GLOBAL CONVERGENCE OF DAMPED NEWTON'S METHOD FOR NONSMOOTH EQUATIONS VIA THE PATH SEARCH

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A natural damping of Newton’s method for nonsmooth equations is presented. This damping, via the path search instead of the traditional line search, enlarges the domain of convergence of Newton’s method and therefore is said to be globally convergent. Convergence behavior is like that of line search damped Newton’s method for smooth equations, including Q-quadratic convergence rates under appropriate conditions.

Applications of the path search include damping Robinson-Newton’s method for nonsmooth normal equations corresponding to nonlinear complementarity problems and variational inequalities, hence damping both Wilson’s method (sequential quadratic programming) for nonlinear programming and Josephy-Newton’s method for generalized equations.

Computational examples from nonlinear programming are given.

1. Introduction. This paper presents a novel, natural damping of (local) Newton’s method—essentially Robinson-Newton’s method (Robinson 1988)—for solving nonsmooth equations. This damping, via the so-called path search instead of the traditional line search, enlarges the domain of convergence of Newton’s method and therefore is said to be globally convergent. It is natural in the sense that the convergence behavior of the method is fundamentally the same as that of the traditional line search damped Newton’s method applied to smooth equations, which, under appropriate conditions, is roughly described as linear convergence far from a solution and superlinear, possibly quadratic, convergence near a solution. Such convergence results are amongst the strongest results of their kind, but also require strong conditions—see further discussion below. We also investigate the nonmonotone path search (§§3, 4) which is an easy extension of the nonmonotone line search (Grippo, Lampariello and Lucidi 1986).

An immediate application is damping (Robinson-) Newton’s method for solving the nonsmooth normal equations (Robinson 1992), yielding a damping procedure for both Wilson’s method (or sequential quadratic programming) (Wilson 1963, Fletcher 1987) for nonlinear programs, and Josephy-Newton’s method (Josephy 1979) for generalized equations. Path search damped Newton’s method likewise applies to the normal equation formulation of variational inequalities and nonlinear complementarity problems, as described in §5.

For the moment, our prototype of a nonsmooth function will be

\[ F_+(x) \overset{\text{def}}{=} F(x^+) + x - x_+, \quad \forall x \in \mathbb{R}^N \]

where \( F: \mathbb{R}^N \to \mathbb{R}^N \) is a smooth function, and \( x_+ \) is the vector in \( \mathbb{R}^N \) whose \( i \)th
component is \( \max\{x_i, 0\} \). \( F_+ \) is only nonsmooth on the boundaries of orthants in \( \mathbb{R}^N \). The system \( F_+(x) = 0 \) is actually the normal formulation of the nonlinear complementarity problem: find \( z \in \mathbb{R}^N \) such that

\[
 z \geq 0, \quad F(z) \geq 0, \quad \text{and} \quad \langle z, F(z) \rangle = 0,
\]

where vector inequalities are taken in a componentwise fashion. Note that if \( x \) solves the normal equation \( F_+(x) = 0 \), then \( z = \text{def} x_+ \) solves the nonlinear complementarity problem; and, conversely, a solution \( z \) of the latter yields a solution \( x = \text{def} z - F(z) \) of the former.

We need some notation to aid further explanation. Suppose \( X \) and \( Y \) are both \( N \)-dimensional normed spaces (though Banach spaces can be dealt with) and \( f: X \to Y \). We wish to solve the equation

\[
f(x) = 0, \quad x \in X.
\]

For the traditional damped Newton’s method (Ortega and Rheinboldt 1970, Burdakov 1980) we assume \( f \) is a continuously differentiable function. Suppose \( x_k \) is the \( k \)th iterate of the algorithm, and \( \hat{x}^{k+1} \) is the zero of the “linearization”

\[
 A_k(x) = f(x^k) + \nabla f(x^k)(x - x^k), \quad x \in X.
\]

Since \( A_k \) approximates \( f \), \( \hat{x}^{k+1} \) at least formally approximates a zero of \( f \). Newton’s method defines \( x^{k+1} = \text{def} \hat{x}^{k+1} \), so \( \hat{x}^{k+1} \) is called the Newton iterate. Newton’s method is damped by a line search; that is by choosing \( x^{k+1} \) on the line segment from \( x^k \) to \( \hat{x}^{k+1} \) such that the decrease in \( \| f(\cdot) \| \) after moving from \( x^k \) to \( x^{k+1} \) is close to the decrease predicted by \( \| A_k(\cdot) \| \). The normed residual \( \| f(\cdot) \| \) is the merit function for determining \( x^{k+1} \). We refer to such methods as line search damped. Some convergence results under different line search strategies are set out by O. P. Burdakov (1980).

The computational success of line search damped Newton’s method relies on uniformly bounded invertibility of the Jacobians \( \nabla f(x^k) \), which yields two key properties of the linearizations \( A_k \) that are independent of \( k \): first, \( A_k \) is a first-order approximation of \( f \) at \( x^k \), i.e., \( f(x) = A_k(x) + o(x - x^k) \) where \( o(x - x^k)/\|x - x^k\| \to 0 \) as \( x \to x^k \); and, secondly, \( A_k \) goes to zero “rapidly” on the path \( p^k(t) = \text{def} x^k + t(\hat{x}^{k+1} - x^k) \) as \( t \) goes from 0 to 1, by which we mean that, given our choice of \( A_k \) and \( \hat{x}^{k+1} \),

\[
 A_k(p^k(t)) = f(x^k) + t\nabla f(x^k)(\hat{x}^{k+1} - x^k) = (1 - t)f(x^k).
\]

We call \( p^k \) the Newton path. Observe that \( f(p^k(t)) = (1 - t)f(x^k) + o(t) \) where \( o(t)/t \to 0 \) as \( t \downarrow 0 \), uniformly for all \( k \). Thus for a fixed \( \sigma \in (0, 1) \) and all sufficiently small positive \( t \) independent of \( k \),

\[
 \| f(p^k(t)) \| \leq (1 - \sigma t)\| f(x^k) \|. 
\]

So there exists a step length \( t_k \in (0, 1] \) such that not only does the previous inequality hold at \( t = t_k \), but there is a positive lower bound on \( t_k \) uniformly for all \( k \). A (monotone) line search determines such a step length; then the damped Newton iterate is \( x^{k+1} = \text{def} p^k(t_k) \). We see that the normed residual \( \| f(\cdot) \| \) is decreased by at least a constant factor after each iteration. As the residual converges to zero, the
iterates converge to a solution and, within a neighborhood of this solution, superlinear convergence of the iterates is observed.

We propose a damping of Newton's method suitable for a nonsmooth equation \( f(x) = 0 \), using first-order approximations \( A_k \) and paths \( p^k \) with the properties summarized above. For example, suppose \( f = F^+ \) from above, and the approximation to \( f \) at \( x^k \) is

\[
A_k(x) \overset{\text{def}}{=} F(x^k) + \nabla F(x^k)(x^- - x^k) + x - x_+, \quad \forall x \in \mathbb{R}^N.
\]

The approximation error \( f(x) - A_k(x) \) is the difference between \( F(x^k) \) and the linearization \( F(x^k) + \nabla F(x^k)(x^- - x^k) \), i.e., the relationship of \( f \) to \( A_k \) is like that of a smooth function to its linearization about a given point. Newton's method—essentially Robinson-Newton's method (Robinson 1988) in this context—defines \( x^{k+1} \) as the Newton iterate \( x^{k+1} = A_k^{-1}(0) \), as in the smooth case. A naive damping of this method is a line search along the interval \([x_k, x_{k+1}]\). Instead, we follow a path \( p^k \) from \( x^k \) \((t = 0)\) to \( x^{k+1} \) \((t = 1)\) on which \( A_k \) goes to zero rapidly: \( p^k(t) \), the Newton path, is defined by the equation

\[
A_k(p^k(t)) = (1 - t)f(x^k).
\]

By assuming \( p^k(t) - x^k = O(t) \), we have that

\[
f(p^k(t)) = A_k(p^k(t)) + o(p^k(t) - x^k) = (1 - t)f(x^k) + o(t);
\]

hence there is a path length \( t_k \in (0, 1] \) such that \( ||f(p^k(t_k))|| \leq (1 - \sigma t)||f(x^k)|| \). A path search determines the path length \( t_k \); then the damped Newton iterate is \( x^{k+1} = p^k(t_k) \). It is not even necessary that \( \hat{x}^{k+1} \) exist as long as \( p^k \) can be defined on an interval \([0, T_k]\) for suitable \( T_k \in (0, 1] \). Assuming each \( A_k \) is invertible near \( x^k \), uniformly in some sense for all \( k \), the path search ensures that the residuals \( f(x^k) \) converge to zero at least linearly, and eventually we obtain superlinear convergence of the damped iterates to a solution. We have outlined path search damped Newton's method.

Since \( A_k \) need not be affine (it is piecewise smooth above), \( p^k \) need not be affine. In this case, there is no basis for the line search on \([x^k, \hat{x}^{k+1}]\). It is even possible that both \( ||A_k(t)|| \) and \( ||f(x^k + t(\hat{x}^{k+1} - x^k))|| \) initially increase as \( t \) increases from 0, causing the line search to fail altogether. The path search, however, may still be numerically and theoretically sound.

The remainder of the introduction will be devoted to related literature. For purposes of comparison, we observe that the approach proposed here is distinguished from most, if not all others by the use of nonsmooth Newton paths \( p^k \). The sections to come are as follows:

\(\S\) 2. Notation and preliminary results.
\(\S\) 4. Path search damped Newton’s method for nonsmooth equations.
\(\S\) 5. Applications.

One of the simplest approaches to (damping) Newton’s method for nonsmooth equations is suggested by J.-S. Pang (1990): set

\[
B_k(x) \overset{\text{def}}{=} f(x^k) + f'(x^k; x - x^k)
\]
where we assume the existence of the directional derivative \( f'(x; d) \) at each \( x \) in each direction \( d \). As before, suppose \( \hat{x}^{k+1} \) solves \( B_k(x) = 0 \). Assuming \( f \) is also locally Lipschitz, \( f'(x; \cdot) \) is a so-called (Bouligand)-derivative (Robinson 1987), and we say the Newton method given by \( x^{k+1} \overset{\text{def}}{=} \hat{x}^{k+1} \) is B-Newton’s method. Now \( f(x) = B_k(x) + o(x - x^k) \), and \( B_k \) is affine on the interval \([x^k, \hat{x}^{k+1}]\), hence damping B-Newton’s method by a line search makes sense. A major difficulty, however, is that the closer \( x^k \) is to a point of nondifferentiability \( f \), the smaller the neighborhood of \( x^k \) in which \( B_k \) accurately approximates \( f \). This difficulty is reflected in the dearth of strong global convergence results of such a scheme. For instance, the global convergence result (Pang 1990, Theorem 6) requires the existence of an accumulation point \( x^* \) of the sequence of iterates \( (x^k) \) at which \( \|f(\cdot)\|_2^2 \) is differentiable; this requirement partly defeats the aim of solving nonsmooth equations. Harker and Xiao (1990) apply the theory of Pang (1990) to the nonsmooth normal mapping \( f = F_+ \) associated with the nonlinear complementarity problem and provide some computational experience of line search damped B-Newton’s method. For further discussion, see §5.2.

A global Newton method with convergence results comparable to ours is given by J.-S. Pang (1991), for the “min” equation formulation of variational inequalities. The min equation for the nonlinear complementarity problem, above, is

\[
0 = f(x) = \min\{x, F(x)\},
\]

where the min operation is taken componentwise. In Pang (1991) a modified B-Newton method, with line search damping, is given for this min equation. This method is globally convergent under a regularity condition at each iterate. The regularity condition at the \( k \)th iterate ensures invertibility of the equation defining the modified B-Newton method, and is quite similar to the local invertibility property of the approximation \( A_k \) of \( F_+ \) needed in path search damped Newton’s method. Nevertheless, the approach and analysis of Pang (1991) are quite different from ours, and have some of the flavor of Han, Pang and Rangaraj (1992), discussed next.

Line search damping of general Newton methods for nonsmooth equations is considered by S.-P. Han, J.-S. Pang and N. Rangaraj (1992). Given a general iteration function \( G(x^k, \cdot) \) from \( \mathbb{R}^n \) to \( \mathbb{R}^n \), the associated Newton method defines \( x^{k+1} \overset{\text{def}}{=} \hat{x}^{k+1} \) where \( \hat{x}^{k+1} \) is such that \( f(x^k) + G(x^k, \hat{x}^{k+1} - x^k) = 0 \). To incorporate a line search on \([x^k, \hat{x}^{k+1}]\), conditions on the iteration function \( G(x^k, \cdot) \) are given to ensure a certain rate of decrease of \( \|f(x^k + t(\hat{x}^{k+1} - x^k))\| \) as \( t \) increases, at least for all small \( t > 0 \) uniformly in \( k \). Such conditions, in general, preclude \( f(x^k) + G(x^k, \cdot) \) from being a good approximation of \( f \) near \( x^k \), uniformly in \( k \). In effect, Han, Pang and Rangaraj (1989) give conditions on \( G(x^k, \cdot) \) such that line search damping is viable on the interval \([x^k, \hat{x}^{k+1}]\), while the approximation properties of \( G(x^k, \cdot) \) with respect to \( f \) are not of importance. In contrast, we must abandon the line search to retain what we believe are the fundamental properties of the approximations \( A_k \) that make the classical Newton’s method successful.

Local and global Newton methods for locally Lipschitz, semismooth functions \( f \) are presented by L. Qi (1993). Such functions are B-differentiable. The proposed Newton iterate \( \hat{x}^{k+1} \) satisfies \( f(x^k) + M_k(\hat{x}^{k+1} - x^k) = 0 \), where the matrix \( M_k \) is any member of a refinement of the Clarke generalized Jacobian (Clarke 1983) of \( f \) at \( x^k \). When \( f \) is piecewise smooth, for example \( f(x) = F_+(x) \) or \( \min\{x, F(x)\} \), \( M_k \) can be chosen very simply: roughly, let \( f_k \) be one of the smooth functions that defines \( f \) on a “piece of smoothness” containing \( x^k \), and take \( M_k \overset{\text{def}}{=} \nabla f_k(x^k) \). The undamped method reduces, in this case, to the method of M. Kojima and S. Shindo (1986). Global convergence properties of the line search damped method are similar to those...
of line search damped B-Newton’s method (Pang 1990), and can be enhanced using
the framework of Han, Pang, and Rangaraj (1992).

P. Marcotte and J.-P. Dussault (1987) have shown global convergence of Josephy-
Newton’s method (Josephy 1979) with line search damping, for monotone variational
inequalities over compact convex sets, using a nonsmooth merit function. Related
work using M. Fukushima’s smooth merit function is given in Fukushima (1992), and
by K. Taji, M. Fukushima and T. Ibaraki (1993), and J. H. Wu, M. Florian and
P. Marcotte (1990), for line search damped Josephy-Newton’s method applied to
strongly monotone variational inequalities over closed convex, but not necessarily
compact sets. Our results are sharper when they apply, i.e., in the strongly monotone
case; see Proposition 11 and the comments after its proof.

It can be argued that line search damping of a Newton method for nonsmooth
equations lacks the elegance of path search damping; putting it differently, path
search damping seems better suited than other damping methods to the standard
merit function, the normed residual. This view, however, may have little bearing on
the ultimate usefulness of the different damping procedures. For example, in §5 we
use a modification of Lemke’s algorithm to determine each path \( p^k \); hence this
implementation might be prey to the same exponential behavior as the original
Lemke’s algorithm for certain pathological problems, a difficulty not observed when
applying line search damped B-Newton’s method to such problems (Harker and Xiao
1990).

It has been pointed out by Liqun Qi that conditions sufficient for global and
superlinear convergence of nonsmooth Newton methods are weaker than the assump-
tions made in this paper, which include a local invertibility property of \( A_k \) (see
Theorem 9 and Proposition 10). For superlinear convergence see F. Bonnans (1990),
however, that substantially weaker conditions require the existence of the solution
point \( a \) priori, whereas our procedure actually proves the existence of such a point. In
fact the same criticism, and rebuttal of that criticism, can be applied to extensions of
the classical Kantorovich-Newton theorem (Ortega and Rheinboldt 1970, Theorem
12.6.2) to generalized equations (Josephy 1979, Theorem 2.2) and nonsmooth equa-
tions (Robinson 1988, Theorem 3.2). Implications of our assumptions are explored
further after Proposition 10.

The weakest assumptions for global convergence, known to the author, are for
Gauss-Newton-like methods, rather than Newton-like methods, such as the method
of J.-S. Pang and S. A. Gabriel (1993). In this basic paper we prefer to make strong
assumptions in order to use principles underlying the classical Newton method, and
obtain strong results. A Gauss-Newton approach to nonsmooth systems prior to Pang
and Gabriel (1993) is given by J.-S. Pang, S.-P. Han and N. Rangaraj (1991). Also,
M. C. Ferris and S. Lucidi (1994) present conditions on merit functions used in line
search damping, that are sufficient for global convergence.

