Minus edge $k$-subdomination numbers in graphs

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Abstract

The closed neighborhood $N_G[e]$ of an edge $e$ in a graph $G$ is the set consisting of $e$ and of all edges having a common end-vertex with $e$. Let $f$ be a function on $E(G)$, the edge set of $G$, into the set $\{-1, 0, 1\}$. If $\sum_{x \in N[e]} f(x) \geq 1$ for at least $k$ edges $e$ of $G$, then $f$ is called a minus edge $k$-subdominating function of $G$. The minimum of the values $\sum_{e \in E(G)} f(e)$, taken over all minus edge $k$-subdominating functions $f$ of $G$, is called the minus edge $k$-subdomination number of $G$ and is denoted by $\gamma'_k(G)$. In this note we initiate the study of minus edge $k$-subdomination numbers in graphs and present some (sharp) bounds for this parameter.

Keywords: minus edge dominating function; minus domination number; minus edge $k$-subdominating function; minus edge $k$-subdomination number.

1 Introduction

Let $G$ be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. We use [8] for terminology and notation which are not defined here. The minimum and maximum vertex degrees in $G$ are respectively denoted by $\delta(G)$ and $\Delta(G)$. The line graph of a graph $G$, written $L(G)$, is the graph whose vertices are the edges of $G$, with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in $G$. It is easy to see that $L(C_n) = C_n$ and $L(P_n) = P_{n-1}$.

Two edges $e_1, e_2$ of $G$ are called adjacent if they are distinct and have a common endvertex. The open neighborhood $N_G(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to $e$. Its closed neighborhood is $N_G[e] = N_G(e) \cup \{e\}$. For a function $f : E(G) \rightarrow \{-1, 0, 1\}$ and a subset $S$ of $E(G)$ we define $f(S) = \sum_{e \in S} f(e)$. If $S = N_G[e]$ for some $e \in E$, then we denote $f(S)$ by $f[e]$. For each vertex $v \in V(G)$ we also define $f(v) = \sum_{e \in E(v)} f(e)$, where $E(v)$ is the set of all edges at vertex $v$. A function $f : E(G) \rightarrow \{-1, 0, 1\}$ is called a minus edge $k$-subdominating function (ME$k$SDF) of $G$, if $f[e] \geq 1$ for at least $k$ edges $e$ of $G$. The minimum of the values $f(E(G))$, taken over all minus edge $k$-subdominating

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functions $f$ of $G$, is called the minus edge $k$-subdomination number of $G$ and is denoted by $\gamma'_{km}(G)$. The minus edge $k$-subdominating function $f$ of $G$ with $f(E(G)) = \gamma'_{km}(G)$ is called $\gamma'_{km}(G)$ - function. For any minus edge $k$-subdominating function $f$ of $G$ we define $P = \{ e \in E(G) \mid f(e) = 1 \}$, $M = \{ e \in E(G) \mid f(e) = -1 \}$, $Z = \{ e \in E(G) \mid f(e) = 0 \}$ and $X = \{ e \in E(G) \mid f[e] \geq 1 \}$.

If $k = m$, then the minus edge $k$-subdomination number is called the minus edge domination number. The minus edge domination number was introduced by Xu and Zhou in [9] and denoted by $\gamma'_m(G)$.

A function $f : E(G) \to \{-1, 1\}$ is called a signed edge $k$-subdominating function (SE$k$SDF) of $G$, if $f[e] \geq 1$ for at least $k$ edges $e$ of $G$. The minimum of the values $f(E(G))$, taken over all signed edge $k$-subdominating functions $f$ of $G$, is called the signed edge $k$-subdomination number of $G$ and is denoted by $\gamma'_k(G)$. The signed edge $k$-subdominating number was introduced by Khodkar et al. in [6]. Since every signed edge $k$-subdominating function of $G$ is a minus edge $k$-subdominating function for $G$, we have

$$\gamma'_k(G) \geq \gamma'_m(G). \quad (1)$$

A minus $k$-subdominating function (M$k$S$F$) for $G$ is defined in [1] as a function $f : V(G) \to \{-1, 0, 1\}$ such that $f(N[v]) \geq 1$ for at least $k$ vertices of $G$ where $N[v]$ is the closed neighborhood of $v$. The minus $k$-subdomination number of a graph $G$, denoted by $\gamma_{ks}^{-101}(G)$, is equal to $\min \{ f(V(G)) \mid f \text{ is a } MksF \text{ of } G \}$. The minus $k$-subdomination number has been studied by several authors (see for example [3, 4, 5]).

