The cyclotomic trace preserves operad actions

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Abstract We prove that the cyclotomic trace preserves operad actions in a wide variety of cases. In particular, the cyclotomic trace for a connective commutative symmetric ring spectrum preserves the $E_\infty$-structure.

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1 Introduction

The cyclotomic trace of Bökstedt, Hsiang and Madsen [4] has proved to give invaluable information about algebraic K-theory, and the question of what structural properties of K-theory it preserves is of utmost importance.

We prove the following result, which has been left as a gap for a long time in the literature. Compare with the announced classification results of Barwick [1] and Blumberg, Gepner and Tabuada [2]. A more detailed version not suitable for the introduction is given as Theorem 2.0.4.

**Theorem 1.0.1** Let $A$ be a connective commutative symmetric ring spectrum and let $F_A$ be the permutative $\text{Sp}^\Sigma$-category of “finitely generated free” $A$-modules defined in 9.2.1. Then

1. the smash product induces an $E_\infty$-structure on $K(A) = K(F_A)$, the algebraic K-theory of $A$,

2. The smash product induces an $E_\infty$-structure on $\text{TC}(A)$, the (integral) topological cyclic homology of $A$.

3. The cyclotomic trace from $K(A)$ to $\text{TC}(A)$ (given by Definition 2.0.3 applied to $C = F_A$ composed with the equivalence $\text{TC}(F_A) \sim \text{TC}(A)$) is a natural $E_\infty$-ring weak map in symmetric spectra.

A *weak map* is a chain of maps where the arrows pointing in the “wrong” direction are weak equivalences.

Note that Theorem 1.0.1 is significantly stronger than the claim that there are $E_\infty$-models for K-theory and topological cyclic homology and a map between them which may be identified with the cyclotomic trace in the homotopy category. We claim that the naturally defined $E_\infty$-structures on the source and target do the trick. As we see in Section 1.1 the weaker claim can be extracted (at least after $p$-completion) from [13] and [6] with much less effort.

Since the model categories of symmetric $E_\infty$-ring spectra and of commutative symmetric ring spectra are Quillen equivalent, Theorem 1.0.1 implies that we may strictify our explicit models for K-theory and topological cyclic homology, and get that the cyclotomic trace is a weak natural transformation in commutative symmetric ring spectra. The key ingredient is the following proposition.
Proposition 1.0.2 Assume that $\mathcal{O}$ is an operad acting on an $\text{Sp}^\Sigma$-category $\mathcal{C}$ with cofibrations and weak equivalences. Then the cyclotomic trace between $K(\mathcal{C})$ and $TC(\mathcal{C})$ is a chain of maps of $\mathcal{O}$-algebras in symmetric spectra. Likewise, if $\mathcal{C}$ is a symmetric monoidal $\text{Sp}^\Sigma$-category.

The proof of Proposition 1.0.2 occupies almost the entire paper. For the spine of the argument and a more detailed statement, see Section 2.

If the operad $\mathcal{O}$ acts on the category $\mathcal{F}_A$ of finitely generated free modules (as defined in Section 9) over a semistable commutative symmetric ring spectrum $A$, then Proposition 1.0.2 entails that the cyclotomic trace from $K(A) = K(\mathcal{F}_A)$ to $TC(\mathcal{F}_A)$ is a weak map of $\mathcal{O}$-algebras. Hence, the leap from Proposition 1.0.2 to Theorem 1.0.1 is bridged by the claim that for commutative $A$, the inclusion of $A$ as the rank one module in $\mathcal{F}_A$ is an $E_\infty$-map and that $TC$ preserves this structure.

The result is needed for the ongoing investigations pertaining to commutative ring spectra and their algebraic $K$-theory in general, and in particular toward the redshift conjecture. We make no particular claim of originality. The structure of the proof is predictable (though the devil is hidden in the details and the author had several scary moments and had at one point to rework all foundations), given the papers [13], [7] and [12] and an observation about nerves along weak equivalences (in [7] we relied on all objects being fibrant, and so the reference there was to the $S$-algebras of [11]).

It should be noted that C. Schlichtkrull has given an alternative definition of the cyclotomic trace for connective commutative symmetric ring spectra [20] using the cyclic nerve, and has announced that he can prove that this model is an $E_\infty$ weak map. Private communications indicate that the line of approach will also be through the use of multicategories, and presumably will give similar results as the above.

The choice of symmetric spectra as technical foundations is somewhat arbitrary and occasionally less than optimal, but was made on the basis of their popularity, but also because some of the constructions naturally produce symmetric spectra.

The reason we cover both Waldhausen’s and the iterated Segal construction is that we have frequently been met with the question why we have used Segal’s model instead of Waldhausen’s, and we hope to make also these readers happy. The added power of the $S$-construction is that this setup works for arbitrary categories with cofibrations (and is probably needed whenever a real theory for derived algebraic geometry finally takes to the wings). The flip side is that it does not work for symmetric monoidal categories where the monoidal structure is different from the coproduct, and requires some proofs that can not be referred away.

There are situations in real life where Waldhausen’s construction will not lead to a weak map to $TC(\mathcal{C})$, but naturally stops at an intermediate stage (see e.g., Remark 2.0.2) so we choose to bare the inner workings of the trace without too many extra assumptions. For the Segal approach, we add the few assumptions necessary to make it a manifest weak map in Definition 2.0.3.

The paper is organized as follows. Section 2 gives a precise definition of the cyclotomic trace, pending on referenced definitions and results verifying that the definitions do what they are claimed to do, giving a refined version of our main result. Section 3 sets up the multicategory framework and gives several important examples used at later stages. In Section 4 and 5 Waldhausen’s and Segal’s approaches to $K$-theory are refined so that the claimed structure becomes apparent. In Section 6 the homotopy nerve is presented. In Section 7 we collect the facts needed to pivot from the spectrum direction provided by $K$-theory to that of topological cyclic homology, which is set up in Section 8. Finally, in Section 9 candidates for categories serving as input to the cyclotomic trace are explored.
1.1 Weaker but simpler results

It is possible to obtain somewhat weaker results with much less effort, and although it is not needed for the rest of the paper we offer an outline that may put the current study in some perspective.

In the appendix of [13], Geisser and Hesselholt prove that the cyclotomic trace, when applied to a commutative discrete ring, is a weak map of symmetric ring spectra. Coupled with our understanding of the relative cyclotomic trace and some observations about the multiplicative structure of TC, one can from this conclude that there is some multiplicative structure on K-theory such that the profinite completion of the cyclotomic trace, when applied to a connective commutative symmetric spectrum $A$, defines a map in the homotopy category of ring spectra. Roughly, this can be achieved by choosing a multiplicative model for TC and letting $K'(A)$ be the homotopy pullback in the category of ring spectra of

$$K(\pi_0 A) \xrightarrow{\text{trc}_{\pi_0 A}} \text{TC}(\pi_0 A; p) \xleftarrow{\text{trc}_{\pi_0 A}} \text{TC}(A; p),$$

where the unmarked map is induced by the canonical map $A \to H\pi_0 A$, where $H$ is the Eilenberg-Mac Lane construction. In view of the homotopy cartesian square [6]

$$\begin{array}{ccc}
K(A) \xrightarrow{\text{trc}_A} & \text{TC}(A; p) \\
\downarrow & \downarrow \\
K(\pi_0 A) \xrightarrow{\text{trc}_{\pi_0 A}} & \text{TC}(\pi_0 A; p),
\end{array}$$

we have a stable equivalence $K(A) \simeq K'(A)$ of spectra (no multiplicativity implied), and so $K'(A)$ is a model for algebraic K-theory with a multiplicative map $K'(A) \to \text{TC}(A; p)$. However, this gives no connection to the natural multiplicative structure on $K(A)$, and does not tell us anything about the $E_\infty$-structure. One should also note that the resolution by means of simplicial rings in [6] is not claimed to have any good properties with respect to commutativity (this is directly related to the difference between commutative and $E_\infty$-simplicial rings).

The setup in [13] is the following: given an exact category $\mathcal{C}$ with a suitably exact tensor $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ giving a symmetric monoidal structure, they observe that Waldhausen’s $S$-construction actually gives a symmetric ring spectrum $\text{ob} \mathcal{S} \mathcal{C}$ and that the “inclusion of fixed points under the circle action” from [9]

$$\text{ob} \mathcal{S} \mathcal{C} \to \text{TC}^0(\mathcal{S} \mathcal{C})$$

becomes a map of symmetric ring spectra (the subscript 0 indicates that this is the “zeroth space” in the TC-direction: the spectrum direction is taken care of by the $S$-construction).

The crucial stumbling block is the following: how does one handle weak equivalences? Isomorphisms have nice inverses, and so [13] runs into no problems. In general, if $\mathcal{C} = (\mathcal{C}, w\mathcal{C}, co\mathcal{C})$ is some category with cofibration and weak equivalences with an enrichment in symmetric spectra so that it makes sense to apply topological cyclic homology to it; then the algebraic K-theory is given by the objects $\text{ob} Nw \mathcal{S} \mathcal{C}$ of the “nerve along the weak equivalences” and we get a map of symmetric ring spectra $\text{ob} Nw \mathcal{S} \mathcal{C} \to \text{TC}^0(Nw \mathcal{S} \mathcal{C})$.

So, what is the trouble? The problems lie in the identification of $\text{TC}^0(Nw \mathcal{S} \mathcal{C})$ and in commutativity issues. The techniques of [7] handle the first problems for commutative $S$-algebras in the sense of [11] (and [15] has to tackle the same issue in a special case). When $\mathcal{C}$ is a category of finitely generated modules over a commutative ring spectrum the key ingredients in identifying $\text{TC}^0(Nw \mathcal{S} \mathcal{C})$ with $\text{TC}(A)$ are the fairly elementary Lemma 9.3.5 and Proposition 9.3.3. So, if all...
we wanted to do was to extend Geisser and Hesselholt’s result that the cyclotomic trace is multiplicative to cover also commutative symmetric ring spectra, large parts of this paper could be dropped.

However, it would be a disservice to the literature to again shirk away from the commutativity of the trace map (which the author has been guilty of for too long), especially since a resolution is not difficult, but just takes some book keeping. The setup we use for our book keeping is showing that all transformations have refinements to multicategories.

1.2 Acknowledgments

This paper uses the ideas of many others, most notably Blumberg, Elmendorf, Geisser, Hesselholt, Madsen, Mandell, McClure, Schwede and Smith, and makes no claim on originality, but is written as a service to the community. Except for the use of the homotopy nerve, the formalism used in this paper was chosen while the author was a visitor at Stanford University fall 2010. Some of the TeXing was undertaken while visiting the University of Virginia and University of Western Ontario the following spring. The author wishes to thank the institutions for their hospitality. The project was scrapped in 2011 due to the perception that other authors had provided simpler and more elegant proofs, most notably Barwick, Blumberg, Gepner and Tabuada. The author wants to thank the people who still have insisted on the value for having a direct proof, in particular Schlichtkrull and Blumberg. The final draft was assembled during the algebraic topology semester at the MSRI, spring 2014.

1.3 Notation

Let $I$ be the category of finite sets of the form $n = \{1, \ldots, n\}$ ($0 = \emptyset$) and injections, endowed with the permutative structure given by concatenation, $\sqcup$. Let $\Delta$ be the category whose objects are the ordered nonempty sets $[n] = \{0 \leq 1 \leq \cdots \leq n\}$ and order preserving functions, so that a simplicial object is functor from $\Delta^o$. If $[q] \in \Delta$, then $\Delta[q]$ is the simplicial set $[n] \mapsto \Delta([n],[q])$.

The category of (pointed) simplicial sets is denoted $S(S_\ast)$, and its objects are referred to as spaces. The one-simplex is written $I = \Delta[1]$, and the circle $S^1 = I/\partial I$. The category of small categories is denoted $Cat$. The adjective “small” may be suppressed when confusion is unlikely. The category of symmetric spectra is denoted $Sp^\Sigma$, and some of its features are touched upon in Section 3.2.

2 The cyclotomic trace

For easy reference we display the chain of maps that make up the cyclotomic trace, although most of its ingredients naturally will only be fully explained later in the paper, as indicated by the forward references. One could say that the entire contents of the paper is that the following definitions actually make sense.

Definition 2.0.1 Let $C = (C,wC,coC)$ be an $Sp^\Sigma$-category with cofibrations and weak equivalences (see Definition 3.2). Then the cyclotomic trace between $K(C) = \text{ob } NwSC$ and $TC(C)$ is the
chain of natural transformations of multifunctors \([3.1.3]\)

\[
\begin{array}{ccccccc}
K(C) & \longrightarrow & TC^0(hoNT_0SC) & \longrightarrow & \mathcal{L}TC(hoNT_0SC) & \longrightarrow & \mathcal{L}TC(M_0SC) & \longrightarrow & TC(C) \\
ob hoNT_0SC & \longrightarrow & L \Sigma^\infty TC^0(hoNT_0SC) & \longrightarrow & \mathcal{L}TC(T_0SC) & \longrightarrow & \mathcal{L}TC(S) & \longrightarrow & \Sigma^\infty TC(C)
\end{array}
\]

Here, the object multifunctor \(ob\) is given in \([3.2.3]\), the nerve \(N\) and the homotopy nerve \(hoN\) (whose underlying spaces of objects are equal) in Section 6, Waldhausen’s \(\Sigma\)-construction in 4.4, the stabilization functors \(M_0\) and \(T_0\) in Definition \([3.4.2]\) topological cyclic homology \(TC\) (and its zeroth space \(TC^0\)) in \([8.3]\) the two inclusions \(\Sigma^\infty\) and \(\Sigma^\infty_l\) of symmetric spectra in bispectra in \([7.1.3]\) and finally \(\mathcal{L}\), which takes a bispectrum to an “average” of the two spectrum directions, is given in \([7.2.6]\).

The natural transformation \(incl: ob \to TC^0\) is the inclusion of the circle fixed points (Dennis trace) of Lemma \([8.3.3]\), \(u_r\) is the positive level equivalence \(u\) of Lemma \([7.2.8]\) and \(u_l\) is its mirror image, \(\sigma_r\) and \(\sigma_l\) are induced by the structure maps of bispectra; \(TM\) and \(1M\) are the stable equivalences given in Lemma \([8.4.3]\) (using Lemma \([7.2.9]\) and \([8.3.5]\)); and lastly, \(deg: 1 \to hoN\) is the inclusion by degeneracies in the homotopy nerve. The link between the simplicial enrichment in \(TC\) for \(incl\) and \(deg\) is handled through Lemma \([8.3.4]\).

**Remark 2.0.2**

1. If the morphism spectra of \(C\) are connective, then Lemma \([7.1.8]\), Lemma \([7.2.9]\) and Lemma \([8.3.4]\) (1) imply that \(\sigma_l\) is a stable equivalence.

2. Under the assumption that \(C\) is \(M7\) (as defined in \([4.1.3]\)) the inclusion \(deg\) by degeneracies in the homotopy nerve induces a stable equivalence by Corollary \([8.3.7]\) and Lemma \([7.2.9]\). Hence, in this case the target \(TC^0(hoNT_0SC)\) of the inclusion of \(S^1\)-fixed points is equivalent to \(\mathcal{L}TC(S)\).

3. Under favorable situations (e.g., if the cofibrations in \(C\) are inclusion into summands) the transformation \(\mathcal{L}TC(S) \xrightarrow{\sigma_r} TC(C)\) is an equivalence, but not in general. In these situations it might be easier to employ Segal’s version of algebraic K-theory for permutative \(Sp^\Sigma\)-categories, which we give below.

**Definition 2.0.3** Let \(C\) be a permutative \(Sp^\Sigma\)-category (see Definition \([5.1.1]\)) with connective morphism spectra. Then the cyclotomic trace for \(C\) is the weak natural transformations of multifunctors from \(K(C) = ob N\tilde{\omega}\tilde{H}C\) to \(TC(C)\)

\[
\begin{array}{ccccccc}
K(C) & \longrightarrow & TC^0(hoN\tilde{\omega}\tilde{H}C) & \longrightarrow & \mathcal{L}TC(hoN\tilde{\omega}\tilde{H}C) & \longrightarrow & \mathcal{L}TC(M_0\tilde{H}C) & \longrightarrow & TC(C) \\
ob hoN\tilde{\omega}\tilde{H}C & \longrightarrow & L \Sigma^\infty TC^0(hoN\tilde{\omega}\tilde{H}C) & \longrightarrow & \mathcal{L}TC(T_0\tilde{H}C) & \longrightarrow & \mathcal{L}TC(H) & \longrightarrow & \Sigma^\infty TC(C)
\end{array}
\]

Here \(\tilde{\omega}\) is the uniform choice of weak equivalences of Definition \([5.4.1]\) and \(\tilde{H}\) is the iterated Segal construction of \([5.2]\). The remaining multifunctors and the natural transformations are given in Definition \([2.0.1]\) with the additional information that \(deg\) is an equivalence by Theorem \([6.2.3]\) Lemma \([8.3.5]\) and Lemma \([5.4.3]\) and \(\sigma_r\) is an equivalence by Lemma \([7.1.8]\) Lemma \([7.2.9]\) and Lemma \([8.3.2]\).
We note that the terminology is totally misleading in that there is no trace whatsoever involved. The only vestige of a trace is in (the classical formula for the homotopy inverse of) the equivalence $\text{TC}(\mathcal{F}_A) \sim \text{TC}(A)$ of [9.2.2].

Since an action by an operad $\mathcal{O}$ is nothing but a multifunctor from $\mathcal{O}$ considered as a multicategory, and all the transformations in the cyclotomic traces are claimed to be natural transformations of multifunctors, we get our main result directly from the referenced claims made in these definitions.

**Theorem 2.0.4** The cyclotomic trace between $K(\mathcal{C})$ and $\text{TC}(\mathcal{C})$ consists of natural transformation of multifunctors to symmetric spectra, both in the case [2.0.1] of $\mathcal{C}$ ranging over $\text{Sp}^\Sigma$-categories with cofibrations and weak equivalences, and in the case [2.0.3] of $\mathcal{C}$ ranging over permutative $\text{Sp}^\Sigma$-categories. Hence the cyclotomic trace preserves operad actions.

In the special case when $\mathcal{C}$ is the permutative $\text{Sp}^\Sigma$-category $\mathcal{F}_A$ of “finitely generated free $A$-modules” for $A$ a connective commutative symmetric ring spectrum, as defined in [9.2.1], the cyclotomic trace from $K(A) = K(\mathcal{F}_A)$ to $\text{TC}(\mathcal{F}_A)$ of Definition 2.0.3 may be composed with the $E_\infty$-map and equivalence $\text{TC}(\mathcal{F}_A) \sim \text{TC}(A)$ of [9.2.2] to give a weak natural $E_\infty$-transformation from $K(A)$ to $\text{TC}(A)$.

The identification of $K(A)$ with more classical formulations of K-theory is summed up in Lemma [92.3].

### 3 A toolbox

We use the language of colored operads, also called multicategories. Their usefulness was driven home to the author through the paper of Elmendorf and Mandell [12], and we are grateful to the authors for promoting this piece of equipment in the mathematical toolbox, making older ad-hoc arguments much more streamlined.

Throughout this section, let $V = (V, \otimes, e)$ be a closed category. For convenience we will assume the existence of limits and colimits without particular mention, since this holds in all our applications.

#### 3.1 Multicategories

**Definition 3.1.1** (The underlying category of a $V$-category) Consider the functor $U$ from $V$ to the category of sets sending $v \in V$ to $Uv = V(e, v)$. Due to the maps $\{id_v\} \subseteq V(e, e)$ and $V(e, v) \times V(e, w) \to V(e \otimes e, v \otimes w) \cong V(e, v \otimes w)$ we get that $U$ is symmetric monoidal. Hence we may define a functor from the category of $V$-categories to categories by applying $U$ to the morphism objects. If $V$ has coproducts, $U$ has a left adjoint, sending a set $S$ to (a choice of) the $S$-fold coproduct of $e$ with itself. This functor often goes without a name, so that a set may suddenly be considered as an object in $V$ and a category as a $V$-category.

**Definition 3.1.2** (Functor categories) If $\mathcal{C}$ and $\mathcal{D}$ are two $V$-categories where $\mathcal{C}$ is small and $\mathcal{D}$ with small $V$-limits, then the $V$-functor category $[\mathcal{C}, \mathcal{D}]$ is the $V$-category whose objects are the $V$-functors $\mathcal{C} \to \mathcal{D}$, and where the $V$-object of morphism between two $V$-functors $F$ and $G$ is the $V$-end $\int_x \mathcal{D}(F(x), G(x))$. If $X$ is a small category and $\mathcal{D}$ is a $V$-category, we use the symbol $[X, \mathcal{D}]$ for the $V$-category whose objects are functors $X \to UD$ and where the $V$-object of morphism between two functors $F$ and $G$ is the end $\int_x \mathcal{D}(F(x), G(x))$. We use the left adjoint of the forgetful
functor $V \to \mathcal{E}ns$ to identify $[X, \mathcal{D}]$ with the $V$-functor category from the $V$-category represented by $X$ to $\mathcal{D}$.

There is an obvious “external tensor product” $[\mathcal{C}, \mathcal{D}] \otimes [\mathcal{C}', \mathcal{D}'] \to [\mathcal{C} \otimes \mathcal{C}', \mathcal{D} \otimes \mathcal{D}']$.

**Definition 3.1.3** A $V$-multicategory $\mathcal{C}$ consists of

1. a class $\text{ob} \mathcal{C}$ of objects,
2. for each $k + 1$-tuple $c_1, \ldots, c_k, c$ of objects in $\mathcal{C}$ there is an object $\mathcal{C}_k(c_1, \ldots, c_k; c)$ in $V$ of “morphisms” from $(c_1, \ldots, c_k)$ to $c$, and for each permutation $\sigma \in \Sigma_k$ a morphism $\sigma^*: \mathcal{C}_k(c_1, \ldots, c_k; c) \to \mathcal{C}_k(c_{\sigma^{-1}(1)}, \ldots, c_{\sigma^{-1}(k)}; c)$ inducing a functor $\Sigma_k \to V$
3. for each object $c$ a morphism $e \to \mathcal{C}_1(c; c)$ in $V$ (the “unit”)
4. for each collection of objects $b_i$ and $a_{m,n}$ with $1 \leq i, m \leq k$ and $1 \leq n \leq n_m$ a unital, equivariant and associative composition morphism $\mathcal{C}_k(b_1, \ldots, b_k; c) \otimes \bigotimes_{m=1}^k \mathcal{C}_{n_m}(a_{m,1}, \ldots, a_{m,n_m}; b_m) \longrightarrow \mathcal{C}_{\sum_{m=1}^k n_m}(a_{1,1}, \ldots, a_{k,n_k}; c)$

where the indices are ordered lexicographically and the (invisible) parentheses for the tensors are nested accordingly.

That the composition is unital, equivariant and associative means that the diagrams corresponding to those in [12, 2.2] commute.

**Definition 3.1.4** Given two $V$-multicategories $\mathcal{C}$ and $\mathcal{D}$, a $V$-multifunctor $f: \mathcal{C} \to \mathcal{D}$ consists of a function $f: \text{ob} \mathcal{C} \to \text{ob} \mathcal{D}$, $c \mapsto f(c)$ and for $c_1, \ldots, c_k, c \in \text{ob} \mathcal{C}$ a morphism $f: \mathcal{C}(c_1, \ldots, c_k, c) \to \mathcal{D}(f(c_1), \ldots, f(c_k); f(c))$ preserving the $\Sigma_k$-action, unit and composition.

Given two $V$-multifunctor $f, g: \mathcal{C} \to \mathcal{D}$, a $V$-multitransformation (or natural transformation of $V$-multifunctors) is a collection of morphisms $\eta_c: f(c) \to g(c)$ in $\mathcal{D}$ such that the induced diagrams

\[
\begin{array}{ccc}
\mathcal{C}(c_1, \ldots, c_k; c) & \longrightarrow & \mathcal{D}(f(c_1), \ldots, f(c_k); f(c)) \\
g \downarrow & & \eta^* \\
\mathcal{D}(gc_1, \ldots, gc_k; gc) & \longrightarrow & \mathcal{D}(f(c_1), \ldots, f(c_k); gc)
\end{array}
\]

commute.