Finally we draw the reader’s attention to the starting point of this research, K. Park
(1989), in which homotopy methods via point-based approximations are considered
for solving nonsmooth equations. Stephen M. Robinson has suggested that, while
Park takes the viewpoint of path following, this paper proposes “multiple path
following”.

2. Notation and preliminary results. Most of the interest for us is in finite
dimensions, when both \( X \) and \( Y \) are \( N \)-dimensional real spaces, \( \mathbb{R}^N \), with arbitrary
norms. Our most fundamental result, however, Theorem 9, is valid for complete
normed spaces \( X, Y \) over \( \mathbb{R} \); hence this generality of \( X \) and \( Y \) will be used unless
otherwise stated. Throughout, \( f \) is function mapping \( X \) to \( Y \), and \( B_X, B_Y \) denote the
closed unit balls in $X, Y$ respectively. The unit ball may be written $\mathbb{B}$ when the context is clear.

We write $t \downarrow 0$ to mean $t \to 0$, $t > 0$. By $o(t)$ (as $t \downarrow 0$) we mean a (vector) function of the scalar $t$ such that $o(t)/t \to 0$ in norm as $t \downarrow 0$. Likewise, by $O(t)$ (as $t \downarrow 0$) we mean $O(t)/t$ is bounded as $t \downarrow 0$. When $d \in X$, $o(d)$ denotes a function of $d$ such that $o(d)/\|d\| \to 0$ as $d \to 0$, $d \neq 0$.

The function $f$ is Lipschitz (of modulus $l \geq 0$) on a subset $X_0$ of $X$ if $\|f(x) - f(x')\|$ is bounded above by a constant multiple ($l$) of $\|x - x'\|$, for any points $x, x'$ in $X_0$. We say $f$ is locally Lipschitz if it is Lipschitz near each $x \in X$. A function $g$ from a subset $X_0$ of $X$ to $Y$ is said to be continuously invertible, or Lipschitzianly invertible (of modulus $l \geq 0$), if it is bijective and its inverse mapping is continuous, or Lipschitz (of modulus $l$), respectively. Such a function $g$ is continuously invertible, or Lipschitzianly invertible (of modulus $l$) near a point $x \in X_0$ if, for some neighborhoods $U$ of $x$ in $X$ and $V$ of $g(x)$ in $Y$, the restricted mapping $g|_{U \cap X_0}: U \cap X_0 \to V$: $x \mapsto g(x)$ is continuously invertible, or Lipschitzianly invertible (of modulus $l$). In defining this restricted mapping it is tacitly assumed that $g(U \cap X_0) \subset V$.

We are interested in approximating $f$ when it is not necessarily differentiable.

**Definition 1.** Let $X_0 \subset X$.

(1) Let $x \in X$. A first-order approximation of $f$ at $x$ is a mapping $\tilde{f}: X \to Y$ such that

$$\tilde{f}(x') - f(x') = o(x - x'), \quad x' \in X.$$

A first-order approximation of $f$ on $X_0$ is a mapping $\mathcal{A}$ on $X_0$ such that for each $x \in X_0$, $\mathcal{A}(x)$ is a first-order approximation of $f$ at $x$.

(2) Let $\mathcal{A}$ be a first-order approximation of $f$ on $X_0$. $\mathcal{A}$ is a uniform first-order approximation (with respect to $X_0$) if there exists $\Delta: (0, \infty) \to [0, \infty]$ with $\Delta(t) = o(t)$, such that for any $x, x' \in X_0$,

$$\|\mathcal{A}(x)(x') - f(x')\| \leq \Delta(\|x - x'\|).$$

$\mathcal{A}$ is a uniform first-order approximation near $x^0 \in X_0$ if, for some $\Delta(t)$ as above, (1) holds for $x, x'$ near $x^0$.

The idea of a path will be needed to define the path search damping of Newton's method.

**Definition 2.** A path (in $X$) is a continuous function $p: [0, T] \to X$ where $T \in [0, 1]$. The domain of $p$ is $[0, T]$, denoted $\text{dom}(p)$.

We note a path lifting result.

**Lemma 3.** Let $\Phi: X \to Y$, $x \in X$ and $\Phi(x) \neq 0$. Suppose the restricted mapping $\hat{\Phi} = \Phi|_{U}: U \to V$ is continuously invertible, where $U$ and $V$ are neighborhoods of $x$ and $\Phi(x)$, respectively. If $U$ is open, and $\epsilon > 0$ is such that $\Phi(x) + \epsilon \mathbb{B}_Y \subset V$, then, for $0 \leq T \leq \min(\epsilon/\|\Phi(x)\|, 1)$, the unique path $p$ of domain $[0, T]$ such that

$$p(0) = x, \quad \Phi(p(t)) = (1 - t)\Phi(x) \quad \forall t \in [0, T]$$

is given by

$$p(t) = \hat{\Phi}^{-1}((1 - t)\Phi(x)) \quad \forall t \in [0, T].$$

**Proof.** Let $U$, $\epsilon$ and $T$ be as above, and observe that $p: [0, T] \to X$: $t \mapsto \hat{\Phi}^{-1}((1 - t)\Phi(x))$ is a well-defined mapping. Suppose $q: [0, T] \to X$ is a path such that $q(0) = x$ and, for $t \in [0, T]$, $\Phi(q(t)) = (1 - t)\Phi(x)$. Clearly $q(t) = p(t)$ if and
only if \( q(t) \in U \), so consider

\[
\hat{t} = \sup \{ t \in [0, T] \mid q(s) \in U, \quad \forall s \in [0, t] \}.
\]

If \( t \in [0, T) \) and \( q(s) \in U \) for each \( s \in [0, t) \), then \( q(t) = p(t) \) by continuity of \( p \) and \( q \); so \( q(t) \) belongs to the open set \( U \). Therefore, by continuity of \( q \), \( q(t') \) belongs to \( U \) for each \( t' \in [0, T] \) near \( t \). It follows that \( \hat{t} \neq T \), hence \( \hat{t} = T \), and \( q(t) = p(t) \) for each \( t \in [0, T] \).

For a nonempty, closed convex set \( C \) in \( \mathbb{R}^N \) and each \( x \in \mathbb{R}^N \), \( \pi_C(x) \) denotes the nearest point in \( C \) to \( x \) with respect to the Euclidean norm. The existence and uniqueness of the projected point \( \pi_C(x) \) is classical in the generality of Hilbert spaces. We refer the reader to Brézis (1973, Example 2.8.2 and Prop. 2.6.1) where we also see that the projection operator \( \pi_C \) is Lipschitz of modulus 1. The normal cone to \( C \) at \( z \) is

\[
N_C(z) = \begin{cases} \{ y \in \mathbb{R}^N \mid \langle y, c - z \rangle \leq 0, \forall c \in C \} & \text{if } z \in C, \\ \emptyset & \text{otherwise}. \end{cases}
\]

The tangent cone to \( C \) at \( z \), denoted \( T_C(z) \), is \( \{ x \in \mathbb{R}^N \mid \langle x, y \rangle \leq 0, \forall y \in N_C(z) \} \) if \( z \in C \), and the empty set otherwise.

Next we present the normal mappings of Robinson (1992). These will be our source of applications (§5).

**Definition 4.** Let \( C \) be a closed, convex, nonempty set in \( \mathbb{R}^N \) and \( F : C \to \mathbb{R}^N \). The normal mapping induced by \( F \) and \( C \) is

\[
F_C = F \circ \pi_C + I - \pi_C
\]

where \( I \) is the identity operator on \( \mathbb{R}^N \).

Our applications will concern finding a zero of a normal mapping such as \( F_C \) above, where \( C \) is usually polyhedral convex. If \( F \) is smooth, its derivative mapping is denoted \( \nabla F \).

**Definition 5.** Let \( F \) and \( C \) be as in Definition 4. \( F \) is continuously differentiable (on \( C \)) if the restriction of \( F \) to the relative interior of \( C \), \( \text{ri} \ C \), is continuously differentiable and, for each relative boundary point \( \tilde{c} \) of \( C \), the limit

\[
\nabla F(\tilde{c}) = \lim_{c \to \tilde{c}} \nabla F(c)
\]

exists.

If \( F \), as above, is continuously differentiable (on \( C \)) then it is locally Lipschitz, just as in the classical case when \( C \) is a linear subspace; hence \( F_C \) is locally Lipschitz.

We relate the normal mapping \( F_C \) to the set mapping \( F + N_C \).

**Lemma 6.** Suppose \( F, C \) are as in the Definition 4, and \( F \) is locally Lipschitz. The mapping \( F_C \) is Lipschitzianly invertible near \( x \) if and only if for some neighborhoods \( U, V \) of \( \pi_C(x) \), \( F_C(x) \) respectively, \( (F + N_C)^{-1} \cap U \) is a (single valued) Lipschitz function when restricted to \( V \).

**Proof.** It is well known (Brézis 1973, Example 2.8.2) that \( c = \pi_C(x) \) if and only if \( c \in C \) and

\[
\langle x - c, c' - c \rangle \leq 0, \quad \forall c' \in C;
\]
thus \( c = \pi_c(\xi + c) \) if and only if \( \xi \in N_c(c) \). It follows that

\[
(2) \quad \xi = F_c(x), \quad c = \pi_c(x) \iff \xi \in (F + N_c)(c), \quad x = \xi + (I - F)(c).
\]

So for \( U \subset \mathbb{R}^N \), \( (F + N_c)(U) \) equals \( F_c(\pi_c^{-1}(U)) \) and, for \( \xi \in \mathbb{R}^N \), we get

\[
(3) \quad (F + N_c)^{-1}(\xi) \cap U = \left[ \pi_c(F_c^{-1}(\xi)) \right] \cap U.
\]

Let \( x^0 \in X \) and \( \xi^0 = \pi_c(x^0) \).

Suppose \( F_c \) is Lipschitzianly invertible near \( x^0 \), so, for some \( \delta > 0 \) and neighborhood \( V^0 \) of \( \xi^0 \), the restriction of \( F_c^{-1} \cap (x^0 + 2\delta B) \) to \( V_0 \) is a Lipschitz function that maps \( V^0 \) onto \( x^0 + 2\delta B \). Let

\[
U = (I - F)^{-1}(x^0 - \xi^0 + \delta B),
\]

\[
V = F_c(\pi_c^{-1}(U)) \cap (\xi^0 + \epsilon B),
\]

where \( \epsilon \in (0, \delta) \) is chosen such that \( \xi^0 + \epsilon B \subset V^0 \). Observe \( U \) is a neighborhood of \( \pi_c(x^0) \), since \( I - F \) is a continuous function which maps \( \pi_c(x^0) \) to \( x^0 - \xi^0 \). Also \( V \) is a neighborhood of \( \xi^0 \), since \( U \) is a neighborhood of \( \pi_c(x^0) \) and \( F_c \) is continuously invertible near \( x_0 \). Moreover for \( \xi \in V \) we have

\[
x \in F_c^{-1}(\xi), \quad \pi_c(x) \in U
\]

\[
\Rightarrow x = \xi + (I - F)(\pi_c(x)), \quad (I - F)(\pi_c(x)) \in x^0 - \xi^0 + \delta B
\]

\[
\Rightarrow x \in x^0 + \xi - \xi^0 + \delta B \subset x^0 + 2\delta B.
\]

Thus

\[
\emptyset \neq \left[ \pi_c(F_c^{-1}(\xi)) \right] \cap U \subset \pi_c\left[ F_c^{-1}(\xi) \cap (x^0 + 2\delta B) \right].
\]

As the set on the right is a singleton, this statement combined with (3) yields

\[
(F + N_c)^{-1}(\xi) \cap U = \pi_c\left[ F_c^{-1}(\xi) \cap (x^0 + 2\delta B) \right].
\]

In particular \( (F + N_c)^{-1} \cap U \), as a mapping on \( V \), is a Lipschitz function.

Conversely suppose the restriction of \( (F + N_c)^{-1} \cap U \) to \( V \) is a Lipschitz function, where \( U \) and \( V \) are respective neighborhoods of \( \pi_c(x^0) \) and \( \xi^0 \). Let \( U^0 \) be a neighborhood of \( \pi_c(x^0) \) in \( U \) such that \( F \) is Lipschitz on \( U^0 \), and \( V^0 = \pi_c(U^0) \cap V \). So the restriction of \( (F + N_c)^{-1} \cap U^0 \) to \( V_0 \) is a Lipschitz function, and \( V^0 \) is a neighborhood of \( \xi^0 \). Let \( U^1 = \pi_c(U^0) \), a neighborhood of \( x^0 \). Using (2), we have

\[
F_c^{-1} \cap U^1 = I + [I - F]\left( (F + N_c)^{-1} \cap U^0 \right).
\]

So the restriction of \( F_c^{-1} \cap U^1 \) to \( V_0 \) is a Lipschitz function, and it follows that \( F_c \) is Lipschitzianly invertible near \( x^0 \). □

Finally, we restate Robinson (1988, Lemma 2.3), a generalization of the Banach perturbation lemma.
Theorem 7. Let $\Omega$ be a set in $X$. Let $g$ and $g'$ be functions from $X$ into $Y$ such that $g|_{\Omega}: \Omega \to g(\Omega)$ is Lipschitzianly invertible of modulus $L > 0$, and $g - g'$ is Lipschitz on $\Omega$ of modulus $\eta > 0$. Let $x^0 \in \Omega$ and $\delta > 0$. If

(a) $\Omega \supset x^0 + \delta B_x$,
(b) $g(\Omega) \supset g(x^0) + (\delta / L) B_y$,
(c) $\eta L < 1$, then $g'|_{\Omega}: \Omega \to g'(\Omega)$ is Lipschitzianly invertible of modulus $L/(1 - \eta L) > 0$, and

$$g'(\Omega) \supset g'(x^0) + (1 - \eta L)(\delta / L) B_y.$$ 

Proof. Define the perturbation function, $h(\cdot) = g'(\cdot) - g(\cdot) - [g'(x^0) - g(x^0)]$. Let $g = \text{def} g|_{\Omega}$ and observe that the Lipschitz property of $g^{-1}$ gives

$$1/L \leq 1/\sup \{\|g^{-1}(y) - g^{-1}(y')\| / \|y - y'\| | y, y' \in g(\Omega), y \neq y'\}$$

$$= \inf \{\|g(x) - g(x')\| / \|x - x'\| | x, x' \in \Omega, x \neq x'\}.$$ 

Thus according to Robinson (1988, Lemma 2.3), $g + h$ satisfies

$$1/L - \eta \leq \inf \{(g + h)(x) - (g + h)(x') / \|x - x'\| | x, x' \in \Omega, x \neq x'\}$$

and $(g + h)(\Omega)$ contains $g(x^0) + (1 - \eta L)(\delta / L) B_y$. Therefore $(g + h)|_{\Omega}: \Omega \to (g + h)(\Omega)$ is invertible and, as above, its inverse is Lipschitz of modulus $1/((1/L) - \eta) = L/(1 - \eta L)$. The claimed properties of $g'$ hold because $g' = g + h + g'(x^0) - g(x^0)$. □

3. Motivation: Line search damped Newton's Method for smooth equations. Let $f: X \to Y$ be smooth, that is continuously differentiable. We wish to solve the nonlinear equation

$$f(x) = 0, \quad x \in X.$$ 

Suppose $x^k \in X (k \in \{0, 1, \ldots \})$ and there exists the Newton iterate $\hat{x}^{k+1}$, i.e., $\hat{x}^{k+1}$ solves the equation

$$f(x^k) + \nabla f(x^k)(x - x^k) = 0, \quad x \in X.$$ 

Newton's method is inductively given by setting $x^{k+1} = \text{def} \hat{x}^{k+1}$.

The algorithm is also called local Newton's method because the Kantorovich-Newton theorem (Ortega and Rheinboldt 1970, Theorem 12.6.2)—probably the best known convergence result for the Newton's method—shows convergence of the Newton iterates to a solution in a ball of radius $\delta > 0$ about the starting point $x^0$. Assumptions include that $\nabla f(x^0)$ is boundedly invertible; then $\delta$ is constructed small enough to ensure, by continuity of $\nabla f$ and the Banach perturbation lemma, that $\nabla f(x)$ is boundedly invertible at each $x \in x^0 + \delta B_x$. Further conditions guarantee $\delta$ can also be chosen large enough such that the sequence of iterates remains in the $\delta$-ball of $x^0$, and convergence follows. It is well known (Burdakov 1980), however, that the domain of convergence of the algorithm can be substantially enlarged using line search damping, reviewed below, which preserves the asymptotic convergence properties of the local method.

Choose line search parameters $\sigma, \tau \in (0, 1)$. As we saw in the introduction, the Newton path $p^k(t) = \text{def} x^k + t(\hat{x}^{k+1} - x^k)$ is such that $f(p^k(t)) = (1 - t)$.
GLOBAL CONVERGENCE OF DAMPED NEWTON’S METHOD

$f(x^k) + o(t)$ for $t > 0$; hence for sufficiently small $t > 0$ and $f(x^k) \neq 0$, we have Monotone Descent of the norm of the residual:

$$(MD) \quad \| f(p^k(t)) \| < (1 - \sigma t) \| f(x^k) \|.$$ 

In the Armijo line search familiar in optimization (McCormick 1983, Chapter 6, §1), the step length $t_k$ is defined as $\tau^l$, where $l$ is the smallest integer $l \in \{0, 1, \ldots \}$ such that $(MD)$ holds at $t = \tau^l$. The (line search) damped Newton iterate is $x^{k+1} = p^k(t_k)$.

