Convergence of Newton’s method and uniqueness of the solution of equations in Banach space

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Under the hypothesis that the derivative satisfies some kind of weak Lipschitz condition, exact estimates of the radii of convergence ball of Newton’s method and of the uniqueness ball for the solution of operator equations are given in Banach space. Relevant results under the assumptions of Kantorovich and Smale types are improved upon in this paper.

1. Introduction

Consider the problem of using Newton’s method

\[ x_{n+1} = x_n - f'(x_n)^{-1}f(x_n), \quad n = 0, 1, \ldots \] (1.1)

to find the approximation solution of the equation

\[ f(x) = 0, \] (1.2)

where \( f \) is an operator mapping from some domain \( D \) in a real or complex Banach space \( X \) to another \( Y \). Ostrowski and Kantorovich once gave a popular formulation which does not presume the existence of the solution (see Wang, 1999). However, it implies the existence of the solution because their hypothesis can result in the existence of \( x^* = \lim x_n \). Therefore, the existence of the solution is a very natural hypothesis. The advantage of such a hypothesis is that it can make us clearly see how big the radius of convergence ball is in the study. For \( x \in X \) and a positive number \( r \), let \( B(x, r) \) denote an open ball with radius \( r \) and centre \( x \) and let \( \overline{B}(x, r) \) denote its closure. For example, under the hypothesis that \( f'(x^*)^{-1}f' \) satisfies the Lipschitz condition

\[ \| f'(x^*)^{-1}(f'(x) - f'(x')) \| \leq L\| x - x' \|, \quad \forall x', x \in B(x^*, r), \] (1.3)

Traub & Wozniakowski (1979) and Wang (1980) independently gave an exact estimate \( r = 2/(3L) \) for the radius of convergence ball \( B(x^*, r) \). Under the hypothesis that \( f \) is analytic and satisfies

\[ \| f'(x^*)^{-1}f^{(k)}(x^*) \| \leq k!\gamma^{k-1}, \quad k = 2, 3, \ldots, \] (1.4)

Smale (1986) gave an exact estimate \( r = (5 - \sqrt{17})/(4\gamma) \) for the corresponding radius. Dedieu (1999) proved that (1.2) has a unique solution in the same ball (see also Blum et al., 1996 and Smale, 1997).

For these two problems, i.e. the convergence domain of Newton’s method and the uniqueness domain of the solution of the equations, we will give in this paper a uniform method to deal with them under more general conditions.
2. Special and generalized Lipschitz conditions

The condition on the operator \( f \)
\[
\| f(x) - f(x') \| \leq L \| x - x' \|, \quad \forall x, x' \in D
\]  
(2.1)
is usually called the Lipschitz condition in the domain \( D \) with constant \( L \). For some kind of domain with centre such as a ball \( B(x_0, r) \), sometimes it is not necessary for the inequality to hold for any \( x, x' \) in the domain: it is only required to hold for any \( x \) and for points lying on the connecting line \( x^\tau = x_0 + \tau(x - x_0) \) between \( x \) and \( x_0 \), where \( 0 \leq \tau \leq 1 \). We refer to this special Lipschitz condition, i.e.
\[
\| f(x) - f(x^\tau) \| \leq L \| x - x^\tau \|, \quad \forall x \in B(x_0, r), \ 0 \leq \tau \leq 1,
\]  
(2.2)
as the radius Lipschitz condition in the ball \( B(x_0, r) \) with the constant \( L \). Sometimes, if it is only required to satisfy
\[
\| f(x) - f(x_0) \| \leq L \| x - x_0 \|, \quad \forall x \in B(x_0, r),
\]  
(2.3)
we call it the centre Lipschitz condition in the ball \( B(x_0, r) \) with the constant \( L \). Furthermore, the \( L \) in the Lipschitz condition need not be a constant, but a positive integrable function. If this is the case, then (2.2) or (2.3) are replaced respectively by
\[
\| f(x) - f(x^\tau) \| \leq \int_{\rho(x)}^{\rho(x_\tau)} L(u) \, du, \quad \forall x \in B(x_0, r), \ 0 \leq \tau \leq 1
\]  
(2.4)
or
\[
\| f(x) - f(x_0) \| \leq \int_0^{\rho(x_0)} L(u) \, du, \quad \forall x \in B(x_0, r),
\]  
(2.5)
where \( \rho(x) = \| x - x_0 \| \). At the same time, the corresponding ‘Lipschitz condition’ is referred to as having the \( L \) average.

