. \square

Let $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$ be S -morphisms. We have the

Proposition 3.2.1.4 *The morphism $(f, g)_S$ is the composition*

$$Z \xrightarrow{\Delta_{Z/S}} Z \times_S Z \xrightarrow{f \times_S g} X \times_S Y$$

Proof. The composition solves the same universal problem as does $(f, g)_S$. \square

The proposition has the immediate

Corollary 3.2.1.5 *If f and g are embeddings, then so is $(f, g)_S$. If they, as well as the diagonal $\Delta_{Z/S}$ are closed embeddings, then so is $(f, g)_S$.*

Now let X and Y be S -schemes, with structure morphisms $f : X \longrightarrow S$ and $g : Y \longrightarrow S$, and let $\varphi : S \longrightarrow T$ be a morphism, by means of which X and Y may also be viewed as T -schemes. Denote by p and q the projection morphisms from $X \times_S Y$ to X and Y , respectively. The structure morphism of the S -scheme $X \times_S Y$ is then $\pi = f \circ p = g \circ q$. We now have the canonical morphism

$$(p, q)_T : X \times_S Y \longrightarrow X \times_T Y.$$

We claim the following:

Proposition 3.2.1.6 *The following diagram is commutative, and is a product diagram over $S \times_T S$:*

$$\begin{array}{ccc}
 X \times_S Y & \xrightarrow{(p,q)_T} & X \times_T Y \\
 \pi \downarrow & & \downarrow f \times_T g \\
 S & \xrightarrow{\Delta_{S/T}} & S \times_T S
 \end{array}$$

Proof. Suppose that we have morphisms h_1 and h_2 making the following diagram commutative:

$$\begin{array}{ccc}
 Z & \xrightarrow{h_1} & X \times_T Y \\
 h_2 \downarrow & & \downarrow f \times_T g \\
 S & \xrightarrow{\Delta_{S/T}} & S \times_T S
 \end{array}$$

Z is then an S -scheme via h_2 and a T -scheme via h_1 and the latter structure is derived from the former by φ . We need to show that there is a unique h making the following commute:

$$\begin{array}{ccc}
 Z & \xrightarrow{h_1} & X \times_T Y \\
 \exists! h \downarrow & & \downarrow f \times_T g \\
 X \times_S Y & \xrightarrow{(p,q)_T} & X \times_T Y \\
 h_2 \downarrow \pi & & \downarrow f \times_T g \\
 S & \xrightarrow{\Delta_{S/T}} & S \times_T S
 \end{array}$$

Now by the universal property of $X \times_T Y$ we have $h_1 = (h_3, h_4)_T$, where h_3 and h_4 are T -morphisms from Z to X and Y , respectively. If we can show that these are actually S -morphisms, then we get h as $h = (h_3, h_4)_S$, and the rest will be obvious. We do this for h_3 only, as h_4 is analogous. As $\text{pr}_1 \circ \Delta_{X/S} = \text{id}_S$, we have the commutative diagram

$$\begin{array}{ccc}
Z & & \\
& \searrow^{h_1} & \\
& & X \times_T Y \\
& \swarrow_{h_2} & \downarrow f \times_T g \\
S & \xrightarrow{\text{pr}_1} & S \times_T S
\end{array}$$

Hence we have the following commutative diagram:

$$\begin{array}{ccc}
Z & \xrightarrow{h_1} & X \times_T Y \\
\downarrow h_2 & \begin{array}{c} \xrightarrow{h_3} \\ \xrightarrow{\text{pr}_X} \end{array} & \downarrow f \\
S & & X
\end{array}$$

In particular it follows that h_3 is not only a T -morphism, but in fact also an S -morphism. \square

We note the

Corollary 3.2.1.7 *The morphism $(p, q)_T$ is an embedding, and if $\Delta_{S/T}$ is a closed embedding, then $(p, q)_T$ is a closed embedding.*

Proof. The claim follows by Proposition 3.2.1.3 and Corollary 3.2.1.5. \square

If we replace S by Y and T by S , then the diagram of Proposition 3.2.1.6 becomes

$$\begin{array}{ccc}
X & \xrightarrow{\Gamma_f = (\text{id}_X, f)_S} & X \times_S Y \\
f \downarrow & & \downarrow f \times_S \text{id}_Y \\
Y & \xrightarrow{\Delta_{Y/S}} & Y \times_S Y
\end{array}$$

We therefore have the

Corollary 3.2.1.8 *Γ_f is an embedding, and if $\Delta_{Y/S}$ is a closed embedding, then so is Γ_f .*

3.2.2 Some concepts from general topology: A reminder

Recall that a topological space X is said to have property T_0 if the following holds:

Definition 3.2.2.1 (Property T_0) *For all $x \neq y \in X$ there either exists an open subset $U \ni x, U \not\ni y$, or there exists an open subset $V \not\ni x, V \ni y$, or both.*

The stronger condition of being T_1 is the following:

Definition 3.2.2.2 (Property T_1) *For all $x \neq y \in X$ there exists an open subset $U \ni x, U \not\ni y$.*

Remark Of course it follows that there also exists an open subset $V \not\ni x, V \ni y$.