More recently, in the context of unconstrained optimization, Grippo, Lampariello and Lucidi (1986) have developed a line search using a Nonmonotone Descent condition that often yields better computational results than monotone damping. Let $M \in \mathbb{N}$, the memory length of the procedure, and relax the progress criterion $(MD)$ to

$$(NmD) \quad \| f(p^k(t)) \| < (1 - \sigma t) \max_{j = 1, \ldots, \min\{M, k + 1\}} \| f(x^{k+1-j}) \|.$$ 

This is identical to $(MD)$ when $M = 1$. The nonmonotone Armijo line search would determine $t_k = \tau^l$ similar to the monotone case.

We abstract the general properties of an unspecified Nonmonotone Line search procedure:

$(NmLs)$. If $(NmD)$ holds at $t = 1$, let $t_k = \text{def} 1$.

Otherwise, choose any $t_k \in [0, 1]$ such that $(NmD)$ holds at $t = t_k$ and

$t_k \geq \tau \sup\{T \in [0, 1] | (NmD) \text{ holds } \forall t \in [0, T]\}.$

It is easy to see that the above nonmonotone Armijo line search produces a step length that fulfills $(NmLs)$. The parameter $\tau$ need not be explicitly used in damped Newton's algorithm, however, so other line search procedures in which $\tau$ is not specified may be valid; only the existence of $\tau$, independent of $k$, is needed to prove convergence.

The formal algorithm is now given.

**Line search damped Newton's method.** Given $x^0 \in X$, the sequence $(x^k)$ is inductively defined for $k = 0, 1, \ldots$ as follows.

If $f(x^k) = 0$, stop.

Find $\tilde{x}^{k+1} = p^k(x^k) = x^k - \nabla f(x^k)^{-1} f(x^k)$.

Line search: Let $p^k(t) = x^k + t(\tilde{x}^{k+1} - x^k)$ for $t \in [0, 1]$. Find $t_k \in [0, 1]$ satisfying $(NmLs)$.

Define $x^{k+1} = p^k(t_k)$.

We present a basic convergence result for the line search damped, or so-called global Newton's method. It is a corollary of Proposition 10.

**Proposition 8.** Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be continuously differentiable, $\alpha_0 > 0$ and

$X_0 = \{x \in \mathbb{R}^N | \| f(x) \| \leq \alpha_0 \}.$

Let $\sigma, \tau \in (0, 1)$ and $M \in \mathbb{N}$ be the line search parameters governing $(NmLs)$.

Suppose $X_0$ is bounded, and $\nabla f(x)$ is invertible for each $x \in X_0$. Then for each $x^0 \in X_0$, line search damped Newton’s method is well defined and the sequence $(x^k)$ converges to a zero $x^*$ of $f$. 

The residuals converge to zero at least at an R-linear rate: for some constant $\rho \in (0, 1)$ and all $k > 0$,
\[
\|f(x^k)\| \leq \rho \max\{\|f(x^{k-j})\| \mid j = 1, \ldots, \min\{M, k\}\} \\
\leq \rho^k \|f(x^0)\|.
\]

The rate of convergence of $(x^k)$ to $x^*$ is Q-superlinear. In particular, if $\nabla f$ is Lipschitz near $x^*$ then $(x^k)$ converges Q-quadratically to $x^*$:
\[
\|x^{k+1} - x^*\| \leq d\|x^k - x^*\|^2
\]
for some constant $d > 0$ and all sufficiently large $k$.

The convergence properties of the method depend both on the uniform accuracy of each of the approximations $A_k$ of $f$ at $x_k$ for all $k$, i.e.,
\[
\frac{\|f(x) - A_k(x)\|}{\|x - x^k\|} \to 0 \quad \text{as} \quad x \to x^k \quad (x \neq x^k), \quad \text{uniformly} \quad \forall k
\]
and on the uniformly bounded invertibility of the approximations $A_k$:
\[
\sup_k \left\| \nabla f(x^k)^{-1} \right\| < \infty.
\]

These uniformness properties are disguised in the boundedness (hence compactness) hypothesis on $X_0$.

4. Path search damped Newton’s method for nonsmooth equations. We want to solve the nonlinear and, in general, nonsmooth equation
\[
f(x) = 0, \quad x \in X
\]
where $f: X \to Y$. We proceed as in the smooth case, the main difference being the use of first-order approximations of the function $f$ instead of linearizations.

Suppose $x^k \in X$ ($k \in \{0, 1, \ldots\}$) and $A_k = \text{def} A_k(x^k)$, where $A_k$ is a first-order approximation of $f$ on $X$. Recall (Definition 2) a path is a continuous mapping of the form $p: [0, T] \to X$ where $T \in [0, 1]$. Assume there exists a path $p^k: [0, 1] \to X$ such that, for $t \in [0, 1]$,
\[
A_k\left(p^k(t)\right) = (1 - t)f(x^k);
\]
$p^k$ is the Newton path. Assume also that $p^k(t) - x^k = O(t)$. Note that $\hat{x}^{k+1} = \text{def} p^k(1)$ is the Newton iterate, a solution of the equation $A_k(x) = 0$.

In nonsmooth Newton’s method, the next iterate is given by $x^{k+1} = \text{def} \hat{x}^{k+1}$ just as in smooth Newton’s method. For nonsmooth functions having a uniform first-order approximation we call this Robinson-Newton’s method, since the ideas behind local convergence results are essentially the same as those employed in the seminal paper (Robinson 1988), although a special uniform first-order approximation called the point-based approximation is required there (see discussion after Proposition 10). (Robinson 1988, Theorem 3.2) is the nonsmooth version of the classical Kantorovich-Newton convergence theorem (Ortega and Rheinboldt 1970, Theorem 12.6.2). Applications of Robinson-Newton’s method include sequential quadratic programming (Fletcher 1987), or Wilson’s method (Wilson 1963) for nonlinear programming; and
Josephy-Newton’s method (Josephy 1979) for generalized equations (see also §5). As in the smooth case, however, convergence of the method is shown within a ball of radius \( \delta > 0 \) about \( x^0 \). The first-order approximation \( A_0 = \text{def} \mathcal{A}(x^0) \) of \( f \) at \( x^0 \) is assumed to be Lipschitzian invertible near \( x^0 \) and then, for small enough \( \delta \), the continuity properties of \( \mathcal{A}(\cdot) \) are used in Lemma 7 to show that \( \mathcal{A}(x) \) is Lipschitzian invertible near each \( x \in x^0 + \delta B_X \). Further assumptions guarantee that Robinson-Newton’s method generates a sequence lying in this \( \delta \)-ball of \( x^0 \), and convergence follows. We propose to enlarge the domain of convergence using a path search.

Now \( A_k \) is a first-order approximation of \( f \) at \( x^k \), and \( o(p^k(t) - x^k) = o(t) \) assuming \( p^k(t) - x^k = O(t) \). With (4) we find that

\[
(5) \quad f(p^k(t)) = (1 - t)f(x^k) + o(t),
\]

i.e., \( f \) moves toward zero rapidly on the path \( p^k \) as \( t \) increases from 0, at least initially.

In the spirit of §3, we fix \( \sigma, \tau \in (0, 1) \). As before, assuming \( f(x^k) \neq 0 \), we have for all sufficiently small positive \( t \),

\[
\|f(p^k(t))\| < (1 - \sigma t)\|f(x^k)\|.
\]

So the nonmonotone descent condition below, formally identical to that given in §3, is valid given any memory size \( M \in \mathbb{N} \) and all small positive \( t \):

\[
(\text{NmD}) \quad \|f(p^k(t))\| < (1 - \sigma t)\max\{\|f(x^k+1-j)\| \mid j = 1, \ldots, \min\{M, k + 1\}\}.
\]

The path search is any procedure satisfying

If (NmD) holds at \( t = 1 \), let \( t_k = \text{def} 1 \).

Otherwise, choose any \( t_k \in [0, 1] \) such that (NmD) holds at \( t = t_k \) and

\[
t_k \geq \tau \sup\{T \in [0, 1] \mid (\text{NmD}) \text{ holds } \forall t \in [0, T]\}.
\]

The (path search) damped Newton iterate is \( x^{k+1} = \text{def} p^k(t_k) \). The path search takes \( t_k = 1 \), hence the Newton iterate \( x^{k+1} = p^k(1) \), if possible, otherwise a path length \( t_k \) large enough to prevent premature convergence under further conditions. Also, as in §3, \( \tau \) need not be specified explicitly.

However, the path search given above is too restrictive in practice; in particular it assumes existence of the Newton iterate \( x^{k+1} \in A_k^{-1}(0) \). Motivated by computation (§5) we only assume the path \( p^k : [0, T_k] \to X \) can be constructed for some \( T_k \in (0, 1] \). The path is constructed by iteratively extending the domain \([0, T_k]\) until either it cannot be extended further (e.g., \( T_k = 1 \)) or the progress criterion (NmD) is violated at \( t = T_k \). The path length \( t_k \) is then chosen with reference to this upper bound \( T_k \). Of course we are still assuming that the path \( p^k \) satisfies the Path conditions:

\[
(\text{P}) \quad p^k(0) = x^k, \quad A_k(p^k(t)) = (1 - t)f(x^k), \quad \forall t \in \text{dom}(p^k)
\]

where \( \text{dom}(p^k) \in [0, T_k] \). The idea of extending the path \( p^k \) is justified by Lemma 3, taking \( \Phi = A_k \) and \( x = p^k(T_k) \), which says that if \( A_k \) is continuously invertible near \( p^k(T_k) \) and \( T_k < 1 \), then \( p^k \) can be defined over a larger domain (i.e., \( T_k \) is strictly increased) while still satisfying (P).
We use the following Nonmonotone Pathsearch in which $p^k: [0, T_k] \to X$ is supposed to satisfy (P).

(NmPs). If (NmD) holds at $t = T_k$, let $t_k = \text{def} T_k$.

Otherwise, choose any $t_k \in [0, T_k]$ such that (NmD) holds at $t = t_k$ and

$$t_k \geq \tau \sup\{T \in [0, T_k] | (\text{NmD}) \text{ holds } \forall t \in [0, T]\}.$$ 

Now we give the algorithm formally. As the choice of $t_k$ may be determined during the construction of $p^k$ (see §5), we have not separated the construction of $p^k$ from the path search in the algorithm.

Path search damped Newton’s method. Given $x^0 \in X$, the sequence $(x^k)$ is inductively defined for $k = 0, 1, \ldots$ as follows.

If $f(x^k) = 0$, stop.

Path search: Let $A_k = \text{def} \mathcal{A}(x^k)$. Construct a path $p^k: [0, T_k] \to X$, $T_k \in [0, 1]$, satisfying (P) such that if $T_k < 1$ then either $A_k$ is not continuously invertible near $p^k(T_k)$, or (NmD) fails at $t = T_k$. Find $t_k \in [0, T_k]$ satisfying (NmPs).

Define $x^{k+1} = \text{def} p^k(t_k)$.

Our main result, showing the convergence properties of global Newton’s method, is now given. The first two assumptions on the first-order approximation $\mathcal{A}$ correspond, in the smooth case, to uniform continuity of $\nabla f$ on $X_0$ and uniformly bounded invertibility of $\nabla f(x)$ for $x \in X_0$, respectively. The purpose of the third, technical assumption is to guarantee the existence of paths used by the algorithm (cf. the continuation property of Rheinboldt (1969)). With the exception of dealing with the paths $p^k$, the proof uses techniques well developed in standard convergence theory of damped algorithms (e.g. (Ortega and Rheinboldt 1970, McCormick 1983, Fletcher 1987)).

**Theorem 9.** Let $f: X \to Y$ be continuous, $\alpha_0 > 0$ and

$$X_0 = \text{def} \{x \in X | \|f(x)\| \leq \alpha_0\}.$$ 

Let $\sigma, \tau \in (0, 1)$ and $M \in \mathbb{N}$ be the parameters governing the path search (NmPs).

Suppose

1. $\mathcal{A}$ is a uniform first-order approximation of $f$ on $X_0$.
2. $\mathcal{A}(x)$ is uniformly Lipschitzianly invertible near each $x \in X_0$, meaning for some constants $\delta, \epsilon, L > 0$ and for each $x \in X_0$, there are sets $U_x$ and $V_x$ containing $x + \delta B_X$ and $f(x) + \epsilon B_Y$ respectively, such that $\mathcal{A}(x)|_{U_x}: U_x \to V_x$ is Lipschitzianly invertible of modulus $L$.
3. For each $x \in X_0$ and $T \in (0, 1]$, if $p: [0, T] \to X$ is such that $p(0) = x$ and, for each $t \in [0, T]$, $\mathcal{A}(x)(p(t)) = (1 - t)f(x)$ and $\mathcal{A}(x)$ is continuously invertible near $p(t)$, then there exists $p(T) = \text{def} \lim_{t \uparrow T} p(t)$ with $\mathcal{A}(x)(p(T)) = (1 - T)f(x)$.

Then for any $x^0 \in X_0$, path search damped Newton’s method is well defined such that the sequence $(x^k)$ converges to a zero $x^*$ of $f$.

The residuals converge to zero at least at an $R$-linear rate: for some constant $\rho \in (0, 1)$ and all $k > 0$,

$$\|f(x^k)\| \leq \rho \max\{\|f(x^{k-1})\| | j = 1, \ldots, \min\{M, k\}\} \leq \rho^k\|f(x^0)\|.$$ 

The rate of convergence of $(x^k)$ to $x^*$ is $Q$-superlinear; indeed, if for $c > 0$ and all points $x$ near $x^*$ we have $\|\mathcal{A}(x)(x^*) - f(x^*)\| \leq c\|x - x^*\|^2$, then the rate of convergence is $Q$-quadratic:

$$\|x^{k+1} - x^*\| \leq cL\|x^k - x^*\|^2$$

for sufficiently large $k$. 

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PROOF. Assume, without loss of generality, that \( f(x^k) \neq 0 \) for each \( k \). Let \( \delta, \epsilon, L \) be the constants given by Hypothesis (2) of the theorem and, for each \( k \), let \( \hat{A}_k \) be the Lipschitzian invertible mapping \( \mathcal{A}(x^k)_{U_k} : U_k \to V_k \) given there. We may also assume without loss of generality that \( U_k \) is open: replace \( U_k \) by its interior, \( \hat{U}_k \), \( \delta \) by any \( \delta' \in (0, \delta) \), \( \epsilon \) by any \( \epsilon' \in (0, \min(\epsilon, \delta/L)) \), and \( V_k \) by \( A_k(\hat{U}_k) \). The only point which may be unclear is that \( A_k(\hat{U}_k) \) contains \( A_k(x^k) + \epsilon B_X \); let \( y \in A_k(x^k) + \epsilon B_X \); then \( x = \operatorname{def} A_k^{-1}(y) \cap U_k \) is an element of \( U_k \) such that

\[
\| x - x^k \| \leq L \| y - A_k(x^k) \| \leq \delta,
\]

hence \( y \in A_k(\hat{U}_k) \). Let \( \Delta(t) = o(t) \) be the uniform bound on the accuracy of \( \mathcal{A}(x) \), \( x \in X_0 \), as given by Definition 1; and \( \Delta(0) = 0 \) for convenience.

We show the algorithm is well defined. Suppose \( x^k \in X_0 \). Lemma 3 can be used to show existence of a (unique) continuous function \( p : I \to X \) of largest domain \( I \), with respect to the conditions

* \( p(0) = x^k \);
* either \( I = [0, 1] \), or \( I = [0, T) \) for some \( T \in (0, 1) \); and
* for each \( t \in I \),

\[
A_k(p(t)) = (1 - t)f(x^k),
\]

\( A_k \) is continuously invertible near \( p(t) \).

If \( I = [0, 1] \), \( p^k = p \) is a path acceptable to the algorithm. If \( I = [0, T) \) then, by Hypothesis (3), we can extend \( p \) continuously to domain \([0, T]\) by \( p(T) = \lim_{t \to T^-} p(t) \), for which \( A_k(p(T)) = (1 - T)f(x^k) \). In this case, by maximality of \( I, A_k \) is not continuously invertible at \( p(T) \), so the extension \( p : [0, T] \to X \) is acceptable as \( p^k \). This shows that existence of the path \( p^k \), required by the algorithm, is guaranteed if \( x^k \in X_0 \). The fact that each \( x^k \) belongs to \( X_0 \) follows from the inequality

\[
\max\{\| f(x^k-j) \| \mid j = 1, \ldots, \min\{M, k\} \} \leq \| f(x^0) \|,
\]

which is easily shown by induction.

Recall that \( p^k : [0, T_k] \to X, 0 \leq T_k \leq 1 \), is the path determined by the algorithm. We will show that \( T_k \) and the path length \( t_k \) are bounded away from zero.

