Some variants of the Chebyshev–Halley family of methods with fifth order of convergence

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In this paper we present some techniques for constructing high-order iterative methods in order to approximate the zeros of a non-linear equation $f(x) = 0$, starting from a well-known family of cubic iterative processes. The first technique is based on an additional functional evaluation that allows us to increase the order of convergence from three to five. With the second technique, we make some changes aimed at minimizing the calculus of inverses. Finally, looking for a better efficiency, we eliminate terms that contribute to the error equation from sixth order onwards.

The paper contains a comparative study of the asymptotic error constants of the methods and some theoretical and numerical examples that illustrate the given results. We also analyse the efficiency of the aforementioned methods, by showing some numerical examples with a set of test functions and by using adaptive multi-precision arithmetic in the computation.

Keywords: non-linear equations; iterative methods; Chebyshev-Halley method; order of convergence; computational efficiency

2000 AMS Subject Classification: 65H05

1. Introduction

There is a great variety of iterative methods for solving a scalar non-linear equation $f(x) = 0$. Without any doubt, Newton’s method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0,$$

is the most famous and studied method in the mathematical literature. In particular, it is well known that Newton’s method is locally quadratically convergent to a root of the aforesaid equation.

When a sequence $\{x_n\}$ tends to a limit $\alpha$ and it satisfies

$$\lim_{n \to \infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} = C,$$

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with \( p \in \mathbb{N} \) and \( C \neq 0 \), it is said that the order of convergence of the sequence is \( p \). \( C \) is known as the asymptotic error constant. As in particular cases, for \( p = 1 \), \( p = 2 \) or \( p = 3 \), the convergence is said linear, quadratic or cubic, respectively.

Most of the well-known one-point cubically convergent iterative methods belong to the one-parameter family

\[
x_{n+1} = \varphi_{\lambda}(x_n) = x_n - \left(1 + \frac{L_f(x_n)}{2(1 - \lambda L_f(x_n))}\right) \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0,
\]

where \( L_f(x) = f(x)f''(x)/f'(x)^2 \) and \( \lambda \in \mathbb{R} \).

This family has been analysed by different authors (see [19], or [1] for a geometrical construction) and it includes Chebyshev’s method (also called Euler–Chebyshev’s method) for \( \lambda = 0 \), Halley’s method for \( \lambda = 1/2 \) and super-Halley method for \( \lambda = 1 \) as in particular cases. For more detailed information about these third-order iterative methods, we recommend the classical books [2,14] or [16].

Our goal in this paper is to present some techniques for constructing new iterative methods starting from the family (Equation (1)). The first variant that we present here involves a new evaluation of the function at another point and it allows an increase in the order of convergence from three to five. In fact, the values of \( f(x_n) \) appearing in Equation (1) are replaced by a linear combination of evaluations of the functions in points previously computed (see [5,6,12]).

Next, with the aim of minimizing the calculus of inverses, we substitute \( 1/(1 - \lambda L_f(x_n)) \) by its approximation \( 1 + \lambda L_f(x_n) \). Finally, looking for the efficiency, we delete the terms that contribute to the error equation from the sixth order onwards.

A comparative study of the asymptotic constants and three different applications to a family of polynomials, the computation of the \( n \)th root of a real number and the calculus of the universal Julia sets for the studied methods are done.

In many problems we need to compute more digits, more quickly and with more precision. We can use an interactive symbolic mathematics system of computation, such as Maple, with adaptive multi-precision arithmetics and by using a floating point representation to obtain up to 2010 decimal places in the mantissa. However, we can obtain better results by increasing the order of the method. This fact allows us to get more precision widening the mantissa at each step.

The preceding procedure has been tested in a set of functions. The numerical results for these functions seem to show that, at least on this set, the new method works better in terms of both order and efficiency.

2. First variant of the Chebyshev–Halley family

In this section, we develop a technique to modify some given iterative methods to approximate the roots of a non-linear univariate function \( f \). Throughout this section, let \( f \) be a univariate function with a simple root \( \alpha \), i.e. \( f(\alpha) = 0 \) and \( f'(\alpha) \neq 0 \).

Let \( \{x_n\} \) be a sequence that tends to the root \( \alpha \). We define the error in the \( n \)th step by \( e_n = x_n - \alpha \). Notice that if we have \( e_{n+1} = C e_n^p + O_{p+1} \), where we denote \( O_{p+1} = O(e_n^{p+1}) \), then \( p \) is the order of convergence of the sequence and \( C \) is the asymptotic error constant.

Let us assume that \( f \) is differentiable enough in \( \alpha \) and let us denote

\[
A_k = \frac{f^{(k)}(\alpha)}{k! f'(\alpha)}, \quad k \geq 2.
\]
2.1 Chebyshev’s method

The technique that we present here has been used in [6] to obtain a modification of the widely used Chebyshev’s method. From Equation (1) if we take $\lambda = 0$, this iterative method can be written as

$$x_{n+1} = \psi_0(x_n) = x_n - \left(1 + \frac{1}{2} L_f(x_n) \right) \frac{f(x_n)}{f'(x_n)},$$

and it is shown that its error equation is $e_{n+1} = x_{n+1} - \alpha = (2A_2^2 - A_3)e_n^3 + O_4$, i.e. Chebyshev’s method has cubic convergence and $2A_2^2 - A_3$ is its asymptotic error constant.

In [5], a new method is considered by substituting $f(x_n)$ by the linear combination $\tilde{f}(x_n) = \gamma_1 f(x_n) + \gamma_2 f(\psi_0(x_n))$ and $L_f(x_n)$ by

$$\tilde{L}_f(x_n) = \frac{f''(x_n) \tilde{f}(x_n)}{f'(x_n)^2}$$

in Equation (3). Then the parameters $\gamma_1$ and $\gamma_2$ are chosen in order to reach the highest order of convergence possible. For $\gamma_1 = \gamma_2 = 1$, it is established that the method

$$x_{n+1} = \psi_0(x_n) - (1 + L_f(x_n)) \frac{f(\psi_0(x_n))}{f'(x_n)} - \frac{f''(x_n) f(\psi_0(x_n))^2}{2 f'(x_n)^3}$$

has fifth order of convergence. In fact, we have $x_{n+1} - \alpha = 3(2A_2^2 - A_3)e_n^5 + O_6$.