By Banach’s theorem (see Banach & Steinhaus, 1927, or Theorem V.4.3 in Kantorovich & Akilov, 1982), the following result can be obtained directly.

LEMMA 2.1 Suppose that \( f \) has a continuous derivative in \( B(\bar{x}, r) \), \( f'(\bar{x})^{-1} \) exists and \( f'(\bar{x})^{-1} f' \) satisfies the centre Lipschitz condition with the \( L \) average:
\[
\| f'(\bar{x})^{-1} f'(x) - 1 \| \leq \int_0^{\rho(x)} L(u) \, du, \quad \forall x \in B(\bar{x}, r),
\]  
(2.6)
where \( L \) is a positive integrable function. Let \( r \) satisfy
\[
\int_0^r L(u) \, du \leq 1.
\]  
(2.7)
Then \( f'(x) \) is invertible in this ball and
\[
\| f'(x)^{-1} f'(\bar{x}) \| \leq \left( 1 - \int_0^{\rho(x)} L(u) \, du \right)^{-1}.
\]  
(2.8)

In the following application of this lemma, we often take \( \bar{x} = x^* \), where \( x^* \) is a zero of \( f \). Further, when we refer to the derivative of a map \( f \), it is meant in the Frechet sense throughout the paper.
3. Convergence ball of Newton’s method

THEOREM 3.1 Suppose that \( f(x^*) = 0 \), \( f \) has a continuous derivative in \( B(x^*, r) \), \( f'(x^*)^{-1} \) exists and \( f'(x^*)^{-1} f' \) satisfies the radius Lipschitz condition with the \( L \) average:

\[
\| f'(x^*)^{-1} (f'(x) - f'(x^*)) \| \leq \int_{\rho(x)}^{\rho(x)} L(u) \, du, \quad \forall x \in B(x^*, r), \ 0 \leq \tau \leq 1, \tag{3.1}
\]

where \( x^* = x^* + \tau (x - x^*) \), \( \rho(x) = \| x - x^* \| \) and \( L \) is nondecreasing. Let \( r \) satisfy

\[
\frac{\int_{0}^{r} L(u) \, du}{r \left( 1 - \int_{0}^{r} L(u) \, du \right)} \leq 1. \tag{3.2}
\]

Then Newton’s method is convergent for all \( x_0 \in B(x^*, r) \) and

\[
\| x_n - x^* \| \leq q^{n-1} \| x_0 - x^* \|, \quad n = 1, 2, \ldots, \tag{3.3}
\]

where

\[
q = \frac{\int_{0}^{\rho(x_0)} L(u) \, du}{\rho(x_0) \left( 1 - \int_{0}^{\rho(x_0)} L(u) \, du \right)} \tag{3.4}
\]

is less than 1.

Proof. Arbitrarily choosing \( x_0 \in B(x^*, r) \), where \( r \) satisfies (3.2), then \( q \) determined by (3.4) is less than 1. In fact, by the monotonicity of \( L \), we obtain

\[
\left( \frac{1}{t_2} \int_{0}^{t_2} - \frac{1}{t_1} \int_{0}^{t_1} \right) L(u) \, du = \left( \frac{1}{t_2} \int_{0}^{t_2} + \frac{1}{t_2} - \frac{1}{t_1} \right) \int_{0}^{t_1} L(u) \, du \\geq L(t_1) \left( \frac{1}{t_2} \int_{0}^{t_2} + \frac{1}{t_2} - \frac{1}{t_1} \right) \int_{0}^{t_1} u \, du \\geq L(t_1) \left( \frac{1}{t_2} \int_{0}^{t_2} - \frac{1}{t_1} \int_{0}^{t_1} \right) u \, du = 0
\]

for \( 0 < t_1 < t_2 \). Thus, \( \frac{1}{t_2} \int_{0}^{t_2} L(u) \, du \) is nondecreasing with respect to \( t \). So we have

\[
q = \frac{\int_{0}^{\rho(x_0)} L(u) \, du}{\rho(x_0)^2 \left( 1 - \int_{0}^{\rho(x_0)} L(u) \, du \right)} \rho(x_0)
\]
\[
\leq \frac{\int_0^r L(u)u du}{r^2 \left(1 - \int_0^r L(u) du\right)} \rho(x_0) \\
\leq \frac{\|x_0 - x^*\|}{r} < 1.
\]

