Surjectivity of Gaussian maps for curves on Enriques surfaces

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Abstract. Making suitable generalizations of known results we prove some general facts about Gaussian maps. These facts are then used, in the second part of the article, to give a set of conditions that insure the surjectivity of Gaussian maps for curves on Enriques surfaces. To do this we also solve a problem of independent interest: a tetragonal curve of genus $g \geq 7$ lying on an Enriques surface and general in its linear system, cannot be, in its canonical embedding, a quadric section of a surface of degree $g - 1$ in $\mathbb{P}^{g-1}$.

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1 Introduction

Gaussian maps have emerged in the mid 1980’s as a useful tool to study the geometry of a given variety $X \subset \mathbb{P}^N$ as soon as one has a good knowledge of the hyperplane sections $Y = X \cap H$.

Let us briefly recall their definition and notation in the case of curves.

Notation 1.1. Let $C$ be a smooth irreducible curve and let $L, M$ be two line bundles on $C$. We denote by $\mu_{L,M} : H^0(L) \otimes H^0(M) \rightarrow H^0(L \otimes M)$ the multiplication map of sections and by $R(L, M) = \text{Ker} \mu_{L,M}$. The Gaussian map associated to $L$ and $M$ will be denoted by

$$\Phi_{L,M} : R(L, M) \rightarrow H^0(\omega_C \otimes L \otimes M).$$

This map can be defined locally by $\Phi_{L,M}(s \otimes t) = sdt - tds$ (see [38]).

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Perhaps the first important result, proved by Wahl, who introduced Gaussian maps, is that if a smooth curve $C$ lies on a K3 surface, then the Gaussian map $\Phi_{\omega_C, \omega_C}$ cannot be surjective. On the other hand, as it was proved by Ciliberto, Harris and Miranda [8], this map $\Phi_{\omega_C, \omega_C}$ is surjective on a curve $C$ with general moduli of genus 10 or at least 12.

The link with the study of higher dimensional varieties was provided, around the same period, by Zak, who proved the following result ([39], see also [2], [28]): If $Y \subset \mathbb{P}^r$ is a smooth variety of codimension at least two with normal bundle $N_{Y/\mathbb{P}^r}$ and $h^0(N_{Y/\mathbb{P}^r}(-1)) \leq r + 1$, then the only variety $X \subset \mathbb{P}^{r+1}$ that has $Y$ as hyperplane section is a cone over $Y$.

Now the point is that, if $Y$ is a curve, we have the formula

$$h^0(N_{Y/\mathbb{P}^r}(-1)) = r + 1 + \text{cork } \Phi_{H_Y, \omega_Y}$$

where $H_Y$ is the hyperplane bundle of $Y$.

On the other hand, if $Y$ is not a curve one can take successive hyperplane sections of $Y$. For example, when $X \subset \mathbb{P}^{r+1}$ is a smooth anticanonically embedded Fano threefold with general hyperplane section the K3 surface $Y$, in [9], Ciliberto, the second author and Miranda were able to compute $h^0(N_{Y/\mathbb{P}^r}(-1))$ by calculating the coranks of $\Phi_{H_C, \omega_C}$ for the general curve section $C$ of $Y$. This then led to recover in [9] and [10], in a very simple way, a good part of the classification of smooth Fano threefolds [18], [19] and of varieties with canonical curve section [29].

To study other threefolds by means of Zak’s theorem, in many cases it is not enough to get down to curve sections and one needs to bound the cohomology of the normal bundle of surfaces. In [25] the following general result was proved:

**Proposition 1.2.** Let $Y \subset \mathbb{P}^r$ be a smooth irreducible linearly normal surface and let $H$ be its hyperplane bundle. Assume there is a base-point free and big line bundle $D_0$ on $Y$ with $H^1(H - D_0) = 0$ and such that the general element $D \in |D_0|$ is not rational and satisfies

(i) the Gaussian map $\Phi_{H_D, \omega_D(D)}$ is surjective;
(ii) the multiplication maps $\mu_{V_D, \omega_D}$ and $\mu_{V_D, \omega_D(D)}$ with $V_D := \text{Im}\{H^0(H - D) \to H^0((H - D)|_D)\}$ are surjective.

Then

$$h^0(N_{Y/\mathbb{P}^r}(-1)) \leq r + 1 + \text{cork } \Phi_{H_D, \omega_D}.$$
(i) \( L^2 = 4 \) and \( h^0(4L_C - M) = 0 \);
(ii) \( L^2 = 6 \) and \( h^0((3L + K_S)|C - M) = 0 \);
(iii) \( L^2 \geq 8 \) and \( h^0(2L_C - M) = 0 \);
(iv) \( L^2 \geq 12 \) and \( h^0(2L_C - M) = 1 \);
(v) \( H^1(M) = 0 \), \( \deg(M) \geq 1 \), \( \frac{1}{2}L^2 + 2 \geq 6 \) and \( h^0(2L_C - M) \leq \text{Cliff}(C) - 2 \).

The proof of this theorem will be accomplished essentially in two steps. We will first prove, in Section 2, some general facts about Gaussian maps, by generalizing some known results. Then, in the second step, in Section 5, we will deal with the specific problem of Gaussian maps for curves on Enriques surfaces. As it turns out, the most difficult point will be to show that a tetragonal curve of genus \( g \geq 7 \) lying on an Enriques surface and general in its linear system, in its canonical embedding, can never be a quadric section of a surface of degree \( g - 1 \) in \( \mathbb{P}^{g-1} \).

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2 Basic results on Gaussian maps

We briefly recall the definition, notation and some properties of gonality and Clifford index of curves.

**Definition 2.1.** Let \( X \) be a smooth surface. We will denote by \( \sim \) (respectively \( \equiv \)) the linear (respectively numerical) equivalence of divisors (or line bundles) on \( X \). We will say that a line bundle \( L \) is primitive if \( L \equiv kL' \) for some line bundle \( L' \) and some integer \( k \), implies \( k = \pm 1 \).

**Definition 2.2.** Let \( C \) be a smooth irreducible curve of genus \( g \geq 2 \). We denote by \( g^r_d \) a linear system of dimension \( r \) and degree \( d \) on \( C \) and say that \( C \) is \( k \)-gonal (and that \( k \) is its gonality) if \( C \) possesses a \( g^1_k \) but no \( g^1_{k-1} \). In particular, we call a 2-gonal curve hyperelliptic, a 3-gonal curve trigonal and a 4-gonal curve tetragonal. We denote by \( \text{gon}(C) \) the gonality of \( C \).