2.2 Halley’s method

Now we are going to construct a variant of another relevant iterative method, Halley’s method that is given by

$$x_{n+1} = \psi_{1/2}(x_n) = x_n - \frac{2}{2 - L_f(x_n)} \frac{f(x_n)}{f'(x_n)}.$$ 

In this case, the error equation is $e_{n+1} = x_{n+1} - \alpha = (A_2^2 - A_3)e_n^3 + O_4$ so the convergence is cubic and the asymptotic error constant is $A_2^2 - A_3$.

As in the case of Chebyshev’s method, a possible improvement can be obtained by substituting $f(x_n)$ by the linear combination $\tilde{f}(x_n) = \gamma_1 f(x_n) + \gamma_2 f(\psi_{1/2}(x_n))$ and $L_f(x_n)$ by $\tilde{L}_f(x_n) = f''(x_n) \tilde{f}(x_n)/f'(x_n)^2$. Then the parameters $\gamma_1$ and $\gamma_2$ can be chosen in order that the new method

$$x_{n+1} = x_n - \frac{2 f'(x_n) (\gamma_1 f(x_n) + \gamma_2 f(\psi_{1/2}(x_n)))}{2 f'(x_n)^2 - f''(x_n)(\gamma_1 f(x_n) + \gamma_2 f(\psi_{1/2}(x_n)))}$$

reaches the highest order of convergence. In the next section, we show that if we take $\gamma_1 = \gamma_2 = 1$ then the corresponding method

$$x_{n+1} = x_n - \frac{2 f'(x_n) (f(x_n) + f(\psi_{1/2}(x_n)))}{2 f'(x_n)^2 - f''(x_n)(f(x_n) + f(\psi_{1/2}(x_n)))}$$

has fifth order of convergence. In addition, the error difference equation in this case is $x_{n+1} - \alpha = 3(A_2^2 - A_3)^2 e_n^5 + O_6$. 
2.3 *Chebyshev–Halley family of methods*

In order to have bigger generality in the procedure introduced for Chebyshev’s and Halley’s methods we consider again the family of iterative methods defined in Equation (1). It can be shown that, for each \( \lambda \in \mathbb{R} \) the error equation is

\[
e_{n+1} = x_{n+1} - \alpha = (2(1 - \lambda)A_2^2 - A_3)e_n^3 + O_4.
\]

Then the methods in this family have cubic convergence. As in particular cases, we have the Chebyshev’s method (for \( \lambda = 0 \)), Halley’s method (for \( \lambda = 1/2 \)) and super-Halley method (for \( \lambda = 1 \)). As we can see, the asymptotic error constants depend on the parameter \( \lambda \).

By following the previously introduced procedure, we can try to improve the order of convergence by defining a new variant of Equation (1):

\[
x_{n+1} = \Psi_{\lambda_1, \lambda_2, \lambda_2}(x_n) = x_n - \left( 1 + \frac{1}{2} \frac{\tilde{L}_f(x_n)}{1 - \lambda \tilde{L}_f(x_n)} \right) \frac{\tilde{f}(x_n)}{f'(x_n)},
\]

where

\[
\tilde{f}(x_n) = \gamma_1 f(x_n) + \gamma_2 f(\psi_\lambda(x_n)), \quad \tilde{L}_f(x_n) = \frac{f''(x_n) \tilde{f}(x_n)}{f'(x_n)^2}
\]

and \( \psi_\lambda(x) \) is given by Equation (1).

Now we have four parameters in order to get the highest order of convergence. As we have said, by using this technique for Chebyshev’s and Halley’s methods we obtain new methods with convergence of order five. We wonder if in this general case this result can be improved. We will show that for these methods the order of convergence five is reached by setting \( \gamma_1 = \gamma_2 = 1 \) and \( \lambda_2 = \lambda_1 = \lambda \), i.e.

\[
x_{n+1} = \Psi_\lambda(x_n) = x_n - \left( 1 + \frac{1}{2} \frac{\tilde{L}_f(x_n)}{1 - \lambda \tilde{L}_f(x_n)} \right) \frac{\tilde{f}(x_n)}{f'(x_n)}.
\]

In this case, we obtain the error equation

\[
e_{n+1} = x_{n+1} - \alpha = 3(2(1 - \lambda)A_2^2 - A_3)^2e_n^5 + O_6,
\]

as we will see in the next section. Consequently, with the same order of convergence, for a given problem, we can choose an adequate method in order to do the asymptotic error constant as small as possible. This fact could have interesting practical applications as we will see in Section 5.

2.4 *Local convergence results for the Chebyshev–Halley family of methods and its first variant*

We have previously introduced the family of iterative methods (Equation (8)) as modifications of some well-known iterative methods. Now we analyse the local convergence of these methods for simple roots \( f(\alpha) = 0, \ f'(\alpha) \neq 0 \). In fact, we show that all of them converge to a root \( \alpha \) of the equation \( f(x) = 0 \) with fifth order of convergence. In addition, we also give the expression of the corresponding error equation.