Now if \(x_n \in B(x^*, r)\), then we have by (1.1)

\[
x_{n+1} - x^* = f'(x_n)^{-1}\left(f(x^*) - f(x_n) + f'(x_n)(x_n - x^*)\right) \\
= f'(x_n)^{-1} f'(x^*) \int_0^1 f'(x^*)^{-1} \left(f'(x_n) - f'(x^*)\right) (x_n - x^*) \, dt,
\]

where \(x^* = x^* + \tau (x_n - x^*)\). Hence, by Lemma 2.1 and condition (3.1) we obtain

\[
\|x_{n+1} - x^*\| \leq \|f'(x_n)^{-1} f'(x^*)\| \int_0^1 \|f'(x^*)^{-1} (f'(x_n) - f'(x^*))\| \cdot \|x_n - x^*\| \, dt \\
\leq \frac{1}{1 - \int_0^{\rho(x_n)} L(u) du} \int_0^1 \int_{\rho(x_n)}^{\rho(x_n)} L(u) du \rho(x_n) \, dt \\
= \frac{\int_0^{\rho(x_n)} L(u) du}{1 - \int_0^{\rho(x_n)} L(u) du}.
\]

Taking \(n = 0\) above, we obtain \(\|x_1 - x^*\| \leq q \|x_0 - x^*\| < \|x_0 - x^*\|.\) Hence, \(x_1 \in B(x^*, r)\). This shows that (1.1) can be continued an infinite number of times. By mathematical induction, all \(x_n\) belong to \(B(x^*, r)\) and \(\rho(x_n) = \|x_n - x^*\|\) decreases monotonically. Therefore, for all \(n = 0, 1, \ldots\), we have

\[
\|x_{n+1} - x^*\| \leq \frac{\int_0^{\rho(x_n)} L(u) du}{\rho(x_n)^2 \left(1 - \int_0^{\rho(x_n)} L(u) du\right)} \rho(x_n)^2 \\
\leq \frac{\int_0^{\rho(x_0)} L(u) du}{\rho(x_0)^2 \left(1 - \int_0^{\rho(x_0)} L(u) du\right)} \rho(x_0)^2 \\
= \frac{q}{\rho(x_0)} \|x_n - x^*\|^2.
\]

Thus (3.3) follows. \(\square\)
4. The uniqueness ball for the solution of equations

Theorem 4.1 Suppose that \( f(x^*) = 0 \), \( f \) has a continuous derivative in \( B(x^*, r) \), \( f'(x^*)^{-1} \) exists and \( f'(x^*)^{-1} f' \) satisfies the centre Lipschitz condition with the \( L \) average:

\[
\| f'(x^*)^{-1} f'(x) - I \| \leq \int_0^{\rho(x)} L(u) \, du, \quad \forall x \in B(x^*, r),
\]

(4.1)

where \( \rho(x) = \| x - x^* \| \) and \( L \) is a positive integrable function. Let \( r \) satisfy

\[
\frac{1}{r} \int_0^r L(u)(r - u) \, du \leq 1.
\]

(4.2)

Then the equation \( f(x) = 0 \) has a unique solution \( x^* \) in \( B(x^*, r) \).

Proof. Arbitrarily choosing \( x_0 \in B(x^*, r) \) and considering the iteration

\[
x_{n+1} = x_n - f'(x_n)^{-1} f(x_n), \quad n = 0, 1, \ldots,
\]

(4.3)

we have

\[
x_1 - x^* = -f'(x^*)^{-1} (f(x_0) - f(x^*) - f'(x^*)(x_0 - x^*))
\]

\[
= -\int_0^1 f'(x^*)^{-1} f'(x^*) - I(x_0 - x^*) \, dr,
\]

where \( x^* = x^* + \tau(x_0 - x^*) \). Thus by condition (4.1) we obtain

\[
\| x_1 - x^* \| \leq \int_0^1 \int_0^\tau \rho(x_0) L(u) \, du \rho(x_0) \, d\tau = \int_0^\rho(x_0) \int_0^L(u) \, du.
\]

(4.4)