We have the following observation:

Proposition 3.2.2.1 *The underlying topological space of any scheme is T_0 , but in general not T_1 . However, the subspace consisting of all the closed points of X is T_1 .*

Proof. We may assume that $X = \text{Spec}(A)$, since if the two points x, y are not contained in the same open affine subset, then the condition T_0 is trivially true for them. So let x, y correspond to the primes $\mathfrak{p}, \mathfrak{q} \subset A$. Since they are different, we either have some $a \in \mathfrak{p}, a \notin \mathfrak{q}$, or some $b \notin \mathfrak{p}, b \in \mathfrak{q}$, or both. Then take $V = D(a)$ and $U = D(b)$. The rest of the claim is obvious. \square

The strongest concept of separation of points lies in the

Definition 3.2.2.3 (Property T_2 : The Hausdorff Axiom) *For all $x \neq y \in X$ there exists an open subset $U \ni x$ and an open subset $V \ni y$ such that $U \cap V = \emptyset$.*

A topological space which satisfies the Hausdorff Axiom is also called a *separated* topological space. It is easily shown that

Proposition 3.2.2.2 *A topological space X is Hausdorff if and only if the diagonal $\Delta \subset X \times X$ is closed in the product topology.*

Proof. Recall that the product topology is the topology given by the base \mathcal{B} consisting of all sets $U \times V$ where U and V are open in X . For the diagonal to be closed, it is necessary and sufficient that $X \times X - \Delta$ be open, thus all points (x, y) in this complement must have an open neighborhood not meeting Δ , or equivalently: Be contained in a set from \mathcal{B} not meeting Δ . If $U \times V$ is this basis open subset, then U and V satisfy the assertion of the Hausdorff Axiom. \square

We finally formulate the

Definition 3.2.2.4 (Quasi-compact and compact spaces) *A topological space is said to be quasi-compact if any open covering of it has a finite sub-covering. If in addition the space is Hausdorff, then it is said to be compact.*

We note the

Proposition 3.2.2.3 *The underlying topological space of $\text{Spec}(A)$ is quasi-compact.*

Proof. Let $\text{Spec}(A) = \cup_{i \in I} U_i$. We wish to show that there is a finite subset $\{i_1, i_2, \dots, i_r\}$ of I such that $\text{Spec}(A) = \cup_{\ell=1}^r U_{i_\ell}$. Covering all the U_i s by basis open sets $D(a)$, we get a covering of $\text{Spec}(A)$ by such open sets, and to find a finite subcovering of the former, we need only find one for the latter. Thus we may assume that $U_i = D(a_i)$. Then $\cap_{i \in I} V(a_i) = \emptyset$, as the complement of this intersection is the union of all the $D(a_i)$ s. Now $\cap_{i \in I} V(a_i) = V(\mathfrak{a})$, where \mathfrak{a} is generated by all the a_i s. But as $V(\mathfrak{a}) = \emptyset$, we must have $1 \in \mathfrak{a}$. So there are elements $a_{i_1}, a_{i_2}, \dots, a_{i_r}$ such that $1 = a_{i_1}b_1 + a_{i_2}b_2 + \dots + a_{i_r}b_r$. Then these $a_{i_1}, a_{i_2}, \dots, a_{i_r}$ generate the ideal A as well, hence reversing the argument above we find that $D(a_{i_1}) \cup D(a_{i_2}) \cup \dots \cup D(a_{i_r}) = \text{Spec}(A)$. \square

The seemingly innocuous condition T_0 does have some apparently substantial consequences:

Proposition 3.2.2.4 *Let*

$$f : X \longrightarrow Y$$

be a surjective mapping of T_0 topological spaces. Assume that X and Y have bases \mathcal{B}_X and \mathcal{B}_Y for their topologies such that the mapping f induces a surjective mapping

$$V \mapsto f^{-1}(V)$$

$$\mathcal{B}_Y \longrightarrow \mathcal{B}_X.$$

Then f is a homeomorphism (i.e., is bijective and bi-continuous).

Proof. It suffices to show that f is injective. Indeed, it then obviously establishes a bijection between the bases \mathcal{B}_X and \mathcal{B}_Y as well, whence is bicontinuous.

So assume that $x_1 \neq x_2$ are mapped to the same point $y \in Y$. By T_0 we get, if necessary after renumbering the x 'es, an open subset U in X such that $x_1 \in U$, and $x_2 \notin U$. We may assume $U \in \mathcal{B}_X$, thus there is a $V \in \mathcal{B}_Y$ such that $U = f^{-1}(V)$. But then we also have $x_2 \in U$, a contradiction. \square

3.2.3 Separated morphisms and separated schemes

In analogy with Proposition 3.2.2.2 we make the following

Definition 3.2.3.1 *An S -scheme X is said to be separated if the diagonal*

$$\Delta_{X/S} : X \longrightarrow X \times_S X$$

is a closed embedding.