We aim to find a positive constant \( \gamma \) such that for each \( k \),

\[
T_k \geq S_k = \min\{\gamma/\| f(x^k) \|, 1\}
\]

and

\[
p^k(t) = \hat{A}_k^{-1}((1 - t)f(x^k)) \quad \forall t \in [0, S_k],
\]

\[
(\text{NmD}) \text{ holds } \forall t \in [0, S_k].
\]

To show these we need several other facts, the first of which is given by Lemma 3 when \( \Phi = \operatorname{def} A_k \) and \( U = \operatorname{def} U_k \):

\[
p^k(t) = \hat{A}_k^{-1}((1 - t)f(x^k)), \quad 0 \leq t \leq \min\{\epsilon/\| f(x^k) \|, T_k\}.
\]
Now if $T_k < 1$ and (NmD) holds at $t = T_k$ then, by choice of $p^k$, $A_k$ is not continuously invertible near $p^k(T_k)$; thus

$$T_k \geq \frac{\epsilon}{\|f(x^k)\|}.$$

Another fact is that

$$(9) \quad \text{(NmD) holds for} \quad 0 \leq t \leq \min \left\{ \frac{\gamma}{\|f(x^k)\|}, T_k \right\}$$

where $\gamma \in (0, \epsilon]$ will be specified below. Recall the path search parameter $\sigma \in (0, 1)$. We choose $\beta > 0$ such that $\Delta(\beta) < \beta(1 - \sigma)/L$ for $0 < \beta \leq \beta$. Then for

$$0 \leq t \leq \min \left\{ \frac{\beta}{L\|f(x^k)\|}, \frac{\epsilon}{\|f(x^k)\|}, T_k \right\},$$

we have

$$\|p^k(t) - x^k\| = \|A_k^{-1}((1 - t)f(x^k)) - A_k^{-1}(f(x^k))\| \quad \text{by (8)}$$

$$\leq L\|((1 - t)f(x^k) - f(x^k)) = tL\|f(x^k)\| \quad \text{by Hypothesis (2)}.$$
Since each iterate $x^k$ belongs to $X_0$, we get

$$t_k \geq \tau \min \{ \gamma / \| f(x^k) \|, 1 \} \geq t = \gamma \min \{ \alpha_0, 1 \} > 0.$$  

Therefore, by a short induction argument, for each $k > 0$

$$\| f(x^k) \| \leq \rho \max \{ \| f(x^{k-j}) \| \mid j = 1, \ldots, \min\{M, k\} \} \leq \rho^k \| f(x^0) \|$$

where $\rho = \text{def} (1 - \sigma \hat{t})^{1/M}$. This validates the claim of linear convergence of the residuals. As a result, for some $K_1 > 0$ and each $k \geq K_1$,

$$\gamma / \| f(x^k) \| \geq 1,$$

whence $S_k = 1$. So $p^k(1) = \hat{A}_k^{-1}(0)$ (by (6)) and (NmD) holds at $t = 1$ (by (7)). Also $T_k = 1$, since $1 = S_k \leq T_k \leq 1$; hence (NmPs) determines $t_k = \text{def} 1$ and the damped Newton iterate is the Newton iterate: $x^{k+1} = \hat{A}_k^{-1}(0)$. For $k \geq K_1$,

$$\| x^{k+1} - x^{k} \| = \| \hat{A}_k^{-1}(0) - \hat{A}_k^{-1}(f(x^k)) \|$$

$$\leq L \| f(x^k) \| \quad \text{by Hypothesis (2)}$$

$$\leq L \rho^k \| f(x^0) \| \quad \text{by (11)}.$$  

So if $K, K' \in \mathbb{N}, K \leq K' \leq K'$, then

$$\| x^K - x^{K'} \| \leq \sum_{k = K}^{\infty} \| x^{k+1} - x^{k} \|$$

$$\leq \sum_{k = K}^{\infty} L \rho^k \| f(x^0) \|$$

$$= \frac{L \| f(x^0) \|}{1 - \rho} \rho^k \rightarrow 0 \quad \text{as } K \rightarrow \infty, K' \geq K.$$  

This shows that $(x^k)$ is a Cauchy sequence, hence convergent in the complete normed space $X$ with limit, say, $x^*$. Since $\| f(x^k) \| \rightarrow 0$, continuity of $f$ yields $f(x^*) = 0$.

Finally note that for all sufficiently large $k$, $x^* \in x^k + \delta \mathbb{B}_X \subset U_{x^k}$. For such $k \geq K_1$, Hypothesis (2) yields

$$\| x^{k+1} - x^* \| = \| \hat{A}_k^{-1}(f(x^*)) - \hat{A}_k^{-1}(A_k(x^*)) \|$$

$$\leq L \| f(x^*) - A_k(x^*) \|$$

$$\leq L \Delta(\| x^k - x^* \|).$$

The second inequality demonstrates $Q$-superlinear convergence. With the first inequality we see that if $\| \varphi(x)(x^*) - f(x^*) \| \leq c \| x - x^* \|^2$ for some $c > 0$ and all $x$ near $x^*$, then

$$\| x^{k+1} - x^* \| \leq cL \| x^k - x^* \|^2$$

for sufficiently large $k$. \quad $\Box$
We can say more about the rate of convergence of the residuals: the inequality
\[ \| f(x^k) \| \leq \rho \max \{ \| f(x^{k-j}) \| \mid j = 1, \ldots, \min\{M, k\} \} \]
shows that convergence is \( M \)-step \( Q \)-linear, and, if \( f \) is Lipschitz near \( x^* \) as it must be in the smooth case, the residuals converge at a \( Q \)-superlinear or \( Q \)-quadratic rate.

We will find the following version of Theorem 9 useful in applications (§5.1).

**Proposition 10.** Let \( f : \mathbb{R}^N \to \mathbb{R}^N \) be continuous, \( \alpha_0 > 0 \) and
\[ X_0 \overset{\text{def}}{=} \{ x \in \mathbb{R}^N \mid \| f(x) \| \leq \alpha_0 \} \]
Let \( \sigma, \tau \in (0, 1) \) and \( M \in \mathbb{N} \) be the parameters governing the path search (NmPs).

Suppose \( X_0 \) is bounded, \( \mathcal{A} \) is a first-order approximation of \( f \) on \( X_0 \), and for each \( x \in X_0 \), the following hold:

1. \( \mathcal{A} \) is a uniform first-order approximation of \( f \) near \( x \).
2. \( \mathcal{A}(x) \) is invertible near \( x \), and continuous.
3. There exists \( l_x > 0 \) such that if \( \mathcal{A}(x) \) is invertible near any \( x' \in \mathbb{R}^N \), then it is Lipschitzianly invertible of modulus \( l_x \) near \( x' \).
4. There are \( \eta_x : [0, \infty) \to [0, \infty] \) and \( U_x \) of \( x \) such that \( \lim_{s \to 0} \eta_x(s) = 0 \) and \( \mathcal{A}(x'^1) - \mathcal{A}(x'^2) \) is Lipschitz of modulus \( \eta_x(\| x'^1 - x'^2 \|) \) on \( U_x \), for \( x'^1, x'^2 \in U_x \).

Then for \( X = Y = \mathbb{R}^N \), the hypotheses (and conclusions) of Theorem 9 hold.

**Proof.** We first strengthen Hypothesis (2): each \( x \in X_0 \) has an open neighborhood \( U_x \) such that

1. \( \mathcal{A}(x') \) is Lipschitziany invertible of modulus \( L_x \), and \( \mathcal{A}(x')(U_x) \) contains \( f(x') + \varepsilon_x B_Y \).

To see this appeal to Hypotheses (2)-(4). There are neighborhoods \( U_x \) and \( V_x \) of \( x \) and \( f(x) \), respectively, and \( l_x > 0 \) for which \( \mathcal{A}(x)|_{U_x} : U_x \to V_x \) is Lipschitziany invertible of modulus \( l_x > 0 \); and there is \( \eta_x : [0, \infty) \to [0, \infty] \) such that \( \lim_{s \to 0} \eta_x(s) = 0 \), \( \eta_x(0) = 0 \), and \( \mathcal{A}(x'^1) - \mathcal{A}(x'^2) \) is Lipschitz of modulus \( \eta_x(\| x'^1 - x'^2 \|) \) for \( x'^1, x'^2 \in U_x \). Also \( \mathcal{A}(x) \) is continuous. So there is \( \delta_x > 0 \) such that
\[ x + 2 \delta_x B_X \subset U_x, \]
\[ \mathcal{A}(y)(x + \delta_x B_X) + (\delta_x/l_x) B_Y \subset V_x, \quad \forall s \in [0, \delta_x]. \]

For \( x' \in x + \delta_x B_X \), we apply Lemma 7 with \( g = \mathcal{A}(x), \quad g' = \mathcal{A}(x'), \quad \Omega = U_x, \quad \mathcal{A}(x')|_{U_x}, \quad L = 2l_x, \quad \eta = \eta_x((\| x - x' \|)), \quad x^0 = x', \quad s = \delta_x/l_x \), and \( s = \mathcal{A}(x')(U_x) \) is Lipschitziany invertible of modulus \( 2l_x \), and \( \mathcal{A}(x')(U_x) \) contains \( f(x') + (\delta_x/(2l_x)) B_Y \). Let \( x = 2l_x x + (\delta_x/(2l_x)) B_Y \). \( \mathcal{A}(x)(U_x) \) contains \( f(x') + \varepsilon_x B_Y \). (2') is confirmed.

By Hypothesis (1) and the above, each \( x \in X_0 \) has a neighborhood \( U_x \) such that for some \( \Delta_x(t) = o(t) \),
\[ \| \mathcal{A}(x')(x^{**}) - f(x^{**}) \| \leq \Delta_x(\| x' - x'' \|), \quad \forall x', x'' \in U_x, \]
and (2') holds. Assume without loss of generality that \( U_x \) contains \( x + \delta_x B_X \), and let
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\[ O_\varepsilon \text{ denote the interior of } x + \delta B_{\varepsilon} y. \] Since \( X_0 \) is compact we may cover it by finitely many neighborhoods \((O_{\varepsilon_i})\) corresponding to a finite sequence \((x^i) \subset X_0\). For each \( i \) and \( x = x^i \) we have \( \Delta_\varepsilon(t) = o(t) \) such that (12) holds, and scalars \( \delta_\varepsilon, \varepsilon_i, L_i \) satisfying (2'). Now each \( x \in X_0 \) is an interior point of \( O_{\varepsilon_i} \) for some \( i \), indeed there exists \( \delta > 0 \), independent of \( x \), such that for \( x + \delta B_{\varepsilon_i} \subset O_{\varepsilon_i} \) for some \( i \); if not, sequential compactness of \( X_0 \) leads to an easy contradiction. Define \( \Delta(t) \) as \( \max \Delta_\varepsilon(t) \) for \( 0 \leq t \leq \delta \) and \( \varepsilon \) for \( t > \delta \), then \( \Delta(t) = o(t) \). Since any two points of \( X_0 \) separated by distance less than \( \delta \) lie in some \( O_{\varepsilon_i} \), (12) yields

\[
\| \mathbf{A}(x')(x'') - f(x'') \| \leq \Delta(\|x' - x''\|) \quad \forall x', x'' \in X_0,
\]

which confirms Hypothesis (1) of Theorem 9.

Hypothesis (2) of Theorem 9 also holds, with \( \delta = \text{def} \delta \), \( \varepsilon = \text{def} \min_i \varepsilon_i \), and \( L = \text{def} \max \max_i L_i \). For each \( x \in X_0 \) we take \( U_i = \text{def} U_{\varepsilon_i} \) for \( i \) such that \( x + \delta B_{\varepsilon_i} \subset O_{\varepsilon_i} \), and \( V_i = \text{def} \mathbf{A}(x)(U_{\varepsilon_i}) \). Note that \( x + \delta B_{\varepsilon_i} \subset x_i + \delta_i B_{\varepsilon_i} \), hence \( x + \delta B_{\varepsilon_i} \subset U_i \) and, by (2'), \( \mathbf{A}(x)|_{U_i} \) is Lipschitzian invertible of modulus \( L \) and \( V_i \supset f(x) + \varepsilon B_Y \).

To show Hypothesis (3) of Theorem 9, let \( x \in X_0 \), \( T \in (0, 1] \) and \( p: [0, T) \to X \) be such that, for each \( t \in [0, T) \), \( \mathbf{A}(x)(p(t)) = (1 - t)f(x) \) and \( \mathbf{A}(x) \) is continuously invertible near \( p(t) \). We only need show \( p \) is Lipschitz on \( (0, T) \), say of modulus \( l \), in which case, first, \( \{ p(t) | t \in (0, T) \} \) is bounded, hence \( p(t) \) has an accumulation point \( y \) as \( t \uparrow T \); secondly, for \( t \in (0, T) \),

\[
\| p(t) - y \| \leq \limsup_{s \uparrow T} \| p(t) - p(s) \| \leq l|t - T|,
\]

so \( \lim_{t \uparrow T} p(t) = y \); and finally, \( \mathbf{A}(x)(y) = (1 - T)f(x) \) by continuity of \( \mathbf{A}(x) \).

Let \( t, t' \in (0, T) \); without loss of generality assume \( t' > t \). Let \( I_x > 0 \) be the Lipschitz constant given by Hypothesis (3), and choose a neighborhood \( U \) of \( p(t) \) such that \( \mathbf{A}(x)^{-1} \cap U \) is Lipschitz of modulus \( I_x \) near \( (1 - t)f(x) \). So there is \( \gamma > 0 \) such that \( (t - \gamma, t + \gamma) \) is contained in \( (0, T) \), and \( p(s) = \mathbf{A}(x)^{-1}[(1 - s)f(x)] \cap U \) for \( s \in (t - \gamma, t + \gamma) \). Then \( p \) is Lipschitz of modulus \( l = \text{def} \|f(x)\| \) on \( (t - \gamma, t + \gamma) \). As \( [t, t'] \) is compact, it can be covered by finitely many open intervals in \( (0, T) \), on each of which \( p \) is Lipschitz of modulus \( l \). Hence there is a finite sequence \( t_0 = t < t_1 < \cdots < t_k = t' \) such that \( p \) is Lipschitz of modulus \( l \) on \( [t_{k-1}, t_k] \) for each \( k = 1, \ldots, K \). We conclude:

\[
\| p(t) - p(t') \| \leq \sum_{k=1}^K \| p(t_{k-1}) - p(t_k) \| = \sum_{k=1}^K (t_{k-1} - t_k)l = (t' - t)l. \quad \square
\]

Hypotheses (1) and (4) of the proposition specify a relaxed local version of the point-based approximation of \( f \) at \( x \), used to define Robinson-Newton’s method (Robinson 1988). A point-based approximation of \( f \) on \( \Omega \subset X \) is a function \( \mathbf{A} \) on \( \Omega \), each value \( \mathbf{A}(x) \) of which is itself a mapping from \( \Omega \) to \( Y \), such that for some \( \kappa > 0 \) and every \( x^1, x^2 \in \Omega \),

(a) \( \| \mathbf{A}(x^1)(x^2) - f(x^2) \| \leq (1/2)\kappa\|x^1 - x^2\|^2 \), and

(b) \( \mathbf{A}(x^1) - \mathbf{A}(x^2) \) is Lipschitz of modulus \( \kappa\|x^1 - x^2\| \).

Suppose \( \mathbf{A} \) is a point-based approximation of \( f \) on a neighborhood \( \Omega \) of \( x \) and, for each \( x^1 \in \Omega \), we extend the domain of \( \mathbf{A}(x^1) \) to \( X \) by arbitrarily defining the values \( \mathbf{A}(x^1)(x^2) \in Y \) for \( x^2 \in \Omega \setminus \Omega \). Then property (a) implies Hypothesis (1), and property (b) implies Hypothesis (4).

Liqun Qi has noted that the conditions of the previous two convergence results preclude from consideration such functions as \( f: \mathbb{R} \to \mathbb{R}: x \mapsto |x| \), because \( f \) does not
have a first-order approximation at 0 that is invertible near 0. This particular example is easily accommodated: Theorem 9 is valid if $X_0$ is replaced by $X_0 \setminus f^{-1}(0)$. A similar recasting of Proposition 10 is possible, though care is needed because compactness of $X_0$ does not imply compactness of $X_0 \setminus f^{-1}(0)$. This leads to a more serious question however: Can we deal with a function that is badly behaved at the current iterate $x^k \in X_0$? In light of Lemma 7, the hypotheses of both Theorem 9 and Proposition 10 imply that $f$ must be Lipschitzianly invertible near each point of $X_0$ (though we can restrict our attention to points of $X_0$ at which $f$ is nonzero), hence at $x^k$. This is an implicit restriction on the kind of mapping to which the analysis applies.

5. Applications.

5.1. Variational inequalities. The Variational Inequality is the problem of finding a vector $z \in \mathbb{R}^N$ such that

\begin{equation}
(VI) \quad z \in C, \\
\langle F(z), c - z \rangle \geq 0, \quad \forall c \in C
\end{equation}

where $F$ is a function from $C$ to $\mathbb{R}^N$ and $C$ is a nonempty closed convex set in $\mathbb{R}^N$. We will also assume that $F$ is continuously differentiable (see Definition 5) and usually that $C$ is polyhedral. See Harker and Pang (1990) for a survey of variational inequalities which includes analysis, algorithms and applications.