First, we give a convergence result for the uniparametric family of methods (Equation (1)). This is a special case of Theorem 2.1 in [13].
THEOREM 2.1  Let $f : I \to \mathbb{R}$ be a real function, where $I$ is a neighbourhood of $\alpha$, a simple root of $f(x)$ ($f'(\alpha) \neq 0$). Assume that $f$ has derivatives at least until fifth order in $I$. Then, for each $\lambda \in \mathbb{R}$, the family of iterative methods defined by Equation (1) is locally convergent with cubic of convergence. In addition, it satisfies the error equation

$$e_{n+1} = x_{n+1} - \alpha = \psi_\lambda(x_n) - \alpha = (2(1 - \lambda)A_2^2 - A_3) e_n^3 + O_4.$$  

Proof  From the following Taylor expansions of $f(x)$, $f'(x)$ and $f''(x)$ we obtain

$$f(x_k) = f'(\alpha) \sum_{j=1}^{5} A_j e^j + O_6, \quad f'(x_k) = \frac{df(x_k)}{de} + O_5, \quad f''(x_k) = \frac{df'(x_k)}{de} + O_4.$$  

Setting $L_1 = f(x_k)/f'(x_k)$ and $L_2 = f''(x_k)/2f'(x_k)$, we get

$$e_{n+1} = \psi_\lambda(x_n) - \alpha = e_n - \left(1 + \frac{L_1 L_2}{1 - 2A_1 L_1 L_2}\right) L_1 = (2(1 - \lambda)A_2^2 - A_3) e_n^3 + O_4, \quad (10)$$  

and the result follows. ■

THEOREM 2.2  Let $f : I \to \mathbb{R}$ be a real function, where $I$ is a neighbourhood of $\alpha$, a simple root of $f(x)$ ($f'(\alpha) \neq 0$). Assume that $f$ has derivatives at least until fifth order in $I$. Then the two-step iteration function defined by Equation (8) has at most local order of convergence five. In addition, if we take $\gamma_1 = \gamma_2 = 1$ and $\lambda_1 = \lambda_2 = \lambda$ the corresponding method (Equation (9)) has maximum local order of convergence equal to five and it satisfies the error equation

$$e_{n+1} = x_{n+1} - \alpha = \Psi_\lambda(x_n) - \alpha = 3(2(1 - \lambda)A_2^2 - A_3) e_n^5 + O_6.$$  

Proof  We denote now $e_n$ the error expressions for the methods (Equation (8)), i.e.

$$e_{n+1} = \Psi_{\gamma_1,\gamma_2,\lambda_1,\lambda_2}(x_n) - \alpha.$$  

We want to find the parameters $\gamma_1$, $\gamma_2$, $\lambda_1$ and $\lambda_2$ involved in Equation (8) in order to reach the highest order of convergence. First, we notice that

$$e_{n+1} = (1 - \gamma_1)e_n + O_2.$$  

Then, we have to choose $\gamma_1 = 1$ in order to have at least quadratic convergence.

With this choice, $\gamma_1 = 1$, and with the coefficients $A_k$ defined in Equation (2), we deduce now

$$e_{n+1} = ((\gamma_2 - 1)A_3 + 2(1 - \gamma_2 + \gamma_2\lambda_1 - \lambda_2)A_2^2) e_n^3 + O_4.$$  

Consequently, we have to choose $\gamma_2 = 1$ and $\lambda_2 = \lambda_1 = \lambda$ to have at least order of convergence four. In fact, in this case we have

$$e_{n+1} = 3(2(1 - \lambda)A_2^2 - A_3) e_n^5 + O_6$$  

and then, for all $\lambda \in \mathbb{R}$, the corresponding method Equation (9) has local order of convergence equal to five. Notice that five is the maximum order of convergence for this family because there is not a value of the parameter $\lambda$ such that the corresponding asymptotic error constant

$$3(2(1 - \lambda)A_2^2 - A_3)^2$$  

vanishes for all function $f$. So, the proof is completed. ■
3. Second variant of the Chebyshev–Halley family

In this section, we introduce a new modification of the five-order method given in Equation (8) by using the following approximation:

$$\frac{1}{1 - \lambda L_f(x_n)} \sim 1 + \lambda L_f(x_n).$$

We can use it when $L_f(x_n)$ is close to zero and this happens, for instance, when $x_n \to \alpha$, with $\alpha$ being a simple root of $f(x)$.

If we substitute Equation (11) in Equations (1) and (9), we obtain the following family of iterative processes:

$$x_{n+1} = \Phi_\lambda(x_n) = x_n - \left(1 + \frac{1}{2} L_f(x_n)(1 + \lambda L_f(x_n))\right) \frac{\tilde{f}(x_n)}{f'(x_n)},$$

where $\tilde{f}(x_n) = f(x_n) + f(\phi_\lambda(x_n))$ and $\phi_\lambda(x_n)$ is given by

$$\phi_\lambda(x_n) = x_n - \left(1 + \frac{1}{2} L_f(x_n) + \frac{\lambda}{2} L_f(x_n)^2\right) \frac{f(x_n)}{f'(x_n)}.$$

This kind of modification is particularly interesting for a possible generalization to the multivariate case, because the calculation of some inverse operators is avoided.

This idea has already been considered for other authors. For instance, in [1] the aforesaid family of methods $x_{n+1} = \phi_\lambda(x_n)$ is considered as a modification of the Chebyshev method (Equation (3)).

All these methods (Equation (13)) have cubic order of convergence and the corresponding error equation is

$$e_{n+1} = x_{n+1} - \alpha = \phi_\lambda(x_n) - \alpha = (2(1 - \lambda)A_2^2 - A_3)e_n^3 + O_4,$$

i.e. they have the same asymptotic error constant (see Theorem 2.1) as the methods given in Equation (1). Now we wonder if the methods introduced in the second modification (Equation (12)) have the same error constant than methods in Equation (9). We show the answer in the following theorem.