In this case we also refer to the structure-morphism $\varphi : X \longrightarrow S$ as being a separated morphism. Thus a morphism $f : X \longrightarrow Y$ is called separated if it makes X into a separated Y -scheme.

In the proof of Proposition 3.2.1.3, that the diagonal is always an embedding, it was noted that for $S = \text{Spec}(A)$ and $X = \text{Spec}(B)$ where B is an A -algebra, the diagonal $\Delta_{X/S} : X \longrightarrow X \times_S X = \text{Spec}(B \otimes_A B)$ corresponds to the multiplication mapping $B \otimes_A B \longrightarrow B$, and is therefore a closed embedding. Hence we have the

Proposition 3.2.3.1 *Any morphism of affine schemes is separated.*

We note the following general

Proposition 3.2.3.2 *A morphism $f : X \longrightarrow Y$ is separated if and only if for all open $U \subset Y$ the restriction $f|_{f^{-1}(U)} : f^{-1}(U) \longrightarrow U$ is separated. For this to be true, it suffices that there is an open covering of Y with this property.*

Proof. With U an open subscheme of Y , we have that $V = f^{-1}(U) \times_U f^{-1}(U)$ is an open subscheme of $X \times_Y X$, and the inverse image of V by the diagonal morphism is U . The claim follows from this. \square

Definition 3.2.3.2 (Local property of a morphism) *Whenever a property of a morphism satisfies the criterion in the proposition above, we say that the property is local on the target scheme.*

We collect some observations on separated morphisms in

Proposition 3.2.3.3 1. *If $\varphi : S \rightarrow T$ is a separated morphism and X, Y are S -schemes, then the canonical embedding $X \times_S Y \rightarrow X \times_T Y$ is a closed embedding.*

2. *If $f : X \rightarrow Y$ is an S -morphism and Y is separated over S , then the graph Γ_f is a closed embedding.*

3. *Let*

$$h : X \xrightarrow{f} Y \xrightarrow{g} Z$$

be a closed embedding where g is separated. Then f is a closed embedding.

4. *Let Z be a separated S -scheme and let $g : X \rightarrow Z$ and $j : X \rightarrow Y$ be S -morphisms, the latter a closed embedding. Then $(j, g)_S$ is a closed embedding.*

5. *If $\varphi : X \rightarrow S$ is a separated morphism and $\sigma : S \rightarrow X$ is a section of φ , i.e. $\varphi \circ \sigma = \text{id}_S$, then σ is a closed embedding*

Proof. 1. is by Proposition 3.2.1.6. 2. follows by Corollary 3.2.1.8. For 3. we have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Gamma_f = (\text{id}_X, f)_Z \Big\downarrow & \text{pr}_Y & \cong \Big\downarrow \text{pr}_Y \\ X \times_Z Y & \xrightarrow{h \times_Z \text{id}_Y} & Z \times_Z Y \end{array}$$

Since g is separated, Γ_f is a closed embedding by 2. h is a closed embedding, thus so is $h_Y = h \times_Z \text{id}_Y$. Finally the right pr_Y is an isomorphism. Thus 3 follows. 4. is shown by applying 3. to the situation

$$j : X \xrightarrow{(j, g)_S} Y \times_S Z \xrightarrow{\text{pr}_Y} Y,$$

and 5. follows by applying 3. to

$$S \xrightarrow{\sigma} X \xrightarrow{\varphi} S.$$

This completes the proof. \square

Remark The proof of 3. above also proves the

Corollary 3.2.3.4 (of proof) *If g is any morphism and h is an embedding, then so is f .*

The property of being separated fits into the following general setup, which holds from a variety of other important properties of morphism. It is generally referred to as *la Sorite*:

Proposition 3.2.3.5 *i) Every monomorphism, in particular every embedding, is separated.*

ii) The composition of two separated morphisms is again separated.

iii) The product $f \times_S g$ of two separated S -morphisms $f : X \rightarrow Y$ and $g : X' \rightarrow Y'$ is again separated.

iv) The property of being separated is preserved by base extensions: If $f : X \rightarrow Y$ is a separated S -morphism, then so is $f_{S'} : X_{S'} \rightarrow Y_{S'}$, for any $S' \rightarrow S$.

v) If the composition $g \circ f$ of two morphisms is separated, then so is f .

vi) $f : X \rightarrow Y$ is separated if and only if $f_{\text{red}} : X_{\text{red}} \rightarrow Y_{\text{red}}$ is separated.