If $k = m$, then the minus $k$-subdomination number is called the minus domination number. The minus domination number was introduced by Dunbar et al. in [2].

In this note we initiate the study of the minus edge $k$-subdomination in graphs and present some (sharp) bounds for this parameter. Here are some well-known results on $\gamma'_m(G), \gamma'_{ks}(G)$ and $\gamma_{ks}^{-101}(G)$.

**Theorem A.** ([7]) Let $G$ be a connected graph of order $n \geq 2$ and size $m$. Then

$$\gamma'_m(G) \geq n - m.$$ 

**Theorem B.** ([9]) For any connected graph $G$ of order $n \geq 2$, $\gamma'_m(G) \geq \frac{(4-n)\Delta}{4}$.

**Theorem C.** ([9]) For any connected graph $G$ of order $n \geq 2$ and size $m$,

$$\gamma'_m(G) \geq \frac{4m - (\Delta - \delta)n^2}{4(2\Delta - 1)}.$$ 

**Theorem D.** ([6]) Let $G$ be a connected graph of order $n \geq 3$, size $m$ and $1 \leq k \leq m - 1$. Then

$$\gamma'_{ks}(G) \geq n + k + 1 - 2m.$$ 

**Theorem E.** ([6]) Let $\Psi(m) = \min \{ \gamma_s'(|G|) \mid G \text{ is a graph of size } m \}$. Then for any simple graph $G$ of order $n \geq 3$, size $m$ and integer $1 \leq k \leq m$,

$$\gamma'_{ks}(G) \geq \Psi(t) - (m - t),$$

for some integer $k \leq t \leq m$.

**Theorem F.** ([2])

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1. For the path $P_n$, $\gamma_{ns}^{-101}(P_n) = \lceil \frac{n}{3} \rceil$.

2. If $n \geq 3$, then $\gamma_{ns}^{-101}(C_n) = \lceil \frac{n}{3} \rceil$.

**Theorem G.** ([1]) For $n \geq 2$ and $1 \leq k \leq n - 1$, $\gamma_{ks}^{-101}(P_n) = \lceil \frac{k}{3} \rceil + k - n + 1$

**Theorem H.** ([4]) If $n \geq 3$ and $1 \leq k \leq n - 1$, then

$$\gamma_{ks}^{-101}(C_n) = \begin{cases} \lceil \frac{n-2}{3} \rceil & \text{if } k = n - 1 \text{ and } k \equiv 0, 1 \pmod{3} \\ 2 \lceil \frac{k+4}{3} \rceil - n & \text{otherwise.} \end{cases}$$

The proof of the following theorem is straightforward and therefore omitted.

**Theorem 1.** For or any graph $G$ of order $n \geq 2$ which has no isolates,

$$\gamma'_{km}(G) = \gamma_{ks}^{-101}(L(G)).$$

Theorems 1, F, G and H lead to:

**Corollary 2.** For $n \geq 2$ and $1 \leq k \leq n$,

$$\gamma'_{km}(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil & \text{if } k = n \\ 2 \lceil \frac{k}{3} \rceil + k - n + 1 & \text{otherwise.} \end{cases}$$

**Corollary 3.** For $n \geq 3$ and $1 \leq k \leq n$,

$$\gamma'_{km}(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil & \text{if } k = n \\ \lceil \frac{n-2}{3} \rceil & \text{if } k = n - 1 \text{ and } k \equiv 0, 1 \pmod{3} \\ 2 \lceil \frac{k+4}{3} \rceil - n & \text{otherwise.} \end{cases}$$

## 2 Lower bounds on the MEkSDNs of graphs

Let $f$ be an MEkSDF of $G$. An edge $e$ is said to be a $+1$ edge if $f(e) = 1$, a $0$ edge if $f(e) = 0$ and it is said to be a $−1$ edge if $f(e) = −1$. In this section we first present a lower bound for $\gamma'_{km}(G)$ in terms of $k$, the size of $G$, the minimum degree and the maximum degree of $G$ and then we find a lower bound for $\gamma'km(G)$ in terms of $k$, the order and the size of $G$. Finally, we generalize Theorem E to the minus edge $k$-subdomination number.