Equivalent to (VI) are the Generalized Equation (Robinson 1979)

\begin{equation}
(GE) \quad 0 \in F(z) + N_C(z)
\end{equation}

where $N_C(z)$ is the normal cone to $C$ at $z$ (see §2), and the Normal Equation (Robinson 1992)

\begin{equation}
(NE) \quad 0 = F(\pi_C(x)) + x - \pi_C(x)
\end{equation}

where $\pi_C(x)$ is the Euclidean projection of $x$ to its nearest point in $C$. So (NE) is just $0 = F_C(x)$. We will work with (NE) because it is a (nonsmooth) equation.

Note that (VI) and (GE) are, but for notation, identical, whereas the equivalence between each of these two problems and (NE) is indirect: if $z$ solves (VI) or (GE) then $x = z - F(z)$ solves (NE), while if $x$ solves (NE) then $z = \pi_C(x)$ solves (VI) and (GE). In fact we have a stronger result from Lemma 6, assuming $F$ is locally Lipschitz: $F_C$ is Lipschitzianly invertible near $x$ if and only if for some neighborhoods $U, V$ of $\pi_C(x), F_C(x)$ respectively, $(F + N_C)^{-1} \cap U$ is a Lipschitz function when restricted to $V$. This result has links to strongly regular generalized equations (see Robinson (1980), and proof of Proposition 13).

A first-order approximation of $f = F_C$ at $x$ is obtained by linearizing $F$ about $c = \pi_C(x)$:

\begin{equation}
(13) \quad A(x)(x') = F(c) + \nabla F(c)(\pi_C(x') - c) + x' - \pi_C(x'), \quad \forall x' \in \mathbb{R}^N.
\end{equation}

So $A(x)(\cdot)$ is $\nabla F(c)(\cdot)$ shifted by the vector $F(c) - \nabla F(c)x(c)$. Therefore $A(x)$ is continuous, and for any $x^1, x^2 \in \mathbb{R}^N$,

\[
\|A(x^1)(x^2) - f(x^2)\| \leq \sup_{0 \leq s < 1} \| \nabla F(\pi_C(x^1)) - \nabla F(s\pi_C(x^1) + (1 - s)\pi_C(x^2)) \|
\times \| \pi_C(x^1) - \pi_C(x^2) \|
\]
by the vector mean value theorem (Ortega and Rheinboldt 1970, Theorem 3.2.3).
Now $\pi_c(x + B)$ is compact, so $\nabla F$ is uniformly continuous on the convex hull of this set, and it follows from the above inequality and Lipschitz continuity of $\pi_c$ that $\mathcal{A}$ is a uniform first-order approximation of $f$ near $x$. Similar arguments ensure that, for $s \geq 0$ and

$$
\eta_s(s) = \sup \left\{ \| \nabla F(\pi_c(\xi^1)) - \nabla F(\pi_c(\xi^2)) \| \| \xi^1, \xi^2 \in x + B, \| \xi^1 - \xi^2 \| \leq s \right\},
$$

if $x^1, x^2 \in x + B$ then $\mathcal{A}(x^1) - \mathcal{A}(x^2)$ is Lipschitz of modulus $\eta_s(\| x^1 - x^2 \|)$, and $\eta_s(s) \downarrow 0$ as $s \downarrow 0$. We have verified the first and fourth assumptions of Proposition 10 for each $x \in \mathbb{R}^N$.

To apply Proposition 10 we still must find a constant $\alpha_0 > 0$ such that the level set

$$
X_0 = \{ x \in \mathbb{R}^N \left| \| f(x) \| \leq \alpha_0 \right. \}
$$

is bounded and only contains points $x$ near which $\mathcal{A}(x)$ is Lipschitzianity invertible, etc. One case we will deal with is when $F$ is strongly monotone of modulus $\lambda > 0$, that is

$$
\langle F(c) - F(c'), c - c' \rangle \geq \lambda \| c - c' \|^2, \quad \forall c, c' \in C.
$$

Another case to consider is when $C$ is polyhedral convex; here $\pi_c$ is piecewise linear, hence, for each $x$, $\mathcal{A}(x)$ is piecewise linear too. In this situation, Robinson’s homeomorphism theorem (Robinson 1992, Theorem 4.3) says $\mathcal{A}(x)$ is homeomorphic if and only if it is coherently oriented, i.e., its determinants in each of its full dimensional pieces of linearity have the same (nonzero) sign. (Robinson 1992, Theorem 5.2) also provides homeomorphism results near a point $x$, via the critical cone (Robinson 1987) to $C$ at $x$, which is the intersection of the tangent cone to $C$ at $c = \pi_c(x), T_c(c)$, with the linear subspace orthogonal to $x - c, (x - c)^\perp$. See also Ralph (1993). These results provide testable conditions for the second assumption of Proposition 10.

**Proposition 11.** Let $C$ be a nonempty closed convex set in $\mathbb{R}^N$, $F: C \to \mathbb{R}^N$ be continuously differentiable, and $\sigma, \tau \in (0, 1)$, $M \in \mathbb{N}$ be the parameters governing the path search (NmPs), and $\alpha_0 > 0$. Let $X_0 = \{ x \in \mathbb{R}^N \left| \| F_c(x) \| \leq \alpha_0 \right. \}$.

Suppose, in addition, either

1. $F$ is strongly monotone; or
2. $X_0$ is bounded; $C$ is polyhedral; and for each $x \in X_0$, $c = \pi_c(x), \nabla F(c)c$ is invertible near $x$, or, equivalently, $\nabla F(c)K$ is coherently oriented where $K = T_c(c) \cap (x - c)^\perp$.

Let path search damped Newton’s method for solving $F_c(x) = 0$ be defined using the first-order approximation (13), starting from $x^0 \in X_0$. Then the damped Newton iterates $x^k$ converge to a zero $x^*$ of $F_c$. The residuals $F_c(x^k)$ and iterates $x^k$ converge at $Q$-superlinear rates to zero and $x^*$, respectively; indeed these rates are $Q$-quadratic if $\nabla F$ is Lipschitz near $\pi_c(x^*)$.

**Proof.** We have seen that Hypotheses (1) and (4) of Proposition 10 hold. It is left to verify the remaining hypotheses. The claim of $Q$-superlinear or $Q$-quadratic convergence of $F_c(x^k)$ is then explained by the comments after Theorem 9, given that $F_c$ is Lipschitz near $x^*$. 

Let $F$ be strongly monotone of modulus $A > 0$. It is well known for such $F$ (Harker and Pang 1990, Corollary 3.2) that the generalized equation

$$y \in F(c) + N_c(c)$$

has a unique solution $c \in C$ for each $y \in \mathbb{R}^N$. Suppose $c^i \in C$ and $y^i \in (F + N_c)c^i$, for $i = 1, 2$. Using the Cauchy-Schwartz inequality, and the definitions of the normal cone and strong monotonicity, we get

$$\|y^1 - y^2\| \geq \langle y^1 - y^2, c^1 - c^2 \rangle$$

$$= \langle y^1 - F(c^1), c^1 - c^2 \rangle + \langle y^2 - F(c^2), c^2 - c^1 \rangle$$

$$+ \langle F(c^1) - F(c^2), c^1 - c^2 \rangle$$

$$\geq A\|c^1 - c^2\|^2.$$ 

Thus $\|y^1 - y^2\| \geq \lambda\|c^1 - c^2\|$. Since $(F + N_c)^{-1}$ maps points to nonempty sets, this bound implies $(F + N_c)^{-1}$ is actually a function on $\mathbb{R}^N$ that is Lipschitz (of modulus $A^{-1}$). So $C_0 = \text{def}(F + N_c)^{-1}(a_0 \mathbb{B})$ is compact and it follows, by continuity of $F$, that $F(C_0)$ is compact too. Recall that from Lemma 6, equation (2), $y = F_c(x)$ if and only if $x = y - F(c) + c$ for some $c \in (F + N_c)^{-1}(y)$. This yields

$$X_0 = \{y - F(c) + c | y \in a_0 \mathbb{B}, c \in (F + N_c)^{-1}(y)\} \subset a_0 \mathbb{B} - F(C_0) + C_0,$$

which shows that $X_0$ is bounded.

It also follows from strong monotonicity of $F$ that for any $x \in \mathbb{R}^n$ and $c = \text{def} \pi_c(x)$, $\nabla F(c)$ is strongly monotone, hence, as above, $(\nabla F(c) + N_c)^{-1}$ is a Lipschitz mapping. Since $\nabla F(c)$ is also Lipschitz on $\mathbb{R}^N$ we see that, as in Lemma 6, the normal mapping $\nabla F(c)_c$ is Lipschitzianly invertible; hence $\mathcal{A}(x)$ is Lipschitzianly invertible. As $\mathcal{A}(x)$ is also continuous, Hypotheses (2) and (3) of Proposition 10 are satisfied.

(2) Under the assumptions of this proposition, we have seen that, by virtue of Robinson (1992), Hypothesis (2) of Proposition 10 holds. We are also given that $X_0$ is bounded, so it only left to show that Hypothesis (3) of Proposition 10 is valid.

Since $C$ is polyhedral convex, $\mathcal{A}(x)(\cdot)$ is piecewise linear for each $x \in \mathbb{R}^N$, that is $\mathcal{A}(x)$ is represented on each of finitely many polyhedral convex sets covering $\mathbb{R}^N$ by an affine mapping $z \mapsto M_i z + b_i$, where $\{M_i\} \subset \mathbb{R}^{N \times N}$ and $\{b_i\} \subset \mathbb{R}^N$. Since $\mathcal{A}(x)$ is invertible near $x$, at least one of the matrices $M_i$ must be invertible; let

$$l_x^{\text{def}} = \max_{M_i \text{ invertible}} \{\|M_i^{-1}\| : M_i \text{ is invertible} \}.$$ 

If $\mathcal{A}(x)$ is invertible near some $x' \in \mathbb{R}^N$, then it has a local inverse $P: V \rightarrow U$ where $V$ is a neighborhood of $\mathcal{A}(x)(x')$ and $U$ is a neighborhood of $x$. We assume without loss of generality that $V$ is convex, and take any $y, y' \in V$. Now the interval joining $P(y)$ and $P(y')$ is covered by finitely many subintervals $\{I_i\}$, on each of which $\mathcal{A}(x)$ is represented by some invertible mapping $M_i z + b_i'$. Hence the interval joining $y$ and $y'$ is covered by the subintervals $\{\mathcal{A}(x)(I_i)\}$, on each of which $P$ is Lipschitz of modulus $l_y$. It follows that $\|P(y) - P(y')\| \leq l_y \|y - y'\|$. □

Fukushima (1992), Taji, Fukushima and Ibaraki (1993), and Wu, Florian and Marcotte (1990) show convergence of line search damped schemes based on Josephy-Newton’s method (Josephy (1979) and below) for strongly monotone varia-
tional inequalities. Superlinear convergence is shown under the additional assumptions of polyhedrality of $C$, and a nondegeneracy condition implying smoothness of $F_C$ at the solution point to which the iterates converge. When $F$ is monotone, but not necessarily strongly monotone, and $C$ is compact as well as being convex and nonempty, Marcotte and Dussault (1987) give a globally convergent damped Josephy-Newton’s method; in this situation, however, it seems that neither our analysis nor the analysis of Fukushima (1992), Taji, Fukushima and Ibaraki (1993) and Wu, Florian and Marcotte (1990) applies.

Josephy-Newton’s method. In Josephy-Newton’s method (Josephy 1979) for solving (GE), given the $k$th iterate $c^k$, the next iterate is defined to be a solution $\hat{c}^{k+1}$ of the linearized generalized equation

$$0 \in F(c^k) + \nabla F(c^k)(c - c^k) + N_C(c), \quad c \in \mathbb{R}^N.$$ 

The equivalence between Josephy-Newton’s method on (GE) and Robinson-Newton’s method on the associated (NE) is well known: if $x^k$ is such that $\pi_C(x^k) = c^k$ and

$$x^{k+1} \overset{\text{def}}{=} \hat{x}^{k+1} - F(c^k) - \nabla F(c^k)(\hat{x}^{k+1} - c^k),$$

then $\hat{x}^{k+1}$ is a zero of $\mathcal{A}(x^k)(\cdot)$ and $\hat{c}^{k+1} = \pi_C(\hat{x}^{k+1})$. Hence the Josephy-Newton iterate is the Euclidean projection onto $C$ of the Robinson-Newton iterate. Path search damping of Robinson-Newton’s method produces $x^{k+1}$ on the Newton path $p^k$ from $x^k$ to $\hat{x}^{k+1}$. The projection $\pi_C(x^{k+1})$, on a path from $c^k$ to $\hat{c}^{k+1}$, is a damped Josephy-Newton iterate.

For practical purposes, suppose $C$ is polyhedral convex. Let us describe the construction of the path $p^k$ given the current iterate $x^k$. From (13), with

$$A_k(x) \overset{\text{def}}{=} \mathcal{A}(x^k)(x) \overset{\text{def}}{=} F(c^k) + \nabla F(c^k)(\pi_C(x) - c^k) + x - \pi_C(x),$$

as $A_k$ is piecewise linear, $p^k$ is also piecewise linear. We construct $p^k$, piece by affine piece, using a pivotal method. Starting from $t = 0$ ($p^k(0) = x^k$), ignoring degeneracy, each pivot increases $t$ to the next breakpoint in the derivative of $p^k$ while maintaining the equation

$$A_k(p^k(t)) = (1 - t)f(x^k),$$

thereby extending the domain of $p^k$. We continue to pivot so long as pivoting is possible and our latest breakpoint $t$ satisfies the nonmonotone descent condition (NmD). If, after a pivot, (NmD) fails, then we line search on the interval $[p^k(t_{old}), p^k(t)]$ to find $x^{k+1}$, where $t_{old}$ is the value of $t$ at the previous breakpoint. The line search makes sense here because $p^k$ is affine between successive breakpoints, hence affine on $[t_{old}, t]$. It is easy to see that the Armijo line search applied with parameters $\sigma, \tau \in (0, 1)$ produces $t_k \in [t_{old}, t]$ that fulfills (NmPs). On the other hand, if (NmD) holds at every breakpoint $t$ then we must eventually stop because $t = 1$ or further pivots are not possible (i.e., $A_k$ is not continuously invertible at $p^k(t)$). In this case we take $T_k = t$ and $x^{k+1} = p(t)$. This procedure can be thought of as a forward path search: we construct the path as $t$ moves forward from 0 to $T_k$, testing (NmD) after each pivot that extends the path. This is an efficient strategy if (NmD) fails after the first few pivots. A backward path search extends the path as far as possible, that is until $T_k = 1$ or $p$ is not
invertible at $T_k$, while recording each pivot point $p(t)$ and the corresponding value of $t$. If (NmD) holds at the final value of $t$, i.e., at $T_k$, take $x^{k+1} \overset{\text{def}}{=} p^k(T_k)$; otherwise search through the list of $(p^k(t), t)$ pairs until one, with $t = t_1$ say, is found such that $t = t_1$ satisfies (NmD), but (NmD) fails at the next value of $t$ seen in the list, say $t = t_2$. A line search is now performed on the interval $[p^k(t_1), p^k(t_2)]$.

For the sake of generality, we mention a class of nonsmooth functions with “natural” first-order approximations that contains each normal mapping $f = F_C$ such that $F$ is smooth and $C$ is closed, convex and nonempty. Consider the class of functions of the form

$$f = H \circ h$$

where $D \subset \mathbb{R}^K$, $h: \mathbb{R}^N \rightarrow D$ is locally Lipschitz, and $H: D \rightarrow \mathbb{R}^N$ is smooth (e.g., $D$ is convex and $H$ is continuously differentiable). A subclass of this class was highlighted in Robinson (1988) in the context of point-based approximations. Similar to above, it is easy to see that

$$\mathcal{A}(x)(x') \overset{\text{def}}{=} H(h(x)) + \nabla H(h(x))[h(x') - h(x)], \quad x, x' \in \mathbb{R}^N$$

defines a first-order approximation of $f$ on $\mathbb{R}^N$ satisfying Assumptions (1) and (4) of Proposition 10. $F_C$ is given by setting $K \overset{\text{def}}{=} 2N$, $D \overset{\text{def}}{=} C \times \mathbb{R}^N$, $h(x) \overset{\text{def}}{=} (\pi_C(x), x - \pi_C(x))$ and $H(a, b) \overset{\text{def}}{=} F(a) + b$.