Proof. i) follows since $f : X \rightarrow Y$ is a monomorphism if and only if $\Delta_{X/Y}$ is an isomorphism. For ii), let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two morphisms. We have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Delta_{X/Z}} & X \times_Z X \\ & & \downarrow j \\ & \Delta_{X/Y} & X \times_Y X \end{array}$$

Here the down-right arrow is a closed embedding since f is separated, and the up-right arrow is a closed embedding since g is separated. Thus the composition is a closed embedding, hence $g \circ f$ is separated. Having i) and ii), iii) and iv) are equivalent. iv) follows since the diagonal of $X_{S'}$ is the extension to S' of the diagonal of X . v) was shown above. Finally, vi)

follows by first observing that $X_{\text{red}} \times_Y X_{\text{red}}$ is canonically isomorphic with $X_{\text{red}} \times_{Y_{\text{red}}} X_{\text{red}}$, $Y_{\text{red}} \hookrightarrow Y$ being a monomorphism. Further, we have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Delta_{X_{\text{red}}}} & X_{\text{red}} \times_Z X_{\text{red}} \\ j \downarrow & & j \times_Y j \downarrow \\ X & \xrightarrow{\Delta_X} & X \times_Y X \end{array}$$

Since the down-arrows are homeomorphisms on underlying topological spaces, the claim follows. \square

We now have the following important criterion for separatedness, which is useful in general since the property is local on the target scheme:

Proposition 3.2.3.6 *A morphism $f : X \rightarrow Y = \text{Spec}(A)$ is separated if and only if for any two open affine $U = \text{Spec}(B_1)$ and $V = \text{Spec}(B_2)$ for which $U \cap V \neq \emptyset$ we have $U \cap V = \text{Spec}(C)$ where the ring homomorphisms corresponding to the inclusions ρ_1 and ρ_2 ,*

$$\begin{array}{c} B_1 \\ \rho_1 \\ C \\ \rho_2 \\ B_2 \end{array}$$

are such that C is generated as an A -algebra by $\rho_1(B_1)$ and $\rho_2(B_2)$. It is sufficient that this holds for an open affine covering of X .

Proof. Assume first that X is a separated S -scheme. Then $\Delta_{X/S} : X \rightarrow X \times_S X$ is a closed embedding. Now $U \times_S V = \text{Spec}(B_1 \otimes_A B_2)$ is an open affine subscheme of $X \times_S X$, hence $\Delta_{X/S}^{-1}(U \times V) = \text{Spec}(C)$, where $C = (B_1 \otimes_A B_2)/\mathfrak{c}$. But we easily see that $\Delta_{X/S}^{-1}(U \times_S V) = U \cap V$, so the statement in the criterion holds. Conversely, assume that there exists an open covering by affine open subschemes so the assertion in the criterion holds for any two members. To show that $\Delta_{X/Y}$ is a closed embedding, we need only check locally on $X \times_S X$: It suffices to show that $\Delta_{X/S}^{-1}(U \times_S V) \rightarrow U \times_S V$ is

a closed embedding for U and V members of the covering given above. But this is clear from the assertion in the criterion. \square

Example 3.2.3.1 Let A be a commutative ring, and put $X_1 = \text{Spec}(B_1)$, where $B_1 = A[t]$ and $X_2 = \text{Spec}(B_2)$, where $B_2 = A[u]$. Of course this is two copies of the affine line over $\text{Spec}(A)$. Further, let $X_{1,2} = D(t)$ and $X_{2,1} = D(u)$. We shall now glue the two affine lines over $\text{Spec}(A)$ in two radically different ways, one way yielding what is known as the *projective line* over $\text{Spec}(A)$, which is a separated scheme over $\text{Spec}(A)$, and the other way of gluing giving us a relatively exotic, non-separated scheme over $\text{Spec}(A)$, which is referred to as the *affine line with the origin doubled*. This is the simplest case of a non-separated scheme over $\text{Spec}(A)$. The first gluing is given by the isomorphisms

$$f_{1,2} : X_{1,2} \longrightarrow X_{2,1}$$

which corresponds to

$$\varphi_{1,2} : A[u, \frac{1}{u}] \longrightarrow A[t, \frac{1}{t}], u \mapsto \frac{1}{t},$$

and

$$f_{2,1} : X_{2,1} \longrightarrow X_{1,2}$$

which corresponds to

$$\varphi_{2,1} : A[t, \frac{1}{t}] \longrightarrow A[u, \frac{1}{u}], t \mapsto \frac{1}{u}.$$

$X_{1,1} = X_1$, $X_{2,2} = X_2$, moreover $f_{1,1}$ and $f_{2,2}$ are the identities. As the cocycle-condition here is trivially satisfied, we obtain a gluing by these data, temporarily denoted by Z . We have the affine covering $Z = X_1 \cup X_2$, where $U = X_1 \cap X_2 = \text{Spec}(C)$ and $C = A[x, \frac{1}{x}]$. The inclusion morphisms from U to X_1 and from U to X_2 are given by $t \mapsto x$ and $u \mapsto \frac{1}{x}$, respectively. Thus the images of B_1 and B_2 generate C as an A -algebra, and Z is separated over $\text{Spec}(A)$.