**Theorem 3.1** Let $f : I \to \mathbb{R}$ be a real function, where $I$ is a neighbourhood of $\alpha$, a simple root of $f(x)(f'(\alpha) \neq 0)$. Assume that $f$ has derivatives at least until fifth order in $I$. Then the two-step iteration function defined by Equation (12) has local order of convergence five and the corresponding error equation is

$$e_{n+1} = x_{n+1} - \alpha = \Phi_\lambda(x_n) - \alpha = 3 \left(2(1 - \lambda)A_2^2 - A_3\right)e_n^5 + O_6.$$

**Proof** The proof follows by taking into account that, for each $\lambda \in \mathbb{R}$, $\Phi_\lambda(\alpha) = \alpha$ and $\Phi_\lambda^{(j)}(\alpha) = 0$ for $j = 1, 2, 3, 4$. Then by using Taylor’s series,

$$\Phi_\lambda(x_n) - \alpha = \frac{1}{5!} \Phi_\lambda^{(5)}(\alpha)(x_n - \alpha)^5 + O_6 = 3 \left(2(1 - \lambda)A_2^2 - A_3\right)^2 e_n^5 + O_6.$$
4. Third variant of the Chebyshev–Halley family

In this section, we try to improve the numerical implementation of the methods given in Theorem 3.1. To illustrate this technique, we consider first the case of the Chebyshev’s method introduced in Section 2.1 and its corresponding variant (Equation (12)) with $\lambda = 0$:

$$x_{n+1} = \Phi_0(x_n) = x_n - \left( 1 + \frac{1}{2} L_f(x_n) \right) \frac{\ddot{f}(x_n)}{f'(x_n)},$$

where $\ddot{f}(x_n) = f(x_n) + f(\phi_0(x_n))$ and

$$\phi_0(x_n) = x_n - \left( 1 + \frac{1}{2} L_f(x_n) \right) \frac{f(x_n)}{f'(x_n)}.$$

First we notice that Equation (14) can be written in the equivalent form

$$x_{n+1} = \phi_0(x_n) - (1 + L_f(x_n)) \frac{f(\phi_0(x_n))}{f'(x_n)} - \frac{f''(x_n) f(\phi_0(x_n))^2}{2 f'(x_n)^3}.$$

Following Kou et al. [11], we notice that the last term in the right-hand side of Equation (15) contributes to the error equation from the sixth order onwards, i.e.

$$\frac{f''(x_n) f(\phi_0(x_n))^2}{2 f'(x_n)^3} = O_6.$$

Therefore, it can be removed without affecting the error equation. In this way, we deduce a new iterative method given by

$$x_{n+1} = \Theta_0(x_n) = \phi_0(x_n) - (1 + L_f(x_n)) \frac{f(\phi_0(x_n))}{f'(x_n)},$$

where the error equation is $x_{n+1} - \alpha = \Theta_0(x_n) - \alpha = 3 \left( 2 A_2^2 - A_3 \right) e_n^5 + O_6$. As we can see, the asymptotic error constant for this method coincides with the one for Equation (14). As Equation (16) requires less calculations per step than Equation (14) we deduce that Equation (16) is more efficient than (14).

Now, by following this idea, we modify the family of methods given by Equation (12) in order to have methods with order of convergence five and with the minimum number of functional evaluations per step.

Notice that by eliminating in Equation (12) the terms that contribute to the error equation in orders bigger than five, we obtain the simpler family of methods:

$$x_{n+1} = \Theta_\lambda(x_n) = \phi_\lambda(x_n) - \left( 1 + L_f(x_n) + \frac{3\lambda L_f(x_n)^2}{2} \right) \frac{f(\phi_\lambda(x_n))}{f'(x_n)},$$

where $\phi_\lambda(x_n)$ is defined in Equation (13). As we can see these methods have local order of convergence equal to five and they have the same asymptotic error constant than Equation (12).

**Theorem 4.1** Let $f : I \to \mathbb{R}$ be a real function, where $I$ is a neighbourhood of $\alpha$, a simple root of $f(x)$ ($f'(\alpha) \neq 0$). Assume that $f$ has derivatives at least until third order in $I$. Then the two-step iteration function defined by Equation (17) has local order of convergence five and the corresponding error equation is

$$e_{n+1} = x_{n+1} - \alpha = \Theta_\lambda(x_n) - \alpha = 3 \left( 2(1-\lambda)A_2^2 - A_3 \right) e_n^5 + O_6.$$
Proof As we have previously indicated for the case of Chebyshev’s method, we notice that the methods defined in Equation (12) can be written in the following fashion:

$$\Phi_\lambda(x_n) = \phi_\lambda(x_n) - \frac{1}{2f'(x)} \left( 2g(x) + g(x)L_f(x) + \lambda g(x)L_f(x)^2 \right.$$ 

$$+ f(x)L_g(x) + g(x)L_g(x) + 2f(x)L_f(x)L_g(x) + 2g(x)L_f(x)L_g(x)$$ 

$$\left. + \lambda f(x)L_f(x)^2 + \lambda g(x)L_g(x)^2 \right)$$

where we denote $g(x) = f(\phi_\lambda(x))$ and $L_g(x) = f''(x)g(x)/f'(x)^2$.

Now, we take into account that $g(x)L_g(x)/f'(x) = O_6$, $g(x)L_f(x)L_g(x)/f'(x) = O_6$, $f(x)L_g(x)^2/f'(x) = O_6$, $g(x)L_g(x)^2/f'(x) = O_6$.

Then, by deleting the terms that contribute to the error formula in terms bigger than five and by writing $L_g(x)f(x) = L_f(x)g(x)$ and $L_f(x)L_g(x)f(x) = L_f(x)^2g(x)$, we obtain, instead of the given method $\Phi_\lambda(x_n)$, a simpler method defined by Equation (17).

5. Examples and comparison of the asymptotic error constants of the methods

In Tables 1 and 2 we show the iterative methods that have been considered in this paper together with their corresponding error expressions.

In Table 2, we have denoted

$$\tilde{f}(x) = f(x) + f(\psi_\lambda(x)), \quad \tilde{L}_f(x) = \frac{f''(x)\tilde{f}(x)}{f'(x)^2},$$

$$\hat{f}(x) = f(x) + f(\phi_\lambda(x)), \quad \hat{L}_f(x) = \frac{f''(x)\hat{f}(x)}{f'(x)^2}.$$

Table 1. Third-order methods and error expressions.