5.2. Nonlinear complementarity problems. We now confine our attention to $C = \mathbb{R}^N_+$, the nonnegative orthant in $\mathbb{R}^N$. The problems (VI), (GE) and (NE) are equivalent to the NonLinear Complementarity Problem:

$$z, F(z) \geq 0,$$

$$\langle z, F(z) \rangle = 0,$$

where vector inequalities are taken componentwise. The normal mapping induced by $F$ and $\mathbb{R}^N$ is denoted by $F_+$, so $F_+ = F_{\mathbb{R}^N_+}$. The normal equation form of (NLCP) is $F_+(x) = 0$ or

$$F(x_+) + x - x_+ = 0$$

where $x_+$ denotes $\pi_{\mathbb{R}^N_+}(x)$. The associated first-order approximation (13) becomes

$$\mathcal{A}(x)(x') \overset{\text{def}}{=} F(x_+) + \nabla F(x_+)(x_+ - x_+) + x' - x_+, \quad \forall x' \in \mathbb{R}^N.$$  

(NLCP) has many applications, for example in nonlinear programming (§5.3) and economic equilibria problems (Harker and Pang 1990, Harker and Xiao 1990, Pang and Gabriel 1993).

More notation is needed. Let $I$ be the identity matrix in $\mathbb{R}^{N \times N}$. Given a matrix $M \in \mathbb{R}^{N \times N}$ and index sets $\mathcal{J}, \mathcal{F} \subset \{1, \ldots, N\}$, let $M_{\mathcal{J},\mathcal{F}}$ be the (possibly vacuous) submatrix of $M$ of elements $M_{i,j}$ where $(i, j) \in \mathcal{J} \times \mathcal{F}$. Also let $\mathcal{J}^c$ be the complement of $\mathcal{J}$, $\{1, \ldots, N\} \setminus \mathcal{J}$, and $M/M_{\mathcal{J},\mathcal{F}}$ be the Schur complement of $M$ with respect to $M_{\mathcal{J},\mathcal{F}}$,

$$M/M_{\mathcal{J},\mathcal{F}} \overset{\text{def}}{=} \begin{cases} M \setminus \mathcal{J}^c, \mathcal{F}^c = M \setminus \mathcal{J}^c, \mathcal{F}^c [M_{\mathcal{J},\mathcal{F}}]^{-1} M_{\mathcal{J},\mathcal{F}} \setminus \mathcal{J}^c, \mathcal{F}^c & \text{if } \emptyset \neq \mathcal{J} \neq \{1, \ldots, N\}, \\
M \underset{\text{vacuous}}{=} M & \text{if } \mathcal{J} = \emptyset, \\
M & \text{if } \mathcal{J} = \{1, \ldots, N\}, \end{cases}$$
assuming $M_{x,\sigma}$ is invertible or vacuous. Also, $M$ is a $P$-matrix if it has positive principal minors.

**Proposition 12.** Let $F: \mathbb{R}^N \to \mathbb{R}^N$ be continuously differentiable, and $\sigma, \tau \in (0, 1)$, $M \in \mathbb{N}$ be the parameters governing the path search (NmPs). Suppose $\alpha_0 > 0$ and

$$X_0^{\text{def}} = \{ x \in \mathbb{R}^N \mid \| F_+(x) \| \leq \alpha_0 \}$$

is bounded. Suppose for each $x \in X_0$ the normal mapping $\nabla F(x_+)$ is Lipschitzian invertible near $x$ or, equivalently, the following (possibly vacuous) conditions hold:

- $\nabla F(x_+)$ is invertible, where $\mathcal{J} = \{ i \mid x_i > 0 \}$;
- $\nabla F(x_+)/(\nabla F(x_+))_{\mathcal{J}}$ is a $P$-matrix, where $\mathcal{J} = \{ i \mid x_i \geq 0 \}$.

Let path search damped Newton's method for solving $F_+(x) = 0$ be defined using the first-order approximation (14). Then for any $x^0 \in X_0$, the damped Newton iterates $x^k$ converge to a zero $x^*$ of $F_+$. The residuals $F_+(x^k)$ and iterates $x^k$ converge at $Q$-superlinear rates to zero and $x^*$, respectively; indeed these rates are $Q$-quadratic if $\nabla F$ is Lipschitz near $x^*$.

**Proof.** If the above conditions on $\nabla F(x_+)$ are equivalent to the assertion that $\nabla F(x_+)$ is Lipschitzian invertible near $x$, then the result is a corollary of Proposition 11.2. The required equivalence follows from Robinson (1992, Proposition 4.1 and Theorem 5.2). □

This result is similar in statement to Harker and Xiao (1990, Theorem 3) on line search damped B-Newton's method, but the conclusions of Harker and Xiao (1990) are weaker. The convergence results of Harker and Xiao (1990) are derived directly from Pang (1990), hence, as noted in the Introduction, require existence of a limit point of the damped Newton iterates at which $\| F_+(\cdot) \|^2_2$ is differentiable. Note that no differentiability properties of the residual, or of its norm squared, are needed in Proposition 12.

On a practical level, let us compare the damped method used in Proposition 12 with line search damped B-Newton's method. In the latter case we use the approximation

$$B_k(x) = F_+(x^k) + F'_+(x^k; x - x^k),$$

where the B-derivative $F'_+(x^k; d)$ equals $\nabla F(x^k)\xi + d - \xi$, and $\xi_i$ is $d_i$ if $x_i^k > 0$, $\max(d_i, 0)$ if $x_i^k = 0$, and 0 otherwise. Solving $B_k(x) = 0$ involves factoring out the linear part of $F'_+(x^k; \cdot)$, which corresponds to the nonzero components of $x^k$, and solving the remaining piecewise linear problem of reduced size. On one hand, this generally requires less work per iteration than using the full piecewise linear approximation (14) to carry out a path search. On the other hand, since the approximation $B_k$ of $F_+$ is generally not as accurate as $A_k$, damped B-Newton's method may require more iterations to reduce the size of the residual to a given level.

5.3. **Nonlinear programming.** Our computational examples are all optimization problems in NonLinear Programming. A general form of the nonlinear programming problem is

$$\min \theta(z) \quad \text{subject to } z \in D, g(z) = 0$$

where $D$ is a nonempty polyhedral convex set in $\mathbb{R}^n$, and $\theta: D \to \mathbb{R}$ and $g: D \to \mathbb{R}^m$.
are smooth functions. Under a constraint qualification the standard first-order conditions necessary for optimality of this problem are of the form (GE), where $N = n + m$,

$$C = D \times \mathbb{R}^m,$$

$$F(z, y) = (V(\theta(z)) + V g(z)^T y, g(z)), \quad \forall (z, y) \in D \times \mathbb{R}^m.$$ 

See, for example, Robinson (1983 §1 and Theorem 3.2). Variable-multiplier pairs $(z, y)$ satisfying the first-order conditions can be rewritten as solutions of (NE) for these $F$ and $C$ (Park 1989, Chapter 3, §4). We will confine ourselves to a more restrictive class of nonlinear programs which contains our computational examples:

(NLP) \hspace{1cm} \min \theta(z) \quad \text{subject to } z \geq 0, g(z) \leq 0.

Again under a constraint qualification such as the Mangasarian-Fromovitz condition (Mangasarian 1969, 11.3.5, McCormick 1983, 10.2.16) the first-order conditions necessary for optimality of (NLP) can be written as either (NLCP), or the corresponding normal equation, where $N = n + m$ and

$$F(z, y) = (\nabla \theta(z) + \nabla g(z)^T y, -g(z)), \quad \forall (z, y) \in \mathbb{R}^{n+m}.$$ 

Kojima (1980) introduced the normal equation formulation for programs with (nonlinear) inequality and equality constraints.

Wilson’s method. In Wilson’s method (Wilson 1963, Fletcher 1987), also known as sequential quadratic programming (SQP), given the $k$th variable-multiplier pair $(a^k, b^k) \in \mathbb{R}^{n+m}$ the next iterate is defined as the optimal variable-multiplier pair $(\hat{a}^{k+1}, \hat{b}^{k+1})$ for the quadratic program:

$$\min_{a \in \mathbb{R}^n} \theta(a^k) + \langle \nabla \theta(a^k), a - a^k \rangle + (1/2)\langle a - a^k, \nabla^2 \theta(a^k)(a - a^k) \rangle$$

subject to $a \geq 0, g(a^k) + \nabla g(a^k)(a - a^k) \leq 0$.

Suppose $x^k = (z^k, y^k) \in \mathbb{R}^{n+m}$ satisfies $(\hat{z}_+^k, \hat{y}_+^k) = (a^k, b^k), F$ is given by (15), and $\mathcal{N}(x^k)$ is given by (14). The SQP iterate $(\hat{a}^{k+1}, \hat{b}^{k+1})$ satisfies the first-order condition for the quadratic program, which is equivalent, by previous discussion, to saying the point

$$\hat{x}^{k+1} = (\hat{z}^{k+1}, \hat{y}^{k+1}) = (\hat{a}^{k+1}, \hat{b}^{k+1}) - F(a^k, b^k)$$

is a zero of $\mathcal{N}(x^k)$. Also $(\hat{z}_+^{k+1}, \hat{y}_+^{k+1}) = (\hat{a}^{k+1}, \hat{b}^{k+1})$. A path search applied to Robinson-Newton’s method for (NE) determines the next iterate $x^{k+1} = (\hat{z}^{k+1}, \hat{y}^{k+1})$ as a point on the Newton path $p^k$ from $x^k$ to $\hat{x}^{k+1}$. The nonnegative part $(\hat{z}_+^{k+1}, \hat{y}_+^{k+1})$, a point on the path from $(a^k, b^k)$ to $(\hat{a}^{k+1}, \hat{b}^{k+1})$, is a damped iterate for SQP.

There are standard conditions that jointly ensure local uniqueness of an optimal variable-multiplier pair $(\hat{z}, \hat{y}) \geq 0$ of (NLP), hence of the solution (also $(\hat{z}, \hat{y})$) of the
associated system (NLCP), and of the solution \((z^*, y^*) = (\tilde{z}, \tilde{y}) - F(\tilde{z}, \tilde{y})\) of the associated equation (NE). At nonsolution points \((z, y)\) of (NE), analogous conditions will guarantee local invertibility of \(F_+ = F_{R^{n+m}_+}\) and its first-order approximation (14) at \((z, y)\).

To specify these conditions at \((z, y) \in R^{n+m}\), let

\[
\mathcal{F}_z = \{i|z_i > 0\}, \quad \mathcal{F}_z^c = \{i|z_i \geq 0\},
\]

\[
\mathcal{F}_y = \{i|y_i > 0\}, \quad \mathcal{F}_y^c = \{i|y_i \geq 0\},
\]

and \(|\mathcal{F}|\) be the cardinality of \(\mathcal{F}_z\), etc. We present the conditions of Linear Independence of binding constraint gradients at \((z, y)\):

\[(LI) \quad \nabla g(z_+) \mathcal{F}_z \mathcal{F}_z^c \text{ has linearly independent rows}
\]

and Strong Second-Order Sufficiency at \((z, y)\):

\[(SSOS) \quad \text{if } \nabla g(z_+) \mathcal{F}_z \mathcal{F}_z^c \hat{d} = 0, \quad 0 \neq \hat{d} \in R^{|\mathcal{F}|},
\]

\[
\text{then } \hat{d}^T \left[ \nabla^2 g(z_+) + y_+^T \nabla^2 g(z_+) \right] \mathcal{F}_z \mathcal{F}_z^c \hat{d} > 0.
\]

We note that if \((z, y)\) is a zero of \(F_+\) then (LI) and (SSOS) correspond to more familiar conditions (e.g., Robinson 1980, §4) defined with respect to \((z_+, y_+)\) rather than \((z, y)\). This connection is explored further in the proof of our next result.

**Proposition 13.** Let \(\theta: R^+ \rightarrow R\), \(g: R^+ \rightarrow R^m\) be twice continuously differentiable functions, and \(F\) be given by (15). Let \(\sigma, \tau \in (0, 1)\) and \(M \in N\) be the parameters governing the path search (NmPs). Suppose \(\alpha_0 > 0\),

\[
X_0 = \{(z, y) \in R^{n+m} | \|F_+(z, y)\| \leq \alpha_0\}
\]

is bounded, and the above (LI) and (SSOS) conditions hold at each \((z, y) \in X_0\).

If path search damped Newton's method for solving \(F_+(z, y) = 0\) is defined using the first-order approximation (14), where \(x = \text{def}(z, y)\), then for any \((z^0, y^0) \in X_0\), the damped Newton iterates \((z^k, y^k)\) converge to a zero \((z^*, y^*)\) of \(F_+\) such that \(z^*_+\) is a local minimizer of (NLP). The residuals \(F_+(z^k, y^k)\) and the iterates \((z^k, y^k)\) converge at \(Q\)-superlinear rates to zero and \((z^*, y^*)\), respectively; indeed these rates are \(Q\)-quadratic if \(\nabla^2 \theta\) and \(\nabla^2 g\) are Lipschitz near \(z^*_+\).

**Proof.** This is essentially a corollary of Proposition 12. Most of the proof is devoted to showing that the (LI) and (SSOS) conditions at a given point \((z^0, y^0) \in R^n \times R^m\) are sufficient for \(\nabla F(z^0, y^0)\) to be Lipschitzianly invertible near that point. Below, \(I\) denotes the \(n\) times \(n\) identity matrix; and the fact that

\[
0 \in u + N_{R^n_+}(z) \iff u, z \geq 0, \quad \langle u, z \rangle = 0 \iff 0 \in z + N_{R^m_+}(u),
\]

will be used without reference. Also Robinson (1980) is needed, so generalized equations rather than normal equations are stressed.
Let \( x^0 = \text{def} (z^0, y^0) \) and \( u^0 = \text{def} - z^0 \) (so that \( u^0_+ = z^0_+ - z^0 \)). Define
\[
\xi^0 = (\xi^0_x, \xi^0_y) = F_+ (x^0).
\]
Consider the nonlinear program, a perturbed version of (NLP).

\[
\text{(NLP)} \quad \min \tilde{\theta}(z) \quad \text{subject to } \tilde{g}(z) \leq 0,
\]
where \( \tilde{\theta}(z) = \text{def} \theta(z) - \langle \xi^0_z, z \rangle \) and \( \tilde{g}(z) = \text{def} (g(z) + \xi^0_y, -z) \). The first-order optimality condition (Robinson 1980) for this problem is the perturbed generalized equation

\[
\text{(GE)} \quad 0 \in (\tilde{F} + N_{\mathbb{R}^n \times \mathbb{R}^{m+n}})(z, y, u)
\]
where
\[
\tilde{F}(z, y, u) = (\nabla \tilde{\theta}(z) + \nabla \tilde{g}(z)^T (y, u), -\tilde{g}(z)) = (F(z, y) - \xi^0 - (u, 0), z).
\]
Now,
\[
\tilde{F}(z^0_+, y^0_+, u^0_+) = (0, y^0_+ - y^0, z^0_+) \in -N_{\mathbb{R}^n \times \mathbb{R}^{m+n}}(z^0_+, y^0_+, u^0_+),
\]
i.e., \((z^0_+, y^0_+, u^0_+)\) solves (GE). This generalized equation is said to be strongly regular (Robinson 1980) at \((z^0_+, y^0_+, u^0_+)\) if the linearized set mapping
\[
T(z, y, u) \text{def} \frac{d}{d\epsilon} F(z^0_+, y^0_+, u^0_+ + \epsilon)
\]
\[
+ \nabla \tilde{F}(z^0_+, y^0_+, u^0_+)(z - z^0_+, y - y^0_+, u - u^0_+)
\]
\[
+ N_{\mathbb{R}^n \times \mathbb{R}^{m+n}}(z, y, u)
\]
is such that for some neighborhoods \( \tilde{U} \) of \((z^0_+, y^0_+, u^0_+)\) and \( \tilde{V} \) of \(0 \in \mathbb{R}^{2n+m}, T^{-1} \cap \tilde{U} \) is a Lipschitz mapping when restricted to \( \tilde{V} \).

Given a subset \( \mathcal{K} \) of the row indices of a matrix \( M \), let \( M_{\mathcal{K}} \) denote the submatrix consisting of the rows of \( M \) of indices in \( \mathcal{K} \). According to Robinson (1980, Theorem 4.1), it is sufficient for strong regularity of (GE) at \((z^0_+, y^0_+, u^0_+)\) that two conditions hold, where, for \( \zeta^0 = \text{def} (y^0_+, u^0_+) \),
\[
\mathcal{K}_+ \text{def} \{ i \mid \tilde{g}(z^0_+)_i = 0, \zeta^0_i > 0 \}, \quad \mathcal{K}_0 \text{def} \{ i \mid \tilde{g}(z^0_+)_i = 0, \zeta^0_i = 0 \}.
\]
The first is linear independence of the binding constraints:
\[
\nabla \tilde{g}(z^0_+)_\mathcal{K}_+ \cup \mathcal{K}_0 \text{ has full row rank;}
\]
and the second, strong second-order sufficiency:
\[
\text{if } z \in \mathbb{R}^n \setminus \{0\} \text{ and } \nabla \tilde{g}(z^0_+)_{\mathcal{K}_+} z = 0, \text{ then } z^T \tilde{\zeta}^n z > 0
\]
where
\[ \tilde{w}'' = \nabla^2 (\theta + (y_0^T, z_0^T) g)(z_0^+) = \nabla^2 (\theta + (y_0^T) g)(z_0^+). \]

In terms of \( g \) and the index sets (16), and \( \mathcal{I}_z = \{ 1, \ldots, n \} \setminus \mathcal{I}_z \), the linear independence condition is
\[ \begin{bmatrix} \nabla g(z_0^+_{\mathcal{I}_z}) \\ -I_{\mathcal{I}_z} \end{bmatrix} \text{ has linearly independent rows;} \]
hence is equivalent to (LI) at \((z_0, y_0)\). Likewise, the strong second-order sufficient condition may be rewritten
\[ \text{if } z \in \mathbb{R}^n \setminus \{0\} \text{ and } \begin{bmatrix} \nabla g(z_0^+_{\mathcal{I}_z}) \\ -I_{\mathcal{I}_z} \end{bmatrix} z = 0, \text{ then } z^T \tilde{w}'' z > 0, \]
which clearly is equivalent to (SSOS) at \((z_0, y_0)\). Therefore the conditions (LI) and (SSOS) at \((z_0, y_0)\) guarantee strong regularity of \((\bar{G}E)\) at \((z_0^+, y_0^+, u_0^+)\).