On the other hand we may glue by defining the isomorphism $f_{1,2}$ as Spec of

$$\psi_{1,2} : A[u, \frac{1}{u}] \longrightarrow A[t, \frac{1}{t}], u \mapsto t,$$

and $f_{2,1}$ as Spec of

$$\psi_{2,1} : A[t, \frac{1}{t}] \longrightarrow A[u, \frac{1}{u}], t \mapsto u.$$

Now the resulting scheme Z' still is the union of two open subschemes (isomorphic to) X_1 and X_2 , and their intersection is still the open affine subscheme $U = \text{Spec}(A[z, \frac{1}{z}]) \cong X_{1,2}$. But now the images of B_1 and B_2 only generate the subring $A[z]$ of $A[z, \frac{1}{z}]$, hence Z' is not a separated $\text{Spec}(A)$ -scheme.

3.3 Further properties of morphisms

3.3.1 Finiteness conditions

We have previously defined *affine spectra of finite type over a field*. This concept is merely a very special case of an extensive set of conditions:

Definition 3.3.1.1 *A morphism $f : X \rightarrow Y$ is said to be:*

1. *Locally of finite type if there exists an open affine covering of Y ,*

$$Y = \bigcup_{i \in I} U_i \text{ where } U_i = \text{Spec}(A_i),$$

such that for all $i \in I$,

$$f^{-1}(U_i) = \bigcup_{j \in J_i} V_{i,j} \text{ where } V_{i,j} = \text{Spec}(B_{i,j}),$$

such that for all i and $j \in J_i$ the restriction of f , $f_{i,j} : V_{i,j} \rightarrow U_i$ is Spec of $\varphi_{i,j} : A_i \rightarrow B_{i,j}$ making $B_{i,j}$ into an A_i -algebra of finite type, i.e., a quotient of a polynomial ring in finitely many variables over A_i .

2. *Of finite type if all the indexing sets J_i in 1. may be taken to be finite sets.*¹⁵

3. *Affine if $f^{-1}(U_i) = \text{Spec}(B_i)$.*

4. *Finite if 3. holds and in addition B_i is finite as an A_i -module.*

The picture emerging from the above definition is completed by

Proposition 3.3.1.1 *If one of the above conditions holds, then the relevant condition on U_i holds for any open affine subscheme of Y .*

¹⁵However, I may be an infinite set.

Proof. This will be provided in the final version of these notes.

The proof of the following proposition is a simple exercise:

Proposition 3.3.1.2 *A morphism $f : X \rightarrow Y$ is an affine morphism if and only if there exists a quasi-coherent \mathcal{O}_X -Algebra \mathcal{A} on X such that X and $\text{Spec}(\mathcal{A})$ are isomorphic over Y .*

3.3.2 The “Sorite” for properties of morphisms

In order to study the different properties of morphisms of schemes, we need a systematic framework. The following simple but clarifying device is due to Grothendieck, [EGA] I 5.5.12:

Proposition 3.3.2.1 *Let \mathbf{P} be a property of morphisms of schemes. We consider the following statements about \mathbf{P} :*

- i) Every closed embedding has property \mathbf{P} .*
- ii) The composition of two morphisms which have property \mathbf{P} again has property \mathbf{P} .*
- iii) If $f : X \rightarrow Y$ is an S -morphism which has property \mathbf{P} , and $S' \rightarrow S$ is any morphism, then the base-extension of f to S' , $f_{S'} : X_{S'} \rightarrow Y_{S'}$ has property \mathbf{P} .*
- iv) The product $f \times_S g$ of two S -morphisms $f : X \rightarrow Y$ and $g : X' \rightarrow Y'$ which have property \mathbf{P} again has property \mathbf{P} .*
- v) If the composition $h = g \circ f$ of two morphisms f and g*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow h & \nearrow g \\ & & Z \end{array}$$

has property \mathbf{P} , and if g is a separated morphism, then f has property \mathbf{P} .

- vi) $f : X \rightarrow Y$ has property \mathbf{P} if and only if $f_{\text{red}} : X_{\text{red}} \rightarrow Y_{\text{red}}$ has property \mathbf{P} .*

Then we have the following: If i) and ii) holds, iii) and iv) are equivalent. Moreover, i), ii) and iii) together imply v) and vi).

Proof. The proof follows the same lines as the proof of Proposition 3.2.3.5. \square

As an application of this proposition we have the

Proposition 3.3.2.2 *The properties for morphisms listed in Definition 3.3.1.1 satisfy i), ii) and iii) and hence i) – vi) in Proposition 3.3.2.1.*

Proof. i) is immediate in each case. ii) is also clear from the definitions. For iii) we may assume that S and S' are affine, in which case the verification of iii) is straightforward. \square

3.3.3 Algebraic schemes over k and k -varieties

We consider schemes over the base $S = \text{Spec}(k)$ where k is a (not necessarily algebraically closed) field, and make the

Definition 3.3.3.1 *A scheme X over $\text{Spec}(k)$ which is separated and of finite type as a $\text{Spec}(k)$ -scheme is called an algebraic scheme. If in addition the scheme $X_{\bar{k}}$ is reduced and irreducible, where \bar{k} denotes the algebraic closure of k , then X is called a k -variety.*

remark It is easily seen that if X is a k -variety, then for all algebraic extensions K of k , X_K is reduced and irreducible.