**Definition 2.3.** Let \( C \) be a smooth irreducible curve of genus \( g \geq 4 \) and let \( A \) be a line bundle on \( C \). The **Clifford index** of \( A \) is the integer

\[
\text{Cliff}(A) := \deg A - 2(h^0(A) - 1).
\]

The **Clifford index** of \( C \) is

\[
\text{Cliff}(C) := \min\{\text{Cliff}(A) : h^0(A) \geq 2, h^1(A) \geq 2\}.
\]

We say that a line bundle \( A \) on \( C \) contributes to the Clifford index of \( C \) if \( h^0(A) \geq 2 \), \( h^1(A) \geq 2 \).
2.1 Preliminaries on Gaussian maps. We recall some well-known facts about Gaussian maps.

Proposition 2.4. [38, Proposition 1.10] Let $C$ be a smooth irreducible nonhyperelliptic curve of genus $g \geq 3$, let $C \subset \mathbb{P}^{g-1}$ be its canonical embedding and let $M$ be a line bundle on $C$. We have two exact sequences

\begin{align*}
0 \rightarrow \text{Coker } \mu_{M, \omega_C} \rightarrow H^1(\Omega^1_{\mathbb{P}^{g-1}}|_C \otimes \omega_C \otimes M) \rightarrow H^1(M)^{\otimes g} & \rightarrow H^1(\omega_C \otimes M) \rightarrow 0 \quad (1) \\
0 \rightarrow \text{Coker } \Phi_{M, \omega_C} \rightarrow H^1(N^*_C/\mathbb{P}^{g-1} \otimes \omega_C \otimes M) \rightarrow H^1(\Omega^1_{\mathbb{P}^{g-1}}|_C \otimes \omega_C \otimes M) & \rightarrow H^1(\omega_C^2 \otimes M) \rightarrow 0. \quad (2)
\end{align*}

In particular

(a) if $H^0(N^*_C/\mathbb{P}^{g-1} \otimes M^{-1}) = 0$ then $\Phi_{M, \omega_C}$ is surjective;
(b) if $H^1(M) = 0$ and $\mu_{M, \omega_C}$ is surjective then $\text{cork } \Phi_{M, \omega_C} = h^0(N^*_C/\mathbb{P}^{g-1} \otimes M^{-1})$.

In the sequel we will collect some results about Gaussian maps of type $\Phi_{M, \omega_C}$ for curves $C$ of low genus or low gonality or with Clifford index higher than $h^0(2K_C - M) + 2$.

We start with an elementary but useful fact.

Lemma 2.5. For $a \geq 2$, let $Q_1, \ldots, Q_a$ be linearly independent homogeneous polynomials of degree 2 in $X_0, \ldots, X_r$. Suppose that the relations among $Q_1, \ldots, Q_a$ are generated by $R_i = [R_{i1}, \ldots, R_{ia}]$, for $1 \leq i \leq b$. If $(c_1, \ldots, c_a) \in \mathbb{C}^a - \{0\}$, then there exists an $i$ such that

$$\sum_{j=1}^{a} c_j R_{ij} \neq 0.$$

Proof. Suppose to the contrary that $\sum_{j=1}^{a} c_j R_{ij} = 0$ for every $i$ with $1 \leq i \leq b$. Without loss of generality assume that $c_1 \neq 0$, so that

$$R_{i1} = - \sum_{j=2}^{a} c_j^{-1} c_j R_{ij}, \quad 1 \leq i \leq b. \quad (3)$$

Claim 2.6. Set $Q'_1 = Q_1, Q'_j = Q_j - c_1^{-1} c_j Q_1$ for $2 \leq j \leq a$. Then

(i) $Q'_1, \ldots, Q'_a$ are linearly independent;
(ii) the relations among $Q'_1, \ldots, Q'_a$ are generated by $S_i = [0, R_{i2}, \ldots, R_{ia}]$, for $1 \leq i \leq b$. 


Consider a relation \( \sum_{j=1}^{a} R'_j Q'_j = 0 \), where the \( R'_j \)'s are polynomials. Then \( R'_1 Q_1 + \sum_{j=2}^{a} R'_j (Q_j - c^{-1}_j c_j Q_1) = 0 \), whence
\[
\left( R'_1 - \sum_{j=2}^{a} c^{-1}_j c_j R'_j \right) Q_1 + \sum_{j=2}^{a} R'_j Q_j = 0. \tag{4}
\]
If all \( R'_j \)'s are complex numbers we get \( R'_j = 0 \) for all \( j \), proving (i).

To see (ii), by (4) and the hypothesis of the lemma we deduce that there are polynomials \( d_j \) such that
\[
\left[ R'_1 - \sum_{j=2}^{a} c^{-1}_j c_j R'_j, R'_2, \ldots, R'_a \right] = \sum_{i=1}^{b} d_i R_i = \left[ \sum_{i=1}^{b} d_i R_{i1}, \sum_{i=1}^{b} d_i R_{i2}, \ldots, \sum_{i=1}^{b} d_i R_{ia} \right]
\]
whence \( R'_j = \sum_{i=1}^{b} d_i R_{ij} \) for \( 2 \leq j \leq a \) and
\[
R'_1 = \sum_{j=2}^{a} c^{-1}_j c_j R'_j + \sum_{i=1}^{b} d_i R_{i1} = \sum_{i=1}^{b} d_i \left( \sum_{j=2}^{a} c^{-1}_j c_j R_{ij} + R_{i1} \right) = 0
\]
by (3). Now
\[
\sum_{i=1}^{b} d_i S_i = \left[ 0, \sum_{i=1}^{b} d_i R_{i2}, \ldots, \sum_{i=1}^{b} d_i R_{ia} \right] = [R'_1, R'_2, \ldots, R'_a]. \tag*{\Box}
\]

Conclusion of the proof of Lemma 2.5. Consider the Koszul relation \([Q'_2, -Q'_1, 0, \ldots, 0]\) among \( Q'_1, \ldots, Q'_a \). By the claim there are polynomials \( d_i \) such that \( \sum_{i=1}^{b} d_i S_i = [Q'_2, -Q'_1, 0, \ldots, 0] \), giving the contradiction \( Q'_2 = 0 \). \tag*{\Box}

In many cases, to compute the corank of Gaussian maps, or, as in Proposition 2.4, to compute a suitable cohomology group involving the normal bundle, it is quite convenient to know some surface containing the given curve. The result below will help to compute the cohomology of the normal bundle with the help of the surface.

**Lemma 2.7.** Let \( Y \subset \mathbb{P}^r \) be an integral subvariety that is scheme-theoretically an intersection of quadrics and let \( X \subset Y \) be a smooth irreducible nondegenerate subvariety. Let \( L = \mathcal{O}_Y(1) \) and \( M \) a line bundle on \( X \). Suppose that either
\[
\begin{align*}
\text{(i)} & \quad h^0(2L|_X - M) = 0 \text{ or} \\
\text{(ii)} & \quad h^0(2L|_X - M) = 1 \text{ and the relations among the quadrics cutting out } Y \text{ are generated by linear ones.}
\end{align*}
\]

Let \( \mathcal{F}_{X,Y} = \text{Hom}_{\mathcal{O}_X}(\mathcal{J}_{Y/\mathbb{P}^r}, \mathcal{O}_X) \). Then \( H^0(\mathcal{F}_{X,Y} \otimes M^{-1}) = 0 \).