**Example 3.1.5** We are particularly interested in the case of $\mathcal{S}$-multicategories, and we will drop the prefix “$\mathcal{S}$” from most concepts. Notice that the nerve identifies $\text{Cat}$-multicategories with a particular kind of multicategories. Similarly, if $V$ is the category of sets, we consider $V$-multicategories as “discrete” multicategories.

A multicategory $\mathcal{O}$ with just one object is nothing but an operad, and if $\mathcal{O} \to \mathcal{C}$ is a multifunctor, the image of the single object is an $\mathcal{O}$-algebra in $\mathcal{C}$. Remember that an $E_{\infty}$-algebra is an object with an action by an operad whose $k$th space is contractible and with free $\Sigma_k$-action for every $k$. 

8
Example 3.1.6 Consider the operad $\Sigma_*$ where the set of $k$-morphisms is $\Sigma_k$ and where composition $\Sigma_k \times \left( \times_{m=1}^k \Sigma_m \right) \to \Sigma_{\sum_{m=1}^k m}$ is given by permuting block sums (see [12] Section 3). Otherwise put, it is the operad for which an algebra in a symmetric monoidal category is exactly a monoid.

Similarly, consider the translation category $E\Sigma_k$ whose objects are permutations in $\Sigma_k$, and where a map from $\sigma_1$ to $\sigma_2$ is a factorization $\sigma_1 = \sigma_2\sigma$. Collecting these categories and using that $E$ preserves products, the operad $\Sigma_*$ gives rise to a $\text{Cat}$-operad $E\Sigma_*$. Since each morphism space in $E\Sigma_*$ is contractible and has free permutation action, an $E\Sigma_*$-algebra in categories has a nerve which is an $E_\infty$-algebra.

Example 3.1.7 One notices that the background closed category $V$ itself is a $V$-multicategory: $V$-object of $k$-morphisms is $V(v_1, \ldots, v_k; v) = V(v_1 \otimes \cdots \otimes v_k, v)$. In general, if $\mathcal{C} = (\mathcal{C}, \otimes, e)$ is a symmetric monoidal $V$-category, $\mathcal{C}$ may be considered as a $V$-multicategory with $k$-morphism object $\mathcal{C}(c_1, \ldots, c_k; c) = \mathcal{C}(c_1 \otimes \cdots \otimes c_k, c)$.

Example 3.1.8 The category $V\text{-Cat}$ of small $V$-categories supports a symmetric monoidal structure, by letting $\mathcal{C} \otimes \mathcal{C}'$ be the $V$-category, whose class of objects is $\text{ob}(\mathcal{C} \otimes \mathcal{C}') = \text{ob}\mathcal{C} \times \text{ob}\mathcal{C}'$, but where $(\mathcal{C} \otimes \mathcal{C}')(\{(c_1, c'_1), (c_0, c'_0)\}) = \mathcal{C}(c_1, c_0) \otimes \mathcal{C}'(c'_1, c'_0)$ and so fits in the framework of Example 3.1.7.

However, for us the enrichment in $\text{Cat}$ (or alternatively $\mathcal{S}$) coming from the $V$-natural transformations are more important, and so we will consider $V\text{-Cat}$ as a $\text{Cat}$-multicategory by declaring that $(V\text{-Cat})(\mathcal{C}_1, \ldots, \mathcal{C}_k; \mathcal{C})$ is the category of $V$-functors from $\mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_k$ to $\mathcal{C}$ and $V$-natural transformations between these.

Example 3.1.9 Assume $V$ has a final object $\ast$. Let $\ast$ also denote the final $V$-category with just one object $\ast$ whose $V$-object of endomorphisms is $\ast$. Let $V\text{-Cat}_\ast$ be the category of small pointed $V$-categories, that is $V$-functors $\ast \to \mathcal{C}$, and where the $V$-object of morphism from $\ast \to \mathcal{C}$ to $\ast \to \mathcal{D}$ is $(V\text{-Cat}_\ast)(\ast \to \mathcal{C}, \ast \to \mathcal{D})$ is the pullback of $(V\text{-Cat})(\mathcal{C}, \mathcal{D}) \to (V\text{-Cat})(\ast, \mathcal{D}) \leftarrow (V\text{-Cat})(\ast, \ast) = \ast$. To simplify the notation, we write $\mathcal{C}$ instead of $\ast \to \mathcal{C}$ and let $\ast$ also denote the distinguished object in $\mathcal{C}$ when no confusion is possible.

Note that if $\ast \otimes v \to \ast$ is an isomorphism for all $v \in V$, then the distinguished object $\ast$ in a pointed $V$-category $\mathcal{C}$ is $V$-final and $V$-initial in the sense that for every $c \in \text{ob}\mathcal{C}$ we have that $\mathcal{C}(\ast, c) \cong \mathcal{C}(c, \ast) \cong \ast$.

More generally, $V\text{-Cat}_\ast$ is a $V$-multicategory: if $c_1, \ldots, c_k; c$ are objects in $\mathcal{C}$, then the $V$-object $(V\text{-Cat}_\ast)(c_1, \ldots, c_k; c)$ of $k$-morphisms from $c_1, \ldots, c_k$ to $c$ is the pullback of

$$\bigotimes_{j=1}^k c_j; c \to \prod_{i=1}^{k-1} (V\text{-Cat})(\bigotimes_{j=1}^i c_j \otimes \ast \otimes \bigotimes_{j=1}^{k-i} c_j; c) \leftarrow \ast.$$  

However, as in Example 3.1.8 to us the $\text{Cat}$-enrichment is most important, so we forget down to sets, but add the pointed $V$-natural transformations to view $V\text{-Cat}_\ast$ as a $\text{Cat}$-multicategory.

### 3.2 Symmetric spectra in spaces and in pointed simplicial categories

As a particular example, we recall some features of the symmetric monoidal category of symmetric spectra which is our chosen framework both as target of our constructions and as spectral enrichment of our input.

Let $S^\Sigma$ be the symmetric monoidal $S_\ast$-category, whose set of objects is the natural numbers thought of as spheres $\{S^n\}_{n \geq 0}$, and where the space of morphism $S^\Sigma(S^n, S^m)$ is the space
I understand, or [10, section 2.6] where you have to do a slight analysis.

A symmetric spectrum in some $S_*$-category $\mathcal{C}$ is simply an $S_*$-functor $S^\Sigma \to \mathcal{C}$. If $X$ is a symmetric spectrum in $\mathcal{C}$, it is customary to write $X_n = X(S^n)$. The $n$-shift $sh_n X$ of $X$ is given by $sh_n X(S^m) = X(S^n \wedge S^m)$. If $\mathcal{C}$ is $S_*$-tensored call the $\Sigma_n \times \Sigma_m$-map $\sigma_{n,m} : X_n \wedge S^m \to X(S^n \wedge S^m) = X_{n+m}$ induced by the $S_*$-structure of $X$ the structure map. If $K \in S_*$ we define $K \wedge X$ by $(K \wedge X)(S^n) = K \wedge X(S^n)$, and note the map $\lambda_n : X^\wedge n \to sh_n X$ given by the simplicial structure, or explicitly in terms of the structure map and the natural isomorphisms:

$$\lambda_{n,m} : S^n \wedge X(S^m) \cong X(S^m) \wedge S^n \xrightarrow{\sigma_{m,n}} X(S^n \wedge S^m) \cong X(S^n \wedge S^m).$$

If $c$ is an object of $\mathcal{C}$, the suspension spectrum $\Sigma^\infty c$ is given by $\Sigma^\infty c(S^n) = c \wedge S^n$.

If we just say symmetric spectrum without specifying the target $S_*$-category, we mean a symmetric spectrum in $S_*$, and write $\text{Sp}^\Sigma = [S^\Sigma, S_*]$. It is a closed symmetric monoidal category – and hence a multicategory – under the smash product of Day [3], and it was shown by Hovey, Shipley and Smith [17] that this closed symmetric structure induces the usual smash product in the stable homotopy category.

The internal morphism objects are denoted simply $\text{Sp}^\Sigma(X, Y)$, with $R\text{Sp}^\Sigma(X, Y)$ for the underlying space and $U\text{Sp}^\Sigma(X, Y)$ for the zero simplices thereof. Explicitly, if $X_1, \ldots, X_k, X$ are symmetric spectra, the space of $k$-morphisms from $X_1, \ldots, X_k$ to $X$ is $R\text{Sp}^\Sigma(X_1 \wedge \ldots \wedge X_k, X)$.

**Definition 3.2.1** If $X$ is a level fibrant symmetric spectrum, consider the map $\lambda_X : X \to R^1 X = S_*(S^1, X(S^1 \wedge -))$ adjoint to the map $\lambda_1$ above. We say that $X$ is an $\Omega$-spectrum if $\lambda_X$ is a level equivalence. Let

$$R^\infty X = \lim \left\{ X \xrightarrow{\lambda_X} R^1 X \xrightarrow{R^1(\lambda_X)} R^1 R^1 X \xrightarrow{R^1 R^1(\lambda_X)} \ldots \right\}.$$

Now, $X$ is said to be semistable if $R^\infty X$ is an $\Omega$-spectrum. In general, a symmetric spectrum $Y$ is said to be an $\Omega$-spectrum or to be semistable if $\text{inf} |Y|$ is.

Another example is symmetric spectra in the category $\text{Cat}_*^{\Delta^\circ}$ of simplicial small pointed categories which is a $S_*$-category with the structure introduced by Quillen quite generally on simplicial objects in a category with coproducts. If $X$ is a space and $\mathcal{C}$ a simplicial category, then $\mathcal{C} \wedge X$ is the simplicial category whose category of $q$-simplices is the (pointed) coproduct of $\mathcal{C}_q$ with itself indexed over $X_q \to \ast$, and the space of functors from $\mathcal{C}$ to $\mathcal{D}$ has $q$-simplices $\text{Cat}_*^{\Delta^\circ}(C \wedge Q[q]_*, D)$. The multicategory of symmetric spectra in $\text{Cat}_*^{\Delta^\circ}$ is then the closed symmetric monoidal (Day again) category $[S^\Sigma, \text{Cat}_*^{\Delta^\circ}]$ of $S_*$-functors from $S^\Sigma$ to $\text{Cat}_*^{\Delta^\circ}$. Remembering the natural transformations as well, we get a multicategory $[S^\Sigma, \text{Cat}_*^{\Delta^\circ}]$ of enriched in bisimplicial sets.

Since $\text{Sp}^\Sigma$ is a closed category, we may consider $\text{Sp}^\Sigma$-categories. The notion that replaces equivalence of categories is the following:

**Definition 3.2.2** An $\text{Sp}^\Sigma$-functor $F : \mathcal{C} \to \mathcal{D}$ is a stable equivalence if

1. for all $c, c' \in \mathcal{C}$ the map $\mathcal{C}(c, c') \to \mathcal{D}(Fc, Fc')$ is a stable equivalence and
2. for all $d \in \mathcal{D}$ there is a $c \in \mathcal{C}$ and an isomorphism $c \cong Fd$ in $\mathcal{D}$.

**Example 3.2.3** Of particular importance to us is the multicategory $[S^\Sigma, \text{Sp}^\Sigma-\text{Cat}_*^{\Delta^\circ}]$ of symmetric spectra in pointed simplicial $\text{Sp}^\Sigma$-categories.
The object functor from pointed $\text{Sp}^\Sigma$-categories (with the Quillen simplicial enrichment) to pointed sets induces a multifunctor

$$\text{ob}: [\text{Sp}^\Sigma, \text{Sp}^\Sigma\text{-Cat}_s^\Delta] \to \text{Sp}^\Sigma.$$  

### 3.3 The category of pairs

For K-theory considerations, the notion of a weak equivalence is of special importance. Just as the set of invertible elements in a ring usually is not closed under addition; the weak equivalences in an $\text{Sp}^\Sigma$-category can not be expected to form a category enriched in $\text{Sp}^\Sigma$. Hence we are forced to consider pairs of categories with different enrichments. We first consider the case where the weak equivalences only form a(n ordinary) category, and then the situation where the weak equivalences form an $\mathcal{S}$-category.

In our context, the category $\mathcal{P}_U$ of pairs has objects $(\mathcal{C}, w)$ where $\mathcal{C}$ is a pointed $\text{Sp}^\Sigma$-category and $w: \mathcal{W} \to UC$ is a functor.

In all of our important examples, $w$ will be an inclusion of a subcategory containing all isomorphisms, and so we will occasionally talk of an object $a$ in $\mathcal{W}$ when we really mean its image $w(a)$ in $\mathcal{C}$.

A map of pairs $(\mathcal{C}, w) \to (\mathcal{C}', w')$ consists of a pointed $\text{Sp}^\Sigma$-functor $F: \mathcal{C} \to \mathcal{C}'$ and a functor pointed $G: \mathcal{W} \to \mathcal{W}'$ such that $w'G = UFw$.

Since the category of small pointed $\text{Sp}^\Sigma$-categories is symmetric monoidal through the smash-product on morphism spectra as in Example 3.1.9, the category of pairs is symmetric monoidal through the product

$$(\mathcal{C}_1, w_1) \wedge (\mathcal{C}_2, w_2) = (\mathcal{C}_1 \wedge \mathcal{C}_2, \mathcal{W}_1 \times \mathcal{W}_2 \xrightarrow{w_1 \times w_2} UC_1 \times UC_2 \to U(C_1 \wedge C_2),$$

where the last map is induced by projecting the product of (pointed) morphism sets to the smash. The unit element is $(S, *)$, where $S$ is the $\text{Sp}^\Sigma$-category with one object $*$ with the sphere spectrum as endomorphism spectrum.

We also will have to consider the variant $\mathcal{P}_R$ of $\mathcal{P}_U$ of pairs $(\mathcal{C}, \mathcal{W})$, where $\mathcal{C}$ is a small pointed $\text{Sp}^\Sigma$-category and $w: \mathcal{W} \to RC$ is a $\mathcal{S}$-functor of small pointed $\mathcal{S}$-categories. Here $RC$ is the underlying $\mathcal{S}$-category of $\mathcal{C}$ (i.e., all morphism spectra are evaluated on $S^0$). As before, the category $\mathcal{P}_R$ is a multicategory.

**Example 3.3.1** Given a $\text{Sp}^\Sigma$-category $\mathcal{C}$ consider the $R$-pair $\omega \mathcal{C} = (\mathcal{C}, \omega \mathcal{C} \subseteq RC)$, where $\omega \mathcal{C}$ is defined by the pullback

$$\begin{array}{ccc}
\omega \mathcal{C} & \xrightarrow{\subseteq} & RC \\
\downarrow & & \downarrow \\
\pi_0RC & \xrightarrow{\subseteq} & \pi_0RC,
\end{array}$$

$\pi_0RC$ is the category you get by applying $\pi$ to each morphism space in $\mathcal{C}$ and $i\pi_0RC$ is the subcategory of isomorphisms therein. In other words, $\omega \mathcal{C}$ is the $\mathcal{S}$ subcategory of $RC$ with all objects, such that given two objects $c, c' \in \text{ob}\mathcal{C}$ the space of morphisms $\omega \mathcal{C}(c, c')$ consists of the invertible components of $RC(c, c')$.

Note that if $\mathcal{C}_1 \wedge \ldots \wedge \mathcal{C}_k \to \mathcal{C}$ is a $k$-morphism in pointed $\text{Sp}^\Sigma$-categories, then $RC_1 \wedge \ldots \wedge RC_k \cong R(C_1 \wedge \ldots \wedge C_k)$ and the composites of isomorphisms are isomorphisms, giving a well-defined map $\omega \mathcal{C}_1 \times \cdots \times \omega \mathcal{C}_k \to \omega \mathcal{C}$, and ultimately a $k$-morphism of pairs from $(\mathcal{C}_1, \omega \mathcal{C}_1), \ldots (\mathcal{C}_k, \omega \mathcal{C}_k)$ to $(\mathcal{C}, \omega \mathcal{C})$.  

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Lemma 3.3.2  The functor $\tilde{\omega}: \SigmaSpCat_* \to \Psi_R$ of Example 3.3.1 has the structure of a multifunctor.

It is convenient to unravel the $R$-pairs in terms of simplicial $U$-pairs as follows. If $(C, w)$ is an $R$-pair, let $(C[n], w[n])$ be the following $U$-pair. The $\SigmaSp$-category $C[n]$ has the same objects as $C$, but the morphism spectrum from $a \in \text{ob}C$ to $b \in \text{ob}C$ is given by $\tilde{\omega}_w(C(n)(a,b)) = S_*(\Delta[n]+, C(a,b))$. The simplicial contraction of $\Delta[n]$ gives that $C(a,b)$ is a deformation retract of $C[n](a,b)$, and so $[n] \to C[n]$ is in every way just a “fat copy” of $C$. Note that $UC[n](a,b)$ is nothing but the set of $n$-simplices in $RC(a,b)$, so we may let $W[n]$ the the $n$-simplices of $W$ viewed as a simplicial category and $w[n]: W[n] \to UC[n]$ be given by the $n$the degree of $Rw$.

Lemma 3.3.3  The functor $(C, w) \mapsto ([n] \mapsto (C[n], w[n]))$ from $R$-pairs to simplicial $U$-pairs is a multifunctor.

Though it plays no rôle in the further development, we note that the image consists of the simplicial $U$-pairs with “constant object datum” (which may be identified with the category of pairs $(C, w)$ where $C$ is a category enriched in simplicial symmetric spectra and $w$ is an $S$-functor with target the $S$-category you get by applying $U$ in every simplicial degree of the morphism spaces of $C$).

The most important examples are the ones where the weak equivalences are reflected in the $\SigmaSp$-enrichment.

Definition 3.3.4  A $U$-pair $(C, w)$ is said to have property $P$ (resp. property $LP$) if for any object $a$ in $C$ and any morphism $b_1 \to b_0$ in $W$, the induced map

$$C(a, wb_1) \to C(a, wb_0)$$

is a stable (resp. level) equivalence of semistable symmetric spectra. An $R$-pair $(C, w)$ is said to have property $P$ or property $LP$ if for each $n$ the $U$-pair $(C[n], w[n])$ has this property.

Lemma 3.3.5  If $C$ is a pointed $\SigmaSp$-category such that each morphism spectrum is a level fibrant, connective $\Omega$-spectrum, then the $R$-pair $\tilde{\omega}C$ has property $LP$.

Proof:  Since $C$ is a deformation retract of $C[n]$, it is enough to prove that $(C, \omega_0 C \subseteq UC)$ has property $P$, where $\omega_0 C = UC \times_{\pi_0 RC} \pi_0 RC$. Furthermore, since a map $X \to Y$ of fibrant connective $\Omega$-spectra is a stable equivalence if and only if the induce map $RX \to RY$ of fibrant spaces is a homotopy equivalence, it is enough to see that if $f: c \to c' \in \omega_0 C(c, c')$ and $a \in \text{ob}C$, then $f_*: RC(a, c) \to RC(a, c')$ is a homotopy equivalence. Since $f \in \omega_0 C$ and the morphism spaces in $RC$ are fibrant, there is a $g: c' \to c \in UC$ and paths $H: \Delta[1]_+ \to RC(c, c)$ and $H': \Delta[1]_+ \to RC(c', c')$ with $Hd_0^1 = \text{id}$, $Hd_1^0 = gf$, $H'd_0^1 = \text{id}$ and $H'd_1^1 = fg$. This defines a homotopy $RC(a, c) \Delta[1]_+ \to RC(a, c)$ between the identity and $(gf)_*$, and likewise for $(fg)_*$.

3.4 Stabilizing morphism spectra

It is often advantageous that our $\SigmaSp$-categories have morphism spectra that are $\Omega$-spectra, and we can eventually force this for symmetric spectra of simplicial pointed $\SigmaSp$-categories. The reader will notice that we use the spectrum direction of our constructions to stabilize morphism spaces without running into Lewis’ paradox with respect to commutative topological monoids.

The multifunctor $T_0$ is a variant of the zero-simplices $THH_0$ of Bökstedt’s topological Hochschild homology. A more detailed discussion of $THH$ will be undertaken in Section 8.1. Classically, the
Varying $\Sigma_n$ symmetric spectrum we get a map through concatenation $\Sigma_n \rightarrow \text{T}_0 \text{Sp}$ defines a strong monoidal functor. An injection $\text{spectra}$, or more precisely, $\text{T}_0 \text{Sp}$ transformation $\varphi$ equivalence $1 \times I \rightarrow \text{T}_0 \text{Sp}$ together with the obvious map $\Sigma_n \rightarrow \text{T}_0 \text{Sp}$, giving a functor $\varphi : P \rightarrow \text{Cat}$ sending $n$ to $\text{T}_n^Q$.

Given a symmetric spectrum $E$, a space $X$ and a finite set $Q$, we define $T_0^Q(E; X)$ to be the homotopy colimit

$$T_0^Q(E; X) = \lim_{x \in I^Q} \Omega^{l_x} X \wedge E(S^{l_x})|.$$ 

Varying $X$ through spheres we get a symmetric spectrum $T_0^Q E = \{T_0^Q(E; S^n)\}$. If $E'$ is another symmetric spectrum we get a map

$$T_0^Q(E; X) \wedge T_0^Q(E'; X') \rightarrow T_0^Q(E \wedge E'; X \wedge X') \rightarrow T_0^Q(E \wedge E'; X \wedge X')$$

through concatenation $\sqcup : \text{T}_0^Q \times \text{T}_0^Q \rightarrow \text{T}_0^Q$, defining a map $T_0^Q E \wedge T_0^Q E' \rightarrow T_0^Q(E \wedge E')$. Together with the obvious map $\text{S} \rightarrow T_0^Q \text{S}$, this assembles to a monoidal functor $T_0^Q : \text{Sp}^\Sigma \rightarrow \text{Sp}^\Sigma$. Unfortunately, the use of concatenation in this definition breaks the symmetry, but we will see how this can be resolved if there is an extra spectral direction to play with.

There is a slight variant of $T_0$, given by

$$M_0^Q(E)_n = \lim_{x \in I^Q} \Omega^{l_x} \text{sh}_n E(S^{l_x})| = \lim_{x \in I^Q} \Omega^{l_x} E(S^n \wedge S^{l_x})|,$$

and the structure maps of $E$ induces natural transformations $E \rightarrow M_0^Q E \leftarrow T_0^Q E$.

**Lemma 3.4.1** If $E$ is semistable and $Q$ a nonempty finite set, then $E \rightarrow M_0^Q E \leftarrow T_0^Q E$ are $\pi_\ast$-isomorphisms.

**Proof:** Since $E$ is semistable, $E \rightarrow \Omega^{l_x} \text{sh}_x E$ is a $\pi_\ast$-isomorphism for all $x \in I^Q$, and so, since $I^Q$ has an initial object, and since homotopy colimits preserve $\pi_\ast$-isomorphisms $E \rightarrow \lim_{x \in I^Q} E \rightarrow \Omega^{l_x} \text{sh}_x E = M_0^Q E$ is a level equivalence followed by a $\pi_\ast$-isomorphism. That $M_0^Q E \leftarrow T_0^Q E$ is a $\pi_\ast$-isomorphism, follows since we may replace $E$ by an $\Omega$-spectrum in which case it follows as in Shipley’s proof of [22, Theorem 3.1.9 part 3] (page 168, third paragraph).

Notice that the map of underlying spaces $R M_0^Q E \leftarrow R T_0^Q E$ is an isomorphism:

$$R M_0^Q E = \lim_{x \in I^Q} \Omega^{l_x} E(S^{l_x})|,$$

and so we get a direct map $RE \rightarrow RT_0^Q E$.