<table>
<thead>
<tr>
<th>Formula</th>
<th>Method</th>
<th>Error equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $\psi_\lambda(x_n) = x_n - \left(1 + \frac{1}{2} \frac{L_f(x_n)}{1 - \lambda L_f(x_n)} \right) \frac{f(x_n)}{f'(x_n)}$</td>
<td>$\left(2(1 - \lambda)A_2^2 - A_3\right) e_n^3 + O_4$</td>
<td></td>
</tr>
<tr>
<td>(13) $\phi_\lambda(x_n) = x_n - \left(1 + \frac{1}{2} \frac{\hat{L}_f(x_n)(1 + \lambda \hat{L}_f(x_n))}{\hat{L}_f(x_n)} \right) \frac{f(x_n)}{f'(x_n)}$</td>
<td>$\left(2(1 - \lambda)A_2^2 - A_3\right) e_n^3 + O_4$</td>
<td></td>
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</tbody>
</table>

Table 2. Five-order methods and error expressions.

<table>
<thead>
<tr>
<th>Formula</th>
<th>Method</th>
<th>Error equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(9) $\Psi_\lambda(x_n) = x_n - \left(1 + \frac{1}{2} \frac{\tilde{L}_f(x_n)}{1 - \lambda \tilde{L}_f(x_n)} \right) \frac{\tilde{f}(x_n)}{f'(x_n)}$</td>
<td>$3 \left(2(1 - \lambda)A_2^2 - A_3\right)^2 e_n^5 + O_6$</td>
<td></td>
</tr>
<tr>
<td>(12) $\Phi_\lambda(x_n) = x_n - \left(1 + \frac{1}{2} \frac{\hat{L}_f(x_n)(1 + \lambda \hat{L}_f(x_n))}{\hat{L}_f(x_n)} \right) \frac{\hat{f}(x_n)}{f'(x_n)}$</td>
<td>$3 \left(2(1 - \lambda)A_2^2 - A_3\right)^2 e_n^5 + O_6$</td>
<td></td>
</tr>
<tr>
<td>(17) $\Theta_\lambda(x_n) = \phi_\lambda(x_n) - \left(1 + L_f(x_n) + \frac{3\lambda L_f(x_n)^2}{2} \right) \frac{f(\phi_\lambda(x_n))}{f'(x_n)}$</td>
<td>$3 \left(2(1 - \lambda)A_2^2 - A_3\right)^2 e_n^5 + O_6$</td>
<td></td>
</tr>
</tbody>
</table>
Let us denote $C_\lambda$, the asymptotic error constant for each method $\Psi_\lambda$, $\Phi_\lambda$ or $\Theta_\lambda$, in Table 2 as:

$$C_\lambda = 3 \left( 2(1 - \lambda)A_2^2 - A_3 \right)^2.$$ 

For iterative methods with the same order of convergence, the smaller is the asymptotic error constant, the faster is the convergence. So, we compare now the asymptotic error constants $C_0$, $C_{1/2}$ and $C_1$ for the modifications of the most famous third-order methods, both in a graphical and analytical way. We have

$$C_0 = 3 \left( 2A_2^2 - A_3 \right)^2, \quad C_{1/2} = 3 \left( A_2^2 - A_3 \right)^2, \quad C_1 = 3A_3^2.$$ 

First, we represent the surfaces $C = C_\lambda (A_2, A_3)$, namely as a function of $A_2$ and $A_3$, in the same graphic (see Figure 1). Because of the symmetry for the values of $A_2$, we have to only consider the case $A_2 \geq 0$.

As we can see in Figure 1, when $A_2 \geq 0$ and $A_3 \geq 0$, these surfaces intersect in the following ways.

- The intersection of the surfaces $C = C_0$ and $C = C_{1/2}$ is the curve $A_3 = 3/2A_2^2$.
- The intersection of the surfaces $C = C_0$ and $C = C_1$ is the curve $A_3 = A_2^2$.
- The intersection of the surfaces $C = C_{1/2}$ and $C = C_1$ is the curve $A_3 = 1/2A_2^2$.

In an analytical way, we can conclude the following results concerning the asymptotic error constants $C_\lambda$ for $\lambda = 0, 1/2, 1$.

- If $A_3 < 1/2A_2^2$, then $C_1 < C_{1/2} < C_0$.
- If $1/2A_2^2 < A_3 < A_2^2$, then $C_{1/2} < C_1 < C_0$.
- If $A_2^2 < A_3 < 3/2A_2^2$, then $C_{1/2} < C_0 < C_1$.
- If $3/2A_2^2 < A_3$, then $C_0 < C_{1/2} < C_1$.

In each case, we can classify the speed of convergence of the method with the same order of convergence in terms of the asymptotic error constant. For instance, in the first case, the modifications obtained from super-Halley method are faster than the modifications of Halley’s method and these are faster than the modifications of Chebyshev’s method. In the other cases, we can deduce a similar classification. We illustrate this fact with a numerical example.

*Figure 1. Asymptotic constant surfaces $C = C_\lambda, \lambda = 0, 1/2, 1.*
Example 5.1 We consider the following polynomials as a test function in order to analyse the asymptotic error constants of the methods given in Table 2:

\[ p_j(x) = 3x^3 + (5 - j)x^2 + 4x, \quad j = 0, 1, 2, 3. \]

All of them have a simple root at \( \alpha = 0 \). For each polynomial, the asymptotic error constants satisfy the following relationships:

For \( p_0(x) \):
\[
C_1 = \frac{27}{16} < C_{1/2} = \frac{507}{256} < C_0 = \frac{1083}{64}.
\]

For \( p_1(x) \):
\[
C_{1/2} = \frac{3}{16} < C_1 = \frac{27}{16} < C_0 = \frac{75}{16}.
\]

For \( p_2(x) \):
\[
C_{1/2} = \frac{27}{256} < C_0 = \frac{27}{64} < C_1 = \frac{27}{16}.
\]

For \( p_3(x) \):
\[
C_0 = \frac{3}{16} < C_{1/2} = \frac{3}{4} < C_1 = \frac{27}{16}.
\]

In the next example, we show the importance in the choice of the parameter in order to get better results among the methods in the same family.