**Theorem 4.** Let $G$ be a simple graph of size $m$, minimum degree $\delta$, maximum degree $\Delta$ and no isolates. Then

$$\gamma'_{km}(G) \geq \frac{2k\delta}{2\Delta - 1} - m.$$  

**Proof.** Let $(d_1, \ldots, d_n)$ be the degree sequence of $G$ where $d_1 \leq d_2 \leq \ldots \leq d_n$. Assume $g$ is a $\gamma'_{km}(G)$-function of $G$ and let $g[e] \geq 1$ for $k$ distinct edges $e$ in $\{ e_{j_1} = u_{j_1}v_{j_1}, \ldots, e_{j_k} = u_{j_k}v_{j_k} \}$. Define $f : E(G) \rightarrow \{ 0, \frac{1}{2}, 1 \}$ by $f(e) = \frac{g(e) + 1}{2}$ for each $e \in E(G)$. We have

$$\sum_{i=1}^{k} f(N_G[e_{j_i}]) \geq \sum_{i=1}^{k} \frac{g(N_G[e_{j_i}]) + \deg(u_{j_i}) + \deg(v_{j_i}) - 1}{2} \geq k\delta + \sum_{i=1}^{k} \frac{g(N_G[e_{j_i}]) - 1}{2} \geq k\delta + \sum_{i=1}^{k} \frac{g(N_G[e_{j_i}]) - 1}{2} \geq k\delta.$$  

(2)
On the other hand,
\[ \sum_{i=1}^{k} f(N_{G[e_i]}) \leq \sum_{e \in E} f(N_{G[e]}) = \sum_{e=uv \in E} (\deg(u) + \deg(v) - 1)f(e) \]
\[ \leq \sum_{e \in E} (2\Delta - 1)f(e) \]
\[ = (2\Delta - 1)f(E(G)). \]

By (1) and (2), \( f(E(G)) \geq \frac{k\delta}{2\Delta - 1}. \) Since \( g(E(G)) = 2f(E(G)) - m, \)
\[ \gamma'_{km}(G) = g(E(G)) \geq \frac{2k\delta}{2\Delta - 1} - m, \]
as desired. \( \square \)

As an immediate consequence of Theorem 4 we have:

**Corollary 5.** For every \( r \)-regular \((r \geq 1)\) graph \( G \) of size \( m \), \( \gamma'_{km}(G) \geq \frac{2rk}{2r - 1} - m. \)
Furthermore, this bound is sharp when \( r = 1. \)

Now we prove that for any simple connected graph \( G \) of size \( m \geq 2 \) and any integer \( 1 \leq k \leq m - 1 \), \( \gamma'_{km}(G) \geq n + k + 1 - 2m. \)

**Theorem 6.** Let \( G \) be a simple connected graph of order \( n \geq 3 \), size \( m \) and \( 1 \leq k \leq m - 1. \) Then
\[ \gamma'_{km}(G) \geq n + k + 1 - 2m. \]
Furthermore, the bound is sharp for each odd \( k \geq 7. \)

**Proof.** The proof is by induction on \( m. \) Obviously, the statement is true for \( m = 2, 3. \)
Assume the statement is true for all simple connected graphs of size less than \( m, \) where \( m \geq 4. \) Let \( G \) be a simple connected graph of size \( m \) and let \( f \) be a \( \gamma'_{km}(G) \)-function.
We may assume \( Z \neq \emptyset, \) for otherwise we have \( \gamma'_{km}(G) = \gamma'_{ks}(G) \) and the result follows by Theorem D. We consider two cases.