**Remark 2.8.** When \( Y \) is smooth we have that \( \mathcal{F}_{X,Y} = N_{Y/\mathbb{P}^r}|_X \). The fact that \( Y \subset \mathbb{P}^r \) is scheme-theoretically an intersection of quadrics certainly holds if \( Y \) satisfies property \( N_1 \), that is \( Y \) is projectively normal and its homogeneous ideal is generated by quadrics.
Let Suppose that If

If Y and let

Also the fact that the relations among the quadrics cutting out Y are generated by linear ones certainly holds if Y satisfies property $N_2$, that is Y satisfies property $N_1$ and the relations among the quadrics generating its homogeneous ideal are generated by linear ones ([26, Def.1.2.5], [16]). The difference, in our case, is that we do not assume Y to be linearly normal.

**Proof of Lemma 2.7.** Let $\{Q_1, \ldots, Q_a\}$ be linearly independent quadrics cutting out Y scheme-theoretically and consider the corresponding beginning of the minimal free resolution of $\mathcal{J}_{Y/X}$:

$$
\bigoplus_{i \geq 0} \mathcal{O}_{\mathbb{P}^r}(-3 - i)^{\oplus b_i} \longrightarrow \mathcal{O}_{\mathbb{P}^r}(-2)^{\oplus a} \longrightarrow \mathcal{J}_{Y/X} \longrightarrow 0.
$$

Applying the left exact functor $\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^r}}(-, \mathcal{O}_X)$ we get an exact sequence

$$
0 \longrightarrow \mathcal{F}_{X,Y} \longrightarrow \mathcal{O}_X(2)^{\oplus a} \longrightarrow \bigoplus_{i \geq 0} \mathcal{O}_X(3 + i)^{\oplus b_i},
$$

whence an exact sequence

$$
0 \longrightarrow H^0(\mathcal{F}_{X,Y} \otimes M^{-1}) \longrightarrow H^0(2L_{|X} - M)^{\oplus a} \xrightarrow{\varphi} \bigoplus_{i \geq 0} H^0((3 + i)L_{|X} - M)^{\oplus b_i}.
$$

Then $H^0(\mathcal{F}_{X,Y} \otimes M^{-1}) = \operatorname{Ker} \varphi.$

If we are under hypothesis (i), then obviously $H^0(\mathcal{F}_{X,Y} \otimes M^{-1}) = 0.$

If we are under hypothesis (ii), then $b_i = 0$ for $i \geq 1$ and we will prove that $\operatorname{Ker} \varphi = 0.$

To this end let $\sigma$ be a generator of $H^0(2L_{|X} - M).$ For $1 \leq i \leq b_0$ let $R_i = [R_{i1}, \ldots, R_{ia}]$ be the linear relations generating all relations among $Q_1, \ldots, Q_a$, so that the map $\varphi$ is given by the matrix $(R_{ij})_{ij \in [1, a]}.$ If $0 \neq (c_1 \sigma, \ldots, c_a \sigma) \in \operatorname{Ker} \varphi$ then, for every $i$ such that $1 \leq i \leq b_0$, we have $\sum_{j=1}^a c_j R_{ij} \mid X = 0$ whence $(\sum_{j=1}^a c_j R_{ij})_{|X} = 0.$ As $X$ is nondegenerate and $\sum_{j=1}^a c_j R_{ij}$ is a linear polynomial, we deduce that $\sum_{j=1}^a c_j R_{ij} = 0$ for all $i$ with $1 \leq i \leq b_0$, contradicting Lemma 2.5.

Now the first general result about Gaussian maps.

**Proposition 2.9.** Let $C$ be a smooth irreducible nonhyperelliptic curve of genus $g \geq 3$ and let $M$ be a line bundle on $C$. We have

(a) If $g = 3$ then $\operatorname{cork} \Phi_{M, \omega_C} \geq h^0(4K_C - M) - \operatorname{cork} \mu_{M, \omega_C} - 3h^1(M)$, with equality if $H^0(-M) = 0$.

(b) If $g = 4$ then $\operatorname{cork} \Phi_{M, \omega_C} \geq h^0(2K_C - M) + h^0(3K_C - M) - \operatorname{cork} \mu_{M, \omega_C} - 4h^1(M)$, with equality if $H^0(-M) = 0$.

(c) If $g = 5$ and $C$ is nontrigonal then $\operatorname{cork} \Phi_{M, \omega_C} \geq 3h^0(2K_C - M) - \operatorname{cork} \mu_{M, \omega_C} - 5h^1(M)$, with equality if $H^0(-M) = 0$.

(d) Suppose that $C$ is a plane quintic and $A$ is the very ample $g_5^2$ on $C$. If $H^0(5A - M) = 0$ then $\Phi_{M, \omega_C}$ is surjective. If $H^1(M) = 0$ and $\mu_{M, \omega_C}$ is surjective then $\operatorname{cork} \Phi_{M, \omega_C} \geq h^0(5A - M)$, with equality holding if in addition $h^0(4A - M) \leq 1.$
Suppose that $C$ is trigonal, $g \geq 5$ and $A$ is a $g^1_3$ on $C$. If $h^0(2K_C - M) \leq 1$ and $H^0(3K_C - (g - 4)A - M) = 0$ then $\Phi_{M,\omega_C}$ is surjective. If $H^1(M) = 0$ and $\mu_{M,\omega_C}$ is surjective then $\text{cork} \Phi_{M,\omega_C} \geq h^0(3K_C - (g - 4)A - M)$, with equality holding if in addition $h^0(2K_C - M) \leq 1$.

**Proof.** Assertions (a), (b) and (c) follow easily from Proposition 2.4.

Let us prove (d). In the canonical embedding $C \subset \mathbb{P}^5$ we have that $C$ is contained in the Veronese surface $Y$, and we have an exact sequence

$$0 \rightarrow N_{C/Y} \otimes M^{-1} \rightarrow N_{C/P5} \otimes M^{-1} \rightarrow N_{Y/P5/C} \otimes M^{-1} \rightarrow 0. \quad (5)$$

Observe that $h^0(N_{C/Y} \otimes M^{-1}) = h^0(5A - M)$. Now if $h^0(5A - M) = 0$ then also $h^0(2K_C - M) = h^0(4A - M) = 0$ and from (5) and Proposition 2.4 (a), we see that to prove (d) we just need to show that $H^0(N_{Y/P5/C} \otimes M^{-1}) = 0$. The latter follows by Lemma 2.7 and Remark 2.8 since, as is well-known, $Y$ satisfies property $N_3$.