We have the following facts on schemes algebraic over a field k :

Proposition 3.3.3.1 *A point $x \in X$ is a closed point if and only if the field $\kappa_X(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ is an algebraic extension of k .*

Proof. Given a finite open covering of X , then x is closed if and only if it is closed in each of the open subsets.

Thus we may assume that $X = \text{Spec}(A)$ where $A = k[X_1, \dots, X_N]/\mathfrak{a}$. The point x is closed if and only if the corresponding prime ideal in A is a maximal ideal, and the claim follows by the Hilbert Nullstellensatz, in the following form:

Theorem 3.3.3.2 (Weak Hilbert Nullstellensatz) *Let A be a finitely generated algebra over a field k . Assume that R is an integral domain. Then R is a field if and only if all its elements are algebraic over k .*

Indeed, if x is a closed point, then $R = A/\mathfrak{p}_x$ is a field, hence an algebraic extension of k . If conversely the point x is such that the quotient field of A/\mathfrak{p}_x is an algebraic extension of k , then in particular all elements of A/\mathfrak{p}_x must be algebraic over k . But then the Hilbert Nullstellensatz applied to $R = A/\mathfrak{p}_x$ shows that this ring must be a field, hence that \mathfrak{p}_x is a maximal ideal, and the claim follows. \square

This proposition has the following

Corollary 3.3.3.3 *The point x is closed in the algebraic scheme X if and only if it is closed in any open subset $U \subset X$ containing it.*

Proof. Closedness is expressed by a property of $\kappa_X(x)$, invariant by passing to an open subscheme containing x . \square

Remark Note that the assertion of the corollary is definitely false without the assumption of X being algebraic over a field. As a counterexample, consider the Spec of a local ring of Krull dimension greater than 0.

Moreover, we have the

Proposition 3.3.3.4 *Let X be an algebraic scheme over the field k . Then the set of closed points in X is dense in X .*

Proof. Assume the converse, and let $Y \subset X$ be the closure of all the closed points of X . Then $X - Y = U$ is a non empty open subscheme, thus contains an open affine subscheme $V = \text{Spec}(A)$, where A is a finitely generated algebra over k . But then V has closed points, and by the corollary above these are also closed points of X , a contradiction. \square

Remark The assertion of this proposition is false without the assumption of the scheme being algebraic, by the same example as for the corollary.

We finally note the

Proposition 3.3.3.5 *A morphism between algebraic schemes maps closed points to closed points.*

Proof. Let $f : X \rightarrow Y$ be the morphism and let $x \in X$ be a closed point. Then the injective k -homomorphism $\kappa(f(x)) \hookrightarrow \kappa(x)$ shows that $\kappa(f(x))$ is an algebraic extension of k . \square

Remark This conclusion also fails without the assumption of X and Y being algebraic. This can be seen, e.g., by taking the Spec of the injective homomorphism associated with localization in a local ring.

3.4 Projective morphisms

3.4.1 Definition of $\text{Proj}(S)$ as a topological space

Let S be a graded A -algebra, where as usual A is a commutative ring with 1. We assume that S is positively graded, that is to say that

$$S = S_0 \oplus S_1 \oplus S_2 \cdots \oplus S_s \oplus \dots,$$

where all the S_d s are A -modules and the multiplication in S satisfies $S_i S_j \subseteq S_{i+j}$. An element $f \in S$ may be written uniquely as

$$f = f_{\nu_1} + \cdots + f_{\nu_r},$$

where $\nu_1 < \cdots < \nu_r$ and $f_{\nu_i} \in S_{\nu_i}$. The elements f_{ν_i} are referred to as the homogeneous components of f . Recall also that an ideal $\mathfrak{a} \subset S$ is called a homogeneous ideal if, equivalently,

1. If $f \in \mathfrak{a}$ then all $f_{\nu_i} \in \mathfrak{a}$
2. *EuFraka* has a homogeneous set of generators

Note that the subset $S_+ = S_1 + S_2 + \cdots \subset S$ is a homogeneous ideal. It is referred to as the irrelevant ideal of S .

Example 3.4.1.1 1. Let $S = A[X_0, X - 1, \dots, X_N]$. Then $S_0 = A$, and S_d is generated as an $A = S_0$ -module by the monomials of degree d .

2. If \mathfrak{J} is a homogeneous ideal in S above then $T = S/\mathfrak{J}$ is another example.

We define the topological space $\text{Proj}(S)$ as the set of all homogeneous prime ideals in $\text{Spec}(S)$ which do not contain S_+ , with the induced topology from $\text{Spec}(S)$.