**Case 1.** There exists a pendant edge \( e = uv \in E(G) \) for which \( f(e) = 0. \)
Let \( \deg(u) = 1 \) and \( G' = G - u. \) First let \( e \notin X. \) If \( k \leq m - 2, \) then obviously \( f, \) restricted to \( G', \) is an MEkSDF of \( G' \) and by the inductive hypothesis, we have
\[ f(E(G)) = f(E(G')) \geq (n - 1) + k + 1 - 2(m - 1) = n + k + 2 - 2m. \]
If \( k = m - 1, \) then by Theorem A we have
\[ f(E(G)) = f(E(G')) \geq (n - 1) - (m - 1) = n + k + 1 - 2m. \]
Now let \( e \in X. \) If \( k = 1, \) then \( f \) must assign +1 to an edge incident to \( v \) and so
\[ f(E(G)) \geq 3 - m \geq n + k + 1 - 2m. \]
We therefore assume \( k \geq 2. \) Then \( f, \) restricted to \( G', \) is an ME(k-1)SDF of \( G' \), and by the inductive hypothesis we have
\[ \gamma'_{km}(G) = f(E(G)) = f(E(G')) \geq (n - 1) + (k - 1) + 1 - 2(m - 1) \geq n + k + 1 - 2m. \]
Subcase 2.2

For any pendant edge \( e \) in \( G \), \( f(e) \neq 0 \).

Since \( Z \neq \emptyset \), there exists a non-pendant edge \( e = uv \in E(G) \) for which \( f(e) = 0 \). First let \( e \) be a non-bridge edge. If \( e \notin X \), then \( f \), restricted to \( G - e \), is an MEkSDF of \( G - e \). If \( k \leq m - 2 \), then by the inductive hypothesis

\[
\gamma'_{km}(G) = f(E(G)) = f(E(G - e)) \geq n + k + 1 - 2(m - 1) = n + k + 3 - 2m.
\]

If \( k = m - 1 \), then \( f \), restricted to \( G - e \), is an MEDF of \( G - e \) and by Theorem A we have

\[
\gamma'_{km}(G) = f(E(G)) = f(E(G - e)) \geq n - (m - 1) = n + m + 2 - 2m.
\]

Let \( e \in X \). If \( k = 1 \), then an argument similar to that described in Case 1 shows that \( \gamma'_{km}(G) \geq n + k + 1 - 2m \). Assume that \( k \geq 2 \). Then \( f \), restricted to \( G - e \), is an ME(k-1)SDF of \( G - e \), by the inductive hypothesis,

\[
\gamma'_{km}(G) = f(E(G)) = f(E(G - e)) \geq n + (k - 1) + 1 - 2(m - 1) = n + k + 2 - 2m.
\]

Now assume \( e \) is a bridge and \( G_1 \) and \( G_2 \) are the connected components of \( G - e \) and \( u \in G_1 \). We consider two subcases.

Subcase 2.1 For \( i = 1, 2 \), \( X \cap E(G_i) \neq \emptyset \). Let \( |X \cap E(G_1)| = k_1 \) and \( |X \cap E(G_2)| = k_2 \). Then for \( i = 1, 2 \), the function \( f \), restricted to \( G_i \), is an MEkSDF for \( G_1 \). Hence, \( \gamma'_{km}(G_i) \leq f(E(G_i)) \) for \( i = 1, 2 \). First let \( E(G_i) \subseteq X \) for \( i = 1, 2 \). By Theorem A,

\[
\gamma'_{km}(G) = f(E(G)) = f(E(G_1)) + f(E(G_2)) \geq n - (m - 1) \geq n + k + 2 - 2m.
\]

Now without loss of generality we assume \( E(G_1) \not\subseteq X \). If \( E(G_2) \not\subseteq X \), then by the inductive hypothesis and Theorem A

\[
\gamma'_{km}(G) = f(E(G)) = f(E(G_1)) + f(E(G_2)) \geq n + k + 3 - 2m.
\]

If \( E(G_2) \not\subseteq X \), then by the inductive hypothesis

\[
\gamma'_{km}(G) = f(E(G)) = f(E(G_1)) + f(E(G_2)) \geq n + k + 3 - 2m.
\]