Now if $H^1(M) = 0$ and $\mu_{M,\omega_C}$ is surjective, we have that $\text{cork} \Phi_{M,\omega_C} = h^0(N_{C/P5} \otimes M^{-1}) \geq h^0(3A - M)$ by Proposition 2.4 (b) and (5). If we also assume that $h^0(4A - M) = h^0(2K_C - M) \leq 1$ then we can apply again Lemma 2.7 and Remark 2.8. We get that $h^0(N_{Y/P5/C} \otimes M^{-1}) = 0$, whence, from (5), that $h^0(N_{C/P5} \otimes M^{-1}) = h^0(5A - M)$.

To see (e) recall that, in the canonical embedding $C \subset \mathbb{P}^{g-1}$, we have [33, 6.1] that $C \in |3H - (g - 4)R|$ on a rational normal surface $Y \subset \mathbb{P}^{g-1}$, where $H$ is its hyperplane bundle and $R$ its ruling. Since, as is well-known, $Y$ satisfies property $N_{g-3}$, applying, as in Case (d), Lemma 2.7 and Proposition 2.4 we get (e). \hfill $\Box$

Note that the Cases (d), (e) of the above proposition and the corollary below are a slight improvement of [35, Theorem 2.4] (because we also consider the case $h^0(2K_C - M) = 1$).

**Corollary 2.10.** Let $C$ be a smooth irreducible curve of genus $g \geq 5$ and let $M$ be a line bundle on $C$. Then the Gaussian map $\Phi_{M,\omega_C}$ is surjective if one of the hypotheses below is satisfied:

(a) $C$ is a plane quintic and $\deg M \geq 25$, $M \neq A$ if equality holds, where $A$ is the very ample $g^3_2$ on $C$;

(b) $C$ is trigonal and $\deg M \geq \max\{4g - 6, 3g + 6\}$, $M \neq 3K_C - (g - 4)A$ if $g \leq 12$ and $\deg M = 3g + 6$, where $A$ is a $g^1_3$ on $C$.

**Proof.** (a) follows immediately from 2.9(d) while (b) is a consequence of 2.9(e) since, if $h^0(2K_C - M) \geq 2$, then $\deg(2K_C - M) \geq 3$, a contradiction. \hfill $\Box$

Another easy but useful consequence of the proof of Lemma 2.7 is the following.

**Proposition 2.11.** Let $C$ be a smooth irreducible curve of genus $g \geq 5$ and let $M$ be a line bundle on $C$. Suppose that either

(i) $\text{Cliff}(C) = 2$ and $h^0(2K_C - M) = 0$ or

(ii) $\text{Cliff}(C) \geq 3$ and $h^0(2K_C - M) \leq 1$.

Then $\Phi_{M,\omega_C}$ is surjective.
Proof. Since \( \text{Cliff}(C) \geq 2 \), by [37], [34], the resolution of the ideal sheaf of the canonical embedding \( C \subset \mathbb{P}^{g-1} \) starts as

\[
\bigoplus_{i \geq 0} \mathcal{O}_{\mathbb{P}^{g-1}}(-3 - i)^{\oplus b_i} \longrightarrow \mathcal{O}_{\mathbb{P}^{g-1}}(-2)^{\oplus a} \longrightarrow \mathcal{J}_{C/\mathbb{P}^{g-1}} \longrightarrow 0
\]

with \( b_i = 0 \) for \( i \geq 1 \) when \( \text{Cliff}(C) \geq 3 \). Restricting to \( C \) and dualizing we get an exact sequence

\[
0 \longrightarrow N_{C/\mathbb{P}^{g-1}} \longrightarrow \mathcal{O}_{C}(2)^{\oplus a} \longrightarrow \bigoplus_{i \geq 0} \mathcal{O}_{C}(3 + i)^{\oplus b_i}
\]

whence an exact sequence

\[
0 \longrightarrow H^0(N_{C/\mathbb{P}^{g-1}} \otimes M^{-1}) \longrightarrow H^0(2K_C - M)^{\oplus a} \xrightarrow{\varphi} \bigoplus_{i \geq 0} H^0((3 + i)K_C - M)^{\oplus b_i}.
\]

As in the proof of Lemma 2.7 we have that \( H^0(N_{C/\mathbb{P}^{g-1}} \otimes M^{-1}) = 0 \) under hypothesis (i) and \( H^0(N_{C/\mathbb{P}^{g-1}} \otimes M^{-1}) = \ker \varphi = 0 \) under hypothesis (ii). Therefore we conclude by Proposition 2.4 (a).

Using an appropriate generalization of the methods of [5, Proof of Theorem 2] we can also get surjectivity when \( h^0(2K_C - M) \geq 2 \).

**Proposition 2.12.** Let \( C \) be a smooth irreducible curve of genus \( g \geq 4 \) and let \( M \) be a line bundle on \( C \). Suppose there exists an integer \( m \geq 1 \) and an effective divisor \( D = P_1 + \cdots + P_m \) such that

(i) \( H^1(M - 2P_i) = 0 \) for \( 1 \leq i \leq m \);

(ii) \( h^0(D) = 1 \) and \( h^0(2K_C - M - D) = 0 \);

(iii) \( m \leq \text{Cliff}(C) - 2 \).

Then \( \Phi_{M, \omega_C} \) is surjective.

**Proof.** As is well-known we have \( \text{Cliff}(C) \leq \left\lfloor \frac{g-1}{2} \right\rfloor \), whence \( m \leq \text{Cliff}(C) - 2 \leq g - 4 \). We start by observing that \( K_C - D \) is very ample. In fact, if \( K_C - D \) is not very ample, there are two points \( Q_1, Q_2 \in C \) such that

\[
h^0(K_C - D - Q_1 - Q_2) = h^0(K_C - D) - 1 = g - 2 - m + h^0(D) = g - 1 - m
\]

whence \( h^1(D + Q_1 + Q_2) = g - 1 - m \geq 3 \) and \( h^0(D + Q_1 + Q_2) = 2 \) by Riemann–Roch. Therefore \( D + Q_1 + Q_2 \) contributes to the Clifford index of \( C \) and we have \( \text{Cliff}(C) \leq \text{Cliff}(D + Q_1 + Q_2) = m \), contradicting (iii).