Example 5.2 In this example, we choose an appropriate value of the parameter \( \lambda \) in the methods in Table 2 in order to solve a given problem. In fact, we consider now the problem of approximating the \( n \)th root of a real number \( R > 1 \) by using an iterative process. This problem has been considered by many other authors (see for instance [4] or [9]) and it has different approaches and several applications.

In this example, we consider the following approach to the problem. We consider the function \( f(x) = x^n - R \) and the methods given in Table 2 to approximate the root \( \alpha = \sqrt[n]{R} \).

As we have said, the methods in Table 2 have, in general, order five. But for this particular problem, we can choose, for each \( n \in \mathbb{N} \), a value of the parameter \( \lambda \) such that the corresponding method has order of convergence at least six.

In fact we are looking for a value of \( \lambda \) such that the asymptotic error constant \( 3(2(1 - \lambda)A_2^2 - A_3)^2 \) vanishes, i.e.

\[
\lambda = 1 - \frac{A_3}{2A_2^2} = 1 - \frac{f'(\alpha)f'''(\alpha)}{3f''(\alpha)^2} = 1 - \frac{n - 2}{3(n - 1)} = \frac{2n - 1}{3(n - 1)}.
\]

Then, for instance, for the calculus of square roots (\( n = 2 \)) we obtain higher-order methods for the modifications of super-Halley method (\( \lambda = 1 \)). For the calculus of cubic roots (\( n = 3 \)), the optimum value of the parameter is \( \lambda = 5/6 \).

In Table 3 we show the precision (\( [\log_{10}|e_4|] \), with \([x]\) denoting the integer part of \( x \)) of the family of iterative methods \( \Theta_\lambda \) introduced in Equation (17) for different values of \( \lambda = 0, 1(1/6), 8 \), for the calculus of the solution of \( x^3 - R = 0 \) with \( R = 7, 8, 9 \). We have considered two different starting points: \( x_0 = 1.75 \) and 2.25.

The order of convergence can be approximated by the computational order of convergence (see [8,18]) defined by

\[
\rho = \frac{\log(|x^{(m+1)} - \alpha|/|x^{(m)} - \alpha|)}{\log(|x^{(m)} - \alpha|/|x^{(m-1)} - \alpha|)}.
\]  

The value of \( \rho \) is equal to the theoretical order of convergence, i.e. 5 for all values of \( \lambda \) except for \( \lambda = 5/6 \). In this case, the computational order of convergence is \( \rho = 7 \).
Table 3. Precision of the methods $\Theta_\lambda, \lambda = 0, 1(1/6)$, for the calculus of cubic roots after four iterations: $[-\log_{10}|e_4|]$.

<table>
<thead>
<tr>
<th>$\lambda \setminus R$</th>
<th>$x_0 = 1.75$</th>
<th>$x_0 = 2.25$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>0</td>
<td>492</td>
<td>371</td>
</tr>
<tr>
<td>1/6</td>
<td>529</td>
<td>412</td>
</tr>
<tr>
<td>1/3</td>
<td>579</td>
<td>472</td>
</tr>
<tr>
<td>1/2</td>
<td>663</td>
<td>587</td>
</tr>
<tr>
<td>2/3</td>
<td>940</td>
<td>676</td>
</tr>
<tr>
<td>5/6</td>
<td>1770</td>
<td>1300</td>
</tr>
<tr>
<td>1</td>
<td>615</td>
<td>462</td>
</tr>
</tbody>
</table>

6. Improvement of the efficiency

The efficiency index, $EFF$ (see [16]), for an iterative method is given by

$$EFF = p^{1/d},$$

where $p$ is the order of the method and $d$ is the information usage, i.e., the number of new pieces of information required per iteration.

Then, for the families of methods $\psi_\lambda, \phi_\lambda$, we have $p = d = 3$; then $EFF = 3^{1/3} \approx 1.442$. However, for the modified methods $\Psi_\lambda, \Phi_\lambda$ and $\Theta_\lambda$ considered in this paper, we obtain an improvement in the efficiency index. In fact, in all these cases we have

$$EFF = 5^{1/4} \approx 1.495.$$  

Notice that if we consider the composition of one of the iterative processes $\psi_\lambda$ or $\phi_\lambda$ with Newton’s method we can reach the sixth order of convergence ($p = 6$). But we need to evaluate the function and its derivative in a new point ($d = 5$). Then we obtain a lower efficiency index: $6^{1/5} \approx 1.431$.

We remark that the family of methods $\Theta_\lambda$ defined in Equation (17) can be considered as a family of two-step methods where the second step is the Newton’s method with the following approximation of the derivative

$$f'(\phi_\lambda(x_n)) \approx \frac{f'(x_n)}{1 + L_f(x_n) + \frac{3\lambda}{2} L_f(x_n)^2}.$$  

In [7], a generalization of the computational efficiency for a multi-precision arithmetic is given in by the formula

$$E = p^{1-p^{-r}}$$

where $p$ is the order of the method, $r$ is the number of functional evaluations per iteration required by the method and $\kappa$ is a constant depending on the arithmetic used. For instance, $\kappa = \infty, 2, \log_2 3 \approx 1.585$, for the classical product with double precision, the classical product with multi-precision (16–230 digits) and Karatsuba method (230–3600 digits), respectively.

Table 4. Efficiencies $E = p^{1-p^{-r}}/r$.

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$\psi_\lambda, \phi_\lambda$</th>
<th>$\Psi_\lambda, \Phi_\lambda, \Theta_\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$</td>
<td>$3^{1/3} \approx 1.442$</td>
<td>$5^{1/4} \approx 1.495$</td>
</tr>
<tr>
<td>2</td>
<td>$3^{2.963} \approx 1.385$</td>
<td>$5^{2.400} \approx 1.472$</td>
</tr>
<tr>
<td>$\log_2 3$</td>
<td>$3^{2.749} \approx 1.353$</td>
<td>$5^{2.305} \approx 1.449$</td>
</tr>
</tbody>
</table>
In Table 4, the values of the efficiency for the different methods and different arithmetics are presented.