We get a monoidal functor $T_0^Q : \text{Sp}^\Sigma \rightarrow \text{Sp}^\Sigma$ through applying $T_0^Q$ to each morphism spectrum. Finally, if $C = \{C^n\}$ is a symmetric spectrum in pointed $\text{Sp}^\Sigma$-categories, we define $T_0^Q C$ as $\{T_0^Q C^n\}$, with structure $\Sigma_n \times \Sigma_m$-map given by the composite

$$(T_0^Q C^n) \wedge S^m \rightarrow (T_0^Q C^n \wedge S^m) \rightarrow T_0^{n,m}(C^n \wedge S^m) \rightarrow T_0^{n,m} C^{n,m}.$$
where the first map is given by the identification $n = n \sqcup \emptyset$ and moving the homotopy colimits and loops outside the smash, the second by $\emptyset \subseteq m$ and the last map is induced by the structure map on $\mathcal{C}$.

We treat $M_0$ is exactly the same manner.

**Definition 3.4.2** If $\{(C^n, W^n \rightarrow R^n)\}_n$ is a symmetric spectrum of simplicial $R$-pairs, then the symmetric spectra in simplicial $R$-pairs $T_0(\mathcal{C}, w)$ is defined as

$$\{(T^n_0 C^n \rightarrow R^n \rightarrow R M^n_0 C^n = R T^n_0 C^n)\}_n.$$  

Likewise for $M_0(\mathcal{C}, w)$.

**Lemma 3.4.3** The functors $T_0, M_0: [S \Sigma, \text{Sp}^\Sigma \text{-Cat}^\Delta_+] \rightarrow [S \Sigma, \text{Sp}^\Sigma \text{-Cat}^\Delta_+]$ have the structure of multifunctors, and the transformations $C \rightarrow M_0 C \leftarrow T_0 C$ respects this structure. Likewise for $R$-pairs.

**Proof:** Let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ be symmetric spectra in pointed simplicial $\text{Sp}^\Sigma$-categories and $n_1, \ldots, n_k$ be sets in $\mathcal{I}$. The isomorphism $I^{n_1} \times \cdots \times I^{n_k} \cong I^{n_1 \sqcup \cdots \sqcup n_k}$ gives rise to the desired map $T^n_0 C_1^{n_1} \wedge \cdots \wedge T^n_0 C_k^{n_k} \rightarrow T^n_0 (C_1^{n_1} \wedge \cdots \wedge C_k^{n_k})$. Likewise for $M_0$, yielding the natural transformations of multifunctors. There is no essential difference when handling pairs. □

**Lemma 3.4.4** If the symmetric spectrum in simplicial $R$-pairs $(\mathcal{C}, w)$ satisfies property $P$ at level $n > 0$, then $T_0(\mathcal{C}, w)$ satisfies property LP at level $n$.

**Proof:** Follows since $T^n_0$ turns stable equivalences of semistable spectra into level fibrations of semistable spectra, whenever $n > 0$. □

The “uniform choice of weak equivalences” is a composite of two multifunctors, and gives the possibility of selecting a $S$-subcategory of weak equivalences directly from a $\text{Sp}^\Sigma$-enrichment.

**Definition 3.4.5** Let $\mathcal{C}$ be a symmetric spectrum in pointed simplicial $\text{Sp}^\Sigma$-categories. Then $\tilde{\omega} \mathcal{C} \in [S \Sigma, \text{Sp}^\Sigma \text{-Cat}^\Delta_+]$ is the result of applying the diagonal and the multifunctor $\mathcal{P}_R \rightarrow \mathcal{P}^\Delta_+$ of 3.3.3 to $\omega T_0 \mathcal{C}$, where $\tilde{\omega}$ is the multifunctor introduced in 3.3.1.

## 4 Waldhausen’s setup

Geisser and Hesselholt [13] use Waldhausen’s construction to define the algebraic K-theory of an exact category as a symmetric spectrum, and shows that the construction has good multiplicative properties. We adapt this idea to give the necessary ingredients for a cyclotomic trace with the same properties. The TC part of the story is postponed to Section 8.

### 4.1 $\text{Sp}^\Sigma$-categories with cofibrations and weak equivalences

**Definition 4.1.1** An $\text{Sp}^\Sigma$-category with cofibrations and weak equivalences is an $\text{Sp}^\Sigma$-category $\mathcal{C}$ together with a structure of a category with cofibrations and weak equivalences on the underlying category $U \mathcal{C}$, satisfying Waldhausen’s axioms [23]: $U \mathcal{C}$ has a final and initial object $0$ and two subcategories $coC$ (whose morphisms are referred to as cofibrations) and $wC$ (whose morphisms are referred to as weak equivalences) such that

1. isomorphisms are both cofibrations and weak equivalences
2. maps from 0 are cofibrations

3. if \( a \to b \) is a cofibration and \( a \to c \in UC \), then the pushout

\[
\begin{array}{c}
a \to b \\
c \to c \amalg_a b
\end{array}
\]

exists, and the lower horizontal map is a cofibration,

4. (the gluing axiom) if the left horizontal maps in the commutative diagram

\[
\begin{array}{c}
d \leftarrow c \to e \\
d' \leftarrow c' \to e'
\end{array}
\]

are cofibrations and the vertical maps are weak equivalences, then the induced map

\[
d \amalg_c d \to d' \amalg_{c'} e'
\]

is also a weak equivalence.

In addition, we will assume that there is a \textit{choice} of the colimits that are required to exist as a part of the data (so that, in particular, the coproduct \( a, b \mapsto a \vee b \) is an \( \text{Sp}^\Sigma \)-functor).

Note that the 0-object is initial in the category of cofibrations, and is both initial and final in its component of weak equivalences.

Occasionally the weak equivalences form a sensible \( S \)-category and there may be sensible interactions between the enrichments and the rest of the structure, but we do not make this a part of the definition, in that the following properties may or may not be satisfied in a given \( \text{Sp}^\Sigma \)-category with cofibrations and weak equivalences.

**Definition 4.1.2** An \( \text{Sp}^\Sigma \)-category with cofibrations and weak equivalences \( C \) is M7 (for short: an \( \text{M7-category} \)) if

1. all morphism spectra are semistable

2. if \( c' \to c \) is a cofibration and \( d \in C \), then \( C(c, d) \to C(c', d) \) is a level fibration, and

3. if \( d' \to d \) is a weak equivalence and \( c \in C \), then \( C(c, d') \to C(c, d) \) is a stable equivalence.

The name M7 is to remind us of Quillen’s axiom SM7 guaranteeing the correct interplay between a model structure and an enrichment. In our case, this indicates that in an M7-category the objects are in some sense not too far from being “fibrant”.

Note that the inclusion of the weak equivalences in an M7 category by definition satisfies “property P” of Definition 3.3.4.
Definition 4.1.3 If $\mathcal{C}$ and $\mathcal{D}$ are $\text{Sp}^\Sigma$-categories with cofibrations and weak equivalences, a $\text{Sp}^\Sigma$-functor $f : \mathcal{C} \to \mathcal{D}$ is exact if its underlying functor of categories with cofibrations and weak equivalences is exact in Waldhausen’s sense.

Exact functors do not need to preserve the chosen colimits (but they are automatically strong monoidal), so the resulting category $\text{Sp}^\Sigma$-Wa of $\text{Sp}^\Sigma$-categories with cofibrations and weak equivalences and exact functors between them is equivalent to the one where no such choices are made.

If $\mathcal{C} \in \text{Sp}^\Sigma$-Wa, we will by abuse of notation use the same letter, $\mathcal{C}$ for its underlying $\text{Sp}^\Sigma$-category, whereas, the category of cofibrations (resp. weak equivalences) will be denoted $\text{co}\mathcal{C}$ (resp. $\text{w}\mathcal{C}$). If there is any chance for confusion (for instance if there are more categories of weak equivalences associated with one $\text{Sp}^\Sigma$-category), we may even specify $\mathcal{C}$ as a pair $(\mathcal{C}, \text{co}\mathcal{C})$ or a triple $(\mathcal{C}, \text{co}\mathcal{C}, \text{w}\mathcal{C})$.

Definition 4.1.4 \cite[6.1]{13} A square

\[
\begin{array}{ccc}
c_0 & \longrightarrow & c_1 \\
\downarrow & & \downarrow \\
c_2 & \longrightarrow & c_{12}
\end{array}
\]

is a cofibration square if $c_0 \to c_1$, $c_0 \to c_2$ and the induced map $c_1 \coprod_{c_0} c_2 \to c_{12}$ are all cofibration. If the square is a cofibration square, a cofiber of $c_1 \coprod_{c_0} c_2 \to c_{12}$ is called an iterated cofiber of the square. We say that a cube $c$ in $\mathcal{C}$ is a strong cofibration cube if all subsquares are cofibration squares.

Definition 4.1.5 (The cube associated to a tuple of morphisms) A morphism $(f_1 : s_1 \to t_1, \ldots, f_k : s_k \to t_k)$ in a product $\mathcal{C}_1 \times \cdots \times \mathcal{C}_k$ of categories give rise to a $k$-cube $f$ in $\mathcal{C}_1 \times \cdots \times \mathcal{C}_k$, with $f_S = (f^j_1, \ldots, f^j_k)$ where

\[
f^j_S = \begin{cases} 
  s_j & \text{if } j \notin S \\
  t_j & \text{if } j \in S
\end{cases},
\]

and if $j \notin S$ then $f_S \to f_{S \cup j}$ is $(1, \ldots, 1, f_j, 1, \ldots, 1)$. For instance, if $k = 2$, the square in question is

\[
\begin{array}{ccc}
(s_1, s_2) & \xrightarrow{(f_1,1)} & (t_1, s_2) \\
\downarrow (1,f_2) & & \downarrow (1,f_2) \\
(s_1, t_2) & \xrightarrow{(f_1,1)} & (t_1, t_2)
\end{array}
\]

4.2 Rig categories with cofibrations and weak equivalences

Multiplicative structures are easy to describe:

Definition 4.2.1 A rig category with cofibrations and weak equivalences is an $\text{Sp}^\Sigma$-category $\mathcal{C}$ with cofibrations and weak equivalences together with an $\text{Sp}^\Sigma$-monoidal structure $\otimes : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ such that the composite $\mathcal{U} \mathcal{C} \times \mathcal{U} \mathcal{C} \to \mathcal{U} \mathcal{C} \otimes \mathcal{U} \mathcal{C} \to \mathcal{U} \mathcal{C}$ satisfies the following properties:

1. given $x \in \mathcal{C}$ the functors $x \otimes -$ and $- \otimes x$ are exact functors $\mathcal{U} \mathcal{C} \to \mathcal{U} \mathcal{C}$ ("$\otimes$ is biexact")
2. if \( a' \rightarrow a \rightarrow a'' \) and \( b' \rightarrow b \rightarrow b'' \) are cofibration sequences in \( C \), then

\[
\begin{array}{ccc}
a' \otimes b' & \rightarrow & a' \otimes b \\
\downarrow & & \downarrow \\
a \otimes b' & \rightarrow & a \otimes b
\end{array}
\]

is a cofibration square with iterated cofiber \( a'' \otimes b'' \).

The monoidal structure \( \otimes \) on a rig category with cofibrations and weak equivalences is referred to as the *multiplicative structure*. Note that distributivity is taken care of by the exactness property of the multiplicative structure, and so a rig category with cofibrations and weak equivalences has an underlying rig \( \operatorname{Sp}^\Sigma \)-category (aka. a bimonoidal \( \operatorname{Sp}^\Sigma \)-category).

**Definition 4.2.2** A *symmetric* rig category with cofibrations and weak equivalences is a rig category with cofibrations and weak equivalences such that the multiplicative structure is symmetric.

A symmetric rig category with cofibrations and weak equivalences is a bipermutative category with cofibrations and weak equivalences if it comes equipped with a strictly associative and unital choice of finite coproducts and the multiplicative structure is strictly associative and unital.

Let \( \mathcal{R} \) and \( \mathcal{S} \) be (symmetric) rig categories with cofibrations and weak equivalences. A map of (symmetric) rig categories with cofibrations and weak equivalences from \( \mathcal{R} \) to \( \mathcal{S} \) is an \( \operatorname{Sp}^\Sigma \)-functor \( \mathcal{R} \rightarrow \mathcal{S} \) that is both exact as a map of \( \operatorname{Sp}^\Sigma \)-categories with cofibrations and weak equivalences and equipped with a structure of a (symmetric) monoidal functors.

A map of (symmetric) rig categories with cofibrations and weak equivalences is *strong* if is strong (symmetric) monoidal with respect to the multiplicative structure (i.e., the structure (symmetry, associativity and unitality morphisms are isomorphism).

If \( Y \) is a small \( \operatorname{Sp}^\Sigma \)-category and \( f : S \rightarrow \operatorname{ob}Y \) is a function, then \( f^*Y \) is the \( \operatorname{Sp}^\Sigma \)-category whose set of objects is \( S \) and where the symmetric spectrum of morphisms from \( s \) to \( t \) (both in \( S \)) is \( Y(fs, ft) \). By abuse of notation, we will call the induced functor \( f^*Y \rightarrow Y \) that is \( f \) on objects and the identity on morphisms also \( f \).

In the particular situation where \( f \) is surjective with a splitting \( \sigma \), we identify \( Y \) and \( \sigma^*f^*Y \) and consider \( \sigma : Y \rightarrow f^*Y \). If \( s \in S \) then \( f^*Y(s, \sigma f(s)) = f^*Y(\sigma f(s), s) = Y(f(s), f(s)) \), amounting to a natural isomorphism \( \sigma f \cong 1_{f^*Y} \). Since \( f \sigma = 1_Y \) we get that \( f \) and \( \sigma \) are inverse equivalences of \( \operatorname{Sp}^\Sigma \)-categories.

**Lemma 4.2.3** There is an endofunctor \( \operatorname{Strig} \) on the category of small symmetric rig categories with cofibrations and weak equivalences and strong maps together with for each rig category with cofibrations and weak equivalences \( \mathcal{R} \) a pair of equivalences \( \pi : \operatorname{Strig}\mathcal{R} \leftrightarrow \mathcal{R} : s \) of rig categories with cofibrations and weak equivalences, such that \( \operatorname{Strig} \) takes values in bipermutative categories with cofibrations and weak equivalences and strict maps. The equivalence \( s \) is natural in \( \mathcal{R} \).

**Proof:** It turns out that May’s rigidification [19, VI, Proposition 3.5] translates nicely to our case. More explicitly, if \( \mathcal{R} \) is a rig category with cofibrations and weak equivalences, consider the free rig \( F \) on \( \operatorname{ob}\mathcal{R} \) modulo the relation that the zero for \( F \) equal to the word 0 of length one and the multiplicative unit is 1 (the unit for \( \otimes \)). One construction is as follows: let \((M, \cdot, e)\) be the free monoid on the set \( \operatorname{ob}\mathcal{R} \), where we call the operation \( \cdot \), modulo the relations \( e = 1 \) and \( 0 \cdot a = a \cdot 0 = 0 \) for all \( a \in \mathcal{R} \). Let \( p : M \rightarrow \operatorname{ob}\mathcal{R} \) be given by \( p(a_1, \ldots, a_n) = a_1 \otimes (a_2 \otimes \cdots \otimes (a_{n-1} \otimes a_n)\ldots) \) for
Next we consider the free monoid \((F, +, n)\) on the set \(M\) modulo the relation \(n = 0\). We extend the product from \(M\) to \(F\) by setting
\[
(a_1 + \cdots + a_m) \cdot (b_1 + \cdots + b_n) = a_1 \cdot b_1 + \cdots + a_m \cdot b_n
\]
(in lexicographical order) for \(a_1, \ldots, a_m, b_1, \ldots, b_n \in M\). Finally we define the function \(\pi: F \to \text{ob} \mathcal{R}\) by declaring that \(\pi(a_1 + \cdots + a_n) = p(a_1) \lor (p(a_2) \lor \cdots \lor (p(a_{n-1}) \lor p(a_n)) \ldots)\) for \(a_1, \ldots, a_n \in M\).

If we need to be precise about it, we will write \(\sigma(a)\) if we think of an object \(a \in \mathcal{R}\) as an object in \(\pi^* \mathcal{R}\) (that is we let \(\sigma: \text{ob} \mathcal{R} \to F\) be the “inclusion of generators”).

As an \(\text{Sp}^\Sigma\)-category we let \(\text{Strig} \mathcal{R}\) be \(\pi^* \mathcal{R}\). Since \(\pi: \pi^* \mathcal{R} \to \mathcal{R}\) is an equivalence of categories it preserves and creates colimits. In particular + is the coproduct in \(\text{Strig} \mathcal{R}\) and is sent to \(\lor\).

We say that a morphism in \(\text{Strig} \mathcal{R}\) is a cofibration (resp. weak equivalence) if it is mapped to one by \(\pi\). We must check that the axioms for cofibrations and weak equivalences are satisfied.

Since \(U\text{Strig} \mathcal{R}\) are isomorphisms if and only if their image under \(\pi\) are, all isomorphisms are cofibrations and weak equivalences. Likewise \(\pi(0)\) = 0 assures that all maps from 0 are cofibrations. If \(a \mapsto b\) is a cofibration and \(a \to c\) is any map in \(\text{Strig} \mathcal{R}\) then the pushout of \(c \leftarrow a \mapsto b\) is represented by \(\pi(c \coprod_a b)\) and the map \(c \to \pi(c \coprod_a b)\) is a cofibration. Similarly, the gluing axiom holds.

It is now also obvious that \(\pi: \text{Strig} \mathcal{R} \to \mathcal{R}\) is an exact functor.

The multiplicative structure on \(\text{Strig} \mathcal{R}\) is lifted from \(\mathcal{R}\) as follows:
\[
\begin{align*}
\text{Strig} \mathcal{R}(a, b) \land \text{Strig} \mathcal{R}(c, d) & \quad \text{Strig} \mathcal{R}(a \cdot c, b \cdot d) \\
\mathcal{R}(\pi(a), \pi(b)) \land \mathcal{R}(\pi(c), \pi(d)) & \quad \mathcal{R}(\pi(a) \otimes \pi(c), \pi(c) \otimes \pi(d)) \quad \mathcal{R}(\pi(a \cdot c), \pi(b \cdot d)).
\end{align*}
\]

We need to see that this structure satisfies the axioms for a rig category with cofibrations and weak equivalences. If \(x\) is an object in \(\text{Strig} \mathcal{R}\), then we divided out by a relation guaranteeing exactly that \(x \cdot 0 = 0 \cdot x = 0\). Since \(\pi\) creates cofibrations, weak equivalences and colimits and sends \(-\otimes\) both \(x \cdot -\) and \(-\cdot x\) preserve cofibrations and weak equivalences and send the distinguished pushout squares to distinguished pushout squares. Lastly, if \(a' \mapsto a \mapsto a''\) and \(b' \mapsto b \mapsto b''\) are cofibration sequences in \(\text{Strig} \mathcal{R}\) (which means that they are sent to cofibration sequence in \(\mathcal{R}\)), then
\[
\begin{align*}
a' \cdot b' & \quad \longrightarrow \quad a' \cdot b \\
\downarrow & \quad \downarrow \\
a \cdot b' & \quad \longrightarrow \quad a \cdot b
\end{align*}
\]
is a cofibration square with iterated cofiber \(a'' \otimes b''\) because this is so after applying \(\pi\).

Finally, the symmetric structure on \(\otimes\) induces one on \(-\cdot\), and \(\pi: \text{Strig} \mathcal{R} \to \mathcal{R}\) becomes a strong monoidal equivalence.

If \(f: \mathcal{R} \to \mathcal{R}'\) is a strong map of rig categories with cofibrations and weak equivalences we define \(\text{Strig} f: \text{Strig} \mathcal{R} \to \text{Strig} \mathcal{R}'\) on objects by sending a formal sum of formal products of objects in \(\mathcal{R}\) to the same formal sum of formal products of \(f\) applied to these objects. So given a collection \(a_1, \ldots, a_{n_1}, a_{2,1}, \ldots, a_{n_1,1}, \ldots, a_{n,n}\) and \(b_1, \ldots, b_{1,m_1}, b_{2,1}, \ldots, b_{m,1}, \ldots, b_{m,m_m}\) of objects in \(\mathcal{R}\) we need to define a map of spectra
\[
\mathcal{R}(V^n_j \otimes_{i=1}^{n_i} a_{i,j}, V^m_l \otimes_{k=1}^{m_k} a_{k,l}) \to \mathcal{R}'(V'^n_j \otimes_{i=1}^{n'_i} f(a_{i,j}), V'^m_l \otimes_{k=1}^{m'_k} f(a_{k,l})).
\]
To do this we need to use that \(f\) is strong, effectively commuting \(f\) with sums of products using the structure isomorphisms. Checking that the rest of the axioms is similar to the above.
Lastly, if \( f: \mathcal{R} \to \mathcal{R}' \) is a strong map of rig categories with cofibrations and weak equivalences we see that the diagram

\[
\begin{array}{ccc}
\mathcal{R} & \xrightarrow{s} & \text{Strig}\mathcal{R} \\
f \downarrow & & \downarrow \text{Strig}f \\
\mathcal{R}' & \xrightarrow{s} & \text{Strig}\mathcal{R}'
\end{array}
\]

commutes, since no instances of the structure isomorphisms of \( f \) are needed in order to define \((\text{Strig}f) \circ s\).

The same proof gives the statement without the symmetry condition.

One should note that the map \( \pi: \text{Strig}\mathcal{R} \to \mathcal{R} \) is only a natural transformation up to a modification: if \( f: \mathcal{R} \to \mathcal{R}' \) is a map of rig categories with cofibrations and weak equivalences, the diagram

\[
\begin{array}{ccc}
\mathcal{R} & \xleftarrow{\pi} & \text{Strig}\mathcal{R} \\
f \downarrow & & \downarrow \text{Strig}f \\
\mathcal{R}' & \xleftarrow{\pi} & \text{Strig}\mathcal{R}'
\end{array}
\]

only commutes up to the isomorphisms given by the fact that \( f \) is strong monoidal in both \( \vee \) and \( \otimes \) (an element \( \sum_{j=1}^{k} \prod_{i=1}^{n_j} a_{ij} \) in the upper right hand corner is sent to \( f(\vee_{j=1}^{k} \otimes_{i=1}^{n_j} a_{ij}) \) or \( \vee_{j=1}^{k} \otimes_{i=1}^{n_j} f(a_{ij}) \) according to which way you move). I do not see how May’s claim about naturality can be anything more than up to such a modification without restricting to strict monoidal \( f \).

4.3 The multicategory of \( \text{Sp}^\Sigma \)-categories with cofibrations and weak equivalences

The category \( \text{Sp}^\Sigma \)-Wa of \( \text{Sp}^\Sigma \)-categories with cofibrations and weak equivalences has a natural structure of a \( \text{Cat} \)-multicategory. A \( k \)-morphism from \( \mathcal{C}_1, \ldots, \mathcal{C}_k \) to \( \mathcal{C} \) is an \( \text{Sp}^\Sigma \)-functor \( e: \mathcal{C}_1 \land \ldots \land \mathcal{C}_k \to \mathcal{C} \) such that the composite

\[
UC_1 \times \cdots \times UC_k \longrightarrow UC_1 \land \ldots \land UC_k = U(\mathcal{C}_1 \land \ldots \land \mathcal{C}_k) \xrightarrow{Ue} UC
\]

is an exact functor of categories with cofibrations and weak equivalences in each variable, sending \( k \)-tuples of cofibrations (considered as \( k \)-cubes) to strong cofibration cubes (Definition 4.1.4). The morphisms are the \( \text{Sp}^\Sigma \)-natural transformations.

The composition

\[
\text{Sp}^\Sigma \text{-Wa}(\mathcal{B}_1, \ldots, \mathcal{B}_k; \mathcal{C}) \times \prod_{j=1}^{k} \text{Sp}^\Sigma \text{-Wa}(\mathcal{A}_{j,1}, \ldots, \mathcal{A}_{j,k_j}; \mathcal{B}_j) \to \text{Sp}^\Sigma \text{-Wa}(\mathcal{A}_{1,1}, \ldots, \mathcal{A}_{k,k_k}; \mathcal{C})
\]

is given by composition of functors. Checking that this is a multicategory is straightforward (and – due to the universality of coproducts – much easier than the analogous checking for permutative categories in [12]).