From \( L(u) > 0 \), we have \( \frac{1}{t} \int_0^t L(u)(t - u) \, du \) is increasing monotonically with respect to \( t \). In fact, for \( 0 < t_1 < t_2 \),

\[
\frac{1}{t_2} \int_0^{t_2} L(u)(t_2 - u) \, du - \frac{1}{t_1} \int_0^{t_1} L(u)(t_1 - u) \, du
\]

\[
= \int_0^{t_1} L(u) \, du - \left( \frac{1}{t_2} \int_0^{t_2} + \left( \frac{1}{t_2} - \frac{1}{t_1} \right) \int_0^{t_1} \right) L(u) u \, du
\]

\[
\geq \int_0^{t_1} L(u) \, du - \int_0^{t_1} L(u) \, du - \left( \frac{1}{t_2} - \frac{1}{t_1} \right) \int_0^{t_1} L(u) u \, du
\]

\[
= \left( \frac{1}{t_1} - \frac{1}{t_2} \right) \int_0^{t_1} L(u) u \, du > 0.
\]

Hence, by (4.2) we obtain

\[
q_0 := \frac{1}{\rho(x_0)} \int_0^{\rho(x_0)} L(u)(\rho(x_0) - u) \, du < \frac{1}{r} \int_0^r L(u)(r - u) \, du \leq 1.
\]

(4.5)

By (4.4),

\[
\| x_1 - x^* \| \leq q_0 \| x_0 - x^* \|.
\]

(4.6)
Hence, the iteration (4.3) can be continued infinitely, and
\[ \|x_n - x^*\| \leq q_n^0 \|x_0 - x^*\|, \quad n = 1, 2, \ldots. \quad (4.7) \]
Therefore, \(\lim x_n = x^*\). But if \(x_0\) satisfies \(f(x_0) = 0\), then \(x_n = x_0\). Thus, it follows that \(x_0 = x^*\). □

5. The optimality of the estimation of the radius

THEOREM 5.1 Suppose that the equality sign holds in the inequality (3.2) in Theorem 3.1. Then the given value \(r\) of the convergence ball is the best possible. Furthermore, \(r\) only depends on \(L\), but is independent of \(f\).

Proof. We notice that when \(r\) is determined by the equality
\[ \frac{\int_0^r L(u) \, du}{r \left(1 - \int_0^r L(u) \, du\right)} = 1, \quad (5.1) \]
there exists \(f\) satisfying (3.1) in \(B(x^*, r)\) and \(x_0\) on the boundary of the closed ball such that Newton’s iteration fails. In fact, the following is an example on the scaled case:
\[ f(x) = \begin{cases} 
  x^* - x + \int_0^{x^* - x_0} (x - x^* - u)L(u) \, du, & x^* - x_0 \leq x < x^* + r; \\
  x^* - x + \int_0^{x^* - r} (x - x^* + u)L(u) \, du, & x^* - r \leq x < x^*,
\end{cases} \quad (5.2) \]
and \(x_0 = r, x_n = (-1)^n r\). □

THEOREM 5.2 Suppose that the equality sign holds in the inequality (4.2) in Theorem 4.1. Then the given value \(r\) of the uniqueness ball is the best possible. Further \(r\) only depends on \(L\), but is independent of \(f\).

Proof. When \(r\) is determined by the equality
\[ \frac{1}{r} \int_0^r L(u)(r - u) \, du = 1, \quad (5.3) \]
there exists \(f\) satisfying (4.1) in \(B(x^*, r)\) and \(x'\) on the boundary of the closed ball such that \(f(x') = 0\). An example of this is (5.2) in which \(x' = r\). □

6. Corollaries of the main results

In the study of Newton’s method, the assumption that the derivative is Lipschitz continuous is considered traditional. Lately, Smale has introduced the assumption of analyticity. In this section we will see how to extend these two kinds of assumption and bring them together into a uniform assumption.

First, for the former kind, combining Theorems 3.1 and 4.1 with Theorems 5.1 and 5.2, and taking \(L\) as a constant, the following two corollaries are obtained directly.
COROLLARY 6.1 Suppose that \( f(x^*) = 0 \), \( f \) has a continuous derivative in \( B(x^*, r) \), \( f'(x^*)^{-1} \) exists and \( f'(x^*)^{-1} f' \) satisfies the radius Lipschitz condition:

\[
\| f'(x^*)^{-1} (f'(x) - f'(x^* + \tau (x - x^*))) \| \leq (1 - \tau)L\| x - x^* \|, \\
\forall x \in B(x^*, r), \ 0 \leq \tau \leq 1.
\]  

(6.1)

where \( L \) is a positive number and \( r = 2/(3L) \). Then Newton’s method (1.1) is convergent for all \( x_0 \in B(x^*, r) \) and for

\[
q = \frac{L\| x_0 - x^* \|}{2(1 - L\| x_0 - x^* \|)},
\]

(6.2)

the inequality (3.3) holds. Moreover, the given \( r \) is the best possible.