Subcase 2.2 \( X \cap E(G_1) = \emptyset \) (the case \( X \cap E(G_2) = \emptyset \) is similar). First let \( e \notin X \) and \( G' = G_2 + uv \). We claim that \( f \) assigns \(-1\) to all edges of \( G_1 \). If \( E(G_1) \cap P \neq \emptyset \), where \( P = \{ e \in E(G) \mid f(e) = 1 \} \), then we define \( g : E(G) \rightarrow \{-1, 0, +1\} \) by \( g(e) = -1 \) if \( e \in E(G_1) \) and \( g(e) = f(e) \) if \( e \in E(G) \setminus E(G_1) \). Then \( g \) is a MEkSDF of \( G \) of weight less than \( f \), a contradiction. This proves our claim. Then \( k \leq k_2 < |E(G')| \) and by the inductive hypothesis we have

\[
\gamma'_{km}(G) = f(E(G)) = f(E(G_1)) + f(E(G')) \geq -|E(G_1)| + |V(G')| + k_2 + 1 - 2|E(G')| \geq n + k_2 + 1 - 2m.
\]
Now assume \( e \in X \). We may assume \( k \geq 2 \) for otherwise the result follows as in Case 1. If \( E(G') \notin X \) and \( f(v) \geq 1 \), then by inductive hypothesis we have

\[
\gamma'_{km}(G) = f(E(G)) = f(E(G_1)) + f(E(G')) \\
\geq |E(G_1)| + |V(G')| + k - 1 - 2|E(G')| \\
= -2m + (n + 1) + k + 1 + |E(G_1)| + |V(G_1)| \\
\geq n + k + 1 - 2m.
\]

If \( E(G') \notin X \) and \( f(v) \leq 0 \), then \( f \) restricted to \( G' \) is an ME\((k - 1)\)SDF of \( G' \) and by inductive hypothesis we have

\[
\gamma'_{km}(G) = f(E(G)) = f(E(G_1)) + f(E(G')) \\
\geq |E(G_1)| + |V(G')| + (k - 1) - 2|E(G')| \\
= -2m + (n + 1) + k + 1 + |E(G_1)| - |V(G_1)| \\
\geq n + k + 2 - 2m.
\]

If \( E(G') \subseteq X \) and \( f(v) \geq 1 \), then \( f \) restricted to \( G' \) is an MEDF of \( G' \) and by Theorem A we have

\[
\gamma'_{km}(G) = f(E(G)) = f(E(G_1)) + f(E(G')) \\
\geq |E(G_1)| + |V(G')| - |E(G')| \\
= n + k + 1 - 2m + (m - k - |V(G_1)|) \\
\geq n + k + 2 - 2m.
\]

Finally, if \( E(G') \subseteq X \) and \( f(v) \leq 0 \), then \( f \) restricted to \( G' \) is an ME\((k - 1)\)SDF of \( G' \) and by inductive hypothesis we have

\[
\gamma'_{km}(G) = f(E(G)) = f(E(G_1)) + f(E(G')) \\
\geq |E(G_1)| + |V(G')| + (k - 1) - 2|E(G')| \\
= n + k + 1 - 2m + |E(G_1)| - |V(G_1)| \\
\geq n + k + 1 - 2m.
\]

To prove sharpness, we consider two cases.

- \( k \geq 7 \) is odd and \( k = m - 1 \). Let \( G \) be obtained from star \( K_{1,k-3} \) with vertex set \( \{v,v_1,\ldots,v_{k-3}\} \) and edge set \( \{vv_i \mid 1 \leq i \leq k - 3\} \) by adding three pendant edges \( v_1v_1',v_2v_2',v_3v_3' \) and an edge \( v_1v_2 \). Define \( f : V(G) \rightarrow \{-1,0,1\} \) by \( f(v_1v_2) = 1 \), \( f(vv_i) = 1 \) if \( 1 \leq i \leq \frac{k-1}{2} \) and \( f(e) = -1 \) otherwise. Then \( f \) is an ME\(k\)SDF of \( G \) with \( f(E(G)) = n + k + 1 - 2m \).

- \( k \geq 7 \) is odd and \( k \leq m - 2 \). Let \( G \) be obtained from star \( K_{1,k-2} \) with vertex set \( \{v,v_1,\ldots,v_{k-2}\} \) and edge set \( \{vv_i \mid 1 \leq i \leq k - 2\} \) by adding pendant edges \( v_1v_i',v_2v_i',v_3v_i' \) for \( j = 1,2,\ldots,m - k - 1 \) and \( v_1v_2 \). Define \( f : V(G) \rightarrow \{-1,0,1\} \) by \( f(v_1v_2) = 1 \), \( f(vv_i) = 1 \) if \( 1 \leq i \leq \frac{k-1}{2} \) and \( f(e) = -1 \) otherwise. Then \( f \) is an ME\(k\)SDF of \( G \) with \( f(E(G)) = n + k + 1 - 2m \).