Consider the embedding \( C \subset \mathbb{P}^{H^0(K_C - D)} = \mathbb{P}^r \), where \( r = g - 1 - m \). We claim that, in the latter embedding, \( C \) has no trisecant lines. As a matter of fact if there exist three points \( Q_1, Q_2, Q_3 \in C \) such that their linear span \( \langle Q_1, Q_2, Q_3 \rangle \) is a line, we have that

\[
1 = \dim(\langle Q_1, Q_2, Q_3 \rangle) = h^0(K_C - D) - 1 - h^0(K_C - D - Q_1 - Q_2 - Q_3)
\]

\[
= g - 1 - m - h^0(K_C - D - Q_1 - Q_2 - Q_3)
\]
whence $h^1(D + Q_1 + Q_2 + Q_3) = g - 2 - m \geq 2$ and again $h^0(D + Q_1 + Q_2 + Q_3) = 2$. Therefore $D + Q_1 + Q_2 + Q_3$ contributes to the Clifford index of $C$ and we get $\text{Cliff}(C) \leq \text{Cliff}(D + Q_1 + Q_2 + Q_3) = m + 1$, contradicting (iii).

Note further that by (ii) and (iii) we have

$$\deg(K_C - D) = 2g - 2 - m \geq 2$$

therefore Green–Lazarsfeld’s theorem [26, Proposition 2.4.2] gives that $C$ is scheme-theoretically cut out by quadrics in $\mathbb{P}^r$. Hence we have a surjection

$$\mathcal{O}_C(2D - 2K_C) \otimes \alpha \to N^*_{C/\mathbb{P}^r} \to 0.$$ 

Setting, as in [5], $R_L = N^*_{C/\mathbb{P}^r} \otimes L$ for any very ample line bundle $L$, we deduce a surjection

$$\mathcal{O}_C(M - K_C + D) \otimes \alpha \to R_{K_C - D} \otimes M \to 0.$$ 

By (ii), we have that $H^1(M - K_C + D) = H^0(2K_C - M - D)^* = 0$ whence

$$H^1(R_{K_C - D} \otimes M) = 0.$$ 

Now there is an exact sequence [5, 2.7], [13, Proof of Theorem 5]

$$0 \to R_{K_C - D} \otimes M \to R_{K_C} \otimes M \to \bigoplus_{i=1}^m \mathcal{O}_C(M - 2P_i) \to 0$$ 

and therefore by (i) we deduce that

$$H^0(N^*_{C/\mathbb{P}^r} \otimes M^{-1}) \cong H^1(N^*_{C/\mathbb{P}^r} \otimes \omega_C \otimes M)^* \cong H^1(R_{K_C} \otimes M)^* = 0.$$ 

Hence we get the surjectivity of $\Phi_{M, \omega_C}$ by Proposition 2.4 (a). 

We will often use the above result in the following simplified version.

**Corollary 2.13.** Let $C$ be a smooth irreducible curve of genus $g \geq 4$ and let $M$ be a line bundle on $C$ such that $H^1(M) = 0$ and $\deg(M) \geq g + 1$. Suppose that $h^0(2K_C - M) \leq \text{Cliff}(C) - 2$.

Then $\Phi_{M, \omega_C}$ is surjective.

**Proof.** Let $m = \text{Cliff}(C) - 2$. Then $m \geq 0$ by hypothesis and when $m = 0$ the surjectivity of $\Phi_{M, \omega_C}$ holds by Proposition 2.11. When $m \geq 1$ choose general points $P_1, \ldots, P_m$ of $C$ and apply Proposition 2.12. 

**Corollary 2.14.** [35, Corollary 1.7] Let $C$ be a smooth irreducible curve of genus $g \geq 5$ nontrigonal and not isomorphic to a plane quintic. Let $M$ be a line bundle on $C$.

Then the Gaussian map $\Phi_{M, \omega_C}$ is surjective if $\deg(M) \geq 4g - 4$ and $M \neq 2K_C$ if equality holds.

**Proof.** Immediate consequence of Corollary 2.13 or of Proposition 2.11.
2.2 Gaussian maps on tetragonal curves. In this subsection we improve Tendian’s results [35] about Gaussian maps on tetragonal curves. Moreover note that, even though the statement in [35, Theorem 2.10] is almost correct, the proof certainly contains a gap (see Remark 2.17).

We start with some generalities on tetragonal curves following again [33, 6.2].

Definition-Notation 2.15. Let $C$ be a smooth irreducible tetragonal curve of genus $g \geq 6$ not isomorphic to a plane quintic. Let $A$ be a $g_5^1$ on $C$ and let $V_A \subset \mathbb{P}^{g-1} = \mathbb{P}^H(\omega_C)$ be the rational normal scroll spanned by the divisors in $|A|$, $H_A$ the hyperplane bundle and $R_A$ a ruling of $V_A$. Let $\mathcal{E}_A$ be the rank 3 vector bundle on $\mathbb{P}^1$ so that $V_A$ is the image of $\mathbb{P}\mathcal{E}_A$ under the morphism given by $|O_{\mathbb{P}\mathcal{E}_A}(1)|$. Let $\bar{H}_A$ and $\bar{R}_A$ be the pull-backs, under this morphism, of $H_A$ and $R_A$ respectively. Then there are two integers $b_{1,A}, b_{2,A}$ such that $b_{1,A} \geq b_{2,A} \geq 0$, $b_{1,A} + b_{2,A} = g - 5$ and there are two surfaces $\bar{Y}_A \sim 2\bar{H}_A - b_{1,A}\bar{R}_A$, $\bar{Z}_A \sim 2\bar{H}_A - b_{2,A}\bar{R}_A$ such that, if $Y_A$, $Z_A$ are their images in $\mathbb{P}^{g-1}$ then $C = Y_A \cap Z_A$. We also define

$$b_2(C) = \min\{b_{2,A}, A g_5^1 \text{ on } C\}.$$ 

We have

Lemma 2.16. The surface $Y_A \subset \mathbb{P}^{g-1}$ has degree $g - 1 + b_{2,A}$ and satisfies property $N_2$.

Proof. We set for simplicity $Y = Y_A$, $\bar{Y} = \bar{Y}_A$, $V = V_A$, $\mathcal{E} = \mathcal{E}_A$, $H = H_A$, $R = R_A$, $\bar{H} = \bar{H}_A$, $\bar{R} = \bar{R}_A$, $b_i = b_{i,A}$, $i = 1, 2$. Note that $\bar{H}^3 = \deg V = g - 3$, $\bar{R}^2 = 0$ and $\bar{H}^2 \cdot \bar{R} = 1$. Let $\bar{X} \in |O_{\bar{X}}(\bar{H})|$ be a general curve. Since $|O_{\bar{Y}}(\bar{H})|$ is not composed with a pencil we have that $\bar{X}$ is irreducible. Moreover $\bar{X}$ is smooth outside $H \cap \text{Sing}(Y)$, whence $\bar{X}$ is also reduced.