**Lemma 4.3.1** Any bipermutative category with cofibrations and weak equivalences is the image of a \( \text{Cat} \)-multifunctor \( E\Sigma_n \to \text{Sp}^\Sigma \)-Wa

**Proof:** Since a bipermutative category with cofibrations and weak equivalences \( \mathcal{R} \) has an underlying bipermutative \( \text{Sp}^\Sigma \)-category (see Definition 5.1.1 below), the only thing to add to the proof
of [12] Theorem 3.8] is to state that the induced $k$-morphisms actually are in $\text{Sp}^\Sigma\text{-Wa}$, which is clear since they only involve permutations of the factors in the source and so do not disturb the cofibrations and weak equivalences.

A similar statement holds without the symmetry condition on the multiplicative structure, giving multifunctors from the associativity operad $\Sigma$.\

4.4 The S-construction

We give a very slight variant on Geisser and Hesselholt’s description of Waldhausen’s S-construction. We will see that the current construction works nicely for $\text{M7}$ categories. If we would want to get good results for more general $\text{Sp}^\Sigma$-categories with cofibrations and weak equivalences we should build in more flab.

Given a category with cofibrations and weak equivalences $\mathcal{C}$, Waldhausen defines $S_\mathcal{C}$ to be the following category with cofibrations and weak equivalences. Consider the functor $\Delta^q \to \text{Cat}$ sending $[q]$ to $[q]^0 = \{q \leftarrow \cdots \leftarrow 1 \leftarrow 0\}$, and the functor $\text{Ar}: \text{Cat} \to \text{Cat}$ which sends a category $D$ to the arrow category $\text{Ar}D$ with objects the morphisms in $D$ and where a morphism from $d \to d'$ to $d_1 \to d_1'$ is a commutative diagram

$$
\begin{array}{ccc}
  d & \longrightarrow & d_1 \\
  \downarrow & & \downarrow \\
  d' & \longrightarrow & d'_1
\end{array}
$$

For a functor $c$ from $\text{Ar}[q]^0$ we write $c_{ij}$ (or $c_{i,j}$ in typographically challenging situations) for the value of $i \leq j$, with unique map $c_{ij} \to c_{ik}$ if $i \leq j \leq k$. For a category with cofibrations, Waldhausen defines $S_\mathcal{C}$ to be the full subcategory of the category of functors $c: \text{Ar}[q]^0 \to \mathcal{C}$ with the property that $c_{ii} = 0$ and if $0 \leq i \leq j \leq k \leq q$ then $c_{ij} \to c_{ik}$ is a cofibration, and the square

$$
\begin{array}{ccc}
  c_{ij} & \longrightarrow & c_{ik} \\
  \downarrow & & \downarrow \\
  c_{jj} & \longrightarrow & c_{jk}
\end{array}
$$

is a pushout cube. The weak equivalences are the natural transformations $c \to c'$ with the property that each $c_{ij} \to c'_{ij}$ is a weak equivalence, and the cofibrations are the natural transformations $c \to c'$ with the property that each $0 < i \leq q$ the map

$$
c'_{0,i-1} \prod_{c_0,i-1} c_{0,i} \to c'_{0,i}
$$

is a cofibration.

Iterating and extending to the enriched situation, we get the following structure: If $T$ is a finite set and $n$ a non-negative integer, let $[n]^T$ the $T$-fold product of $[n]$ (Geisser and Hesselholt allow products of different elements, obtaining a multisimplicial functor, but we simplify since we are only interested in the diagonal). If $i \leq j \leq k \in [n]^0$ and $U \subseteq T$, let $(ijk)_U$ be the object in $\text{Ar}[n]^0$ whose $t$th component is $i_t \leq k_t$ if $t \in U$ and $i_t \leq j_t$ otherwise. This defines a $T$-cube.

If $\mathcal{C} \in \text{Sp}^\Sigma\text{-Wa}$, the $\text{Sp}^\Sigma$-category $S^T\mathcal{C}$ is the full $\text{Sp}^\Sigma$-subcategory of $[\text{Ar}[n]^0, \mathcal{C}]$ of whose objects are the functors $c: \text{Ar}[n]^0 \to U\mathcal{C}$ satisfying

1. if $i \leq j \in [n]^0$ is such that $i_t = j_t$ for some $t \in T$, then $c_{ij} = 0$
2. for every $i \leq j \leq k \in [n]^o$ the $T$-cube sending $U \subseteq T$ to $c_{i j k}^T$ is a strong cofibration cube
3. for every $i \leq j \leq k \in [n]^o$ the canonical map

$$0 \prod_{t \in T} c_{i j k} \to c_{j k}$$

is an isomorphism (i.e., $c_{j k}$ is an “iterated cofiber” of the cube $U \mapsto c_{i j k}^T$).

Lastly, $wS^T T \mathcal{C}$ is the subcategory of $US^T T \mathcal{C}$ of natural transformations $c \to c'$ where each $c_t \to c'_t$ is in $w \mathcal{C}$. We see that $(S^T T \mathcal{C}, wS^T T \mathcal{C}) \in \mathcal{P}_U$, and varying $n$, we get a simplicial pair $(S^T \mathcal{C}, wS^T \mathcal{C}) \in \mathcal{P}_U^n$. When it will not cause any confusion, we will often refer to the simplicial pair as simply $S^T \mathcal{C}$.

Consider an element in $Sp^\Sigma \text{-Wa}(\mathcal{C}_1, \ldots, \mathcal{C}_k; \mathcal{C})$ given by an $Sp^\Sigma$-functor $f : \mathcal{C}_1 \wedge \ldots \wedge \mathcal{C}_k \to \mathcal{C}$ such that the composite

$$UC_1 \times \ldots \times UC_k \to UC_1 \wedge \ldots \wedge UC_k \xrightarrow{Uf} \mathcal{C}$$

is multiexact (exact in each variable). For every $n$ we want to define an $Sp^\Sigma$-functor

$$S^{T_k}_{n} \mathcal{C}_1 \wedge \ldots \wedge S^{T_k}_{n} \mathcal{C}_k \to S^{T_{1 \ldots l} T_k}_{n} \mathcal{C},$$

Notice that if $X$ and $Y$ are small categories and $\mathcal{C}$ and $D$ are $Sp^\Sigma$-categories, there is a natural and fully faithful $Sp^\Sigma$-functor of $Sp^\Sigma$-categories $[X, \mathcal{C}] \wedge [Y, D] \to [X \times Y, \mathcal{C} \wedge D]$ given by taking the suspension spectra of the domains and using the external tensor product

$$[X, \mathcal{C}](c, d) \wedge [Y, D](e, f) = \int_x c(c_x, d_x) \wedge \int_y d(e_y, f_y) \cong \int_{(x,y)} (c \wedge d)((c_x, e_y), (d_x, f_y))$$

$$\cong [X \times Y, \mathcal{C} \wedge D]((c, e), (d, f)).$$

Using the canonical isomorphism $(Ar[n]^o)^{T_{1 \ldots l} T_k} \cong \prod_{t=1}^k (Ar[n]^o)^{T_t}$, this gives rise to the desired map

$$S^{T_k}_{n} \mathcal{C}_1 \wedge \ldots \wedge S^{T_k}_{n} \mathcal{C}_k \to S^{T_{1 \ldots l} T_k}_{n} \mathcal{C},$$

and likewise on the weak equivalences, giving a map

$$Sp^\Sigma \text{-Wa}(\mathcal{C}_1, \ldots, \mathcal{C}_k; \mathcal{C}) \to \mathcal{P}_U^n(S^{T_k}_{1} \mathcal{C}_1, \ldots, S^{T_k}_{1} \mathcal{C}_k; S^{T_{1 \ldots l} T_k}_{1} \mathcal{C}).$$

Explicitly, on objects this sends the $k$-tuple $(c^i : (Ar[n]^o)^{T_i} \to \mathcal{C}_i)_{1 \leq i \leq n}$ to the object $f(c^1, \ldots, c^k)$ as the composite

$$(Ar[n]^o)^{T_{1 \ldots l} T_k} \cong \prod_{t=1}^k (Ar[n]^o)^{T_t} \xrightarrow{\prod_{t=1}^k f} \prod_{t=1}^k U \mathcal{C}_t \xrightarrow{\wedge_{t=1}^k U \mathcal{C}_t} U \mathcal{C}.$$
Now, there is an element in \( \text{Sp}^\Sigma\text{-Wa}(\Gamma^p, C; C) \) given by \( \Gamma^p \wedge C \to C \) given by sending \((n_+, e)\) to the \( n \)-fold coproduct \( \coprod_n C \). Using this and the map \( S^n = S^1 \wedge \ldots \wedge S^1 \to S^0 \wedge \ldots \wedge S^0 \to S^n \Gamma^p \) we obtain the element \( \mathcal{P}_U^\Delta^v(S^n, (S^mC, wS^mC); (S^{n+m}C, wS^{n+m}C)) \). We observe that it is appropriately \( \Sigma_n \times \Sigma_m \)-equivariant, and so establishes \( S^n \mapsto (S^nC, wS^nC) \) as an \( S_+ \)-functor \( S^\Sigma \to \mathcal{P}_U^\Delta^v \) — that is, a symmetric spectrum in the category of simplicial pairs.

**Definition 4.4.1** Let \( C \in \text{Sp}^\Sigma\text{-Wa} \). Then \( \overline{SC} \) is the symmetric spectrum in \( \mathcal{P}_U^\Delta^v \) given through the construction above, with \( n \)-th level \( S^nC \).

Since the composition of multifunctors is a multifunctor we get

**Lemma 4.4.2** If \( C \in \text{Sp}^\Sigma\text{-Wa} \), then \( \overline{S} \) is a symmetric spectrum in \( \mathcal{P}_U^\Delta^v \), and the structure above endows \( \overline{S} : \text{Sp}^\Sigma\text{-Wa} \to \left[S^\Sigma, \mathcal{P}_U^\Delta^v \right] \) with the structure of a \( \text{Cat} \)-multifunctor.

### 4.5 Homotopical properties of \( SC \).

Assuming the input is \( M7 \) (see [4.1.2]), the enrichment cooperates nicely with the \( S \)-construction. For instance, if \( c = \{c_01 \mapsto c_02 \mapsto c_12\} \) and \( d = \{d_01 \mapsto d_02 \mapsto d_12\} \) are objects in \( S_nC \), then the induced diagram

\[
\begin{align*}
S_2C(c, d) & \longrightarrow C(c_{01}, d_{01}) \\
\downarrow & \quad \downarrow \\
C(c_{02}, d_{02}) & \longrightarrow C(c_{01}, d_{02})
\end{align*}
\]

is both a categorical and level homotopy cartesian diagram (categorically since the colimit and enriched colimit agree and homotopically since the right vertical map is a level fibration). Furthermore, the diagram

\[
\begin{align*}
wS_2C(c, d) & \longrightarrow wC(c_{01}, d_{01}) \\
\downarrow & \quad \downarrow \\
wC(c_{02}, d_{02}) & \longrightarrow wC(c_{01}, d_{02})
\end{align*}
\]

is cartesian by the gluing lemma.

**Lemma 4.5.1** Let \( C \) be an \( M7 \)-category. Let \( c \sim c' \in wS^{T_n}\Gamma^p \) and \( d \in \text{ob}S^{T_n}\Gamma^p \). Then the induced map \( S^{T_n}C(c', d) \to S^{T_n}C(c, d) \) is a stable equivalence.

Together with Proposition [6.2.3] we get

**Corollary 4.5.2** Let \( C \) be an \( M7 \)-category. Then the face maps \( N_qS^{T_n}C \to N_0S^{T_n}C \cong S^{T_n}C \) are stable equivalences in the sense of [3.2.2]

Applying the transformations of multifunctors of Definition [3.4.2] (restricted to \( U \)-pairs) we get stable equivalences

\[
\{(S_nC, wS^nC)\} \leftrightarrow \{(M^n_0S^nC, wS^nC)\} \leftrightarrow \{(T^n_0S^nC, wS^nC)\} = \overline{SC}.
\]

**Lemma 4.5.3** If \( C \) is an \( M7 \)-category, then \( \overline{SC} \) (\( T_0\overline{SC} \)) satisfies property \( P \) (LP) of Definition [3.3.4] at positive levels.

**Proof:** Since \((C, wC)\) is assumed to be \( M7 \), Lemma 4.5.1 states that \((S^nC, wS^nC)\) satisfies property \( P \) for each \( n \). Then we are done by Lemma [3.4.4] □
5 Segal’s setup

Elmendorf and Mandell [12] use the iterated Segal construction to define the algebraic K-theory of a permutative category as a symmetric spectrum, and shows that the construction assembles to a multifunctor. In extending their work to $\text{Sp}^\Sigma$-enriched categories, one is faced with the dilemma that writing everything out requires a prohibitive amount of repetition, but just saying “[12] extends trivially to the enriched setting” is not totally fair. We have chosen the problematic compromise of relying on the reader having a copy of [12] easily available and only point out the places needed to understand the extension. On the other hand, we have resisted the temptation of stating everything in full generality, and we hope the reader is grateful for that.

5.1 Permutative $\text{Sp}^\Sigma$-categories

Much of the following are easier versions of what we underwent for rig categories with cofibrations and weak equivalences, and we allow ourselves to be a bit briefer, focusing on the differences. We choose the formalism as close as possible to [12] both to ease the comparison and also to shorten the presentation by just referring most of the tedious definitions that are helpfully spelled out nicely there.

Definition 5.1.1 A permutative $\text{Sp}^\Sigma$-category is a monoid $(\mathcal{C}, \oplus, 0)$ in the cartesian category of $\text{Sp}^\Sigma$-categories together with a $\text{Sp}^\Sigma$-natural isomorphism $\gamma_{a,b}: a \oplus b \to b \oplus a$ such that

- $\gamma_{a,b}^2 = \text{id}_{a \oplus b}$
- $\gamma_{0 \oplus a} = \text{id}_a$ and
- $\gamma_{a \oplus b, c} = (\gamma_{a, c} \oplus \text{id}_b)(\text{id}_a \oplus \gamma_{b, c})$.

This definition is slightly subtle. The composition in the underlying $\text{Sp}^\Sigma$-category uses the smash product in $\text{Sp}^\Sigma$, but the monoidal operation $\oplus$ uses the categorical product in $\text{Sp}^\Sigma$: it is a $\text{Sp}^\Sigma$-functor from $\mathcal{C} \times \mathcal{C}$, not from $\mathcal{C} \wedge \mathcal{C}$. The reason is that we are interested in having the categorical coproduct in $\mathcal{C}$ (if it exists) as an example.

Definition 5.1.2 Let $\mathcal{C}$ and $\mathcal{D}$ be permutative $\text{Sp}^\Sigma$-categories. A lax map from $\mathcal{C}$ to $\mathcal{D}$ is a $\text{Sp}^\Sigma$-functor $f: \mathcal{C} \to \mathcal{D}$ together with a $\text{Sp}^\Sigma$-natural transformation $\phi: \oplus (f \times f) \Rightarrow f \oplus$ such that the underlying structure on categories define a lax map of permutative categories in the sense of [12, 3.1]. Similarly, a strong map is a lax map $(f, \phi)$ such that $\phi$ is a $\text{Sp}^\Sigma$-isomorphism. We let $\text{Sp}^\Sigma\text{Prm}^{\text{stg}}$ be the resulting category of permutative $\text{Sp}^\Sigma$-categories and strong maps.

A lax $\text{Sp}^\Sigma$-natural transformation between two lax maps $(f_1, \phi_1), (f_2, \phi_2): \mathcal{C} \to \mathcal{D}$ is a $\text{Sp}^\Sigma$-natural transformation $\eta: f_1 \Rightarrow f_2$ defining a lax natural transformation on the underlying lax maps. The $\text{Sp}^\Sigma$-natural transformation $\eta$ is a $\text{Sp}^\Sigma$-natural isomorphism if $\eta$ is a $\text{Sp}^\Sigma$-isomorphism.

More generally, if $\mathcal{C}_1, \ldots, \mathcal{C}_k$ and $\mathcal{D}$ are symmetric monoidal $\text{Sp}^\Sigma$-categories, we let

$$\text{Sp}^{\Sigma\text{Prm}^{\text{stg}}}_k(\mathcal{C}_1, \ldots, \mathcal{C}_k; \mathcal{D})$$

be the category defined as in [12, 3.2] with the following minor adjustments. An object consists of
a $\text{Sp}^\Sigma$-functor $f : C_1 \wedge \ldots \wedge C_k \to D$ and a collection of $\text{Sp}^\Sigma$-natural distributivity isomorphisms

\[
C_1 \wedge \ldots \wedge (C_i \times C_i) \wedge \ldots \wedge C_k \xrightarrow{id \wedge \oplus \wedge id} C_1 \wedge \ldots \wedge C_k \quad \text{and} \quad (C_1 \wedge \ldots \wedge C_k) \times (C_1 \wedge \ldots \wedge C_k) \xrightarrow{\delta_i} \quad f
\]

for $1 \leq i \leq k$, satisfying the conditions of [12, p. 173]. The morphism symmetric spectra are predictably given by formulating the condition on the top of page 174 of [12] as an equalizer.

**Remark 5.1.3** Apart from the enrichment, the difference from [12] is that we require the distributivity transformations $\delta_i$ to be isomorphisms. Letting the $\delta_i$s be arbitrary maps can mess up the enrichment, and to ensure the homotopy type of the Segal construction given in the next section, at the very least we would need $\delta_i$ to be a $\text{Sp}^\Sigma$-natural transformation that induces $\pi_*$-isomorphisms on morphism spectra.

Repeating the work done by Elmendorf and Mandell we get the expected result.

**Lemma 5.1.4** Definition 5.1.2 endows the category $\text{Sp}^\Sigma \text{Prm}^{\text{stg}}$ with a structure of a multicategory.

**Definition 5.1.5** An $\text{Sp}^\Sigma$ EM-bipermutative category is a multifunctor $E \Sigma \to \text{Sp}^\Sigma \text{Prm}^{\text{stg}}$.

An important source of examples are the rig categories with cofibrations and weak equivalences of Definition 4.2.1. Noting that the $\text{Sp}^\Sigma$-enrichment does in no way interfere with the proof, the following is a special case of [12, 3.8].

**Lemma 5.1.6** Let $\mathcal{R}$ be a symmetric rig category with cofibrations and weak equivalences. Then $\text{Strig}\mathcal{R}$ has the structure of a $\text{Sp}^\Sigma$ EM-bipermutative category.

## 5.2 The $\vec{H}$-construction

The iterated Segal construction gives a convenient model for the algebraic K-theory of a permutative $\text{Sp}^\Sigma$-category, and again we choose to import notation from [12], commenting on the few cases where the $\text{Sp}^\Sigma$-enrichment is of interest.

If $\mathcal{C}$ is a small permutative $\text{Sp}^\Sigma$-category and $A_1, \ldots, A_n$ are finite based sets, then the set $\text{ob}\mathcal{C}(A_1, \ldots, A_n)$ is the set defined in [12, 4.4] with the small modification that the structure maps are required to be isomorphisms. This is a matter of convenience, and we prefer this version since it allows the construction to serve as input also to functors that depend not only on the weak homotopy type of the result of composing with a nerve-construction. In slightly more detail, an object is a “system” $(C, \rho) = \{C(s), \rho(s); i, T, U\}$, with $C(s) \in \text{ob}\mathcal{C}$, where $\langle S \rangle = \langle S_1, \ldots, S_n \rangle$ runs through all $n$-tuples of basepoint-free subsets $S_i \subset A_i$, and $\rho(\langle S \rangle; i, T, U)$ is an isomorphism

\[
C_{\langle S_1, \ldots, T, \ldots, S_n \rangle} \oplus C_{\langle S_1, \ldots, U, \ldots, S_n \rangle} \to C_{\langle S_1, \ldots, S_{i-1}, S_i, S_{i+1}, \ldots, S_n \rangle} = C(s),
\]

for $T \cup U = S_i$, $T \cap U = \emptyset$ (with $T$ and $U$ inserted at the $i$th slot), satisfying certain conditions. The symmetric spectrum of maps from an object $(C, \rho)$ to another $(C', \rho')$ is given by the equalizer
in symmetric spectra
\[ \overline{\mathcal{C}}(A_1, \ldots, A_n)((C, \rho), (C', \rho')) \to \prod_{(S)} C(C_{(S)}, C'_{(S)}) \Rightarrow \prod_{(S):i,T,U} C(C_{(S_1, \ldots, T, \ldots, S_n)}, C'_{(S_j)}) \]
as defined for discrete categories at the bottom of page 183 in [12].

Requiring the \( \rho_s \) to be isomorphism has the pleasant effect of making the construction “special” in the \( \Gamma \)-space terminology: everything is determined by the \( C_{(S)} \)'s with \( S \) being an \( n \)-tuple of singletons.

**Lemma 5.2.1** There is an \( \text{Sp}^\Sigma \)-equivalence of categories
\[ \overline{\mathcal{C}}(k_1^1, \ldots, k_n^n) \to C^{k_1 \times \cdots \times k_n}, \]
sending an object \( \{C_{(S)}, \rho_{(S):i,T,U}\} \) to the tuple \( (C_{(\{i_1\} \ldots \{i_n\}))}_{ij \in k^j} \) and restricting morphisms to these objects.

This has important consequences, of which the following corollary is an example.

**Corollary 5.2.2** If the morphism spectra of \( \mathcal{C} \) all share a property that is closed under finite products, like being semistable or \( \Omega \)-spectra, the morphism spectra of \( \overline{\mathcal{C}}(k_1^1, \ldots, k_n^n) \) also have this property.

Varying \( (A_1, \ldots, A_n) \) one allows \( \overline{\mathcal{C}} \) also to have inputs finite pointed simplicial sets (take the diagonal to avoid the number of simplicial directions to depend on \( n \)). Also allowing permutations of the input and extensions of the type \( \overline{\mathcal{C}}(A_1, \ldots, A_n) \to \overline{\mathcal{C}}(A_1, \ldots, A_n, 0^+) \) as on page 184-198 of [12] one gets a multifunctor
\[ \overline{H} : \text{Sp}^\Sigma \text{Prm}^{stg} \to [S^\Sigma, \text{Sp}^\Sigma \text{-} \text{Cat}_{\Delta^o}], \]
with \( n \)th level \( \overline{H}(C)_n = \overline{\mathcal{C}}(S^n, \ldots, S^n) \), analogous to Elmendorf and Mandell’s \( K^{\text{new}} \). For a given \( \mathcal{C} \), the underlying spectrum \( HC \) of \( \text{Sp}^\Sigma \)-categories is exactly the spectrum you get by iterating Segal’s \( \Gamma \)-space machine and evaluating on circles. We will refer to the multifunctor \( \overline{H} \) as the (iterated) Segal construction.

### 5.3 Permutative \( \text{Sp}^\Sigma \)-categories with weak equivalences

This section is not used in later, but is included for completeness, since for some applications the freedom it offers is of value. For simplicity (and necessity in the application to our tiny category of free modules) we have chosen a model where the weak equivalences are extracted from the enrichment, see Section 5.4.

We consider \( \text{Sp}^\Sigma \)-categories with cofibrations and weak equivalences to have an ordinary category of weak equivalences, and for most applications this is good enough. On reason this works so well is that Waldhausen’s \( S \)-construction knows how to deal with nonsplit extensions, so in the presence of a cylinder functor we immediately get that the suspension represents “multiplication by \(-1\)”. For the Segal construction it is crucial to work with \( S \)-categories of weak equivalences. This represents little extra complexity.