Let $f \in S_d$ be a homogeneous element. Define

$$D_+(f) = D(f) \cap \text{Proj}(S) \text{ and } V_+(f) = V(f) \cap \text{Proj}(S).$$

We have the following

Proposition 3.4.1.1 *As h runs through the set L of all homogeneous elements in S the subsets $D_+(h)$ constitutes a basis for the topology on $\text{Proj}(S)$.*

Proof. If \mathfrak{p} is a homogeneous ideal $f \in \mathfrak{p} \Leftrightarrow$ all the homogeneous components of f are $\in \mathfrak{p}$. \square

As usual we let S_f denote the localization of S in the multiplicatively closed set

$$\Delta(f) = \{1, f, f^2, \dots, f^r, \dots\}$$

when f is not a nilpotent element. If f is a homogeneous element of S , say $f \in S_d$, then S_f is a graded A -algebra, but in this case graded by \mathbb{Z} , in the sense that

$$S_f = \left\{ \frac{g}{f^n} \mid g \in S_m, m = 0, 1, 2, \dots, n = 0, 1, 2, \dots \right\} = \\ \dots \oplus (S_f)_{-2} \oplus (S_f)_{-1} \oplus (S_f)_0 \oplus (S_f)_1 \oplus (S_f)_2 \oplus \dots$$

The homogeneous piece of degree zero is of particular interest, we put

$$S_{(f)} = (S_f)_0 = \left\{ \frac{g}{f^n} \mid g \in S_{nd} \right\}$$

We now define a mapping of topological spaces

$$\psi_f : D_+(f) \longrightarrow \text{Spec}(S_{(f)})$$

by

$$\mathfrak{p} \mapsto \mathfrak{q} = \left\{ \frac{g}{f^n} \mid g \in \mathfrak{p}_{nd} \right\}$$

We have to show that \mathfrak{q} is a prime ideal in $S_{(f)}$. It clearly is a subset of $S_{(f)}$, to show it's an additive subgroup it suffices to show it's closed under subtraction. Let $\frac{g_1}{f^n}, \frac{g_2}{f^m} \in \mathfrak{q}$. Then $g_1 \in \mathfrak{p}_{nd}$ and $g_2 \in \mathfrak{p}_{md}$, thus $f^m g_1 - f^n g_2 \in \mathfrak{p}_{dm+dn}$ hence

$$\frac{f^m g_1 - f^n g_2}{f^{m+n}} = \frac{g_1}{f^n} - \frac{g_2}{f^m} \in \mathfrak{q}.$$

The multiplicative property is also immediate, thus $\text{EuFrak}\mathfrak{q}$ is at least an ideal in $S_{(f)}$. To show primality, assume that

$$\left(\frac{g_1}{f^n}\right)\left(\frac{g_2}{f^m}\right) = \frac{g_1 g_2}{f^{m+n}} = \frac{G}{f^N} \text{ where } G \in \mathfrak{p}_{Nd}.$$

Then there exists r such that

$$f^r(f^N g_1 g_2 - f^{m+n} G) = 0,$$

thus $f^{r+N} g_1 g_2 = f^{m+n} G \in \mathfrak{p}$. Since $f \notin \mathfrak{p}$, we get $g_1 g_2 \in \mathfrak{p}$, thus g_1 or $g_2 \in \mathfrak{p}$, and the claim follows.

We now have the

Proposition 3.4.1.2 ψ_f is a homeomorphism of topological spaces.

Proof. By Proposition 3.2.2.4 it suffices to show that ψ_f is a surjective mapping, and that it establishes a surjection from a basis for the topology of $\text{Spec}(S_{(f)})$ to a basis for the topology on $D_+(f)$. We first show surjectivity of ψ_f .

So let \mathfrak{q} be a prime ideal in $S_{(f)}$. For all $n \geq 0$ let

$$\mathfrak{p}_n = \left\{ g \in S_n \mid \frac{g^d}{f^n} \in \mathfrak{q} \right\}$$

To show is that

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_d \oplus \cdots$$

is a homogeneous prime such that $\psi_f(\mathfrak{p}) = \mathfrak{q}$.

We first show that \mathfrak{p} is, equivalently that for all n \mathfrak{p}_n is, an additive subgroup of S_+ , and do so by showing that it is closed under subtraction. It may come as a slight surprise that this argument needs \mathfrak{q} to be a radical ideal, which is OK as it is actually a prime. Let $g_1, g_2 \in \mathfrak{p}_n$, i.e., $\frac{g_1}{f^n}$ and $\frac{g_2}{f^n} \in \mathfrak{q}$. Expanding by the binomial formula we then find that $\frac{(g_1 - g_2)^{2d}}{f^{2n}} \in \mathfrak{q}$, thus $\frac{(g_1 - g_2)^d}{f^n} \in \mathfrak{q}$, as \mathfrak{q} is prime and hence radical. Thus $g_1 - g_2 \in \mathfrak{p}_n$.

For the multiplicative property, it evidently suffices to show that $\mathfrak{p}_n S_m \subset \mathfrak{p}_{n+m}$. This is completely straightforward.