Let $\mathcal{L} = O_{\bar{X}}(\bar{H})$, $X = Y \cap H$, so that $\varphi_{\mathcal{L}}(\bar{X}) = X$. We will first prove that $X$ satisfies property $N_2$. To this end by [3, Theorem A] it is enough to show that

$$\deg X \geq 2p_a(X) + 3.$$  

(6)

Taking intersections in $\mathbb{P}\mathcal{E}$ we have

$$\deg X = \deg \mathcal{L} = \bar{H}^2 \cdot \bar{Y} = \bar{H}^2 \cdot (2\bar{H} - b_1\bar{R}) = 2g - 6 - b_1 = g - 1 + b_2.$$  

(7)

On the other hand, using the cohomology of the scroll, we get

$$p_a(X) = 1 - \chi(O_X) = 1 - \chi(O_Y) + \chi(O_Y(-1))$$

$$= 1 - \chi(O_Y) + \chi(O_Y(-2H + b_1R)) + \chi(O_Y(-1)) - \chi(O_Y(-3H + b_1R))$$

$$= g - 4 - b_1.$$ 

Now $2p_a(X) + 3 = 2g - 5 - 2b_1 \leq 2g - 6 - b_1$ if and only if $b_1 \geq 1$. The latter holds because $b_1 \geq b_2 \geq 0$ and $g \geq 6$. Therefore (6) is proved.

Again using the cohomology of the scroll it is easy to prove that $H^1(J_{Y/p^{g-1}}(j)) = 0$ for every $j \in \mathbb{Z}$ and that $H^1(O_Y(j)) = 0$ for every $j \geq 0$. Applying [16, Theorem 2.a.15 and Theorem 3.b.7] (that hold for any scheme) we deduce that $Y$ satisfies property $N_2$ since $Y \cap H$ does. \qed
Remark 2.17. In Tendian’s paper it is assumed that a general hyperplane section \( Y \cap H \) is smooth, but in fact it can be singular [33, 6.5] when the \( g^1_\delta \) of elliptic or hyperelliptic curve.

Proposition 2.18. Let \( C \) be a smooth irreducible tetragonal curve of genus \( g \geq 6 \) not isomorphic to a plane quintic. Let \( A \) be a \( g^1_\delta \), set \( b_2 = b_{2,A} \) and let \( M \) be a line bundle on \( C \). We have

(i) If \( h^0(2K_C - M) \leq 1 \) and \( h^0(2K_C - M - b_2A) = 0 \), then \( \Phi_{M,\omega_C} \) is surjective;

(ii) If \( H^1(M) = 0 \) and \( \mu_{M,\omega_C} \) is surjective, then cork \( \Phi_{M,\omega_C} \geq h^0(2K_C - M - b_2A) \), with equality holding if \( h^0(2K_C - M) \leq 1 \).

Proof. Let \( Y \) be the surface arising in the scroll defined by \( A \) and set, as in Lemma 2.7, \( \mathcal{F}_{C,Y} = \text{Hom}_{\mathcal{O}_{p^{g-1}}}(\mathcal{J}_{Y/P^{g-1}}, \mathcal{O}_C) \). Applying the left exact functor \( \text{Hom}_{\mathcal{O}_{p^{g-1}}}(-, \mathcal{O}_C) \) to the exact sequence

\[
0 \rightarrow \mathcal{J}_{Y/P^{g-1}} \rightarrow \mathcal{J}_{C/P^{g-1}} \rightarrow \mathcal{J}_{C/Y} \rightarrow 0
\]

we get an exact sequence

\[
0 \rightarrow N_{C/Y} \otimes M^{-1} \rightarrow N_{C/P^{g-1}} \otimes M^{-1} \rightarrow \mathcal{F}_{C,Y} \otimes M^{-1}.
\] (8)

Observe that \( h^0(N_{C/Y} \otimes M^{-1}) = h^0(2K_C - M - b_2A) \). Now if \( h^0(2K_C - M - b_2A) = 0 \), from (8) and Proposition 2.4 (a), we see that to prove (i) we just need to show that

\[
\text{if } h^0(2K_C - M) \leq 1 \text{ then } H^0(\mathcal{F}_{C,Y} \otimes M^{-1}) = 0.
\] (9)

On the other hand, under the hypotheses in (ii), we have that cork \( \Phi_{M,\omega_C} = h^0(N_{C/P^{g-1}} \otimes M^{-1}) \) by Proposition 2.4 (b). Now from (8) we get that \( h^0(N_{C/P^{g-1}} \otimes M^{-1}) \geq h^0(2K_C - M - b_2A) \) and to prove equality we need again to prove (9).

To conclude we just note that (9) holds by Lemmas 2.7 and 2.16. \( \square \)

3 Linear series on quadric sections of surfaces of degree \( g - 1 \) in \( P^{g-1} \)

In this section we will use some well-known vector bundle methods ([26], [36]) to study linear series on curves of genus \( g \) that are, in their canonical embedding, a quadric section of a surface of degree \( g - 1 \) in \( P^{g-1} \). We recall that when the surface is a smooth Del Pezzo the gonality and Clifford index of such curves are known by [31], [21]. Most of the results we prove are probably known, at least in the smooth case, but we include them anyway for completeness’ sake.

Lemma 3.1. Let \( X \) be a smooth surface with \( -K_X \geq 0 \). Let \( C \subset X \) be a smooth irreducible curve of genus \( g \) and let \( A \) be a base-point free \( g^1_k \) on \( C \). Suppose that \( 2g - 2 - K_C \geq 4k \geq \max\{0, 3 - 4\chi(\mathcal{O}_X)\} \) and, if \( h^1(\mathcal{O}_X) \geq 1 \), that \( h^0(N_{C/X} \otimes A^{-1}) \geq 2h^1(\mathcal{O}_X) + 1 \). Then there exist two line bundles \( L, M \) on \( X \) and a zero-dimensional subscheme \( Z \subset X \) such that the following hold:
there exists an effective divisor $D$ on $C$ of degree $M.L + L^2 - k \geq 0$ such that $A \cong L_{(-D)}$;

(iv) if $L^2 = 0$ then $M.L = k$ and $A \cong L_{(-D)}$;

(v) $L$ is base-component free and nontrivial;

(vi) if $C \sim -2K_X$ then $3L^2 + M.L \in 4\mathbb{Z}$.