Just as we talked about permutative \( \text{Sp}^\Sigma \)-categories, we may talk about permutative \( S \)-categories; as a matter of fact, a permutative \( S \)-category can be viewed as a special kind of simplicial permutative category where the simplicial structure maps on the object level are identities. The
evaluation on the zero sphere $R: \text{Sp}^\Sigma \to \mathcal{S}$ (please forget the basepoint) applied to morphism spectra gives a multifunctor from the category of permutative $\text{Sp}^\Sigma$-categories to the category of permutative $\mathcal{S}$-categories.

Definition 5.3.1 A permutative $\text{Sp}^\Sigma$-category with weak equivalences consists of a permutative $\text{Sp}^\Sigma$-category $\mathcal{C}$ together with a permutative $\mathcal{S}$-subcategory $\mathcal{W} \subseteq \mathcal{R}\mathcal{C}$ containing all isomorphisms and satisfying the two-out-of-three property. A map of permutative $\text{Sp}^\Sigma$-category with weak equivalences consists of a strong map $\mathcal{C}_1 \to \mathcal{C}_2$ that restricts to a map on the weak equivalences. We let $\text{Sp}^\Sigma\text{Prm}^{\text{stg},w}$ be the resulting multicategory of permutative $\text{Sp}^\Sigma$-categories with weak equivalences.

The Segal construction applies equally well to permutative $\mathcal{S}$-categories, and if $(\mathcal{C}, \mathcal{W})$ is a permutative $\text{Sp}^\Sigma$-category with weak equivalences, we get a pair $(\bar{\mathcal{H}}\mathcal{C}, \bar{\mathcal{H}}\mathcal{W}) \in [\text{S}^\Sigma, \mathcal{P}\Delta^o]$. Composing with the multifunctor $\mathcal{P}_R \to \mathcal{P}\Delta^o$ of Lemma 3.3.3 and the taking the diagonal we get a multifunctor

$$\text{Sp}^\Sigma\text{Prm}^{\text{stg},w} \to [\text{S}^\Sigma, \mathcal{P}\Delta^o],$$

which could be considered to be the Segal construction for permutative $\text{Sp}^\Sigma$-category with weak equivalences.

5.4 A uniform choice of weak equivalences

The “uniform choice of weak equivalences” is a composite of the two multifunctors, and gives the possibility of selecting a $\mathcal{S}$-subcategory of weak equivalences directly from a $\text{Sp}^\Sigma$-enrichment. The first is the stabilization functor $T_0$ of Section 3.4, which is an endo-multifunctor on the category of symmetric spectra in pointed simplicial $\text{Sp}^\Sigma$-categories, and the second is the multifunctor $\omega: \text{Sp}^\Sigma\text{-Cat}_* \to \mathcal{P}_R$ of Example 3.3.1. The reader will notice that we use the spectrum direction of our constructions of K-theory to stabilize morphism spaces without running into Lewis’ paradox with respect to commutative topological monoids.

Definition 5.4.1 Let $\mathcal{C}$ be a symmetric spectrum in pointed simplicial $\text{Sp}^\Sigma$-categories. Then $\tilde{\omega}\mathcal{C} \in [\text{S}^\Sigma, \text{Sp}^\Sigma\text{-Cat}^\Delta_*]$ is the result of applying the diagonal and the multifunctor $\bar{\mathcal{P}}_R \to \mathcal{P}\Delta^o$ of 3.3.3 to $\bar{\omega}T_0T_0\mathcal{C}$, where $\bar{\omega}$ is the multifunctor introduced in 3.3.1.

Definition 5.4.2 The composite multifunctor

$$\tilde{\omega}\bar{\mathcal{H}}: \text{Sp}^\Sigma\text{Prm}^{\text{stg}} \to [\text{S}^\Sigma, \mathcal{P}\Delta^o],$$

where $\tilde{\omega}$ was defined in 5.4.1, is referred to as the Segal construction with uniform choice of weak equivalences.

Lemma 5.4.3 If $\mathcal{C}$ is a permutative $\text{Sp}^\Sigma$-category with connective morphism spectra, then the positive levels of $\tilde{\omega}\bar{\mathcal{H}}\mathcal{C}$ satisfy property LP.

Proof: Follows from Lemma 3.3.5 (where the connectivity is used) and Shipley’s result that $T_0E$ of a symmetric spectrum $E$ is semistable and $T_0T_0E$ is a stably fibrant. 

Remark 5.4.4 The Segal construction with uniform choice of weak equivalences of a permutative $\text{Sp}^\Sigma$-category can detect weak equivalences that seem to be inaccessible from the category itself. More precisely, there may be maps in the stabilized categories $T_0^2\mathcal{C}$ that ought to count as weak equivalences, but which do not actually come from maps in $\mathcal{C}$. These will be detected as we move in the spectrum direction of the Segal construction.


6 The homotopy nerve

A crucial part of the cyclotomic trace in the current setup is that at a certain stage a nerve has to be “deconstructed”. The amount of difficulty this will entail depends on the generality of the weak equivalences allowed. For exact categories, where the relevant notion of weak equivalences is that of isomorphisms, this is simple. It essentially boils down to the following observation: if $G$ is a groupoid, any natural transformation $\eta: f \Rightarrow g$ is of functors $I \to J$ induces a natural *isomorphism*

$$[J, G] \xrightarrow{f^* \eta} [I, G],$$

which in turns means that if $h: J \to I$ is such that the identities of $I$ and $J$ are connected by natural transformations to $hf$ and $fh$, then $f^*$ is an equivalence of categories. The example in question is when $f$ is the canonical functor $[q] \to [0]$: the induced functor $G \cong [[0], G] \to [[q], G]$ is an equivalence of categories.

6.1 The nerve as a multifunctor

The nerve is often thought of as a functor from small categories to spaces. However, the notion we need to generalize is the functor

$$N: \text{Cat} \to \text{Cat}^{\Delta^o}$$

given by setting $N_qC$ to be the functor category $[[q], C]$. This functor is strong symmetric monoidal with respect to the product, and so defines a multifunctor. The simplicial enrichment deserves some mention; a functor $C \times [q] \to D$ (i.e., a string of $q$ natural transformations) is taken to $NC \times N[q] \cong N(C \times [q]) \to N(D)$, and the two inclusions $[q] \subseteq N[q]$ (inclusion of the zero simplices) and $\Delta[q] = \text{ob}N[q] \subseteq N[q]$ shows that the nerve is simplicially enriched under either of the two usual enrichments of $\text{Cat}^{\Delta^o}$ (set of $q$-simplices of maps from $X$ to $Y$ are either $\text{Cat}^{\Delta^o}(X \times [q], Y)$ or $\text{Cat}^{\Delta^o}(X \times \Delta[q], Y)$).

If $\mathcal{V} = (\mathcal{V}, \otimes, e)$ is a small closed category with pullbacks, this construction extends to give a nerve

$$N^\mathcal{V}: \mathcal{V}\text{-Cat} \to (\mathcal{V}\text{-Cat})^{\Delta^o},$$

where $\text{ob}N_q^\mathcal{V}C = \text{ob}N_qUC = \{\text{functors } c: [q] \to UC\} = \{c_0 \leftarrow \cdots \leftarrow c_q \in UC\}$, and where the morphism objects are given by the $\mathcal{V}$-natural transformations

$$N^\mathcal{V}_qC(c, d) = \int_{0 \leq i, j \leq q} \mathcal{C}(c_i, d_j) = \lim \mathcal{C}(c_0, d_0) \to \mathcal{C}(c_1, d_0) \leftarrow \mathcal{C}(c_1, d_1) \to \cdots \leftarrow \mathcal{C}(c_q, d_q).$$

The monoidal structure on (simplicial) $\mathcal{V}$-categories of Example 3.1.9 makes $N^\mathcal{V}$ lax monoidal: forgetting down to objects this is the strong monoidality of the ordinary nerve, and on morphism level the map from

$$(N_q^\mathcal{V}C_1) \otimes (N_q^\mathcal{V}C_1)((c^1, c^2), (d^1, d^2)) = (N_q^\mathcal{V}C_1(c^1, d^1)) \otimes (N_q^\mathcal{V}C_2(c^2, d^2)) = \int_{i \leq j} C_1(c^1_i, d^1_j) \otimes \int_{k \leq l} C_2(c^2_k, d^2_l)$$

to

$$\int_{i \leq j, k \leq l} C_1(c^1_i, d^1_j) \otimes C_2(c^2_k, d^2_l)$$
given by the universality of the end, followed by the restriction to the diagonal indexing
\[
\int_{m \leq n} C_1(c^1_m, d^n_1) \otimes C(c^2_m, d^n_2) = N_q^{V}(C_1 \otimes C_2)((c^1, c^2), (d^1, d^2))
\]
gives the lax structure map \((N_q^{V}C_1) \otimes (N_q^{V}C_1) \to N_q^{V}(C_1 \otimes C_2)\). In particular \(N^V: \mathcal{V} \text{-Cat} \to (\mathcal{V} \text{-Cat})^{\Delta^v}\) is a multifunctor (with simplicial enrichments handled as above).

We will need an extension of this in two directions. First we are interested in \(\mathcal{V}\)-full subcategories \(N_q^{V}(C, W)\) of \(N_q^{V}C\) where the objects belong to the nerve of some subcategory (of “weak equivalences”) \(W \subseteq UC\), and secondly we need to replace the limits defining the morphism objects by some more homotopy responsible constructions (but with a well defined composition). We will mostly be interested in the case \(\mathcal{V} = \text{Sp}^S\), and we will stop decorating the nerve by the enrichment.

6.2 The homotopy nerve

We learned about the existence of the homotopy nerve from Blumberg and Mandell, and it appears in \([3]\). With the current application in mind we worked out the details in \([8, \text{V.4}]\), and we will refer away much of the technicality.

Recall from \([8, \text{V.4}]\) that if \(Y\) is a topological space and \(0 \leq i \leq j \leq n\), then the space of “Moore singular simplices” \(P^{i \leq j}Y\) consisting of pairs \((r, f)\) where \(r\) is a non-negative real number and \(f\) is a continuous function from \(\Delta^n_r = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | x_k \geq 0, \sum_{k=0}^n x_k = \sum_{k=i}^j x_k = r\}\) to \(Y\).

The Moore singular simplices fit together to give strictly associative versions of homotopy pullbacks. In particular, if \(C\) is a \(S\)-category, then we can define the homotopy nerve \(\text{hoNC}\) to be the simplicial \(S\)-category with the same objects as the usual nerve, but with morphism space from \(c\) to \(d \in \text{ob} N_C\) given by
\[
\text{hoNC}(c, d) = \lim_{\leftarrow i\leq j \leq k \leq l} P^{i \leq j \leq k \leq l}_n |C(c_i, d_i)|.
\]

Quite concretely, \(\text{hoNC}(c, d)\) is a model for the homotopy limit of the diagram
\[
C(c_0, d_0) \to C(c_1, d_0) \leftarrow C(c_1, d_1) \rightarrow \cdots \leftarrow C(c_q, d_q)
\]
used to define the usual nerve, but the size of the involved simplices is “remembered not fixed”, just as in Moore loops, allowing a strictly associative composition.

We will need the following strengthening of \([3, \text{Lemma V.4.1.10}]\). New feature is the use of right properness to see that it is enough that just the morphism moving the target give weak equivalences. This is the core of the usefulness of the property \(P\).

**Lemma 6.2.1** Given an \(S\)-category \(C\) and \(c, d \in \text{ob hoN}_q C\) with the property that for each \(i\) and \(j \leq k \in [n]\) the map \(C(c_i, d_k) \to C(c_i, d_j)\) is a weak equivalence. Then the projection \(\text{hoN}_q C(c, d) \to \text{hoN}_0 C(c_0, d_0)\) is a weak equivalence.
Proof: In the diagram defining \( \text{ho}N_n \mathcal{C}(c,d) \), use that the projections \( P_i^{s,t} Y \to P_{s,t} Y \) are acyclic fibrations (cf. [S V.4.1.8]) for \( i \leq j \leq k \leq l \) and use right properness in \( \mathcal{S} \). For \( n = 2 \) the diagram takes the form

\[
\begin{array}{cccccc}
P^{00} \mathcal{C}(c_0, d_0) & \xrightarrow{=} & P^{01} \mathcal{C}(c_1, d_0) & \xrightarrow{=} & P^{11} \mathcal{C}(c_1, d_1) & \xrightarrow{=} \ \ x \\
& & \downarrow & & \downarrow & \ \\
P^{00} \mathcal{C}(c_2, d_0) & \xrightarrow{=} & P^{01} \mathcal{C}(c_2, d_0) & \xrightarrow{=} & P^{11} \mathcal{C}(c_2, d_1) & \xrightarrow{=} \ \ x \\
& & \downarrow & & \downarrow & \ \\
P^{02} \mathcal{C}(c_2, d_0) & \xrightarrow{=} & P^{12} \mathcal{C}(c_2, d_0) & \xrightarrow{=} & P^{22} \mathcal{C}(c_2, d_1) & \xrightarrow{=} \ \ x \\
& & \downarrow & & \downarrow & \ \\
P^{22} \mathcal{C}(c_2, d_2) & \xrightarrow{=} & P^{22} \mathcal{C}(c_2, d_2) & \xrightarrow{=} & P^{22} \mathcal{C}(c_2, d_1) & \xrightarrow{=} \ \ x \\
& & & & & \downarrow \\
& & & & & P^{22} \mathcal{C}(c_2, d_2)
\end{array}
\]

(realization and \( \leq \) suppressed).

Again we see that the homotopy nerve is lax monoidal with structure map

\[
(\text{ho}N_1 \mathcal{C}) \times (\text{ho}N_2 \mathcal{C}) \to \text{ho}N(\mathcal{C}_1 \times \mathcal{C}_2)
\]

induced by the monoidality of the Moore simplices (cf. [S V.4.1.4]: for \( i \leq j \leq k \leq l \), addition embeds \( \Delta^n_{k}(k \leq l) \times \Delta^n_{i}(i \leq j) \) in \( \Delta^n_{j+k}(i \leq l) \)).

Similarly we get a lax monoidal homotopy nerve of small \( \mathcal{S}_\bullet \)-categories, dividing appropriately out by the contractible space \( A_n = \lim_{\leq i \leq k \leq l} \sin P^i_{n} \) on the morphism spaces in order for accommodate for the pointed composition:

\[
\text{ho}N_n \mathcal{C}(c,d) = (\lim_{\leq i \leq k \leq l} \sin P^j_{n})/A_n
\]

(unsing that composition with the basepoint map always gives the basepoint map). Extending to enrichment over symmetric spectra represents no difficulty: if \( \mathcal{C} \) is a small \( \text{Sp}^{\Sigma} \)-category and \( c,d \in \text{ob}N \mathcal{C}_q \), then \( \text{ho}N_n \mathcal{C}(c,d) \) is on level \( s \) simply given by \( (\text{ho}N_n \mathcal{C}(c,d))(S^s) = \text{ho}N_n \mathcal{C}(c,d)(S^s) \) (the pointed version). Again, the monoidality of the Moore singular simplices give us an associative and unital composition \( \text{ho}N_n \mathcal{C}(c,d)(S^s) \wedge \text{ho}N_n \mathcal{C}(c,d)(S^t) \to \text{ho}N_n \mathcal{C}(c,d)(S^s \wedge S^t) \).

With no essential difference, we get that \( \text{ho}N \) becomes a lax symmetric monoidal functor from the category of small \( \text{Sp}^{\Sigma} \)-categories to the category of simplicial small \( \text{Sp}^{\Sigma} \)-categories.

Similarly, if \( (\mathcal{C}, w: \mathcal{W} \to \mathcal{U} \mathcal{C}) \) is a \( \mathcal{U} \)-pair, then \( \text{ho}N(\mathcal{C}, w) \) is the simplicial \( \mathcal{U} \)-pair consisting of the nerve \( NW \), together with the simplicial \( \text{Sp}^{\Sigma} \)-category which in degree \( n \) is the full \( \text{Sp}^{\Sigma} \)-subcategory \( \text{ho}N^w \mathcal{C} \) of \( \text{ho}N_n \mathcal{C} \) with objects in the image of \( \text{ob}N_n \mathcal{W} \to \text{ob}N_n \mathcal{U} \mathcal{C} = \text{ob} \text{ho}N_n \mathcal{C} \).

**Definition 6.2.2** The *homotopy nerve* of \( \mathcal{U} \)-pairs is the multifunctor \( \text{ho}N: \mathfrak{P}_U \to (\mathfrak{P}_U)^{\Delta^0} \).

Again, in this definition the simplicial enrichment is ambiguous on purpose, since we will use the same notation with respect to the enrichment via natural transformations and via Quillen’s enrichment of simplicial objects.

**Theorem 6.2.3** If the pair \( (\mathcal{C}, w) \in \mathfrak{P}_U \) has property LP (c.f. [3.3.4]), then the face maps \( \text{ho}N^w \mathcal{C} \to \text{ho}N^w \mathcal{C} \) are stable equivalences in the sense of [3.2.2].
Proof: Let \((\mathcal{C}, w : \mathcal{W} \to UC)\) have property LP. Let \(c = \{c_0 \leftarrow c_1 \leftarrow \cdots \leftarrow c_n\}\) be an arbitrary object of \(\text{ho}N_n\mathcal{C}\) and let \(w = \{w_0 \leftarrow w_1 \leftarrow \cdots \leftarrow w_n\}\) be in the image of \(\text{ob}N_n\mathcal{W} \to \text{ob}N_n\mathcal{U}C = \text{ob}\text{ho}N_n\mathcal{C}\). Now, for a given level \(m\), the maps \(\mathcal{C}(c_i, w_j)_m \to \mathcal{C}(c_i, k)_m\) are weak equivalences, and so the argument of Lemma 6.2.1 gives exactly that the projection

\[ \text{ho}N_n\mathcal{C}(c, w)_m \to \text{sin} \left| P^{0=0}\mathcal{C}(c_0, w_0)_m \right| = \text{ho}N_0\mathcal{C}(c_0, w_0)_m \]

are stable (even level) equivalences. The second condition for the face map to be a stable equivalence is obviously satisfied: if \(c'\) is an object of \(\text{ho}N_0\mathcal{C}\) (i.e., an object of \(\mathcal{C}\) in the image of \(w\)), then \(c' = \cdots = c'\) is an object of \(\text{ho}N_0 w\mathcal{C}\) being sent (identically) to \(c'\).

\[ \square \]

### 6.3 Algebraic K-theory

We are now ready for the official definitions of algebraic K-theory, appropriately enriched to allow as inputs in functors like topological cyclic homology. Note that there are many possible variants. We have chosen two that seem to be representative and also suit our present needs. The reader is invited to modify these definitions to fit her applications.

**Definition 6.3.1** If \(\mathcal{C} = (\mathcal{C}, w\mathcal{C}, co\mathcal{C})\) is an \(\text{Sp}^\Sigma\)-category with cofibrations and weak equivalences, then \(K(\mathcal{C})\) is defined to be the simplicial symmetric spectrum in small pointed categories \(\text{ho}NS\mathcal{C}\). The associated symmetric spectrum \(\text{ob} K(\mathcal{C})\) is the *algebraic K-theory spectrum of \(\mathcal{C}\).

If \(\mathcal{C}\) is a permutative \(\text{Sp}^\Sigma\)-category, then \(K(\mathcal{C})\) is defined to be the simplicial symmetric spectrum in small pointed categories \(\text{ho}NS\mathcal{C}\). The associated symmetric spectrum \(\text{ob} K(\mathcal{C})\) is the *algebraic K-theory spectrum of \(\mathcal{C}\) (with the uniform choice of equivalences).

By the fact that both these definitions of algebraic K-theory results as a composite of multi-functors we get.

**Proposition 6.3.2** Let \(\mathcal{C}\) be an \(\text{Sp}^\Sigma\)-category with cofibration and weak equivalences with an exact monoidal structure; or a permutative \(\text{Sp}^\Sigma\)-category. Then \(\text{ob} K(\mathcal{C}, w)\) is a symmetric ring spectrum, which is \(E_\infty\) if \(\mathcal{C}\) is. More generally, if \(\mathcal{C}\) is an \(\mathcal{O}\)-algebra, then so is \(\text{ob} K(\mathcal{C}, w)\).

This should also be compared with [13, 6.1.1].

**Note 6.3.3** Note that the construction shrinks considerably when taking the objects. For instance, if \(\mathcal{C} = (\mathcal{C}, w\mathcal{C}, co\mathcal{C})\) is an \(\text{Sp}^\Sigma\)-category with cofibrations and weak equivalences, then

\[ \text{ob} K(\mathcal{C}) = \text{ob} \text{ho}NS\mathcal{C} = \{n \mapsto \text{ob}NwS^{(n)}UC\}, \]

where \(\text{ob}NwS^{(n)}UC\) is Waldhausen’s construction applied to the underlying category with cofibrations and weak equivalences \((UC, w\mathcal{C}, co\mathcal{C})\) (no \(\text{Sp}^\Sigma\)-enrichment and the categorical nerve).

Similarly with the Segal construction: if \(\mathcal{C}\) is a permutative \(\text{Sp}^\Sigma\)-category, then if we let \(w\mathcal{C}\) be the \(S\)-subcategory \(R\tilde{\omega}T_0T_0\mathcal{C}\) that become invertible in \(\pi_0R\tilde{\omega}T_0T_0\mathcal{C} = \pi_0\mathcal{C}\), then \(\text{ob} \text{ho}NS\mathcal{C}\) is a group completion of \(\text{ob}Nw\mathcal{C}\), simply because \(\tilde{H}\) gives special output (in the sense of \(\Gamma\)-objects), \(T_0\) preserves products and \(T_0T_0E \to M_0M_0E \leftarrow E \to M_0^aM_0^aE \leftarrow T_0^aT_0^aE\) is a natural chain of stable equivalences for any \(n > 0\).
7 Bispectra

We will need a few basic facts about bispectra. These can presumably be adapted from published sources, but since we need access to the inner combinatorics we spell out the details we need explicitly.

7.1 The building blocks

**Definition 7.1.1** Let $\mathcal{N} \subseteq \mathcal{S}$ be the $\mathcal{S}$-subcategory with objects $S^0, S^1, S^2, \ldots$ (with $S^n = S^1 \land \ldots \land S^1$) and with morphism spaces

$$\mathcal{N}(S^m, S^n) = \begin{cases} * & \text{if } n < m \\ S^{n-m} & \text{if } n \geq m. \end{cases}$$

The inclusion $\mathcal{N} \subseteq \mathcal{S}$ is given by the identity on objects and on morphisms it is given by the inclusion

$$\mathcal{N}(S^m, S^n) = S^{n-m} \to \mathcal{S}(S^m, S^m \land S^{n-m}) \cong \mathcal{S}(S^m, S^n)$$

when $m \leq n$, and where the map is the adjoint of the identity on $S^m \land S^{n-m}$.

Consider a factorization of $\mathcal{N} \subseteq \mathcal{S}$ through a symmetric monoidal $\mathcal{S}$-functor $\mathcal{J} \to \mathcal{S}$, where $(\mathcal{J}, \land, S^0)$ is a skeletonally small symmetric monoidal $\mathcal{S}$-category. Let $\mathcal{O}$ be a set of objects in $\mathcal{J}$.

Of particular importance is $\mathcal{J} = S^2$ (the smallest symmetric monoidal $\mathcal{S}$-subcategory of $\mathcal{S}$ containing $\mathcal{N}$), defined in giving symmetric spectra. Another important example is $\mathcal{J} = f \mathcal{S}$, the category of finite pointed simplicial sets, giving rise to the so-called simplicial functors.

Notice that, even though $\mathcal{J}$ is an $\mathcal{S}$-category, it is not pointed, so when we write $\mathcal{J} \land \mathcal{J}$ we mean the $\mathcal{S}$-category with set of objects $\text{ob} \mathcal{J} \times \text{ob} \mathcal{J}$, but with morphism spaces $(\mathcal{J} \land \mathcal{J})(i, j), (i', j') = \mathcal{J}(i, i') \land \mathcal{J}(j, j')$.