We now have that \mathfrak{p} is a homogeneous ideal in S , to show that it is prime we need to show that the graded A -algebra $T = S/\mathfrak{p}$ is without zero-divisors. Clearly, it suffices to show that there are no homogeneous ones.

For this, assume g_1 and g_2 to be elements of S_m and S_n , respectively, such that $g_1 g_2 \in \mathfrak{p}_{m+n}$ while $g_1 \notin \mathfrak{p}_m$ and $g_2 \notin \mathfrak{p}_n$. But then $\mathfrak{g}_1^m f^m$ and $\mathfrak{g}_1^n f^n$ are not in \mathfrak{q} , while their product is, a contradiction.

Finally, we show that $\psi_f(\mathfrak{p}) = \mathfrak{q}$. We have

$$\psi_f(\mathfrak{p}) = \left\{ \frac{g}{f^n} \mid g \in \mathfrak{p}_{nd} \right\}$$

and by definition

$$g \in \mathfrak{p}_{nd} \iff \frac{g^d}{f^{nd}} \in \mathfrak{q}$$

which as \mathfrak{q} is radical is equivalent to $\frac{g}{f^n} \in \mathfrak{q}$, and the claim follows.

Finally we show that the mapping

$$V \mapsto \psi_f^{-1}(V)$$

maps the basis for $\text{Spec}(S_{(f)})$

$$\mathcal{B}_1 = \left\{ D\left(\frac{g}{f^n}\right) \mid g \in S_{dn}, n = 0, 1, 2, \dots \right\}$$

onto the basis for the topology on $D_+(f)$,

$$\mathcal{B}_2 = \{ D_+(gf) \mid g \in S_{dn}, n = 0, 1, 2, \dots \}$$

As evidently $\psi_f^{-1}(D(\frac{g}{f^n})) = D_+(gf)$, we need only show that \mathcal{B}_2 is a basis for the topology on $D_+(f)$. Now all the sets $D_+(h)$, as h runs through the homogeneous elements on S , form a base for the topology on $\text{Proj}(S)$. Thus the sets $D_+(hf)$ constitute a base for the topology on $D_+(f)$. As $D_+(h) = D_+(h^d)$, the claim follows.

This completes the proof of the proposition. \square

3.4.2 The scheme structure on $\text{Proj}(S)$ and \widetilde{M} of a graded S -module

Let M be a graded module over the graded A -algebra S . We define a sheaf, temporarily only as a sheaf of $\mathcal{A}b$, on the topological space $\text{Proj}(S)$ defined in the previous paragraph. We proceed as follows:

Let $f \in S_s$. On $\text{Spec}(S_{(f)})$ we put $\mathcal{M}_f = \widetilde{M_{(f)}}$, where $M_{(f)}$ is the homogeneous part of degree zero in M_f , evidently an $S_{(s)}$ -module. By the homeomorphisms ψ_f this sheaf is transported to $D_+(f)$, denoted by \widetilde{M}_f . The canonical isomorphisms for $f \in S_d$ and $g \in S_e$,

$$S_{(fg)} \cong (S_{(f)})_{\frac{g^d}{f^e}} \text{ and } M_{(fg)} \cong (M_{(f)})_{\frac{g^d}{f^e}}$$

identifies \widetilde{M}_{fg} with the restriction of \widetilde{M}_f to $D(\frac{g^d}{f^e})$. Thus we may glue the \widetilde{M}_f to a sheaf on all of $\text{Proj}(S)$, which we denote by \widetilde{M}_f ¹⁶

We now define $\mathcal{O}_{\text{rmProj}(S)} = \widetilde{S}$. We obtain a scheme in this way, also denoted by $\text{Proj}(S)$. \widetilde{M} is an $\mathcal{O}_{\text{rmProj}(S)}$ -module on $\text{Proj}(S)$.

Also the morphisms $\pi_f : \text{Spec}(S_{(f)}) \rightarrow \text{Spec}(A)$ glue to a morphism $\pi : \text{Proj}(S) \rightarrow \text{Spec}(A)$.

We have the

Proposition 3.4.2.1 $\pi : \text{Proj}(S) \rightarrow \text{Spec}(A)$ is a separated morphism.

Proof. By the affine criterion for separatedness. \square

3.4.3 Proj of a graded \mathcal{O}_X -Algebra on X

3.4.4 Definition of projective morphisms

3.4.5 The projective N -space over a scheme

3.5 Proper morphisms

3.5.1 Definition of proper morphisms

3.5.2 Basic properties and examples

3.5.3 Projective morphisms are proper

4 Some general techniques and constructions

4.1 The concept of blowing-up

4.2 The conormal scheme

4.3 Kähler differentials and principal parts

¹⁶Strictly speaking we have to use the gluing lemma for sheaves here, as we are dealing with isomorphisms rather than with equalities. But the canonical nature of these isomorphisms secure that the cocycle condition holds.

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