7. Numerical results

We have tested the preceding methods with seven functions. We have computed the root of each function for an initial approximation $x_0$, and we have defined at each step of the iterative method the length of the floating point arithmetic with multi-precision by

$$
\text{Digits} := p \times [-\log |e_n| + 1],
$$

where $p$ is the order of the method which extends the length of the mantissa of the arithmetic, and $[x]$ is the largest integer least or equal to $x$.

The iterative method is stopped when $|x_k - \alpha| < 10^{-2000}$, and $\alpha$ is the exact root computed with 2010 significant digits. If in the last step of any iterative method it is necessary to increase the number of digits beyond 2000, then this is done. In these methods, it is necessary to begin with one initial approximation $x_0$. Table 5 shows the expression of the test functions, the initial approximation $x_0$ which is the same for all the methods and the root $\alpha$ with seven significant digits.

Table 6 shows the number of necessary iterations for some methods and each function to compute the root with the described precision. In Table 5 we see that the best results, lowest number of iterations and lowest total number of functional evaluations (TNFE), are obtained by using the iterative method $\Theta_1(x)$.

Although the results for the improved methods, $\Psi_\lambda$, $\Phi_\lambda$ and $\Theta_\lambda$, are very similar for all the examples considered in this paper, we can conclude that, in general, the best method amongst them is $\Theta_1$. In fact, $\Theta_1$ has the smallest cost for all the methods considered in this paper.

---

### Table 5. Test functions, their roots and their initial points.

<table>
<thead>
<tr>
<th>Function</th>
<th>$\alpha$</th>
<th>$x_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1(x) = x^3 - 3x^2 + x - 2$</td>
<td>2.893289</td>
<td>2.5</td>
</tr>
<tr>
<td>$f_2(x) = x^3 + \cos x - 2$</td>
<td>1.172578</td>
<td>1.5</td>
</tr>
<tr>
<td>$f_3(x) = 2\sin x + 1 - x$</td>
<td>2.380061</td>
<td>2.5</td>
</tr>
<tr>
<td>$f_4(x) = (x + 1)e^{-x} - 1$</td>
<td>0.557146</td>
<td>1.0</td>
</tr>
<tr>
<td>$f_5(x) = e^{x^2+7x-30} - 1$</td>
<td>3.0</td>
<td>2.94</td>
</tr>
<tr>
<td>$f_6(x) = e^{-x} + \cos(x)$</td>
<td>1.746140</td>
<td>1.5</td>
</tr>
<tr>
<td>$f_7(x) = x - 3\ln x$</td>
<td>1.857184</td>
<td>2.0</td>
</tr>
</tbody>
</table>

---

### Table 6. Iteration number and total number of function evaluations (TNFE).

<table>
<thead>
<tr>
<th>$\psi_0$</th>
<th>$\psi_{1/2}$</th>
<th>$\psi_1$</th>
<th>$\phi_1$</th>
<th>$\Theta_0$</th>
<th>$\psi_{1/2}$</th>
<th>$\psi_1$</th>
<th>$\Theta_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1(x)$</td>
<td>9</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>$f_2(x)$</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>6</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>$f_3(x)$</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$f_4(x)$</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>$f_5(x)$</td>
<td>9</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>$f_6(x)$</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$f_7(x)$</td>
<td>8</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Iteration</td>
<td>56</td>
<td>53</td>
<td>53</td>
<td>52</td>
<td>40</td>
<td>39</td>
<td>38</td>
</tr>
<tr>
<td>TNFE</td>
<td>168</td>
<td>159</td>
<td>159</td>
<td>156</td>
<td>160</td>
<td>156</td>
<td>152</td>
</tr>
</tbody>
</table>
Table 7. Rational functions conjugated with the methods in Table 2.

<table>
<thead>
<tr>
<th>Method</th>
<th>Rational function</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Psi_0, \Phi_0$</td>
<td>$z^5 \frac{P_1(z)}{Q_1(z)}$</td>
</tr>
<tr>
<td>$\Theta_0$</td>
<td>$z^5 \frac{P_2(z)}{Q_2(z)}$</td>
</tr>
<tr>
<td>$\Psi_{1/2}$</td>
<td>$z^5 \frac{P_3(z)}{Q_3(z)}$</td>
</tr>
<tr>
<td>$\Phi_{1/2}$</td>
<td>$z^5 \frac{P_4(z)}{Q_4(z)}$</td>
</tr>
<tr>
<td>$\Theta_{1/2}$</td>
<td>$z^5 \frac{P_5(z)}{Q_5(z)}$</td>
</tr>
<tr>
<td>$\Psi_1$</td>
<td>$z^7 \frac{P_6(z)}{Q_6(z)}$</td>
</tr>
<tr>
<td>$\Phi_1$</td>
<td>$z^7 \frac{P_7(z)}{Q_7(z)}$</td>
</tr>
<tr>
<td>$\Theta_1$</td>
<td>$z^7 \frac{P_8(z)}{Q_8(z)}$</td>
</tr>
</tbody>
</table>

$P_1(z) = z^{11} + 14z^{10} + 90z^9 + 350z^8 + 910z^7 + 1650z^6 + 2112z^5 + 1890z^4 + 1152z^3 + 460z^2 + 110z + 12,$

$P_2(z) = z^3 + 8z^2 + 27z + 48z^2 + 42z + 12,$

$P_3(z) = z^2 + 2z + 3,$

$P_4(z) = z^{11} + 34z^{10} + 560z^9 + 5950z^8 + 45815z^7 + 272275z^6 + 1298633z^5 + 5103098z^4 + 16829477z^3 + 47206979z^2 + 113737897z + 237080400z^{10} + 429767426z^9 + 679956076z^8 + 941089208z^7 + 114076524z^6 + 1211358569z^5 + 1126099062z^4 + 915078269z^3 + 648460362z^2 + 399372779z + 121278482z + 97467608z^9 + 38070076z^8 + 12541526z^7 + 3433896z^6 + 765961z^5 + 135539z^4 + 18177z^3 + 173z^2 + 105z + 3,$