Furthermore, assuming \((z^*, y^*)\) is a zero of \( F_+ \), if \((z_0^+, y_0^+) = (z^*, y^*)\) then \( \xi_0 = (0, 0) \) and (NLP) is just the original problem (NLP). So, as shown above, (LI) and (SSOS) at \((z^*, y^*)\) are respectively equivalent to the usual conditions of linear independence and strong second order sufficiency for (NLP) with respect to the variable \( z^* \), the multipliers \( y^* \) corresponding to the nonlinear constraints \( g(z^*) \leq 0 \), and the multipliers \( z^* - z^0 \) corresponding to the constraints \( z^0 > 0 \). Applying McCormick (1983, 10.4.1) we find that \( z^* \) is a strict local minimizer of (NLP).

We now show that strong regularity of \((\bar{G}E)\) implies \( \nabla F(z_0^+, y_0^+) \) is Lipschitzianly invertible near \((z_0^+, y_0^+)\). Let \( M = \text{def} \nabla F(z_0^+, y_0^+) \) and \( \bar{M} = \text{def} \nabla F(z_0^+, y_0^+, u_0^+) \). Observe that
\[ M = \begin{bmatrix} \tilde{w}'' & G^T \\ -G & 0 \end{bmatrix}, \quad \bar{M} = \begin{bmatrix} \tilde{w}'' & G^T & -I \\ -G & 0 & 0 \\ I & 0 & 0 \end{bmatrix}. \]

where \( \tilde{w}'' = \text{def} \nabla^2 (\theta + (y_0^T) g)(z_0^+) \) (\( = \tilde{w}'' \)), \( G = \text{def} \nabla g(z_0^+) \). Define
\( (\xi_z^{00}, \xi_y^{00}) = \xi_0 - (F - M)(z_0^+, y_0^+); \)
then
\[ (\xi_z^{00}, \xi_y^{00}, 0) = -(\bar{F} - \bar{M})(z_0^+, y_0^+, u_0^+) \in (\bar{M} + N_{R^* \times R^* + m} \bar{K})(z_0^+, y_0^+, u_0^+). \]

Note \( T(\cdot) + (\xi_z^{00}, \xi_y^{00}, 0) = (\bar{M} + N_{R^* \times R^* + m} \bar{K})(\cdot) \).

For any \((z, y), (\xi_z, \xi_y) \in \mathbb{R}^n \times \mathbb{R}^m\),
\[ (\xi_z, \xi_y) \in (M + N_{R^* + m})(z, y) \]
\[ \exists u \in \mathbb{R}^n, \quad \xi_z = \tilde{w}'' z + G^T y - u, \]
\[ \xi_y \in -G z + N_{R^*}(y), \]
\[ 0 \in z + N_{R^+}(u), \]
\[ \exists u \in \mathbb{R}^n, \quad (\xi_z, \xi_y, 0) \in (\bar{M} + N_{R^* \times R^* + m})(z, y, u). \]
Hence if \( \tilde{U}, \tilde{V} \) are respective neighborhoods of \( (z^0, y^0, u^0) \), \( (x^{00}, \xi_{y}^{00}, 0) \) such that 
\[
(M + N_{R^{n} \times R^{m+n}})^{-1} \cap \tilde{U}
\]
is a Lipschitz mapping when restricted to \( \tilde{V} \), then \( (M + N_{R^{n} \times R^{m+n}})^{-1} \cap U \) is a Lipschitz map when restricted to \( V \), where
\[
V \overset{\text{def}}{=} \{ (\xi_z, \xi_y) \in R^{n+m} \mid (\xi_z, \xi_y, 0) \in \tilde{V} \},
\]
\[
U \overset{\text{def}}{=} \{ (z, y) \in R^{n+m} \mid (z, y, u) \in \tilde{U} \text{ for some } u \in R^n \}.
\]

In this case \( U \) and \( V \) are neighborhoods of \( (z^0, y^0) \) and \( (x^{00}, \xi_y^{00}) \) respectively, so Lemma 6 (and equation (2)) assures us that \( M^+ \) is also Lipschitzianly invertible near
\[
(z^0, x^0) + (I - M)(z^0, x^0) = (z^0, x^0).
\]

To summarize: \( (z^0, y^0, u^0) \) solves \( (GE) \) and
\[
(LI) \text{ and } (SSOS) \text{ hold at } (z^0, y^0)
\]

\[
\Rightarrow T^{-1} \cap \tilde{U} \text{ is Lipschitz when restricted to } \tilde{V},
\]

for some neighborhoods \( \tilde{U}, \tilde{V} \) of \( (z_+^0, y_+^0, u_+^0) \), \( 0 \in R^{2n+m} \) respectively
\[
\Leftarrow (M^+ + N_{R^{n} \times R^{m+n}})^{-1} \cap \tilde{U} \text{ is Lipschitz when restricted to } \tilde{V},
\]

for some neighborhoods \( \tilde{U}, \tilde{V} \) of \( (z_+^0, y_+^0, u_+^0) \), \( (\xi_z^{00}, \xi_y^{00}, 0) \) respectively
\[
\Rightarrow (M^+ + N_{R^{n} \times R^{m+n}})^{-1} \cap U \text{ is Lipschitz when restricted to } V,
\]

for some neighborhoods \( U, V \) of \( (z_+^0, y_+^0), (\xi_z^{00}, \xi_y^{00}) \) respectively
\[
\Leftarrow M^+ \text{ is Lipschitzianly invertible near } (z^0, y^0).
\]

The proof is complete. \( \Box \)

We show that Proposition 13 applies to convex programs.

**Lemma 14.** Let \( \theta, g, F, \sigma, \tau \), and \( M \) be as in Proposition 13. Let \( \theta \) and each component function \( g_i \) \( (i = 1, \ldots, m) \) of \( g \) be convex. Suppose there is \( z > 0 \) with 
\[
g(z) < 0 \text{ (Slater constraint qualification) and } (NLP) \text{ has a (global) minimum at } \hat{z} \in R^n.
\]
Then there exists \( \hat{y} \in R^m \) such that \( (z^*, y^*) = (\hat{z}, \hat{y}) - F(\hat{z}, \hat{y}) \) is a zero of \( F_+ \).

If \( (LI) \) and \( (SSOS) \) hold at \( (z^*, y^*) \) then there is \( \alpha_0 > 0 \) such that
\[
X_0 \overset{\text{def}}{=} \{ (z, y) \in R^{n+m} \mid \|F_+(z, y)\| \leq \alpha_0 \}
\]
is bounded, and \( \nabla F(z_+, y_+) \) is Lipschitzianly invertible near each \( (z, y) \in X_0 \).
PROOF. It is a classical result in optimization (Mangasarian 1969, 7.3.7, McCormick 1983, 10.2.16) that there exists a Lagrange multiplier \((\hat{y}, \hat{u}) \in \mathbb{R}^m \times \mathbb{R}^n\) such that

\[
0 = \nabla \theta(\hat{z}) + \nabla g(\hat{z})^T \hat{y} - \hat{u},
\]

\[
0 \leq \hat{u}, \quad \langle \hat{u}, \hat{z} \rangle = 0,
\]

\[
0 \leq \hat{y}, \quad \langle \hat{y}, g(\hat{z}) \rangle = 0.
\]

Since \(\hat{z}\) is also feasible \((\hat{z} \geq 0 \text{ and } g(\hat{z}) \leq 0)\), \((z, y, u) = (\hat{z}, \hat{y}, \hat{u})\) is a solution of a generalized equation \((GE^*)\)

\[
0 \in \left(\tilde{F}^* + N_{\mathbb{R}^m \times \mathbb{R}^n}(z, y, u)\right)
\]

where \(\tilde{F}^*(z, y, u) = \text{def}(F(z, y) - (u, 0), z)\). \((GE^*)\) is just the generalized equation \((GE)\) from the proof of Proposition 13 when \(\xi^0 = 0 \in \mathbb{R}^{n+m}\). It is easy to see that \((\hat{z}, \hat{y})\) solves \((GE)\),

\[
0 \in (F + N_{\mathbb{R}^m}(\hat{z}, \hat{y}) ,
\]

hence \((z^*, y^*) = \text{def}(\hat{z}, \hat{y}) - F(\hat{z}, \hat{y})\) is a zero of \(F_+\); and \((z^*_+, y^*_+, z^*_X - z^*)\) equals \((\hat{z}, \hat{y}, \hat{u})\), a solution of \((GE^*)\).

Since \((LI)\) and \((SSOS)\) hold at \(x^* = \text{def}(z^*_+, y^*_+)\), the proof of Proposition 13 demonstrates that \(\nabla F(x^*_+)\) is Lipschitzianly invertible near \(x^*\). Lemma 7 with \(g = \text{def}\nabla F(x^*_+), g' = \text{def}\nabla F(0)\) can be used to show that \(F_+\) is Lipschitzian invertible near \(x^*\). Let \(U^*\) be a neighborhood of \(x^*\) and \(\alpha > 0\) be such that

\[
F_+|_{U^*}: U^* \to \alpha_+\mathbb{B}
\]

is Lipschitzian invertible. So \(U^*\) is bounded. For any element \(x^0\) of \(U^*\), another application of Lemma 7, this time with \(g = \text{def}\nabla F(x^0), g' = \text{def}\nabla F(x^0)\), shows that \(\nabla F(x^0)\) is Lipschitziany invertible near \(x^0\). It only remains to be seen that for some \(\alpha_0 > 0\),

\[
X^0_0 = F_+^{-1}(\alpha_0 \mathbb{B}) \subset U^*,
\]

because then \(X^0_0\) is bounded and, for each \(x^0 \in X^0_0\), \(\nabla F(x^0)\) is Lipschitziany invertible near \(x^0\).

It is basic exercise in convex analysis to show that \(F\) is monotone on \(\mathbb{R}^{n+m}\):

\[
\langle F(x) - F(x'), x - x' \rangle \geq 0, \quad \forall x, x' \in \mathbb{R}^{n+m}.
\]

By Brézis (1973, Proposition 2.10), \(F + N_{\mathbb{R}^m}\) is a maximal monotone operator, meaning there is no monotone operator whose graph strictly contains the graph of \(F + N_{\mathbb{R}^m}\). Hence \((F + N_{\mathbb{R}^n})^{-1}\) is also maximal monotone and, as a result (see Brézis 1973, Example 2.1.2), \((F + N_{\mathbb{R}^m})^{-1}(\xi)\) is a convex set for each \(\xi \in \mathbb{R}^{n+m}\).

We have established that \(F_+\) is Lipschitziany invertible near \(x^*\), so Lemma 6 yields neighborhoods \(U^1\) of \(\pi_+(x^*)\) and \(V^1\) of \(0 \in \mathbb{R}^{n+m}\) such that \((F + N_{\mathbb{R}^m})^{-1} \cap U^1\) is a Lipschitz function when restricted to \(V^1\). We may assume \(V^1 \subset \alpha_+\mathbb{B}\) and \(U^1\) is open. Let \(\xi \in V^1, x = (F + N_{\mathbb{R}^m})^{-1}(\xi) \cap U^1\) and \(x' \in (F + N_{\mathbb{R}^m})^{-1}(\xi)\), so \(x\) need not lie in \(U^1\). Then \((1 - t)x + tx'\) lies in \((F + N_{\mathbb{R}^m})^{-1}(\xi) \cap U^1\) for all sufficiently small \(t \in [0, 1]\), because \((F + N_{\mathbb{R}^m})^{-1}(\xi)\) is convex and \(U^1\) is open. This contradicts uniqueness of \(x\) in \(U^1\), unless \(x' = x\).
Therefore \((F + N_{\beta + \alpha m})^{-1}|_{V^1}\) is a function. It follows that, by using (2) from the proof
of Lemma 6, \(F_+^{-1}|_{V^1}\) is also a function. Recall \(F_+^{-1} \cap U^*\) is a function when restricted
to \(\alpha + v\), and \(\alpha + v\) contains \(V^1\); therefore \(F_+^{-1}|_{V^1} \cap U^*\) and \(F_+^{-1}|_{V^1}\) are identical, in
particular \(F_+^{-1}(V^1) \subset U^*\). To conclude, choose \(\alpha_0 > 0\) such that \(\alpha_0 \subset V^1\), and
observe
\[X_0 \overset{\text{def}}{=} F_+^{-1}(\alpha_0 \beta) \subset F_+^{-1}(V^1) \subset U^*.\]

5.4. Implementation of the path search. We will outline how the Newton path \(p^k\)
is obtained computationally at iteration \(k + 1\), for solving (NE) when \(C = \text{def} R_+^N\). We
are actually thinking of solving (NLP), that is letting \(N = \text{def} n + m\) and \(F\) be given by
(15). First Lemke's algorithm (Cottle and Dantzig 1974), a standard pivotal method
for solving linear complementarity problems, is reviewed.

The Linear Complementarity Problem is a special case of the nonlinear complementarity problem when the function \(F\) defining (NLCP) is affine: fine \(v, w \in R^N\)
such that
\[(LCP) \quad w = Mv + q, \quad 0 \leq v, w, \quad 0 = \langle v, w \rangle,\]
where \(M \in R^{N \times N}\) and \(q \in R^N\). In Lemke's algorithm an artificial variable \(v_0 \in R\) is
introduced. Let \(e = \text{def}(1, \ldots, 1)^T \in R^N\). At each iteration of the algorithm we have a
basic feasible solution or BFS (Chvátal 1983), \((v, w, v_0)\), of the system
\[(17) \quad w = Mv + q + ev_0, \quad 0 \leq (v, w, v_0),\]
which is almost complementary, i.e., for each basic variable \(v_i\) (respectively \(w_i\),
\(1 \leq i \leq N, w_i\) (respectively \(v_i\) is nonbasic. The variable \(w_i\) is called the complement
of \(v_i\), and \textit{vice versa}. In particular \(\langle v, w \rangle = 0\), hence if \(v_0 = 0\) then \((v, w)\) solves
\((LCP)\). Given \(v_0\), \((v, w)\) solves the parametric linear complementarity problem
\[w = Mv + q + ev_0, \quad 0 \leq v, w, \quad 0 = \langle v, w \rangle.\]

The initial almost complementary BFS is given by taking \(v_0\) large and positive, and
\((v, w) = (0, q + ev_0)\). So Lemke's algorithm may be viewed as a method of construct-
ing a path of solutions \((v(v_0), w(v_0))\) to the parametric problem as \(v_0\) moves down
toward zero.\(^1\) We use this idea, but with a different parametric problem, to construct
the path \(p^k\) we need for iteration \(k + 1\) of damped Newton's method.

Let \(N = \text{def} n + m\), \(F\) be given by (15) and \(x^k = \text{def}(z^k, \lambda^k)\). Let the first-order
approximation \(A\) of \(F_+\) on \(R^N\) be given by (14); and denote \(A(x^k)\) by \(A_k\). Now
\(p^k(0) = \text{def} x^k\) and, for each \(t \in \text{dom}(p^K), p^k(t)\) is the solution \(x\) to
\[(1 - t) F_+(x^k) = A_k(x)\]
\[= F(x^k) + \nabla F(x^k) (x - x^k) + x - x^k\]
\[= \nabla F(x^k) (x) + [F(x^k) - \nabla F(x^k) x^k].\]

\(^1\) Actually, \(v_0\) need not strictly decrease as a result of a pivot in Lemke's algorithm, in which case the
 corresponding solution \((v, w)\) of the parametric problem is not a \textit{function} of \(v_0\). In any case \((v, w)\) traces
out a path.
Equivalently, \((v, w) = (p^k(t)_+, p^k(t)_- - p^k(t))\) solves the parametric linear complementarity problem:

\[(\text{LCP})(t) \quad w = M^k v + q^k - (1 - t)r^k, \quad 0 \leq v, w, \quad 0 = \langle v, w \rangle,\]

where

\[M^k \overset{\text{def}}{=} \nabla F(x^k_+) = \begin{bmatrix} \nabla ^m \theta \nabla g(z^k_+) \\ -\nabla g(z^k_+) \end{bmatrix},\]

\[q^k \overset{\text{def}}{=} F(x^k_+ - \nabla F(x^k_+)x^k_+ = \begin{bmatrix} \nabla \theta(z^k_+) - \partial ^m \theta z^k_+ \\ -g(z^k_+) + \nabla g(z^k_+)z^k_+ \end{bmatrix},\]

\[r^k \overset{\text{def}}{=} F_+(x^k) = M^k v^k + q^k - w^k,\]

and

\[(v^k, w^k) \overset{\text{def}}{=} (x^k_+, x^k_+ - x^k_+).\]

Clearly \((v^k, w^k)\) solves \((\text{LCP})(0)\), and a solution \((v, w)\) of \((\text{LCP})(1)\) yields a zero \(v - w\) of \(A_k\), i.e., the Newton iterate.