**Definition 7.1.2** A $\mathcal{J}$-bispectrum is an $\mathcal{S}$-functor $\mathcal{J} \land \mathcal{J} \to \mathcal{S}$, that is, an object in $[\mathcal{J} \land \mathcal{J}, \mathcal{S}]$.

Precomposition with the smash $\mathcal{J} \land \mathcal{J} \to \mathcal{J}$ gives an $\mathcal{S}$-adjoint pair

$$[\mathcal{J}, \mathcal{S}] \xrightarrow{L} [\mathcal{J} \land \mathcal{J}, \mathcal{S}]$$

with $UX(R, S) = X(R \land S)$ and $LY(T) = \int^{(R, S)} \mathcal{J}(R \land S, T) \land Y(R, S)$.

**Definition 7.1.3** If $i \in \mathcal{J}$ precomposition with the inclusion $in_\bullet : \mathcal{J} \to \mathcal{J} \land \mathcal{J}$ sending $j \in \mathcal{J}$ to $(i, j)$ gives an $\mathcal{S}$-adjoint pair

$$[\mathcal{J}, \mathcal{S}] \xrightarrow{in_\bullet} [\mathcal{J} \land \mathcal{J}, \mathcal{S}]$$

with $in_\bullet X(j) = X(i, j)$ and $in_\bullet Y(j, k) = \mathcal{J}(i, j) \land Y(k)$. Likewise for the inclusion to the other factor $in_{\bullet j}$, giving an adjoint pair $(in_{\bullet j}, in_{\bullet j}^*)$.

The special situation $i = S^0$ is used so often so we allow the more memorable name $(\Sigma^\infty, R)$ for $(in_{S^0 \bullet}, in_{S^0 \bullet}^*)$. In Section we also used $\Sigma_i^\infty$ for $in_{\bullet S^0}$.

**Definition 7.1.4** Let $j \in \mathcal{J}$ and for $i = 0, 1$, let $i_v : S^0 \to I_+$ be the inclusion of the $v$th vertex.

1. If $c : K \to L \in \mathcal{S}$, let $c_j$ be the induced functor $1 \land c : \mathcal{J}(j, -) \land K \to \mathcal{J}(j, -) \land L$. 31
2. Let $k_j: \mathcal{J}(j \wedge S^1, -) \wedge S^1 \rightarrow M_j$ be the mapping cylinder factorization of the evaluation $\mathcal{J}(j \wedge S^1, -) \wedge S^1 \rightarrow \mathcal{J}(j, -)$ (in detail, $M_j$ is the pushout of $\mathcal{J}(j \wedge S^1, -) \wedge S^1 \xrightarrow{1 \wedge i_1} \mathcal{J}(j \wedge S^1, -) \wedge S^1 \xrightarrow{ev} \mathcal{J}(j, -)$, and $k_j$ is induced by $i_0$).

3. If $i, j \in \mathcal{J}$ and $c: K \rightarrow L \in \mathcal{S}_s$, let $c_{i,j}$ be the induced functor $c_{i,j} = 1 \wedge c; \mathcal{J}(i, -) \wedge \mathcal{J}(j, -) \wedge K \rightarrow \mathcal{J}(i, -) \wedge \mathcal{J}(j, -) \wedge L$.

4. If $f: X \rightarrow Y$ and $f': X' \rightarrow Y'$ are maps of pointed spaces, the pushout product $f \square f'$ is the canonical map $X \wedge Y' \coprod_{X \wedge X} Y \wedge X' \rightarrow Y \wedge Y'$.

Let $I_s$ and $J_s$ be the standard sets of generating cofibrations and generating acyclic cofibrations.

**Definition 7.1.5** (Generating structure for spectra) Let

$$I = \{c_j \mid j \in \mathcal{O}, c \in I_s\}, \quad J = \{c_j \mid j \in \mathcal{O}, c \in J_s\}, \quad \text{and} \quad K = \{c \square k_j \mid j \in \mathcal{O}, c \in I_s\}.$$  

We call the elements in $I$ the generating $\mathcal{O}$-level cofibrations (but also the generating $\mathcal{O}$-stable cofibrations), the elements in $J$ the generating $\mathcal{O}$-level acyclic cofibrations and the elements in $J \cup K$ the generating $\mathcal{O}$-stable acyclic cofibrations.

**Definition 7.1.6** (Generating structure for bispectra) Let $\mathcal{O}_2 = \mathcal{O} \times \text{ob}\mathcal{J} \cup \text{ob}\mathcal{J} \times \mathcal{O}$ and

$$I_2 = \{c_{i,j} \mid (i, j) \in \mathcal{O}_2, c \in I_s\}, \quad J_2 = \{c_{i,j} \mid (i, j) \in \mathcal{O}_2, c \in J_s\},$$

$$K^r = \{c_i \square k_j \mid (i, j) \in \mathcal{O}_2, c \in I_s\}, \quad K^l = \{k_i \square c_j \mid (i, j) \in \mathcal{O}_2, c \in I_s\}, \quad \text{and} \quad K = K^l \cup K^r.$$  

The elements in $I_2$ and $J_2$ are the generating $\mathcal{O}$-level (acyclic) cofibrations, the elements in $I_2$ and $J_2 \cup K^r$ are the generating right $\mathcal{O}$-stable (acyclic) cofibrations and the elements in $I_2$ and $J_2 \cup K^l$ are the generating $\mathcal{O}$-stable (acyclic) cofibrations.

With appropriate choices for $\mathcal{J}$ and $\mathcal{O}$ and usual naming conventions on level and stable equivalences and (co)fibrations we get cofibrantly generated module structures on spectra and bispectra.

Letting $\mathcal{J} = \mathcal{O} = \mathcal{S}^\Sigma$ we get the correspondence between symmetric spectra, $\text{Sp}^\Sigma = [\mathcal{S}^\Sigma, \mathcal{S}_s]$ and bispectra $[\mathcal{S}^\Sigma \wedge \mathcal{S}^\Sigma, \mathcal{S}_s] \cong [\mathcal{S}^\Sigma, [\mathcal{S}^\Sigma, \mathcal{S}_s]]$.

**Theorem 7.1.7** The pairs $(I, J \cup K)$ and $(I_2, J_2 \cup K_2)$ give a cofibrantly generated model structures on $\text{Sp}^\Sigma = [\mathcal{S}^\Sigma, \mathcal{S}_s]$ and $[\mathcal{S}^\Sigma \wedge \mathcal{S}^\Sigma, \mathcal{S}_s]$ such that both $(\Sigma^\infty, R)$ and $(L, U)$ are Quillen equivalences.

**Proof:** Since $L\Sigma^\infty$ is an isomorphism of categories and Quillen equivalences satisfy the two-out-of-three property it is enough to prove that $\Sigma^\infty$ is a Quillen equivalence. This follows through Hovey’s argument [16] proof of Theorem 5.1.9.1.

With $\mathcal{O} = \{S^1, S^2, \ldots\}$ we get the positive structures on spectra and bispectra. We will not need the full theory, so for brevity we prove just what our applications will need. A map $E \rightarrow F$ of symmetric spectra is a positive level equivalence if for all $n > 0$ the map $E(S^n) \rightarrow F(S^n)$ is a weak equivalence. A bispectrum $X$ is positively stably fibrant if for $(m, n) \neq (0, 0)$ the structure map $X(S^m, S^n) \rightarrow \mathcal{S}_s(S^a \wedge S^b, X(S^m, S^n))$ is a weak equivalence of fibrant spaces. More generally, $X$ is right positively stably fibrant if these structure maps are weak equivalences of fibrant spaces for $n \neq 0$. 

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Lemma 7.1.8 If \( X \) is a right positively stably fibrant bispectrum, then the counit \( \Sigma^\infty RX \to X \) is a stable equivalence.

Proof: Let \( E \) be any symmetric spectrum with cofibrant replacement \( CE \sim \to E \). Since \( \Sigma^\infty \) preserves level equivalences we get that the induced map \( \Sigma^\infty CE \to \Sigma^\infty E \) is a level equivalence. Since \( X \) is right positively stably fibrant, the stably fibrant replacement \( X \to \tilde{X} \) gives a positive level equivalence \( RX \to R\tilde{X} \) which implies that \( \Sigma^\infty RX \to \Sigma^\infty R\tilde{X} \) is also a positive level equivalence. The claim now follows since the fact that \( (\Sigma^\infty, R) \) is a Quillen equivalence implies that \( \Sigma^\infty CR\tilde{X} \to \tilde{X} \) is a stable equivalence.

Bispectra appear in the cyclotomic trace because they are good pivots to exchange different spectral directions as can be seen from the following corollary, which unfortunately is not strong enough for our purposes.

Corollary 7.1.9 On the category of positively stably fibrant bispectra \( X \) there is a natural chain of stable equivalences connecting \( RX \) and \( R(X \circ tw) \), where \( tw: S^\Sigma \wedge S^\Sigma \to S^\Sigma \wedge S^\Sigma \) switches the two factors.

Proof: Since \( X \) is sphere stably fibrant if and only if \( X \circ tw \) is, it is enough to display a natural chain of sphere stable equivalences between \( RX \) and \( LC_2X \), where \( C_2 \) is the sphere cofibrant replacement of \( J \)-bispectra. Here goes:

\[
RX \xleftarrow{\sim} CRX \cong LC_2\Sigma^\infty CRX \xleftarrow{\sim} LC_2\Sigma^\infty RX \Rightarrow LC_2X,
\]

where the first map is the sphere stable cofibrant replacement, the second and third maps are sphere stable equivalences since \( \Sigma^\infty \) and \( L \) are left Quillen functors and the last since \( X \) is positively stably fibrant.

The only problem with this in our situation is that we do not know of symmetric monoidal cofibrant replacement functors. Hence we approximate the left Kan extension \( L \) by another functor \( \mathcal{L} \) with better homotopy properties.

7.2 The average of two spectral directions

Before we give the details of \( \mathcal{L} \) it is convenient to fix some language on cyclic nerves of permutative \( S_* \)-categories.

Definition 7.2.1 Let \( \mathcal{C} \) be an \( S_* \)-category and \( M \) a \( \mathcal{C} \)-bimodule, i.e., an \( S_* \) functor \( M: \mathcal{C}^o \wedge \mathcal{C} \to S_* \). The cyclic nerve of \((\mathcal{C}, M)\) is the space

\[
N_q^{cy}(\mathcal{C}, M) = \bigvee_{a_0, \ldots, a_q \in \text{ob} \mathcal{C}} M(a_0, a_q) \wedge \bigwedge_{i=1}^q \mathcal{C}(a_i, a_{i-1})
\]

with Hochschild-style face and degeneracy maps.

The cyclic nerve obviously defines a functor from the category of \( \mathcal{C} \)-bimodules to spaces. Furthermore, the end \( \int^a M(a, a) \) is nothing but the coequalizer of the two face maps \( N_0^{cy}(\mathcal{C}, M) \rightleftarrows N_1^{cy}(\mathcal{C}, M) \).

We also need a variant of the cyclic nerve, taking more structured input and yielding symmetric spectra as output and having the important feature of giving rise to a lax symmetric monoidal functor.
Definition 7.2.2 Let $\mathcal{C} = (\mathcal{C}, \land, e)$ be a permutative $S_\ast$-category. A $\mathcal{C}$-bimodule with values in $\text{Sp}^\Sigma$ is an $S_\ast$-functor $P: \mathcal{C}^\ast \land \mathcal{C} \to \text{Sp}^\Sigma$. The cyclic nerve of $(\mathcal{C}, P)$ is the simplicial symmetric spectrum $N^\Sigma(\mathcal{C}, P)$ defined as follows. The $q$-simplices of the $n$th level is the space

$$N_q^\Sigma(\mathcal{C}, P)(S^n) = \bigvee_{\{a_{ij}\}_{i \in [q], j \in n}} P(\land_{j=1}^n a_{0,j}, \land_{j=1}^n a_{q,j})(S^n) \land \bigwedge_{i=1}^q \mathcal{C}(\land_{j=1}^n a_{i,j}, \land_{j=1}^n a_{i-1,j}),$$

with the usual simplicial structure and with $S_\ast$-functoriality in $S^n$ through the functoriality of $P$ and through injective maps on the $j$-coordinates, using the permutative structure of $\mathcal{C}$.

Lemma 7.2.3 Let $\mathcal{C} = (\mathcal{C}, \land, e)$ be a permutative $S_\ast$-category. Then, for $n > 0$ the space $N^\Sigma(\mathcal{C}, P(S^n))$ is naturally a simplicial deformation retract of $N^\Sigma(\mathcal{C}, P)(S^n)$.

Proof: Consider the map $r_q: N_q^\Sigma(\mathcal{C}, P)(S^n) \to N_q^\Sigma(\mathcal{C}, P(S^n))$ that sends the $\{a_{ij}\}_{i \in [q], j \in n}$-summand identically to the $\{\land_{j=1}^n a_{ij}\}_{i \in [q]}$-summand. This map is a retract of a map $i_q$ sending the $\{a_{ij}\}_{i \in [q]}$-summand to the $\{a_{ij}\}_{i \in [q], j \in n}$-summand with $a_{i1} = a_i$ and $a_{ij} = e$ for $j > 1$. The maps $r_q$ and $i_q$ gather to simplicial maps $r$ and $i$, and to see that $ir$ is homotopic to the identity one constructs a simplicial homotopy as follows. If $\phi \in \Delta([q], [1])$ has $\phi^{-1}(0) = \{0, \ldots, k\}$, then the $\{a_{ij}\}_{i \in [q], j \in n}$-summand of $N_q^\Sigma(\mathcal{C}, P(S^n))$ is sent to the $\{b_{ij}\}_{i \in [q+1], j \in n}$-summand of $N_{q+1}^\Sigma(\mathcal{C}, P)(S^n)$ via the unit map $S^0 \to C(\land_{j=1}^n a_{kj}, \land_{j=1}^n a_{kj})$, where $b_{ij} = a_{ij}$ for $i \leq k$, and for $i > k$ $b_{i1} = \land_{j=1}^n a_{i-1,j}$ and $b_{ij} = e$ if $j > 1$.

If $\mathcal{C}$ is a permutative $S_\ast$-category and $P$ and $Q$ are two $\text{Sp}^\Sigma$-valued bimodules, we produce a product $N^\Sigma(\mathcal{C}, P) \land N^\Sigma(\mathcal{C}, Q) \to N^\Sigma(\mathcal{C}, P \land Q)$ as follows (where $P \land Q$ is the usual Day product: $(P \land Q)(a, b) = \int_{(a,b_1),(a_2,b_2)} \mathcal{C}^\ast(a_1 \land a_2, a) \land \mathcal{C}(b_1 \land b_2, b) \land P(a_1) \land Q(a_2 \land b_2)$). For $S^m, S^n$ in $\text{Sp}^\Sigma$ and $[q] \in \Delta$ we need to produce a map

$$N_q^\Sigma(\mathcal{C}, P)(S^m) \land N_q^\Sigma(\mathcal{C}, Q)(S^n) \to N^\Sigma(\mathcal{C}, P \land Q)(S^m \land S^n).$$

If we start in the $(\{a_{ij}\}_{i \in [q], j \in m}, \{b_{ij}\}_{i \in [q], j \in n})$ summand on the left hand side, the symmetric monoidal structure of $\mathcal{C}$ induces a map to the $(\{c_{ij}\}_{i \in [q], j \in m \land n})$ summand on the right, with $c_{ij}$ equal $a_{ij}$ for $j \leq m$ and $b_{ij} - m$ if $j > m$, using the structure map

$$P(\land_{j=1}^m a_{q,j}, \land_{j=1}^m a_{0,j})(S^m) \land Q(\land_{j=1}^n b_{q,j}, \land_{j=1}^n b_{0,j})(S^n) \to (P \land Q)(\land_{j=1}^m a_{q,j} \land \land_{j=1}^n b_{q,j}, \land_{j=1}^m a_{0,j} \land \land_{j=1}^n b_{0,j})(S^m \land S^n)$$

of the Day product.

The point of letting the sum in $N_q^\Sigma(\mathcal{C}, P)(S^m)$ run over $m$-tuples of functions $[q] \to \text{ob} \mathcal{C}$ (and then smashing them all together) is to remember what level you are at, leaving enough room for the construction to be symmetric:

Lemma 7.2.4 Let $\mathcal{C}$ be a permutative $S_\ast$-category. The functor $N^\Sigma(\mathcal{C}, -)$ from $\mathcal{C}$-bimodules with values in $\text{Sp}^\Sigma$ to simplicial symmetric spectra is lax symmetric monoidal.

We are particularly interested the case $\mathcal{C} = \Sigma^\ast \land \Sigma^\ast$ and bimodules of the form

$$P_X((a, a'), (b, b'))(S^n) = \Sigma(a \land a', S^n) \land X(b, b'),$$

for $X: \Sigma^\ast \land \Sigma^\ast \to S_\ast$ a bispectrum.

Lemma 7.2.5 Considered as a functor $P$ from bispectra to $\Sigma^\ast \land \Sigma^\ast$-bimodules, $X \mapsto P_X$ is lax symmetric monoidal.
**Definition 7.2.6** Let $\mathcal{L}: [S^\Sigma \wedge S^\Sigma, S_*] \to \text{Sp}^\Sigma$ be the lax symmetric monoidal functor which to a symmetric bispectrum $X$ assigns the symmetric spectrum $\mathcal{L}X = N^\text{cy}(S^\Sigma \wedge S^\Sigma, P_X)$.

**Remark 7.2.7** From Lemma 7.2.3 we get that if $X$ is a bispectrum, then in positive levels the spectrum $LX$ of Definition 7.1.3 is a coequalizer of the two face maps $L_0X \rightrightarrows L_1X$, giving a lax symmetric monoidal natural transformation $u: \mathcal{L} \to L$.

Also, given the form of the $S^\Sigma \wedge S^\Sigma$-bimodule $P_X$ as the smash of the $(S^\Sigma \wedge S^\Sigma)^o$-module $S^\Sigma(\bullet \wedge \bullet, S^n)$ with the $(S^\Sigma \wedge S^\Sigma)$-module $X$, we note that $LX(S^n)$ is equivalent to the (diagonal of the) two-sided bar construction

$$N^\text{cy}(S^\Sigma \wedge S^\Sigma, P_X(S^n)) \cong B(S^\Sigma(\bullet \wedge \bullet, S^n), S^\Sigma \wedge S^\Sigma, X).$$

A way of thinking about this is to consider the two-sided bar construction as a rope of maps in $S^\Sigma$ from one of the variables of $X$ to the other, suspended over the pulley $S^\Sigma(\bullet \wedge \bullet, S^n)$. The next lemma illustrates what happens if one of the slots of $X$ is free to move through the pulley and gives that the isomorphism $L\Sigma^\infty E \cong E$ is mirrored by $\mathcal{L}$.

**Lemma 7.2.8** Let $E$ be a symmetric spectrum. Then the lax symmetric monoidal natural transformation $u_{\Sigma^\infty E}: \mathcal{L}\Sigma^\infty E \to L\Sigma^\infty E \cong E$ is a positive level equivalence.

**Proof:** Rewrite $N^\text{cy}(S^\Sigma \wedge S^\Sigma, S^\Sigma(\bullet \wedge \bullet, S^n) \wedge S^\Sigma(S^0, \bullet) \wedge E(\bullet)) \cong \mathcal{L}\Sigma^\infty E(S^n)$ as the diagonal of an iterated two-sided bar-construction:

$$B(B(S^\Sigma(\bullet \wedge \bullet, S^n), S^\Sigma(S^0, \bullet)), S^\Sigma, E(\bullet))$$

(the $\bullet$s are for the innermost bar, the $\ast$s for the outermost). Now, the innermost bar-construction contracts to $S^\Sigma(S^0 \wedge \bullet, S^n) \cong S^\Sigma(\ast, S^n)$ and $B(S^\Sigma(\ast, S^n), S^\Sigma, E(\bullet))$ contracts in turn to $E(S^n)$.

**Lemma 7.2.9** The functor $\mathcal{L}$ preserves stable equivalences.

**Proof:** It is clear the $\mathcal{L}$ preserves level equivalences, and so it is enough to prove that $\mathcal{L}$ preserves stable acyclic cofibrations between cofibrant bispectra. Hence it is enough to show it for cellular extensions and ultimately is is enough to prove it for the generating stable acyclic cofibrations of the form $c_i \Box k_j$ for $i, j \in S^\Sigma$ and $c: K \subseteq L \subseteq I_{S_*}$ as in Definition 7.1.6.

As in the proof of Lemma 7.2.8 we get that if $E$ is a symmetric spectrum, then $\mathcal{L}(S^\Sigma(i, \bullet) \wedge E)(S^n)$ is equivalent to $B(S^\Sigma(i \wedge \bullet, S^n), S^\Sigma(\bullet), E)$, and so $\mathcal{L}(S^\Sigma(i, \bullet) \wedge S^\Sigma(j, \bullet))(S^n)$ is equivalent to $S^\Sigma(i \wedge j, S^n)$ and $\mathcal{L}(S^\Sigma(i, \bullet) \wedge S^\Sigma(j \wedge S^1, \bullet) \wedge S^1)(S^n)$ is equivalent to $S^\Sigma(i \wedge j \wedge S^1, S^n) \wedge S^1$. If $M_j$ is the mapping cylinder of the map $S^\Sigma(j \wedge S^1, \bullet) \wedge S^1 \to S^\Sigma(j, \bullet)$, then $\mathcal{L}(S^\Sigma(i, \bullet) \wedge M_j) \to \mathcal{L}(S^\Sigma(i, \bullet) \wedge S^\Sigma(j, \bullet))$ is a level equivalence.

Consequently, since $\mathcal{L}$ preserves colimits, the proof of $\mathcal{L}(c_i \Box k_j)$ being a stable equivalence reduces to observing that

$$K \wedge S^\Sigma(i \wedge j \wedge S^1, \bullet) \wedge S^1 \longrightarrow L \wedge S^\Sigma(i \wedge j \wedge S^1, \bullet) \wedge S^1$$

is homotopy cocartesian (since both vertical maps are stable equivalences).

Using Lemma 7.2.8 and Lemma 7.1.8 we get that Lemma 7.2.9 has the following corollary.

**Corollary 7.2.10** Let $X$ be a right positively stably fibrant bispectrum. Then the natural chain $RX \leftarrow \mathcal{L}\Sigma^\infty RX \to \mathcal{L}X$ of multifunctors from symmetric bispectra to symmetric spectra consists of stable equivalences.
8 Topological cyclic homology

Hesselholt and Madsen’s definition \[14\] 2.7.1 (and further developed in \[13\]) of topological cyclic homology as a symmetric spectrum with a sensible multiplicative structure adopts to our situation with little work. Due to the strong similarities, we give only an outline of the construction adjusted to our somewhat more general context.

8.1 Topological Hochschild homology

Recall the discussion of the category \(\mathcal{I}\) from Section 3.4. Let \(\mathcal{C}\) be an \(\text{Sp}^\Sigma\)-category. If \(x = (x_0, \ldots, x_q) \in \mathcal{I}^{q+1}\), define \(V(\mathcal{C})(x)\) to be the space

\[
\bigvee_{c_0, \ldots, c_q \in \mathcal{C}} \mathcal{C}(c_0, c_q)(S^{x_0}) \wedge \bigwedge_{i=1}^q \mathcal{C}(c_i, c_{i-1})(S^{x_i}),
\]

and let

\[
G(\mathcal{C}; X)_q : \mathcal{I}^{q+1} \to \mathcal{S}_*
\]

be the functor sending \(x\) to \(\Omega^X(X \vee V(\mathcal{C})(x))\). As \(q\) varies, the \(G(\mathcal{C}; X)_q\) assemble into a natural transformation from the cyclic category \([q] \mapsto \mathcal{I}^{q+1}\) to the constant functor with value \(\mathcal{S}_*\), and

\[
[q] \mapsto \text{THH}^n(\mathcal{C}; X)_q = \text{holim}_{x \in (\mathcal{I}^{n})^{q+1}} G(\mathcal{C}; X)_q \circ (\sqcup^n)
\]

defines an epicyclic pointed space \(\text{THH}^n(\mathcal{C}; X)_q\). This construction is functorial in \(n \in \mathcal{I}\) (it is not functorial with respect to surjections of sets).