COROLLARY 6.2 Suppose that \( f(x^*) = 0 \), \( f \) has a continuous derivative in \( B(x^*, r) \), \( f'(x^*)^{-1} \) exists and \( f'(x^*)^{-1} f' \) satisfies the centre Lipschitz condition:

\[
\| f'(x^*)^{-1} f'(x) - I \| \leq L\| x - x^* \|, \\
\forall x \in B(x^*, r),
\]

(6.3)

where \( L \) is a positive number and \( r = 2/L \). Then the equation \( f(x) = 0 \) has a unique solution \( x^* \) in the open ball \( B(x^*, r) \). Moreover, the given \( r \) is the best possible and is independent of \( f \).

Under the common Lipschitz condition, Corollary 6.1 is obtained by Traub & Wozniakowski (1979) and Wang (1980), but it seems that Corollary 6.2 has not appeared in the literature.

For the latter kind of assumption, by combining Theorems 3.1 and 4.1 with Theorems 5.1 and 5.2, and taking

\[
L(u) = \frac{2\gamma}{(1 - \gamma u)^3},
\]

(6.4)

we obtain the following two corollaries.

COROLLARY 6.3 Suppose that \( f(x^*) = 0 \), \( f \) has a continuous derivative in \( B(x^*, r) \), \( f'(x^*)^{-1} \) exists and \( f'(x^*)^{-1} f' \) satisfies:

\[
\| f'(x^*)^{-1} (f'(x) - f'(x^* + \tau (x - x^*))) \| \leq \frac{1}{(1 - \gamma\| x - x^* \|)^2} - \frac{1}{(1 - \tau\gamma\| x - x^* \|)^2}, \\
\forall x \in B(x^*, r), \ 0 \leq \tau \leq 1,
\]

(6.5)

where \( \gamma \) is a positive number and \( r = (5 - \sqrt{17})/(4\gamma) \). Then Newton’s method (1.1) is convergent for all \( x_0 \in B(x^*, r) \) and for

\[
q = \frac{\gamma\| x_0 - x^* \|}{1 - 4\gamma\| x_0 - x^* \| + 2(\gamma\| x_0 - x^* \|)^2},
\]

(6.6)

the inequality (3.3) holds. Moreover, the given \( r \) is the best possible and is independent of \( f \).
COROLLARY 6.4 Suppose that \( f(x^*) = 0 \), \( f \) has a continuous derivative in \( B(x^*, r) \), \( f'(x^*) \) exists and \( f'(x^*)^{-1} f' \) satisfies:

\[
\| f'(x^*)^{-1} f'(x) - 1 \| \leq \frac{1}{(1 - \gamma \| x - x^* \|^2)^2} - 1, \quad \forall x \in B(x^*, r),
\]

(6.7)

where \( \gamma \) is a positive number and \( r = 1/(2\gamma) \). Then the equation \( f(x) = 0 \) has a unique solution \( x^* \) in the open ball \( B(x^*, r) \). Moreover, the given \( r \) is the best possible and is independent of \( f \).

When \( f \) is analytic and satisfies (1.4), Corollary 6.3 is proved following Smale (1986); Corollary 6.4 is proved following Dedieu (1999) for a value of \( r = (5 - \sqrt{17})/(4\gamma) \), but this is not the best possible \( r \).

REMARK Using Theorems 3.1 and 4.1, we also derive some new properties in essence about the convergence of Newton’s method and the uniqueness of the solutions of the equation. In the following examples, let \( c \) be an arbitrary positive constant.