To construct \(p^k\) we use a modification of Lemke’s algorithm in which the role of the auxiliary variable \(v_0\) is played by \(t\). Assume \((v, w, t)\) is an almost complementary BFS of

\[(18) \quad w = M^k v + q^k + (t - 1)r^k, \quad 0 \leq v, w, \quad 0 \leq t \leq 1,\]

such that \(t\) is basic. This defines the current point on \(p^k\): \(p^k(t) = \overset{\text{def}}{=} v - w\). As there are exactly \(N\) basic variables, there is some \(i\) for which both \(v_i\) and \(w_i\) are nonbasic. One of these nonbasics is identified as the entering variable. The modified algorithm iterates like Lemke’s algorithm, in the following way.

Increase the entering variable—altering the basic variables as needed to maintain the equation \(w = M^k v + q^k + (t - 1)r^k\)—until (at least) one of the basic variables is forced to a bound. We choose one of these basic variables to be the leaving variable, that is the variable to be replaced in the basis by the entering variable. This transformation of the entering and basic variables is called a pivot operation, and corresponds to moving along an affine piece of the path \(p^k\) from the value of the parameter \(t\) before the pivot, \(t_{\text{old}}\), to the current parameter value. After a pivot, \((v, w, t)\) is still an almost complementary BFS for \((18)\); the leaving variable is now nonbasic, and, assuming it is not \(t\), its complement is nonbasic too. The next entering variable is defined as the complement of the leaving variable. This completes an iteration of the modified algorithm, and we are ready to begin the next iteration.

The modified algorithm is initialized at \((v, w, t) = (v^k, w^k, 0)\) with \(t\) as the nonbasic entering variable (so \(p^k(0) = v^k - w^k\) as needed).

Now the algorithm cannot continue if either \(t\) becomes the leaving variable (because no entering variable is defined) or no leaving variable can be found (because increasing the entering variable causes no decrease in any basic variable). In the latter case we have detected a ray. Other stopping criteria are needed to reflect the
aim of the exercise, namely to path search. We stop iterating if \( t = 1 \) (the entire path has been traversed), or \( t \), with \( p^k(t) = \text{def} u - w \), does not satisfy the descent condition \((\text{NmD})\), or \( t \) strictly decreases as a result of the last pivot—satisfaction of \((\text{NmD})\) after each pivot is the basic requirement of a forward path search. In practice, an upper bound on the number of pivots is also enforced.

It is a subtle point that, when solving \((\text{NLP})\), the first-order approximation at \( x^k = (z^k, y^k) \in \mathbb{R}^{n+m} \) used to define the path \( p^k \) is \textit{exact} for points \( x = (z, y) \) such that \( z = z^+ : A_k(x) = F(x^+) + x - x^+ \). Using this idea, we may save a little work by only checking \((\text{NmD})\) if the positive part of the variables \( z \) differ from \( z^k \). This corresponds in the modified Lemke’s algorithm to only checking \((\text{NmD})\) if the subvector of the first \( n \) components of \( v \) differs from the corresponding subvector of \( v^k \).

Suppose the method halts because a ray is detected or \( t \) decreases. In both cases we know \( A_k \) is not invertible at \( p^k(t_{\text{old}}) \), otherwise it would have been possible to perform a pivot yielding \( t > t_{\text{old}} \), so we take \( x^{k+1} = \text{def} p^k(t_{\text{old}}) \). Another possibility is that the method halts because \((\text{NmD})\) fails at \( t \), in which case we use the Armijo line search on \([p(t_{\text{old}}), p(t)]\) to determine a path length \( t_k \in [t_{\text{old}}, t] \) satisfying \((\text{NmPs})\), and take \( x^{k+1} = \text{def} p^k(t_k) \).

We have described our implementation of a forward path search. A backward path search was also implemented as follows. We start by recording the first tuple \((v^k, w^k, 0)\), and then perform pivots as described previously, recording the tuple \((v, w, t)\) after each successful pivot. As before, pivoting stops if a ray is detected, \( t \) leaves the basis, or \( t = 1 \); we do not, however, use the condition \((\text{NmD})\) or the sign of the change in \( t \) to decide whether to continue pivoting. Suppose \( J \) is the number of successful pivots. If the \((J + 1)\)th tuple \((v, w, t)\) is accurate, meaning \((\text{NmD})\) is satisfied at \( t \) with \( p^k(t) = \text{def} u - w \), then take \( x^{k+1} = \text{def} u - w \). If not, take \( \hat{J} = 1 \), \( \hat{J} = J + 1 \) and, while \( \hat{J} + 1 \neq \hat{J} \), do the following bisection search on the list: if the tuple at position \( J' = \text{def} [(\hat{J} + \hat{J})/2] \) in the list is accurate, then let \( \hat{J} = J' \) and \( \hat{J} \) be unchanged, otherwise let \( \hat{J} \) be unchanged and \( \hat{J} = J' \). After the bisection search, the tuple at position \( \hat{J} \) is accurate and the tuple at position \( \hat{J} + 1 \) is not accurate. An Armijo line search is now carried out on the interval between the points \( v - w \) defined by these two tuples, and \( x^{k+1} \) defined accordingly.

One difficulty we have not discussed is the possibility that \((v^k, w^k)\) is not a basic solution of

\[
    w = M^k v + q^k - r^k, \quad 0 \leq v, w,
\]

i.e., \((v^k, w^k, 0)\) with \( t \) nonbasic cannot be used as a starting point of the modified Lemke’s algorithm. To overcome this, we performed a modified Cholesky factorization \((\text{Gill, Murray and Wright 1981})\) of the Hessian of the Lagrangian, \( L_k'' \) above, changing it if necessary into a positive definite matrix. This operation—somewhat crude because it attacks the entire Hessian of the Lagrangian, rather than just the “basis” columns—may not always be sufficient to correct the problem since columns of \( M^k \) not corresponding to columns of \( L_k'' \) are generally present in the initial basis. However it was sufficient in the problems we tried, below. It has the added advantage of being a heuristic for preventing convergence to a Karush-Kuhn-Tucker point of the nonlinear program which corresponds to a local maximum or saddle point instead of a local minimum of the program.

5.5. Computational examples. Our computational examples are of global, or path search damped Newton’s method applied to the nonlinear programs of the form
(NLP):

$$\min \theta(z) \quad \text{subject to } z \geq 0, \ g(z) \leq 0.$$  

The computer programs were written in C and the computation carried out on a Sun 4 (Sparcstation). Most problems are taken from Ferris (1990), with starting variables $z^0$ close to solutions and initial multipliers $y^0$ set to zero. We note that these problems all have small dimension: no more than 15 variables and 10 nonlinear constraints. Results include performance of global Newton’s method using both the forward and backward path search procedures. Other problems, showing convergence of the damped method in spite of the cycling (nonconvergence) that occurs without damping, are also tested.

For comparison we have also tested local (Robinson- or Josephy-)Newton’s method on the same problems, using Lemke’s algorithm to find the Newton iterate at each iteration. This method was found to be somewhat fragile with respect to starting points; quite often the method failed because a ray was detected by Lemke’s algorithm, i.e., the Newton iterate $\hat{x}^{k+1}$ could not be determined. To make the method more robust we altered it, allowing it continue after Lemke’s algorithm detects a ray by setting $x^{k+1} = v - w$, where $(v, w, v_0)$ is the last almost complementary BFS produced by Lemke’s algorithm. The altered method was usually successful in solving test problems.

Our limited computational results, below, bear out what intuition suggests. First, the global method is more robust than the local method especially in avoiding cycling, discussed below. Second, the global method requires more stringent conditions on the current approximation $A_k$ than does the local method, and enforcing these heuristically can degrade convergence. Third, the computational cost of carrying out a path search is seen in the increase the number of evaluations of $F$. For these problems, the backward path search was more efficient than the forward procedure because, at each iteration, the Newton iterate $p^k(1) \in A^{-1}(0)$ usually existed and satisfied the descent criterion (NmD), so that $x^{k+1} = \text{def} p^k(1)$.

In higher dimensions we expect the value of path searching to be even greater, a view partly supported by the following observations. One peculiarity of Newton’s method is the possibility of cycling, depending on the starting point, as in the smooth case. This is seen for the altered Newton’s algorithm when testing the Colville 2 problem with a feasible starting point (Table 1). It is easy to find a real function of one variable, which, when minimized over $[0, \infty)$ by Newton’s method, demonstrates cycling of (unaltered) Newton’s method (Example 15 below). It turns out that for any problem (NLP), the new problem formed by adding such a function in an $(n + 1)$th variable to the objective function $\theta$ cannot be solved by Newton’s method for some starting points (Example 16): at best it will converge in the first $n$ variables and cycle in the $(n + 1)$th variable. So it can be argued that the likelihood of cycling in Newton’s method increases as the dimension of the problem increases. Global Newton’s method converges for otherwise well behaved problems.

Table 1 summarizes our results on the problems from Ferris (1990). The following parameters were set: in the path search, $\sigma = \text{def} 0.1$, $\tau = \text{def} 0.5$, $M = \text{def} 4$. The problem was considered solved when the Euclidean norm of the residual satisfied

$$\| F(x^k) + x^k - x^k_+ \|_2 \leq 10^{-5}.$$  

\footnote{Fortran subroutines and initial values of variables supplied by Michael C. Ferris, Computer Sciences Department, University of Wisconsin-Madison, Madison, WI 53706.}
TABLE 1

<table>
<thead>
<tr>
<th>Problem</th>
<th>Size</th>
<th>Pivots</th>
<th>Evaluations</th>
<th>Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>I</td>
<td>II</td>
<td>III</td>
</tr>
<tr>
<td>Rosenbrock</td>
<td>4 x 2</td>
<td>20</td>
<td>19</td>
<td>21</td>
</tr>
<tr>
<td>Himmelblau</td>
<td>3 x 4</td>
<td>25</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>Wright</td>
<td>3 x 5</td>
<td>99</td>
<td>31</td>
<td>31</td>
</tr>
<tr>
<td>Colville 1</td>
<td>10 x 5</td>
<td>31</td>
<td>41</td>
<td>41</td>
</tr>
<tr>
<td>Colville 2 (feasible)</td>
<td>5 x 15</td>
<td>*</td>
<td>23</td>
<td>24</td>
</tr>
<tr>
<td>Colville 2 (infeasible)</td>
<td>5 x 15</td>
<td>113</td>
<td>40</td>
<td>40</td>
</tr>
</tbody>
</table>

Starting points $x^0 = \text{def}(z^0, y^0) \in \mathbb{R}^{n+m}$ were taken with the variables $z^0$ used in Ferris (1990) and the multipliers $y^0 = \text{def} 0$. The first column of Table 1 gives the name of the problem, and the second column gives the number of constraints $m$ in the vector inequality $g(z) < 0$, and the number of variables $n$, i.e., the respective dimensions of multipliers $y$ and variables $z$. Each of the remaining columns contains three subcolumns corresponding to three implementations of Newton’s method: I denotes the (altered) local Newton’s method, II, the global method with a forward path search, and III, the global method with a backward path search. The ‘Pivots’ column lists the total number of pivots required by the three implementations to solve a given problem. The ‘Evaluations’ column lists the total number of evaluations of $F$ used by the three implementations. The final column, ‘Iterations’, shows the number of iterations needed by the implementations to solve each problem. An asterisk * indicates failure to solve the problem.

REMARKS ON TABLE 1.

Iterations. The most important feature of the results is the number of iterations needed, i.e., the number of evaluations of the Hessian of the Lagrangian, $\mathcal{L}''$. Three problems where the number of iterations differ widely are discussed further.

The objective function of the Rosenbrock problem is highly nonlinear causing any damping of Newton’s method (even in the unconstrained case) to take very short steps, hence many iterations, unless the current iterate is very close to a minimizer. The nonmonotone line search alleviates this problem: with memory length $M = 4$ only 9 iterations were needed, while more than 500 iterations were needed in another test using the monotone path search ($M = 1$). In other tests using the monotone path search, the remaining problems required the same or slightly fewer iterations than listed in Table 1.

In the Wright problem, the local method does much better (7 iterations) than both implementations of the global method (27 iterations) because it uses the actual Hessian of the Lagrangian whereas the global implementations introduce error by using a modified Cholesky version of this. In another test, 27 iterations were also required by the local method using the modified Cholesky version of the Hessian of the Lagrangian. This highlights a difficulty of path search damping: care is needed to ensure that the path is well defined near current iterate. On the other hand, this has advantages (other than damping) as we see in the remarks on pivots below.

For the Colville 2 problem with a feasible starting point, the local Newton’s method cycled while the global Newton implementations solved the problem easily. In the local implementation, after the first iteration the pair $(v^k, w^k)$ alternated between two different vector pairs for which the associated residual $\|F(v^k) - w^k\|$ took values 98 and 5 respectively, the former corresponding to an unbounded ray.

Evaluations. The number of evaluations of $F$, i.e., of $\nabla \theta$, $g$ and $\nabla g$, is higher for the global implementations than for the local implementation, because the former
compare \( \| F(v) - w \| \) to \( (1 - t)\| F(v^k) - w^k \| \) after perhaps several pivots during iteration \( k + 1 \), whereas the latter only checks the norm of the residual at the start of the iteration. For global Newton’s method, the backward path search (III) generally uses fewer function and derivative evaluations than the forward version (II) because, near the solution, the Newton iterate usually exists and is acceptable as the next iterate.

The difference between the number in the ‘Pivots’ column and the number in the ‘Evaluations’ column for method II (global Newton with a forward path search) is the savings in function/derivative evaluations obtained by only checking \( NmD \) when the first \( n \) components of \( v \) differ from the first \( n \) components of \( v^k \).

**Pivots.** The global implementations usually require fewer pivots per iteration than does the local implementation. This is not surprising: the damped method requires the current iterate \( x^k \) to correspond to a basis of \( (LCP)(0) \). We initiate the modified Lemke’s algorithm at this basis by explicitly factorizing the corresponding matrix. By starting with this ‘warm’ basis, both the forward and backward implementation of the damped method generally take many fewer pivots than the local implementation to find the Newton iterate, the solution of \( (LCP)(1) \).

**Example 15.** Let \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable function whose gradient is

\[
\nabla \phi(z) = \text{arctan}(z - 10),
\]

the shifted inverse trigonometric tangent function. Integrating and setting the constant of integration to be zero we obtain

\[
\phi(z) = (z - 10)\text{arctan}(z - 10) - (1/2)\log\left(1 + (z - 10)^2\right).
\]

Now consider a simple problem of the form (NLP):

\[
\min_{z \in \mathbb{R}} \phi(z) \quad \text{subject to } z > 0.
\]

The unique solution is \( z = 10 \).

It can be easily shown that for any starting point \( |z^0 - 10| \geq 2 \) Newton’s method cycles infinitely. This has been observed in computation for several such starting points. Global Newton’s method using the forward path search, with the same parameters as used to obtain the results of Table 1, converged in 4–33 iterations over a variety of starting points \( 2 \leq |z^0 - 10| \leq 100 \).

**Example 16.** Derive a problem from (NLP) by including an extra variable \( z_{n+1} \). The new objective function is

\[
\tilde{\phi}(z_1, \ldots, z_{n+1}) \overset{\text{def}}{=} \phi(z_1, \ldots, z_n) + \phi(z_{n+1}),
\]

where \( \phi \) is defined in the last example. The constraints are \( g(z_1, \ldots, z_n) \leq 0 \) as before, and \( 0 \leq z = \text{def}(z_1, \ldots, z_{n+1}) \). It is not hard to see that for any starting point \( (z^0, y^0) \in \mathbb{R}^{n+1} \times \mathbb{R}^m \) in which \( |z^0_{n+1} - 10| \geq 2 \), Newton’s method cannot converge in \( z_{n+1} \).

We have tested this for the Himmelblau (NLP), where \( n = \text{def}4, (z_0^1, \ldots, z_0^4) \) are the initial variables used in Ferris (1990), \( z_0^5 = 0 \), and all multipliers are initially zero. Newton’s method cycles infinitely; damped Newton’s method using the forward path search, with the same parameters as used for Table 1, converges in 33 iterations.

In other tests using the monotone path search, damped Newton’s method using the forward path search only required 4–7 iterations to solve the problem in Example 15.
from various starting points; and 9 iterations for the augmented problem of Example 16 using the starting point specified there. In both cases this is an improvement on nonmonotone damping. It is interesting that nonmonotone damping, which aids convergence for highly nonlinear problems by accepting iterates that need not decrease the size of the residual, slows convergence here because it perpetuates almost cyclic behavior of the iterates.

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