This completes the proof. \( \square \)

Xu and Zhou in [9] defined \( \eta(m) = \min \{ \gamma'_m(G) \mid G \text{ is a graph of size } m \} \) for any positive integer \( m \). The proof of the following Lemma is straightforward and therefore omitted.

**Lemma 7.** Let \( \eta \) be as above. Then
1. \( m \geq \eta(m) \) for every positive integer \( m \), and

2. \( \eta(a) + \eta(b) \geq \eta(a + b) \) for each pair of positive integers \( a \) and \( b \).

The proof of the following theorem is essentially similar to the proof of Theorem 6 of [6].

**Theorem 8.** For any simple graph \( G \) of order \( n \geq 3 \), size \( m \) and integer \( 1 \leq k \leq m \),

\[
\gamma'_{km}(G) \geq \eta(t) - (m - t),
\]

for some integer \( k \leq t \leq m \). Furthermore, this bound is sharp when \( t = k \).

**Proof.** The statement holds for all simple graphs of size \( m = 1, 2, 3 \). Now assume \( m \geq 4 \).

Let, to the contrary, \( G \) be a simple graph of size \( m \geq 4 \) such that \( \gamma'_{km}(G) < \eta(t) - (m - t) \) for every integer \( k \leq t \leq m \). Choose such a graph \( G \) with as few edges as possible for which \( \lambda(G) + |T(G)| \) is maximum, where \( \lambda(G) \) denotes the number of components of \( G \) and \( T(G) = \{ u \in V(G) \mid \deg(u) \leq 2 \} \). Without loss of generality we may assume \( G \) has no isolated vertices. Let \( f \) be a \( \gamma'_{km}(G) \)-function. We may assume \( Z \neq \emptyset \), for otherwise we have \( \gamma'_{km}(G) = \gamma_{ks}(G) \) and the result follows by Theorem E. Let \( G_1, \ldots, G_{\lambda(G)} \) be the connected components of \( G \). If \( G_i \simeq K_2 \) for each \( 1 \leq i \leq \lambda(G) \), then obviously

\[
\gamma'_{km}(G) = k - (m - k) \geq \eta(k) - (m - k).
\]

Let \( G \) have component \( H \) of size at least 2.

**Claim 1.** \( E(H) \cap (M \cup Z) \subseteq X \).

Let \( e \in E(H) \cap M \) (the case \( e \in E(H) \cap Z \) is similar). Suppose that, to the contrary, \( e \notin X \). Assume \( G' \) is obtained from \( G - e \) by adding a new component \( u_0v_0 \). Define \( g : E(G') \mapsto \{-1, 0, 1\} \) by \( g(u_0v_0) = -1 \) and \( g(x) = f(x) \) if \( x \in E(G) \setminus \{e\} \).

Obviously, \( g \) is an MEkSDF of \( G' \) with \( g(E(G')) = f(E(G)) \) and \( \lambda(G') + |T(G')| > \lambda(G) + |T(G)| \). This contradicts the assumptions on \( G \). Thus \( e \in X \).

**Claim 2.** There is no non-pendant edge \( e = uv \in E(H) \cap Z \).

Let \( e = uv \in E(H) \cap Z \) be a non-pendant edge. Since \( e \in X \), \( f(u) \geq 1 \) or \( f(v) \geq 1 \). Let without loss of generality \( f(u) \geq 1 \) and let \( G' \) be obtained from \( G - e \) by adding a pendant edge \( uv' \). Then obviously \( g : E(G') \mapsto \{-1, 0, 1\} \), which is defined by \( g(uv') = 0 \) and \( g(x) = f(x) \) if \( x \in E(G) \setminus \{e\} \), is an MEkSDF of \( G' \) with \( g(E(G')) = f(E(G)) \) and \( \lambda(G') + |T(G')| > \lambda(G) + |T(G)| \). This contradicts the assumptions on \( G \).