EXAMPLE 1 Taking

\[
L(u) = 2c\gamma(1 - \gamma u)^{-3},
\]

(6.8)

we obtain that if the right-hand side in (6.5) is replaced by

\[
\frac{c}{(1 - \gamma \| x - x^* \|^2)^2} - \frac{c}{(1 - \gamma \| x - x^* \|^2)^2},
\]

we have

\[
r = \frac{3c + 2 - \sqrt{c(9c + 8)}}{2(c + 1)\gamma},
\]

and

\[
q = \frac{c\gamma \| x_0 - x^* \|}{1 - 2(c + 1)\gamma \| x_0 - x^* \| + (c + 1)(\gamma \| x_0 - x^* \|^2)}.
\]

(6.9)

If the right-hand side in (6.7) is replaced by

\[
\frac{c}{(1 - \gamma \| x - x^* \|^2)^2} = c,
\]

then

\[
r = \frac{1}{(c + 1)\gamma}.
\]

EXAMPLE 2 Taking

\[
L(u) = 2c\gamma(1 - \gamma u)^{-3/2},
\]

(6.10)

we obtain that if the right-hand side in (6.5) is replaced by

\[
\frac{c}{\sqrt{1 - \gamma \| x - x^* \|^2}} - \frac{c}{\sqrt{1 - \gamma \| x - x^* \|^2}},
\]

then

\[
r = \frac{\sqrt{(3c - 1)^2 + 16c} - (3c - 1)}{2(c + 1)\gamma}
\]
and
\[ q = \frac{c \gamma \| x_0 - x^* \|}{2 - (c + 2) \gamma \| x_0 - x^* \| + (2 - (c + 1) \gamma \| x_0 - x^* \|) \sqrt{1 - \gamma \| x_0 - x^* \|}}. \] (6.11)

If the right-hand side in (6.7) is replaced by
\[ \frac{c}{\sqrt{1 - \gamma \| x - x^* \|}} - c, \]
then
\[ r = \begin{cases} \frac{1}{\gamma}, & \text{if } c \leq 1, \\ \frac{4 c}{(c + 1)^2 \gamma}, & \text{if } c > 1. \end{cases} \]

**Example 3**

Taking
\[ L(u) = \frac{c \gamma}{2} (1 - \gamma u)^{-1/2}, \] (6.12)
we obtain that if the right-hand side in (6.5) is replaced by
\[ c \sqrt{1 - \gamma \| x - x^* \|} - c \sqrt{1 - \gamma \| x - x^* \|}, \]
then
\[ r = \frac{1}{2 \gamma} \left\{ \frac{3}{c} + \left( \frac{3}{4} \left( 1 - \frac{1}{c} \right) \right)^2 - \left( \frac{3}{4} \left( 1 - \frac{1}{c} \right) \right)^2 \right\} \]
and
\[ q = \frac{c}{3} \left( 1 - \sqrt{1 - \gamma \| x_0 - x^* \|} + \gamma \| x_0 - x^* \| \right) - c \gamma \| x_0 - x^* \|. \] (6.13)

If the right-hand side in (6.7) is replaced by
\[ c \left( 1 - \sqrt{1 - \gamma \| x - x^* \|} \right), \]
then
\[ r = \begin{cases} \frac{1}{\gamma}, & \text{if } c \leq 3, \\ \frac{1}{\gamma} \left( \frac{3}{8} + \frac{9}{4 c} - \frac{9}{8 c^2} - \sqrt{\left( \frac{3}{8} + \frac{9}{4 c} - \frac{9}{8 c^2} \right)^2 - \frac{9}{4 c}} \right), & \text{if } c > 3. \end{cases} \]

7. Applications to the determination of an approximation zero

Smale (1981) first proposed the definition of an approximation zero of Newton’s method. Afterwards, it was found that it did not describe the property of quadratic convergence of Newton’s method and was inconvenient to apply in the study of the computational complexity of zeros. Hence, he proposed new definitions of the approximation zeros of two kinds (Smale, 1986) and one of the second kind is taken as the definition of the approximation zero in his new works (Blum et al., 1996; Smale, 1997). This definition is as follows (the same definition was also given in Wang & Xuan, 1987).
DEFINITION 7.1 If $x_0 \in X$ such that Newton’s iteration (1.1) for $f : D \subset X \to Y$ is well defined and (3.3) is satisfied with $q = \frac{1}{2}$, then $x_0$ is called an approximation zero of the adjoint zero $x^*$ of $f$.