The assignment \(S^n \mapsto \text{THH}^n(\mathcal{C}; S^n)\) (with diagonal \(\Sigma_n\)-action) and with \(\Sigma_n \times \Sigma_n\) maps

\[
\text{THH}^n(\mathcal{C}; S^n) \wedge S^m \to \text{THH}^n(\mathcal{C}; S^{m \vee m}) \to \text{THH}^{m \vee m}(\mathcal{C}; S^{m \vee m})
\]

(where the first map is just the simplicial structure and the last map uses the functoriality in \(\mathcal{I}\)) defines an epicyclic (and so cyclotomic) symmetric spectrum \(\text{THH}(\mathcal{C})\).

By the arguments in \[8\] 4.3.5.3 adapted to symmetric spectra (using \[22\] to replace Bökstedt’s approximation lemma for the case where the functor is the identity on objects and \(\pi_*\)-isomorphisms on morphisms), topological Hochschild homology is well behaved with respect to stable equivalences in the sense of Definition 3.2.2.

**Lemma 8.1.1** If \(\mathcal{C} \to \mathcal{D}\) is a stable equivalence of small \(\text{Sp}^\Sigma\)-categories with semistable morphism spectra, then the induced map \(\text{THH}(\mathcal{C}) \to \text{THH}(\mathcal{D})\) is a level equivalence.

**Corollary 8.1.2** If \(\mathcal{C}\) is a small \(\text{Sp}^\Sigma\)-category with connective semistable morphism spectra, then \(\text{sin } \text{THH}(\mathcal{C})\) is positively fibrant.

**Proof:** The map \(\mathcal{C} \to T_0\mathcal{C}\) is a stable equivalence to a \(\text{Sp}^\Sigma\)-category with morphism spectra connective \(\Omega\)-spectra, which consequently satisfy the stabilization criteria used to define “FSPs defined on spheres” as in \[14\]. By Lemma 8.1.1 it is thus enough to prove the corollary for such \(\text{Sp}^\Sigma\)-categories, which follows the usual stabilization argument giving that \(\text{THH}^n(\mathcal{C}, S^n) \to \Omega \text{THH}^{n+1}(\mathcal{C}, S^{n+1})\) is an equivalence for \(n > 0\).

Similarly, if \(\mathcal{C} = \{S^n \mapsto C^n\} : S^\Sigma \to \text{Sp}^\Sigma-\text{Cat}_{\Delta^v}\) is a symmetric spectrum in simplicial pointed \(\text{Sp}^\Sigma\)-categories, the assignment \((S^m, S^n) \mapsto \text{THH}^{m \vee m}(\mathcal{C}^m, S^n)\) defines a simplicial epicyclic (in one
direction) symmetric bispectrum $\text{THH}(C)$. Taking the diagonal of the simplicial directions we get an epicyclic symmetric bispectrum which we will also call $\text{THH}(C)$.

If $C_1, \ldots, C_k$ are $\text{Sp}^\Sigma$-categories and $n_1, \ldots, n_k \in I$, we get a map $V(C_1) \circ \sqcup n_1 \wedge \ldots \wedge V(C_k) \circ \sqcup n_k \rightarrow V(C_1 \wedge \ldots \wedge C_k) \circ \sqcup n_1 \mid_n \ldots \mid_n n_k$ by distributivity and rearranging smash factors, leading to a map.

$$\text{THH}^n(C_1; X_1)_q \wedge \ldots \wedge \text{THH}^n(C_k; X_k)_q \rightarrow \text{THH}^n(\sqcup n_1 \mid_n \ldots \mid_n n_k(C_1 \wedge \ldots \wedge C_k; X_1 \wedge \ldots \wedge X_k)_q$$

Using this with $X_j = S^{n_j}$, we get that

**Lemma 8.1.3** The structure above defines a multifunctor $\text{THH}: \text{Sp}^\Sigma\text{-Cat} \rightarrow \text{Sp}^\Sigma$ which to an $\text{Sp}^\Sigma$-category $C$ assigns the $\text{S}_\ast$-functor $\text{THH}(C): S^\Sigma \rightarrow \text{S}_\ast$ sending $S^n$ to $\text{THH}^n(C; S^n)$, with $\Sigma_n$ acting diagonally. This structure extends to a multifunctor to the multicategory of bispectra

$$\text{THH}: [S^\Sigma, \text{Sp}^\Sigma\text{-Cat}_\ast^\Sigma] \rightarrow [S^\Sigma \wedge S^\Sigma, \text{S}_\ast],$$

whose value at the symmetric spectrum $C$ of pointed simplicial $\text{Sp}^\Sigma$-categories sends $(S^m, S^n) \in S^\Sigma \times S^\Sigma$ to $\text{THH}^m \otimes \text{THH}^n(C^m, S^n)$.

**Lemma 8.1.4** Let $C$ be a symmetric spectrum $C$ of pointed simplicial $\text{Sp}^\Sigma$-categories with semistable morphism spectra.

1. If for a given $m$ the morphism spectra of $C^m$ are connective, then the spectrum $\{n \mapsto \sin |\text{THH}^m \otimes \text{THH}^n(C^m, S^n)|\}$ is positively stably fibrant, and if $m > 0$, then it is stably fibrant.

2. Assume $C$ comes from a special $\Gamma$-object in pointed $\text{Sp}^\Sigma$-categories (in the sense that for every $k > 0$ the canonical projection $C(k_+) \rightarrow C \times \cdots \times C$ (coming from the $k$ maps $k_+ \rightarrow 1_+$ sending everything but one element to the basepoint) is a stable equivalence of $\text{Sp}^\Sigma$-categories). Then for fixed $n$ the spectrum $\{m \mapsto \sin |\text{THH}^m \otimes \text{THH}^n(C^m, S^n)|\}$ is positively stably fibrant, and if $n > 0$, then it is stably fibrant.

**Proof:** The first statement is a special case of Corollary 8.1.2. For $m > 0$ we get that $m \sqcup n$ is nonempty, also when $n = 0$, and so the argument in Corollary 8.1.2 works even in this case.

If $C$ comes from a special $\Gamma$-object, then since $\text{THH}^m \otimes \text{THH}^n$ preserves stable equivalences and products up to stable equivalence whenever $m \sqcup n$ is nonempty, we get that $\{m \mapsto \text{THH}^m \otimes \text{THH}^n(C^m, S^n)\}$ itself comes from a special $\Gamma$-object in symmetric spectra (and hence “very special”) and consequently is (positively) stably fibrant.

### 8.2 The Dennis trace

Recall the $\text{S}_\ast$-functor $R: \text{Sp}^\Sigma \rightarrow \text{S}_\ast$ evaluating a symmetric spectrum at $S^0$. Given a small $\text{Sp}^\Sigma$-category $C$, the **Dennis trace map** is the composite

$$\text{ob} C \subseteq \bigvee_{c \in \text{ob} C} R\mathcal{C}(c, c) \rightarrow \text{THH}^0(C; S^0) = \text{THH}^0(C) = R\text{THH}(C)$$

where the first map sends an object $c$ to its identity morphism in $R\mathcal{C}(c, c)$, the second map is inclusion of degeneracies.

If $C$ has an initial object 0, then the Dennis trace map is a map of pointed simplicial sets.

**Lemma 8.2.1** The Dennis trace map defines a natural transformation $\text{trc}: \text{ob} \Rightarrow R\text{THH}$ of $\text{S}_\ast$-multifunctors $\text{Sp}^\Sigma\text{-Cat}_\ast \rightarrow \text{S}_\ast$.
**Proof:** For pointed Sp$^\Sigma$-categories $C_1, \ldots C_k$ and $C$ we have to prove that the diagram

\[
\begin{array}{ccc}
\text{Sp}^\Sigma\text{-Cat}_c(C_1, \ldots, C_k; C) & \xrightarrow{\text{ob}} & \mathcal{S}_*(\text{ob}C_1, \ldots, \text{ob}C_k; \text{ob}C) \\
\mathbb{R} \text{THH} & \downarrow & \\
\mathcal{S}_*(\text{THH}(C_1), \ldots, \text{THH}(C_k); \text{THH}(C)) & \xrightarrow{\text{trc}_*} & \mathcal{S}_*(\text{ob}C_1, \ldots, \text{ob}C_k; \text{THH}(C))
\end{array}
\]

commutes, which is easily checked. □

### 8.3 Topological cyclic homology

If $X$ is a simplicial object and $r$ an integer, the $r$th *edgewise subdivision* $sd^r X$ is the precomposition of $X$ by the functor $sd^r: \Delta^o \to \Delta^o$ sending a finite ordered set $S$ to the $r$-fold concatenation $sd^r S = S \sqcup \cdots \sqcup S$, ordered so that elements in later sets are greater. If $X$ is a cyclic object in the sense of Connes, then the cyclic structure defines a $C_r$ structure on $sd^r X$. If $X$ is a cyclic set, then the realization $|X|$ carries a natural action by the circle $|S^1| = S(C)$. Recall from [1] the homeomorphism $D_r: \{sd^r X\} \cong |X|$ given by sending $(x, u) \in X_{rj-1} \times \Delta^{j-1}$ to $(x, \frac{j}{n}(u, \ldots u)) \in X_{rj-1} \times \Delta^{rj-1}$, where $\Delta^n = \{(u_0, \ldots, u_n \mid \sum u_j = 1\}$ is the standard topological $n$-simplex. The two $C_r$-actions on $|sd^r X|$ one coming as the realization of the $C_r$-action on $sd^r X$ and the other coming from the restriction of the circle action on the realization of the cyclic set $sd^r X$, agree, and $D_r$ induces a homeomorphism $|sd^r X^C_r| \cong |X|^C_r$.

If $X$ is a cyclic set, the *Frobenius map* $F^a$ is the composite

\[
|sd^a X^C_{ab}| \xrightarrow{D_{ab}} |X|^C_{ab} \subseteq |X|^C_a \xrightarrow{D_b} |sd^b X^C_b| .
\]

Notice that the fixed point space under the circle action on $|X|$ is the discrete space $|X|^{S^1} = \{x \in X_0 \mid s_0 x = ts_0 x\}$ where $t$ is the cyclic operator. If $x \in X_0$ is in this fixed point space, $s_0^{ab} x \in |sd^a X^C_{ab}| \subseteq |sd^b X^C_{ab}|$, and $F^a(s_0^{ab} x)$ is represented by $(s_0^{ab} x, (\frac{1}{r}, \ldots, \frac{1}{r})) \in X_{ab-1} \times \Delta^{b-1}$, which is equivalent to $(s_0^r, 1) \in X_{b-1} \times \Delta^b$. Hence the inclusion from the circle fixed points into $|X|$ factors over the limit over the Frobenii.

**Definition 8.3.1** If $C$ is an Sp$^\Sigma$-category and $X$ a space, let

\[
T^n(a)(A; X) = \sin |sd^a \text{THH}^n(C; X)^C_a| .
\]

By the observations above, we know that the inclusion $\text{ob}C \to \text{THH}^n(C; S^0)$ factors over all the Frobenii $F^a: T^n(ab)(C; S^0) \to T^n(b)(C; S^0)$.

The other map of importance for defining topological cyclic homology, the so-called *restriction map* is the map $R_b: T^n(ab)(C; X) \to T^n(a)(C; X^C_{ab})$ is defined by restricting the $C_{ab}$-maps $(\bigwedge_{i=1}^n S^{n_i})^{ab} \to X \wedge (V(C)(n_1, \ldots, n_j))^a$ appearing when writing out the $C_{ab}$ fixed points of $sd^a \text{THH}^n(C; X)$ to the $C_a$ fixed points $\bigwedge_{i=1}^n S^{n_i} \to X^C_a \wedge V(C)(n_1, \ldots, n_j)$.

Again, since the Dennis trace map includes everything by identity morphisms, it factors over the restriction maps as well. The restriction and Frobenius maps commute, and so the Dennis trace factors over the homotopy limit over the diagram given by using both restriction and Frobenius.
Topological cyclic homology of an $\text{Sp}^\Sigma$-category $\mathcal{C}$ is the symmetric spectrum $\text{TC}(\mathcal{C}) = \{S^n \mapsto \text{TC}^n(\mathcal{C}; S^n)\}$ most effectively defined as in [8], by a cartesian square

$$\begin{array}{ccc}
\text{TC}^n(\mathcal{C}; X) & \longrightarrow & \sin|\text{THH}^n(\mathcal{C}; X)|_{hT} \\
\downarrow & & \downarrow \\
\left( \text{holim}_{R,F} T^n(a)(\mathcal{C}; X) \right) & \longrightarrow & \left( \text{holim}_{F} |\text{THH}^n(\mathcal{C}; X)|_{hC_a} \right) \\
\end{array}$$

where $X$ is a space. The lower horizontal map in the defining square for $\text{TC}$ is obtained by first projecting onto the homotopy limit along the Frobenii only, then using the homeomorphisms $D_a$ and finally mapping from the $C_a$-fixed points to the homotopy fixed points (note that this last map would not make sense before we peeled away the restriction maps). The rightmost vertical map is given by the restriction from the homotopy fixed points of all of $T$ to its finite subgroups, and is an equivalence after profinite completion.

Similarly, if $\mathcal{C} = \{S^n \mapsto \mathcal{C}^n\}: S^\Sigma \to \text{Sp}^\Sigma$ is a symmetric spectrum in simplicial pointed categories, the assignment $(S^m, S^n) \mapsto \text{TC}^{m|n}(\mathcal{C}^m, \mathcal{C}^n)$ defines a symmetric bispectrum $\text{TC}(\mathcal{C})$. Taking the diagonal of the simplicial directions we get a symmetric bispectrum which we will also call $\text{TC}(\mathcal{C})$.

Since (homotopy) limits of stably fibrant symmetric spectra are fibrant we get directly from these definitions and Lemma 8.1.4 that topological cyclic homology takes fibrant values in the following sense.

**Lemma 8.3.2** Let $\mathcal{C}$ be a symmetric spectrum $\mathcal{C}$ of pointed simplicial $\text{Sp}^\Sigma$-categories with semistable morphism spectra.

1. For fixed $\mathbf{m}$ the spectrum $\{ \mathbf{n} \mapsto \text{TC}^{m|\mathbf{n}}(\mathcal{C}^m; S^n) \}$ is positively stably fibrant, and if $m > 0$, then it is stably fibrant.

2. Assume $\mathcal{C}$ comes from a special $\Gamma$-object in pointed $\text{Sp}^\Sigma$-categories (in the sense that for every $k > 0$ the canonical projection $\mathcal{C}(k+) \to \mathcal{C} \times \cdots \times \mathcal{C}$ (coming from the $k$ maps $k+ \to 1_+$ sending everything but one element to the basepoint) is a stable equivalence of $\text{Sp}^\Sigma$-categories). Then for fixed $\mathbf{n}$ the spectrum $\{ \mathbf{m} \mapsto \text{TC}^{m|\mathbf{n}}(\mathcal{C}^m; S^n) \}$ is positively stably fibrant, and if $n > 0$, then it is stably fibrant.

The map

$$\text{THH}^{m_1}(\mathcal{C}_1; X_1)_q \wedge \cdots \wedge \text{THH}^{m_k}(\mathcal{C}_k; X_k)_q \to \text{THH}^{n_1|\cdots|m_k}(\mathcal{C}_1 \wedge \cdots \wedge \mathcal{C}_k; X_1 \wedge \cdots \wedge X_k)_q$$

defined in the previous paragraph when establishing that $\text{THH}$ is a multifunctor is cyclic and restricts to a map

$$sd_r \text{THH}^{m_1}(\mathcal{C}_1; X_1)^{C_r} \wedge \cdots \wedge sd_r \text{THH}^{m_k}(\mathcal{C}_k; X_k)^{C_r} \to sd_r \text{THH}^{n_1|\cdots|m_k}(\mathcal{C}_1 \wedge \cdots \wedge \mathcal{C}_k; X_1 \wedge \cdots \wedge X_k)^{C_r},$$

which commutes with the restriction map, in the sense that the diagram

$$\begin{array}{ccc}
sd_{rp} \text{THH}^{m_1}(\mathcal{C}_1; X_1)^{C_{rp}} \wedge \cdots \wedge sd_{rp} \text{THH}^{m_k}(\mathcal{C}_k; X_k)^{C_{rp}} & \longrightarrow & sd_{rp} \text{THH}^{n_1|\cdots|m_k}(\mathcal{C}_1 \wedge \cdots \wedge \mathcal{C}_k; X_1 \wedge \cdots \wedge X_k)^{C_{rp}} \\
\downarrow_{R} & & \downarrow_{R} \\
sd_r \text{THH}^{m_1}(\mathcal{C}_1; X_1)^{C_r} \wedge \cdots \wedge sd_r \text{THH}^{m_k}(\mathcal{C}_k; X_k)^{C_r} & \longrightarrow & sd_r \text{THH}^{n_1|\cdots|m_k}(\mathcal{C}_1 \wedge \cdots \wedge \mathcal{C}_k; X_1 \wedge \cdots \wedge X_k)^{C_r} \\
\end{array}$$

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commutes. A similar property holds for the Frobenius maps, and pulling back along the defining square of TC one gets a map

\[ TC(C_1 \land \ldots \land TC(C_k) \to TC(C_1 \land \ldots \land C_k) \]

giving the central ingredient in

**Lemma 8.3.3** The structure above endows TC: \( Sp^{\Sigma}_{-} \to Sp^{\Sigma}_{-} \) with the structure of a multifunctor. Likewise, considered as a functor from \( [S^\Sigma, Sp^{\Sigma}_{-} \to Sp^{\Sigma}_{-}] \), we get a multifunctor TC to symmetric bispectra. Furthermore, the inclusion of fixed points \( ob \Rightarrow RTC \) is a natural transformation of multifunctors \( Sp^{\Sigma}_{-} \to S_* \).

In the above, the multicategory \( [S^\Sigma, Sp^{\Sigma}_{-} \to Sp^{\Sigma}_{-}] \) is by default \( S \)-enriched by the structure used to define the spectral direction and the enrichment of TC is straight forward (the functor is applied degewise). However, we could also have used the enrichment coming from \( Sp^{\Sigma}_{-} \)-natural isomorphisms: topological cyclic homology sends \( Sp^{\Sigma}_{-} \)-natural isomorphisms to homotopies (c.f. \( \$ 4.3.5.2 \)). We need to compare these two since they both are used when composing with the nerve.

Let \( f_{01}: f_1 \Rightarrow f_0 \) be an \( Sp^{\Sigma}_{-} \)-natural isomorphism of \( Sp^{\Sigma}_{-} \)functors \( C \to D \) with inverse \( f_{10} \). The composite \( THHhoN: Sp^{\Sigma}_{-} \to Sp^{\Sigma}_{-} \) sends \( f_{01} \) to a homotopy \( THHhoNC^\Delta[1]_+ \to THHhoN \) in two different ways, depending on whether we use THH or hoN to convert the isomorphism to a simplicial homotopy. Concretely, the space of \( q \)-simplices of \( THH^{nhoNC} \) is built out of (paths spaces, pullbacks, wedges and smash of) mapping spectra \( C(c_{ij}, c_{kl}) \) with \( 0 \leq i, j, k, l \leq q \) where we think of \( i \) and \( k \) as moving in the THH-direction and \( j \) and \( l \) in the hoN-direction. If \( \phi: [q] \to [1] \in \Delta \), we use the composite of \( f_0: C(c_{ij}, c_{kl}) \to D(f_0c_{ij}, f_0c_{kl}) \) with either

\[ D(f_0c_{ij}, f_0c_{kl}) \xrightarrow{D(f_0\phi(|i|), f_0\phi(|k|))} D(f_0\phi(i)c_{ij}, f_0\phi(k)c_{kl}) \text{ or } D(f_0c_{ij}, f_0c_{kl}) \xrightarrow{D(f_0\phi(|i|), f_0\phi(|k|))} D(f_0\phi(j)c_{ij}, f_0\phi(l)c_{kl}) \]

to get the two homotopies. Likewise, given an \( s \)-simplex \( f_s \Rightarrow \cdots \Rightarrow f_1 \Rightarrow f_0 \) in \( Sp^{\Sigma}_{-} \) we get \( s \)-simplices in \( S_{\ast}(THHhoNC, THHhoN \Delta) \) by the same formula, so that THHhoN has two different enrichments. However these two can be compared through the functor \( C \mapsto H(C) = THHhoNC^{\Delta[1]_+} \) with the following enrichment. With \( f_s \Rightarrow \cdots \Rightarrow f_1 \Rightarrow f_0 \) and \( \phi \) as above and \( \psi: [q] \to [1] \in \Delta \), the required map

\[ (THH^{nhoNC})_{q}^{\Delta([q], [1])_+} \land \Delta([(q), [s])]_+ \to (THH^{nhoN\Delta})_{q}^{\Delta([q], [1])_+} \]

is induced by the composite (where \( pr_0(i, j) = i, \ pr_1(i, j) = j \))

\[ C(c_{ij}, c_{kl}) \xrightarrow{f_0} D(f_0c_{ij}, f_0c_{kl}) \xrightarrow{D(f_0\psi(i, j), f_0\psi(k, l))} D(f_0\psi(i, j)c_{ij}, f_0\psi(k, l)c_{kl}). \]

The two different enrichments of THHhoN are embedded in \( H \) at the two end points of \( \Delta[1] \). Ultimately, this gives that it is not essential which one we use:

**Lemma 8.3.4** The two simplicial enrichments for THHhoN are linked by natural weak equivalences of multifunctors to \( H \). Likewise for TChoN.

From Lemma 8.1.1 we get that topological cyclic homology preserves stable equivalences in the sense of Definition 3.2.2.
Lemma 8.3.5 If $C \to D$ is a stable equivalence of $\text{Sp}^{\Sigma}$-categories with semistable morphism spectra, then the induced map $TC(C) \to TC(D)$ is a level equivalence.

Together with Lemma 4.5.1 and Proposition 6.2.3, Lemma 8.3.5 implies the following corollary, where the simplicial enrichment for $TC$ is through the $\text{Sp}^{\Sigma}$-natural isomorphisms.

Corollary 8.3.6 Let $C$ be an $M7$-category. Then the inclusion by degeneracies induces a level equivalence $TC(SC) \to TC(\text{hoNSC})$.

Also, using Lemma 3.4.1 we get that the $M_0$ and $T_0$-constructions do not change $TC$.

Corollary 8.3.7 Let $C$ be a semistable $\text{Sp}^{\Sigma}$-category and $Q$ a non-empty finite set. Then the transformations $C \to M^Q_0C \leftarrow T^Q_0C$ induce stable equivalences $TC(C) \to TC(M^Q_0C) \leftarrow TC(T^Q_0C)$.

9 Categories of finitely generated free modules

If $A$ is a symmetric ring spectrum (a monoid in $(\text{Sp}^{\Sigma}, \wedge, S)$), and $M$ and $N$ are $A$-modules, the morphism object $A\text{-mod}(M, N)$ is defined as the equalizer

$$\text{Sp}^{\Sigma}(M, N) \longrightarrow \text{Sp}^{\Sigma}(A\wedge M, N)$$

of the maps $\text{Sp}^{\Sigma}(M, N) \to \text{Sp}^{\Sigma}(A\wedge M, N)$ and $\text{Sp}^{\Sigma}(M, N) \to \text{Sp}^{\Sigma}(A\wedge M, A\wedge N) \to \text{Sp}^{\Sigma}(A\wedge M, N)$ induced by the multiplication in $M$ and $N$ respectively. If $N$ is an $A$-module and $M$ an $A^o$-module, then the smash $M\wedge A N$ is the coequalizer of the two maps

$$M\wedge A\wedge N \longrightarrow M\wedge N$$

induced by the actions. If $A = A^o$, that is, if $A$ is a commutative symmetric ring spectrum, then the category $A\text{-mod}$ of $A$-modules is a rig $\text{Sp}^{\Sigma}$-category under coproduct (wedge) and $\wedge_A$.