**Claim 3.** For every non-pendant edge \( e = uv \in E(H) \cap M \) we have \( \deg(u) = \deg(v) = 2 \).

If \( f(u) \geq 1 \) (the case \( f(v) \geq 1 \) is similar), then an argument similar to that described in claim 2 leads to a contradiction. Hence, \( f(u) = f(v) = 0 \). Since \( e \) is a non-pendant edge, \( \deg(u), \deg(v) \geq 2 \). Let \( \deg(u) \geq 3 \) (the case where \( \deg(v) \geq 3 \) is similar). Then there is a +1 edge \( e' = uv \) at \( u \). Assume \( G' \) is obtained from \( G - \{e, e'\} \) by adding a new vertex \( z \) and two new edges \( vz \) and \( wz \). Define \( g : E(G') \mapsto \{-1, 0, 1\} \) by \( g(vz) = -1 \), \( g(wz) = 1 \) and \( g(x) = f(x) \) if \( x \in E(G) \setminus \{e, e'\} \). Obviously, \( g \) is an MEkSDF of \( G' \) with \( g(E(G')) = f(E(G)) \) and \( \lambda(G') + |T(G')| > \lambda(G) + |T(G)| \), a contradiction. Hence, \( \deg(u) = \deg(v) = 2 \).

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Claim 4. Let \( e = uv \in E(H) \cap M \) be a non-pendant edge and \( uu', vv' \in E(G) \). Then \( uu', vv' \in X \).

Let, to the contrary, \( uu' \notin X \) (the case \( vv' \notin X \) is similar). Since \( e \in X \), \( f(uu') = f(vv') = 1 \). Suppose that \( \text{deg}(u') = 1 \) and \( G' \) is obtained from \( G - \{e, uu'\} \) by adding a pendant edge \( vv_1 \) and a new component \( u_0v_0 \). Define \( g : E(G') \rightarrow \{-1, 0, 1\} \) by \( g(vv_1) = -1 \), \( g(u_0v_0) = 1 \) and \( g(x) = f(x) \) if \( x \in E(G) \setminus \{e, uu'\} \). Then \( g \) is an MEkSDF of \( G' \) with \( g(E(G')) = f(E(G)) \) and \( \lambda(G') + |T(G')| > \lambda(G) + |T(G)| \), a contradiction. Therefore \( \text{deg}(v') \geq 2 \).

First let \( u' = v' \). Since \( uu' \notin X \), we have \( vv' \notin X \). Suppose that there exists a -1 or 0 pendant edge \( u'z \) at \( u' \). By Claim 1, \( u'z \in X \), which implies that \( f(u') \geq 1 \).

Let \( G' \) be the graph obtained from \( G - \{e\} \) by adding a new component \( u'v' \). Define \( g : E(G') \rightarrow \{-1, 0, 1\} \) by \( g(u_0v_0) = -1 \) and \( g(x) = f(x) \) if \( x \in E(G) \setminus \{e\} \). Obviously, \( g \) is an MEkSDF of \( G' \) with \( g(E(G')) = f(E(G)) \) and \( \lambda(G') + |T(G')| > \lambda(G) + |T(G)| \), a contradiction. Therefore, there is no -1 or 0 pendant edge at \( u' = v' \). If there exists a -1 non-pendant edge at \( u' \), then an argument similar to that described in Claim 3 shows that \( \text{deg}(v') = 2 \), a contradiction. Thus every edge at \( u' \) is +1 edge. This forces \( uu' \notin X \), a contradiction.

Now let \( u' \notin v' \). Since we have assumed \( uu' \notin X \), it follows that \( f(u') \leq 1 \). If there is a -1 or 0 pendant edge \( u'w \) at \( u' \), then by Claim 1 we have \( u'w \in X \) and hence, \( f(u') = f(N[u'w]) \geq 1 \). If there is a -1 non-pendant edge at \( u' \), then \( \text{deg}(u') = 2 \) by Claim 3 and hence, \( f(u') = 0 \). It follows that \( f(u') = 0, 1 \).