By Example 1 of Section 6, solving $\delta$ from the equation

\[ q = \frac{c\delta}{1 - 2(c + 1)\delta + (c + 1)^2} \quad (7.1) \]

yields

\[ \delta = \frac{(2q + 1)(c + 1) - 1 - \sqrt{c(2q + 1)^2(c + 1) - 1}}{2q(c + 1)} \quad (7.2) \]

Thus we have

THEOREM 7.2 Suppose that $f(x^*) = 0$, $f$ has a continuous derivative in $B(x^*, \delta/\gamma)$, $f'(x^*)^{-1}$ exists and $f'(x^*)^{-1}f'$ satisfies:

\[ \| f'(x^*)^{-1}(f'(x) - f'(x^* + \tau(x - x^*)))\| \leq \frac{c}{(1 - \gamma \|x - x^*\|^2)} - \frac{c}{(1 - \tau \gamma \|x - x^*\|)^2}, \quad \forall x \in B(x^*, \delta/\gamma), \quad 0 \leq \tau \leq 1, \quad (7.3) \]

where $c, \gamma$ and $q$ are positive numbers with $0 < q < 1$ and $\delta$ is determined by (7.2). If $x_0$ satisfies

\[ \gamma \|x_0 - x^*\| < \delta, \quad (7.4) \]

then Newton’s method (1.1) is well defined and (3.3) is satisfied. In particular, if

\[ \gamma \|x_0 - x^*\| < \frac{2c + 1 - \sqrt{c(4c + 3)}}{c + 1}, \quad (7.4a) \]

$x_0$ is an approximation zero of the adjoint zero $x^*$.

When $f$ is analytic in $B(x^*, \delta/\gamma)$, condition (7.3) is satisfied by $f$ such that

\[ \left\| \frac{1}{n!} f'(x^*)^{-1} f^{(n)}(x^*) \right\| \leq c\gamma^{n-1}, \quad n \geq 2. \quad (7.5) \]

In fact, using (7.5) we have

\[ \left\| f'(x^*)^{-1} f''(x) \right\| \leq \sum_{n=2}^{\infty} n(n - 1)c\gamma^{n-2} \|x - x^*\|^{n-2} = \frac{2c\gamma}{(1 - \gamma \|x - x^*\|)^3}. \]

Hence (7.3) holds.

Setting $c = 1$, then for $q = \frac{1}{2}$ (7.2) gives the value

\[ \delta = \frac{3 - \sqrt{7}}{2}, \quad (7.6) \]

which was obtained by Smale (1986).
Besides $q = \frac{1}{2}$, there are other choices for $q$ (see Chen, 1994). But $q$ must be fixed, otherwise the uniform quadratic convergence disappears and the definition is the same as the one in Smale (1981).

Besides $c = 1$, there are also other choices for $c$. An interesting combination between $c$ and $q$ can give a very simple result. In fact, if $c$ is taken to be

$$c = \frac{1 - 2q + q^2}{1 + 2q - q^2}$$

(7.7)

in condition (7.3) or (7.5) in Theorem 7.2, then the $\delta$ on the right-hand side of (7.4), which is determined by (7.2), may be taken as

$$\delta = q.$$  

(7.8)

Hence, we have

**Corollary 7.3** Suppose that $f(x^*) = 0$, $f$ is analytic in $B(x^*, 1/\gamma)$, $f'(x^*)^{-1}$ exists and for some $q \in (0, 1)$ there holds

$$\left\| \frac{1}{n!} f'(x^*)^{-1} f^{(n)}(x^*) \right\| \leq \frac{1 - 2q + q^2}{1 + 2q - q^2} \gamma^{n-1}, \quad n \geq 2.$$  

(7.9)

If $x_0$ satisfies

$$\gamma \|x_0 - x^*\| < \gamma \|x_0 - x^*\| < q,$$  

(7.10)

then Newton’s method (1.1) is well defined and (3.3) is satisfied. In particular, if

$$\left\| \frac{1}{n!} f'(x^*)^{-1} f^{(n)}(x^*) \right\| \leq \frac{\gamma^{n-1}}{7}$$

and

$$\gamma \|x_0 - x^*\| < \frac{1}{2},$$

$x_0$ is an approximation zero of the adjoint zero $x^*$.

Examples 2 and 3 in Section 6 also give other forms of the bound on the right-hand side of (7.5) which can be applied to study the computational complexity. For example, Example 2 gives a convergent ball of Newton’s method under the condition

$$\left\| \frac{1}{n!} f'(x^*)^{-1} f^{(n)}(x^*) \right\| \leq 2 \frac{(2n - 3)!!}{2n!!} \gamma^{n-1}, \quad n \geq 2.$$  

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REFERENCES