$P_5(z) = z^{11} + 14z^{10} + 90z^9 + 350z^8 + 910z^7 + 1641z^6 + 2048z^5 + 1711z^4 + 902z^3 + 281z^2 + 46z + 3,$

$P_6(z) = z^9 + 4z^8 + 10z^7 + 20z^6 + 28z^5 + 32z^4 + 30z^3 + 20z^2 + 12z + 4,$

$P_7(z) = z^{29} + 34z^{28} + 560z^{27} + 5950z^{26} + 45815z^{25} + 272275z^{24} + 1298528z^{23} + 510460z^{22} + 16813880z^{21} + 47103580z^{20} + 113225022z^{19} + 235090408z^{18} + 423548480z^{17} + 663992380z^{16} + 906983660z^{15} + 1079595728z^{14} + 1118766867z^{13} + 1007463890z^{12} + 786240860z^{11} + 529825190z^{10} + 306781077z^9 + 151611968z^8 + 63362060z^7 + 22106380z^6 + 6322580z^5 + 1443904z^4 + 253086z^3 + 31940z^2 + 2580z + 100,$

$P_8(z) = z^9 + 14z^8 + 90z^7 + 350z^6 + 910z^5 + 1638z^4 + 2002z^3 + 1530z^2 + 605z + 100,$

and

$$\Omega_k(z) = z^\rho P_k\left(\frac{1}{z}\right), \quad 1 \leq k \leq 8,$$

where $\rho = \deg(P_k(z))$.  


We finish this section of numerical examples with a comparison of the dynamics of the methods given in Table 2. We see that all of them have a dynamical behaviour completely different although they have the same order of convergence and even the same asymptotic error constant.

**Example 7.1** In this example, we consider the dynamics of the methods given in Table 2 for the case of a quadratic complex polynomial with two simple roots: \( f(z) = (z - \alpha)(z - \beta) \) with \( \alpha \neq \beta \). There are also many papers related to the problem of the dynamics of iterative processes (see [10] or [15], for instance). All of them have their origins in the Cayley’s problem [3] that consists in characterizing the global basins of attraction for each root \( \alpha \) and \( \beta \). For the case of Newton’s method applied to a quadratic polynomial the problem was successfully solved by Cayley in 1879. But for other methods or other functions, Cayley’s problem is not easy to solve as the foregoing mathematical development has shown. The notions of Julia and Fatou sets were introduced in this topic (see [3] for more information).

For the quadratic polynomial \( f(z) = (z - \alpha)(z - \beta) \) with \( \alpha \neq \beta \), the methods defined in Table 2 are conjugated, via the Möbius transform

\[
M(z) = \frac{z - \alpha}{z - \beta}
\]

with some rational functions. A function \( g(z) \) is conjugated with \( f(z) \) by a Möbius transform \( M(z) \) if \( g(z) = M \circ f \circ M^{-1}(z) \).

We can have an idea of the dynamical behaviour of an iterative method by analysing its universal Julia set. This concept was introduced by Kneisl [10] and, in a few words, consists of assigning to a family of functions (in our example, quadratic polynomials) a rational function. Then the basins of attraction for the polynomials are conformally equivalent to the basin of the rational function and this allows us to elucidate the structure of the Julia set for the set of quadratic polynomials.

In this example, we analyse the dynamical behaviour of the methods \( \Psi_\lambda, \Phi_\lambda \) and \( \Theta_\lambda \) for \( \lambda = 0, 1/2, 1 \). The conjugated rational functions associated with these methods when they are applied to a quadratic polynomial are given in Table 7.

Figure 2. The first graphic shows the universal Julia set for the methods \( \Psi_0 \) and \( \Phi_0 \). The second one is the universal Julia set for the method \( \Theta_0 \).
The rational functions (Table 7) have two super attracting fixed points: 0 and \(\infty\). In Figures 2 and 3 we assign the colours white and black to the attraction basins of 0 and \(\infty\), respectively. We have made the graphics by following the instructions given by Varona in [17].

In Figures 2 and 3, the universal Julia sets for the rational functions related to the methods defined in Table 2 are shown. As we can see, all of them give rise to highly intricate figures that show the complexity of the problem. With this example we want to show that, in general, the global convergence behaviour of the methods in Table 2 could be very different.

8. Concluding remarks

In this paper, we have presented three different techniques to construct new iterative methods starting from a well-known family of third-order iterative processes. In fact, we give some families of methods depending on a real parameter \(\lambda\) (see Table 2). All of these new methods have local order of convergence five for simple roots. We have also given the asymptotic error constants depending on the parameter \(\lambda\).

We have compared the asymptotic error constants of the methods both in an analytical and graphical way. In addition, we have shown some examples where we can choose the parameter \(\lambda\) in order to have some advantages in the solution of particular problems, such as the calculus of the \(n\)th root of a positive number. We have compared the dynamics of the methods and we have seen that all of them have a completely different behaviour.

The new methods introduced in this paper improve, not only the order of convergence of the original family, but the efficiency too. We have used different measures of the computational efficiency for this purpose.

Finally, we have numerically compared different methods given in Tables 1 and 2 by using a set of test functions. With this numerical experiment, we see that the optimal method among these given in this paper is the one defined by function \(\Theta_1\) that gives rise to the following iterative...
scheme:

\[
\begin{align*}
&x_n \text{ given} \\
\phi_1(x_n) &= x_n - \left(1 + \frac{1}{2}L_f(x_n) + \frac{1}{2}L_f(x_n)^2\right) \frac{f(x_n)}{f'(x_n)} \\
x_{n+1} &= \Theta_1(x_n) = \phi_1(x_n) - \left(1 + L_f(x_n) + \frac{3L_f(x_n)^2}{2}\right) \frac{f(\phi_1(x_n))}{f'(x_n)}
\end{align*}
\]

where \(L_f(x) = f(x)f''(x)/f'(x)^2\).

The advantage of using these high-order convergence methods can be revealed in problems where a high precision in the results is needed.

Acknowledgements

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References