When \( f(u') = 1 \), define \( G' \) to be the graph obtained from \( G - \{e\} \) by adding a new component \( u_0v_0 \). Then \( g : E(G') \rightarrow \{-1, 0, 1\} \) defined by \( g(u_0v_0) = -1 \) and \( g(x) = f(x) \) if \( x \in E(G) \setminus \{e\} \) is an MEkSDF of \( G' \) with \( g(E(G')) = f(E(G)) \) and \( \lambda(G') + |T(G')| > \lambda(G) + |T(G)| \), a contradiction. Therefore \( f(u') = 0 \) and hence, there exists a -1 edge \( u'u'' \) at \( u' \). By claim 1 we have \( \text{deg}(u'') \neq 1 \). Hence, \( \text{deg}(u'') = 2 \) (see Claim 3). Let \( G' \) be obtained from \( G - \{e, uu', uu''\} \) by adding a new component \( u_0v_0 \) and a new vertex \( z \) along with two edges \( u''z, vz \). Then \( g : E(G') \rightarrow \{-1, 0, 1\} \) defined by \( g(u_0v_0) = -1 \), \( g(u''z) = -1 \), \( g(zv) = 1 \) and \( g(x) = f(x) \) if \( x \in E(G) \setminus \{e, uu', uu''\} \) is an MEkSDF of \( G' \) with \( g(E(G')) = f(E(G)) \) and \( \lambda(G') + |T(G')| > \lambda(G) + |T(G)| \), a contradiction. Therefore \( uu' \notin X \), a contradiction.

Claim 5. \( E(H) \cap P \subseteq X \).

Let \( e = uv \in E(H) \cap P \). If there is a -1 non-pendant edge at \( u \) or at \( v \), then by Claim 4 we have \( e \in X \). If there exists a -1 or 0 pendant edge \( e' \) at \( u \) and no 0 or -1 pendant edge at \( v \), then since \( e' \in X \), \( f(u) \geq 1 \). Since there is not any -1 edge at \( v \), \( f(v) \geq 1 \). Hence \( f(N[uw]) \geq 1 \) and \( e \in X \). If there exist -1 or 0 pendant edges at \( u \) and \( v \) then \( f(u), f(v) \geq 1 \). Thus \( e \in X \). Obviously, if there is not any -1 or 0 pendant edges at \( u \) and \( v \), we see that \( e \in X \).

Let \( G_1, \ldots, G_s \) be the connected components of \( G \) for which \( E(G_i) \subseteq X \). Thus, \( f|_G \) is a \( \gamma_{m}^{i} \)-function on \( G_i \) for each \( 1 \leq i \leq s \). Now by Claims 1 and 5, \( X \cap \bigcup_{i=s+1}^{\infty} E(G_i) = \emptyset \).
Let $|E(G_i)| = m_i$ for each $1 \leq i \leq \lambda(G)$. Then $|X| = \sum_{i=1}^{s} m_i \geq k$ and $\sum_{i=s+1}^{\lambda(G)} m_i \leq m - k$. Then by Lemma 7,

$$\gamma_{k\text{m}}'(G) = \sum_{i=1}^{s} \gamma_{k\text{m}}'(G_i) - \sum_{i=s+1}^{\lambda(G)} m_i$$

$$\geq \sum_{i=1}^{s} \eta(m_i) - \sum_{i=s+1}^{\lambda(G)} m_i$$

$$\geq \eta(\sum_{i=1}^{s} m_i) - \sum_{i=s+1}^{\lambda(G)} m_i$$

$$= \eta(t) - (m - t).$$

Where $t = \sum_{i=1}^{s} m_i \geq k$

In order to prove that the lower bound is sharp when $t = k$, let $H_1$ be a graph of size $k$ with $\gamma_{k\text{m}}'(H_1) = \eta(k)$ and let $H_2$ be a graph of size $m - k$ such that $V(H_1) \cap V(H_2) = \emptyset$. Suppose $G = H_1 \cup H_2$ and $f$ is a $\gamma_{k\text{m}}'(H_1)$-function. Then $g : E(G') \rightarrow \{-1, 0, 1\}$ defined by $g(e) = f(e)$ if $e \in E(H_1)$ and $g(e) = -1$ if $e \in E(H_2)$, is an MEkSDF of $G$ with $g(E(G)) = \eta(k) - (m - k)$. This completes the proof. 

\section*{References}