For the remainder of this section, let $A$ be a semistable connective commutative symmetric spectrum. Semistability is no restriction, since all symmetric spectra coming from orthogonal spectra (through forgetting down to symmetric spectra based on topological spaces and applying the singular functor) are known to be semistable, and any commutative ring spectrum has a strictly commutative model in orthogonal spectra.

The level structure on the category of $A$-modules is such that a map is a weak equivalence or a fibration if the underlying map of symmetric spectra is so in the level structure. The cofibrations are the maps having the left lifting property with respect to the maps that are level acyclic fibrations. This structure is cofibrantly generated by inducing up the generating (acyclic) cofibrations by smashing with $A$.

9.1 Symmetric spectra in topological spaces

Let $\text{Top}_*$ be the $S_*$-category of (weak Hausdorff) compactly generated based spaces: if $X$ and $Y$ are compactly generated, then the set of $q$-simplices of $\text{Top}_*(X, Y)$ is the set of continuous based maps $X\wedge|\Delta[q]| \to Y$. In particular, a symmetric spectrum in $\text{Top}_*$ is an $S$-functor $S^\Sigma \to \text{Top}_*$.

The singular functor/geometric realization adjunction gives a monoidal Quillen equivalence between any of the various model structures on $\text{Sp}^{\Sigma}$ and a corresponding structure on $[S^\Sigma, \text{Top}_*]$. To us, an important aspect is that in the level structure, any object in $[S^\Sigma, \text{Top}_*]$ is fibrant.
For the sake of this paper, a *(commutative)* ring spectrum based on $\text{Top}_*$ is a (symmetric) monoid in $[S^+,\text{Top}_*]$. The category of modules over such a (commutative) ring spectrum $B$ inherits (symmetric monoidal) model structures by insisting that the forgetful functor to any of the many model structures on $[S^+,\text{Top}_*]$ reflects weak equivalences and fibrations.

If $N$ is a $B$-module, we let as before $R^1 N = \Omega N(S^1 \wedge -)$, and we note that the singular functor/realization adjunction gives a natural isomorphism $R^1(B\text{-mod}(M,N)) \cong B\text{-mod}(M,R^1 N)$, where $M$ is a $B$-module. Hence, if $M$ is small we get that the natural map $R^\infty(B\text{-mod}(M,N)) \cong B\text{-mod}(M,R^\infty N)$ is an isomorphism. If the $B$-modules $N$ and $N'$ are semistable, then a map $N \to N'$ is a stable equivalence if and only if the induced map $R^\infty N \to R^\infty N'$ is a level equivalence. This has the following consequence, where we note the missing fibrancy conditions on the targets.

**Lemma 9.1.1** Let $B$ be a ring spectrum based on $\text{Top}_*$, $M$ a small cofibrant $B$-module and $N \to N'$ a stable equivalence of semistable $B$-modules, then the induced map

$$B\text{-mod}(M,N) \to B\text{-mod}(M,N')$$

is a stable equivalence.

**Definition 9.1.2** A $B$-module $M$ is said to have property P if for all stable equivalences $X \xrightarrow{\sim} Y$ of semistable $B$-modules, the induced map

$$B\text{-mod}(M,X) \to B\text{-mod}(M,Y)$$

is a stable equivalence of semistable symmetric spectra.

The trivial module 0 and the rank 1-module $B$ are examples of $B$-modules that have property P.

**Lemma 9.1.3** Let $\mathcal{C} \subseteq B\text{-mod}$ be an $\text{Sp}^\Sigma$-full subcategory of semistable small cofibrant $B$-modules. Then the smallest $\text{Sp}^\Sigma$-full subcategory with cofibrations and weak equivalences $\mathcal{M}(\mathcal{C})$ of $B\text{-mod}$ (with the stable structure) containing $\mathcal{C}$ is $\mathcal{M}7$. In addition, all objects in $\mathcal{M}(\mathcal{C})$ are small, cofibrant and semistable.

In the case of a commutative $B$; if $\mathcal{C}$ is closed under $\wedge_B$, then so is $\mathcal{M}(\mathcal{C})$.

**Proof:** The only thing missing for $\mathcal{C}$ to be a category with cofibrations and weak equivalences is that it might not contain 0 and it might not be closed under pushouts along cofibrations. Adding 0 to $\mathcal{C}$ poses no problems, so assume

$$\begin{array}{ccc}
M' & \xrightarrow{\sim} & M \\
\downarrow & & \downarrow \\
N' & \xrightarrow{\sim} & N
\end{array}$$

is a pushout along a cofibration, with $M'$, $M$ and $N'$ in $\mathcal{C}$. First note that $N$ is automatically small, cofibrant and semistable. If $X$ is any semistable $B$-module, then the horizontal map in the induced cartesian square

$$\begin{array}{ccc}
& B\text{-mod}(M',X) & \leftarrow B\text{-mod}(M,X) \\
\downarrow & & \downarrow \\
B\text{-mod}(N',X) & \leftarrow B\text{-mod}(N,X)
\end{array}$$

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are level fibrations, and so the square also homotopy cartesian in the level structure on \( \text{Sp}^\Sigma \). Since three of the vertices are semistable by assumption, so is the remaining vertex \( B\text{-mod}(N,X) \). Consequently, if you apply \( R^\infty \) to the diagram you get a homotopy cartesian square in the stable model structure on \( \text{Sp}^\Sigma \), since \( R^\infty \) commutes with finite limits. Furthermore, the \( R^\infty \) of the square is naturally isomorphic to the square you get by replacing all occurrences of \( X \) with \( R^\infty X \). Hence, \( B\text{-mod}(N,-) \) preserves all stable equivalences between semistable \( B \)-modules. By induction, any \( N \) in \( \mathcal{M}(\mathcal{C}) \) will have this property – in addition to being small, cofibrant and semistable.

The axiom (2) of an M7-category is automatic since all \( B \)-modules are level fibrant.

The claim about \( \mathcal{M}(\mathcal{C}) \) being closed under smash is immediate since smashing preserves pushout (distributivity).

Likewise, we have the easier results:

**Lemma 9.1.4** Let \( \mathcal{C} \subseteq B\text{-mod} \) be an \( \text{Sp}^\Sigma \)-full subcategory of semistable small cofibrant \( B \)-modules \( M \) and let \( \mathcal{K} \) be a class of finite spaces. Let

1. \( \mathcal{C}(\mathcal{K}) \) be the \( \text{Sp}^\Sigma \)-full subcategory of \( B\text{-mod} \) with objects \( M \wedge |K^+| \), for \( M \in \mathcal{C} \) and \( K \in \mathcal{K} \)
2. \( \mathcal{C}^{\text{cycl}} \) be the smallest \( \text{Sp}^\Sigma \)-full subcategory of \( B\text{-mod} \) containing \( \mathcal{C} \) and closed under mapping cylinders, and
3. \( \mathcal{C}_\vee \) be the smallest \( \text{Sp}^\Sigma \)-full subcategory of \( B\text{-mod} \) containing \( \mathcal{C} \) and closed under finite coproducts.

Then all objects in \( \mathcal{C}(\mathcal{K}) \), in \( \mathcal{C}^{\text{cycl}} \) and in \( \mathcal{C}_\vee \) are small, semistable and cofibrant. If \( B \) is commutative, \( \mathcal{C} \) closed under smash and \( \mathcal{K} \) closed under product, then \( \mathcal{C}(\mathcal{K}), \mathcal{C}^{\text{cycl}} \) and \( \mathcal{C}_\vee \) are closed under smash.

**Example 9.1.5** Given a semistable symmetric ring spectrum \( A \), this gives examples of well behaved categories of “finitely generated free/projective modules”. For Let \( fS_\ast \) be the category of finite spaces and \( \mathcal{K} \) the category of contractible finite spaces. We let \( \{ |A| \} \) be the \( \text{Sp}^\Sigma \)-category containing only the rank 1-module.

1. Categories with sums and weak equivalences whose objects are cofibrant, semistable and satisfy property P: \( \{ |A| \}_\vee, (\{ |A| \}_\vee)^{\text{cycl}}, \{ |A| \}(\mathcal{K})_\vee \)
2. M7-categories: \( \mathcal{M}((\{ |A| \}), \mathcal{M}((\{ |A| \}(\mathcal{K})), \) and \( \mathcal{M}((\{ |A| \}(fS_\ast))). \)

If \( w|A|mod \) consists of the stable equivalences and \( v|A|mod \) consists of the acyclic stable cofibrations, then \( \{ |A| \}(\mathcal{K})_\vee, \mathcal{M}((\{ |A| \}(\mathcal{K}))) \) and \( \mathcal{M}((\{ |A| \}(fS_\ast))) \) have \( vw \)-cylinder functors in the sense of Section 9.3 below.

The minimal example, \( \{ |A| \}_\vee \), consist of the \( |A| \)-modules that are “free on finite sets”, whereas the maximal example, \( \mathcal{M}((\{ |A| \}(fS_\ast))) \), consists of all the “finite cell \( |A| \)-modules”.

Any object in \( \{ |A| \}(\mathcal{K})_\vee \) (resp. \( \mathcal{M}((\{ |A| \}(\mathcal{K}))) \)) is stably equivalent to an object in \( \{ |A| \}_\vee \) (resp. \( \mathcal{M}((\{ |A| \})) \)), and similarly for the cylinder versions.

Also, there would not be any harm done by considering also retracts, giving access to “projective” modules. We might consider restricting the morphisms, for instance so that all pushout diagrams in the K-theory constructions would be along split cofibrations. In the case where \( A \) is commutative, all the examples above give rig \( \text{Sp}^\Sigma \)-categories, and applying the strictification functor \( \text{Strig} \) of Lemma 4.2.3 give us strict versions.
9.2 The cyclotomic trace of a commutative symmetric ring spectrum

We now present the leanest explicit version of a multiplicative cyclotomic trace map for commutative ring spectra we are aware of:

**Definition 9.2.1** If \( A \) is a semistable commutative symmetric ring spectrum, then \( \mathcal{F}_A \) is the \( \text{Sp}^\Sigma \)-rig category with sum and weak equivalences obtained by applying the strictification \( \text{Strig} \) of Lemma 4.2.3 to the category with sum \( \{ |A| \}_\vee \) of Example 9.1.5.

Standard Morita equivalence arguments [9] give

**Lemma 9.2.2** Let \( A \) be a semistable symmetric ring spectrum. Then the inclusion of the rank 1-module \( |A| \subseteq \{ |A| \}_\vee \) induces equivalences

\[
\text{TC}(A) \xrightarrow{\sim} \text{TC}(\sin|A|) \xrightarrow{\sim} \text{TC}(\{ |A| \}_\vee).
\]

If \( A \) is commutative, then the inclusions are rig \( \text{Sp}^\Sigma \)-maps, and so the induced maps in topological cyclic homology are \( E_\infty \).

Finally, the the homotopy type of the K-theory side conforms with the usual definition.

**Lemma 9.2.3** There is a weak equivalence \( \Omega^{\infty}\text{obhoN}\tilde{\text{H}}\mathcal{F}_A \simeq \mathbb{Z} \times \hat{GL}(A)^+ \).

**Proof:** We saw in Note 6.3.3 that \( \Omega^{\infty}\text{obhoN}\tilde{\text{H}}\mathcal{F}_A \) is a group completion of \( \text{obN}w\{ |A| \}_\vee \). Here the morphism space \( w\{ |A| \}_\vee \{ |A| \}_\vee \) is empty if \( m \neq n \) and consists of the invertible components of

\[
RT_0T_0\sin \prod_n \bigvee_n A(S^n)| = \holim \Omega^x \holim \Omega^y (S^x \wedge \sin \prod_n \bigvee_n A(S^n)|),
\]

which is just a rewriting of Waldhausen’s \( \hat{GL}_n(\sin |A|) \simeq \hat{GL}_n(A) \).

9.3 Categories with \( vw \)-cylinder functors

For “backward compatibility” we include a short discussion of \( vw \)-cylinders. These are useful in that in their presence, the homotopy nerve can be replaced by the simpler categorical nerve, and the current formalism can be used to transfer the ideas of [7] from the EKMM-setup.

Let \( \nabla \) be the category

\[
\begin{array}{ccc}
\bullet & \xrightarrow{i} & \bullet \\
\downarrow & & \downarrow \\
\bullet & \xrightarrow{j} & \bullet
\end{array}
\]

and, if \( \mathcal{C} \) is a \( V \)-category for some \( V \), let \( \int_\nabla \mathcal{C} \) be the \( V \)-functor category from \( \nabla \) to \( \mathcal{C} \).

**Definition 9.3.1** Let \( \mathcal{C} \) be a \( V \)-category, and \( v\mathcal{C} \subseteq w\mathcal{C} \subseteq U\mathcal{C} \). A \( vw \)-cylinder functor is a \( V \)-functor \( N_1(\mathcal{C}, w\mathcal{C}) \to \int_\nabla \mathcal{C} \) which sends \( c_0 \xleftarrow{i} c_1 \in w\mathcal{C} \) to

\[
\begin{array}{ccc}
c_1 & \xrightarrow{i(f)} & T(f) \xrightarrow{j(f)} & c_0 \\
\downarrow f & & \downarrow \rho(f) \\
c_0 & \xrightarrow{p(f)} \end{array}
\]

where \( i(f) \) and \( j(f) \) are in \( v\mathcal{C} \) and \( p(f) \) is in \( w\mathcal{C} \).
Definition 9.3.2 A porsche-category is an $\text{Sp}^\Sigma$-category $\mathcal{C}$ together with subcategories $v\mathcal{C} \subseteq w\mathcal{C} \subseteq U\mathcal{C}$ containing all objects and a $vw$-cylinder functor, such that

1. all morphism spectra are semistable
2. if $c' \to c \in v\mathcal{C}$ and $d \in \mathcal{C}$, then $\mathcal{C}(c, d) \to \mathcal{C}(c', d)$ is a level fibration, and
3. if $d' \to d \in w\mathcal{C}$ and $c \in \mathcal{C}$, then $\mathcal{C}(c, d') \to \mathcal{C}(c, d)$ is a stable equivalence.

We see that in a porsche-category, the maps in $w\mathcal{C}$ are similar to stable equivalences and the maps in $v\mathcal{C}$ are similar to cofibrations, and we may write $\twoheadrightarrow$ (resp., $\to$) when talking about maps in $v\mathcal{C}$ ($w\mathcal{C}$) to keep reminding ourselves of this behavior.

Proposition 9.3.3 Let $(\mathcal{C}, w\mathcal{C}, v\mathcal{C})$ be a porsche-category. Then the inclusion by degeneracies induces a level equivalence $TC(\mathcal{C}) \to TC(N(\mathcal{C}, w\mathcal{C}))$.

Proof: The presence of the $vw$-cylinder guarantees by the argument of [7, Corollary 2.6] that the map $TC(N(\mathcal{C}, v\mathcal{C})) \to TC(N(\mathcal{C}, w\mathcal{C}))$ is a level equivalence, so all we have to show is that the inclusion by degeneracies $\mathcal{C} = N_0(\mathcal{C}, v\mathcal{C}) \to N_q(\mathcal{C}, v\mathcal{C})$ induces a level equivalence $TC(\mathcal{C}) \to TC(N_q(\mathcal{C}, v\mathcal{C}))$.

So, if

$$c = \{c_0 \leftarrow c_1 \leftarrow \cdots \leftarrow c_q\} \in N_q v\mathcal{C}$$

and

$$d = \{d_0 \leftarrow d_1 \leftarrow \cdots \leftarrow d_q\} \in N_q w\mathcal{C},$$

then $N_q(\mathcal{C}, w\mathcal{C})(c, d)$ is isomorphic to the limit $\bigcap_i \mathcal{C}(c_i, d_i)$ of

$$\begin{array}{ccc}
\mathcal{C}(c_0, d_0) & \to & \mathcal{C}(c_1, d_0) \\
& \searrow & \downarrow \sim \\
& & \mathcal{C}(c_1, d_1) \\
& \to & \cdots
\end{array}$$

where the horizontal arrows are level fibrations and the vertical maps are stable equivalences. Now, the $R^\infty$-construction commutes with finite limits, so $R^\infty N_q(\mathcal{C}, w\mathcal{C})(c, d)$ is isomorphic to the limit $\bigcap_i R^\infty \mathcal{C}(c_i, d_i)$ of the diagram above with $R^\infty$ applied everywhere. Now, $R^\infty$ of semistable spectra transforms stable equivalences to level equivalences and preserves level fibrations. Since the level structure on $\text{Sp}^\Sigma$ is right proper, this means that the canonical map $R^\infty N_q(\mathcal{C}, w\mathcal{C})(c, d) \to R^\infty \mathcal{C}(c_0, d_0)$ is a level equivalence, and since $\mathcal{C}(c_0, d_0)$ is semistable this means that $N_q(\mathcal{C}, w\mathcal{C})(c, d) \to \mathcal{C}(c_0, d_0)$ is a stable equivalence.

Since any object in $\mathcal{C}$ is $c_0$ for some chain $c$ in $N_q v\mathcal{C}$ and $v\mathcal{C} \subseteq w\mathcal{C}$, this means that the face maps induce a stable equivalence $[3.2.2] N_q(\mathcal{C}, v\mathcal{C}) \to \mathcal{C}$, and so by Lemma 8.3.5 the induced map $TC(N_q(\mathcal{C}, v\mathcal{C})) \to TC(\mathcal{C})$ is a level equivalence.

Example 9.3.4 We display a porsche-category $\mathcal{F}_A'$ of cell $A$-modules. In the case of a commutative symmetric ring spectrum we replace this construction to the equivalent bipermutative category with cofibrations and weak equivalences we get by applying the Strig-construction of Lemma 4.2.3.
The category $\mathcal{F}_A'$ of “finitely generated free” $A$-modules is an $\text{Sp}^\Sigma$-category with cofibrations and weak equivalences, where we take care that the $\text{Sp}^\Sigma$-enrichment and the cofibrations and weak equivalences are delicately balanced. The objects $\text{ob}\mathcal{F}_A'$ is the smallest set of objects of $|A|$-modules satisfying

1. $0$ and $|A|$ are in $\text{ob}\mathcal{F}_A'$,
2. if $c$ is in $\text{ob}\mathcal{F}_A'$, then the cylinder $c \land I_+$ is in $\text{ob}\mathcal{F}_A'$ and
3. if $c', c$ and $d$ in $\text{ob}\mathcal{F}_A'$ and $d \leftarrow c' \rightarrow c$ is a diagram of $A$-modules, with $c' \rightarrow c$ a level cofibration, then $c \amalg d$ is in $\text{ob}\mathcal{F}_A'$.

If $c,d \in \text{ob}\mathcal{F}_A'$ we define $\mathcal{F}_A'(c,d)$ as $\text{sin}(|A|\cdot \text{mod}(c,d))$. This defines an $\text{Sp}^\Sigma$-category. We define a cofibration (resp. weak equivalence) in $\mathcal{F}_A'$ to be a level cofibration (resp. stable equivalence) of $|A|$-modules between objects in $\mathcal{F}_A'$, defining a category $\text{co}\mathcal{F}_A'$ (resp. $\text{w}\mathcal{F}_A'$).

Let $v\mathcal{F}_A' = w\mathcal{F}_A' \cap \text{co}\mathcal{F}_A'$. If $f : c' \xrightarrow{\sim} d \in w\mathcal{F}_A'$, we define the $vw$-cylinder $T(f)$ to be the pushout of

$$
\begin{array}{c}
d \leftarrow f^c \xrightarrow{\sim} \xrightarrow{i_{\nu}} c' \land I_+
\end{array}
$$

where $i_{\nu}$ is the level acyclic cofibration given by the inclusion $d^0 : \Delta[0] = * \subseteq I = \Delta[1]$. We will decorate the resulting natural morphisms accordingly:

$$
\begin{array}{ccc}
c & \xrightarrow{i(f)} & T(f) & \xrightarrow{j(f)} & d \\
\downarrow f & & \downarrow \rho(f) & & \downarrow d
\end{array}
$$

(all maps are stable equivalences, but we have marked only the simplicial homotopy equivalence).

**Lemma 9.3.5** Let $A$ be a semistable symmetric ring spectrum. Then the category $\mathcal{F}_A'$ of Example 9.3.4 is a porsche-category.

**Proof:** To see that the morphism spectra are semistable and preserve stable equivalences in the target variable, we argue inductively. Given $d \xrightarrow{\sim} d' \in \text{ob}\mathcal{F}_A'$, let $C$ be the set of objects such that $\mathcal{F}_A'(c,d') \rightarrow \mathcal{F}_A'(c,d)$ is a stable equivalence of semistable symmetric spectra. Obviously, both 0 and $A$ are in $C$. Also, if $c \in C$, then the natural (in $d$) homotopy equivalence $\mathcal{F}_A'(c \land I_+, d) \cong S_\pi(I_+, \mathcal{F}_A'(c,d)) \xrightarrow{\sim} \mathcal{F}_A'(c,d)$ gives that $c \land I_+$ is in $C$. Lastly, assume $x$ is the pushout of $x' \leftarrow c' \rightarrow c \in A\cdot \text{mod}$, where $x, c'$ and $c$ are in $C$ and $c' \rightarrow c$ is a cofibration. Then the fact that

$$
\begin{array}{ccc}
\mathcal{F}_A'(x,d) & \longrightarrow & \mathcal{F}_A'(c,d) \\
\downarrow & & \downarrow \\
\mathcal{F}_A'(x',d) & \longrightarrow & \mathcal{F}_A'(c',d)
\end{array}
$$

is cartesian with vertical maps level fibrations, implies that $R^\infty \mathcal{F}_A'(x,d)$ is an $\Omega$-spectrum, and so $\mathcal{F}_A'(x,d)$ is semistable, and also that $R^\infty \mathcal{F}_A'(x,d) \rightarrow R^\infty \mathcal{F}_A'(x,d')$ is a level equivalence, and since $\mathcal{F}_A'(x,d)$ is semistable, that $\mathcal{F}_A'(x,d) \rightarrow \mathcal{F}_A'(x,d')$ is a stable equivalence. Hence $c$ is in $C$, and we are done by the definition of $\text{ob}\mathcal{F}_A'$.

If $c' \xrightarrow{\sim} c \in v\mathcal{F}_A'$ and $d \in \text{ob}\mathcal{F}_A'$, then the level structure in $|A|\cdot \text{mod}$ gives that $\mathcal{F}_A'(c,d) \rightarrow \mathcal{F}_A'(c',d)$ is a level fibration. \qed
Lemma 9.3.6 The smash product over $A$ induces a symmetric monoidal structure on $\mathcal{F}'_A$ compatible with the $\text{Sp}^\Sigma$ enrichment and the structure of cofibrations and weak equivalences inherited from the stable structure on $\mathcal{M}_A$, making $\mathcal{F}'_A$ a symmetric rig category with cofibrations and weak equivalences.

Proof: Follows from the symmetric monoidal structure on the category of $A$-modules, together with the fact that smashing over $A$ commutes with colimits and tensors and the fact that all objects are cofibrant.

Lemma 4.3.1 tells us that after an application of the strictification functor $\text{Strig}$ of Lemma 4.2.3, $\mathcal{F}'_A$ is the image of a multifunctor from the operad $E\Sigma_n$ of to the multicategory of $\text{Sp}^\Sigma$-categories with cofibrations and weak equivalences. Being porsche is preserved under the equivalence $1 \to \text{Strig}$, and so Lemma 9.3.5 guarantees that

Lemma 9.3.7 If $A$ is a semistable commutative symmetric ring spectrum, then $\text{Strig}\mathcal{F}'_A$ is porsche.

References