Proof. Let $\mathcal{F} = \text{Ker}\{H^0(A) \otimes \mathcal{O}_X \rightarrow A\}$ and $E = \mathcal{F}^*$. As is well-known ([26]) $E$ is a rank two vector bundle sitting in an exact sequence

$$0 \rightarrow H^0(A)^* \otimes \mathcal{O}_X \rightarrow E \rightarrow N_{C/X} \otimes A^{-1} \rightarrow 0 \quad (10)$$

and moreover $c_1(E) = C$ and $c_2(E) = k$, so that $\Delta(E) := c_1(E)^2 - 4c_2(E) = C^2 - 4k = 2g - 2 - K_X.C - 4k \geq 0$. Let $H$ be an ample line bundle on $X$ and suppose that $E$ is $H$-stable. Then $h^0(E \otimes E^*) = 1$ by [14, Corollary 4.8] and $h^2(E \otimes E^*) = h^0(E \otimes E^*(K_X)) \leq h^0(E \otimes E^*) = 1$, therefore $2 \geq h^0(E \otimes E^*) + h^2(E \otimes E^*) = h^1(E \otimes E^*) + \chi(E \otimes E^*) \geq 4\chi(O_X) + \Delta(E) \geq 3$, a contradiction. Hence $E$ is not $H$-stable and if $M$ is the maximal destabilizing subbundle we have an exact sequence

$$0 \rightarrow M \rightarrow E \rightarrow \mathcal{J}_{Z/X} \otimes L \rightarrow 0 \quad (11)$$

where $L$ is another line bundle on $X$ and $Z$ is a zero-dimensional subscheme of $X$. Computing Chern classes in (11) we get (i) and the equality in (ii). Since the destabilizing condition reads $(M - L).H \geq 0$ and since $(M - L)^2 = \Delta(E) + 4\text{length}(Z) \geq 0$, we see that $M - L$ belongs to the closure of the positive cone of $X$. We now claim that $E$ is globally generated off a finite set. In fact if $h^1(O_X) \geq 1$ we have by hypothesis that $h^0(N_{C/X} \otimes A^{-1}) \geq 2h^1(O_X) + 1$ and the claim follows by (10) since the map $\psi : H^0(E) \rightarrow H^0(N_{C/X} \otimes A^{-1})$ is nonzero. On the other hand if $h^1(O_X) = 0$ we have that $\psi$ is surjective, whence, again by (10), we just need to prove that $h^0(N_{C/X} \otimes A^{-1}) \geq 1$. Since $g \geq 2k + 1 + \frac{1}{2}K_X.C$ we get $\text{deg}(N_{C/X} \otimes A^{-1}) = 2g - 2 - K_X.C - k \geq g$. Therefore $h^0(N_{C/X} \otimes A^{-1}) \geq 1$ by Riemann–Roch and the claim is proved.

Since $E$ is globally generated off a finite set then so is $L$. It follows that $L \geq 0$, $L$ is base-component free and $L^2 \geq 0$. Now the signature theorem [4, VIII.1] implies that $(M - L).L \geq 0$ thus proving (ii). To see (iii) and (iv) note that if $M.L > 0$ then the nefness of $L$ implies that $H^0(-M) = 0$. On the other hand if $M.L = 0$ then $L^2 = C.L = 0$ whence $L \equiv 0$ by the Hodge index theorem and therefore $C \equiv M$. Then $M.H = C.H > 0$ whence again $H^0(-M) = 0$. Twisting (10) and (11) by $-M$ we deduce that $h^0(L_{(-C)} \otimes A^{-1}) \geq h^0(E(-M)) \geq 1$. This proves (iii) and also (v). Moreover it gives $\text{deg}(L_{C} \otimes A^{-1}) \geq 0$, whence, if $L^2 = 0$, we get that $M.L \geq k$. By (ii) it follows that $M.L = k$ and therefore $\text{deg}(L_{C} \otimes A^{-1}) = 0$, whence $L_{(-C)} \equiv A$. This proves (iv).

Finally suppose that $C \sim -2K_X$. We have $\chi(L) = \chi(O_X) + \frac{1}{2}L.(L - K_X)$ whence $2L.(L - K_X)$ is divisible by 4. But $2L.(L - K_X) = 2L^2 + L.C = 3L^2 + M.L$, giving (vi).\[\square\]

We now analyze linear series on curves on surfaces of degree $r$ in $\mathbb{P}^r$. We will use the following
Definition-Notation 3.2. For \(1 \leq n \leq 9\) we denote by \(\Sigma_n\) the blow-up of \(\mathbb{P}^2\) at \(n\) possibly infinitely near points, by \(\tilde{H}\) the strict transform of a line and by \(G_i\) the total inverse image of the blown-up points. Let \(Q \subset \mathbb{P}^3\) be a quadric cone with vertex \(V\). We denote by \(\text{Bl}_V Q\) the blow-up of \(Q\) along \(V\) and by \(\tilde{H}\) the strict transform of a plane. Let \(C_n \subset \mathbb{P}^n\) be the cone over a smooth elliptic curve in \(\mathbb{P}^{n-1}\) and let \(V\) be the vertex. We denote by \(\text{Bl}_V C_n\) the blow-up of \(C_n\) along \(V\), by \(C_0\) the inverse image of \(V\) and by \(f\) the numerical class of a fiber.

Remark 3.3. We recall that by [30, Theorem 8] a linearly normal integral surface \(Y \subset \mathbb{P}^r\) of degree \(r\) is either the anticanonical image of \(\Sigma_{9-r}\) or \(C_0\) or the 2-Veronese embedding in \(\mathbb{P}^8\) of an irreducible quadric in \(\mathbb{P}^3\) or the 3-Veronese embedding in \(\mathbb{P}^9\) of \(\mathbb{P}^2\).

Proposition 3.4. Let \(X\) be a surface among \(\Sigma_n\), \(\text{Bl}_V Q\) or \(\text{Bl}_V C_n\) as in Definition 3.2 and let \(C\) be a smooth irreducible curve such that, if \(X = \Sigma_n\) or \(\text{Bl}_V Q\) then \(C \sim -2K_X\). While if \(X = \text{Bl}_V C_n\) then \(C \equiv -2K_X - 2C_0\). We have:

(a) if \(X = \Sigma_1\) then \(C\) has no complete base-point free \(g^1_0\);
(b) if \(X = \Sigma_2\) then every complete base-point free \(g^1_0\) on \(C\) is \((\tilde{H} - G_i)_{|C}\), \(i = 1, 2\);
(c) if \(X = \Sigma_2\) then every complete base-point free \(g^1_0\) on \(C\) is \((2\tilde{H} - G_1 - G_2)_{|C} - P_1 - P_2\), where \(P_1, P_2\) are two points of \(C\);
(d) if \(X = \Sigma_3\) then every complete base-point free \(g^1_0\) on \(C\) is \((\tilde{H} - G_i)_{|C}\), \(i = 1, 2, 3\);
(e) if \(X = \Sigma_3\) then every complete base-point free \(g^1_0\) on \(C\) is either \(\tilde{H}_{|C} - P\) or \((2\tilde{H} - G_1 - G_2 - G_3)_{|C} - P\), for some point \(P\) in \(C\);
(f) if \(X = \Sigma_3\) and \(A\) is a complete base-point free \(g^1_0\) on \(C\) then either \(A \equiv (2\tilde{H} - G_i - G_j)_{|C} - P_1 - P_2\), for \(1 \leq i < j \leq 3\) and \(P_1, P_2\) are two points of \(C\) or \((-K_X)_{|C} - A\) is another complete base-point free \(g^1_0\) on \(C\) different from \((2\tilde{H} - G_i - G_j)_{|C} - P_1 - P_2\);
(g) if \(X = \text{Bl}_V C_0\) then \(C\) has no complete base-point free \(g^1_0\) and every complete base-point free \(g^1_0\) on \(C\) is \((f_1 + f_2)_{|C}\), where \(f_1, f_2\) are two fibers;
(h) if \(X = \text{Bl}_V Q\) then \(C\) has a unique complete base-point free \(g^1_0\), namely \(f_{|C}\), where \(f\) is the pull-back of a line of the cone \(Q\);
(i) if \(X = \text{Bl}_V Q\) then every complete base-point free \(g^1_0\) on \(C\) is \(\tilde{H}_{|C} - P_1 - P_2\), where \(P_1, P_2\) are two points of \(C\);
(j) if \(X = \text{Bl}_V Q\) then there is no effective divisor \(Z \subset C\) such that \(f_{|C} + Z\) is a complete base-point free \(g^1_0\) on \(C\).

Proof. We record, for later use, the following fact on \(X = \Sigma_n\). Let \(\mathcal{L}\) be a nef line bundle on \(X\) with \(\mathcal{L} \sim a\tilde{H} - \sum_{i=1}^n b_i G_i\). Then

\[ a = \mathcal{L}.\tilde{H} \geq 0, \quad b_i = \mathcal{L}.G_i \geq 0, \quad \mathcal{L}^2 = a^2 - \sum_{i=1}^n b_i^2, \quad \mathcal{L}.(-K_X) = 3a - \sum_{i=1}^n b_i \]  

(12)

and the Cauchy–Schwartz inequality \((\sum_{i=1}^n b_i)^2 \leq n \sum_{i=1}^n b_i^2\) implies that

\[ (3a + \mathcal{L}.K_X)^2 \leq n(a^2 - \mathcal{L}^2). \]  

(13)

We will now apply Lemma 3.1 to a base-point free \(g^1_0\) indicated in (a)-(i) and we will set \(z = \text{length}(Z)\).
(a) We have $K_X^2 = 8$ whence $C^2 = 32, k = 6$ and from (ii) of Lemma 3.1 we deduce that $6 = M.L + z \geq M.L \geq L^2 \geq 0$. Now if $3 \leq L^2 \leq 6$ we have a contradiction by the Hodge index theorem applied to $C$ and $L$. The same theorem implies, for $L^2 = 2$, that $C \equiv 4L$. But $C \sim 6\tilde{H} - 2G_1$ whence the contradiction $4L.\tilde{H} = C.\tilde{H} = 6$. If $L^2 = 1$ write $L \sim a\tilde{H} - b_1G_1$. Then $a^2 = b_1^2 + 1$ therefore $a = 1, b_1 = 0$ and $L \sim \tilde{H}$. Then $\deg(L(C) \otimes A^{-1}) = \tilde{H}.C - 6 = 0$, whence $A \cong H_i$ by (iii) of Lemma 3.1. Therefore we have the contradiction $h^0(A) = 3$. If $L^2 = 0$ by (iv) of Lemma 3.1 we have that $M.L = 6$ whence $3L^2 + M.L = 6$, contradicting (vi) of Lemma 3.1. This proves (a).

(b) We have $K_X^2 = 7, C^2 = 28$ and $k = 4$. By (ii) of Lemma 3.1 and the Hodge index theorem applied to $C$ and $L$ we see that we are left with the case $L^2 = 0$ whence $A \cong L_i$. By (12), (13) we deduce that $L \sim \tilde{H} - G_i$ for $i = 1, 2$. This proves (b).

(c) We have $K_X^2 = 7$ whence $C^2 = 28$ and $k = 6$. From (ii) of Lemma 3.1 and the Hodge index theorem applied to $C$ and $L$ we get $0 \leq L^2 \leq 2$. The same theorem implies, for $L^2 = 2$, that $z = 0, M.L = 6$. By (iii) of Lemma 3.1 there are two points $P_1, P_2 \in C$ such that $A \cong L_i - P_1 - P_2$. By (12), (13) we deduce that $L \sim 2\tilde{H} - G_1 - G_2$. If $L^2 = 1$ again by (ii) of Lemma 3.1 and the Hodge index theorem applied to $C$ and $L$ we get that $0 \leq z \leq 1$ and $5 \leq M.L \leq 6$. By (vi) of Lemma 3.1 we have that $M.L = 5$ whence $\deg(L(C) \otimes A^{-1}) = 0$, so that $A \cong L_i$ by (iii) of Lemma 3.1. By (12), (13) we deduce that $L \sim \tilde{H}$, giving the contradiction $h^0(A) = 3$. If $L^2 = 0$ we have that $M.L = 6$ by (iv) of Lemma 3.1 contradicting (vi) of Lemma 3.1. This proves (c).

(d) We have $K_X^2 = 6$ whence $C^2 = 24$ and $k = 4$. From (ii) of Lemma 3.1 and the Hodge index theorem applied to $C$ and $L$ we get $0 \leq L^2 \leq 1$. The same theorem implies, for $L^2 = 1$, that $z = 0, M.L = 4$, contradicting (vi) of Lemma 3.1. Therefore $L^2 = 0$ and (iv) of Lemma 3.1 implies that $M.L = 4$ and $A \cong L_i$. By (12), (13) we deduce that $L \sim \tilde{H} - G_i$. This proves (d).

(e) We have $K_X^2 = 6$ whence $C^2 = 24$ and $k = 5$. From (ii) of Lemma 3.1 and the Hodge index theorem applied to $C$ and $L$ we get $L^2 \leq 2$ with equality only when $z = 0, M.L = 5$, contradicting (vi) of Lemma 3.1. When $L^2 = 1$, the same theorem together with (vi) of Lemma 3.1 implies that $z = 0, M.L = 5$, whence $A \cong L_i - P$ by (iii) of Lemma 3.1. By (12), (13) we deduce that either $L \sim \tilde{H}$ or $L \sim 2\tilde{H} - G_1 - G_2 - G_3$. If $L^2 = 0$ then (iv) of Lemma 3.1 implies that $M.L = 5$, contradicting (vi) of Lemma 3.1. This proves (e).

(f) We have $K_X^2 = 6$ whence $C^2 = 24$ and $k = 6$. From (ii) of Lemma 3.1 and the Hodge index theorem applied to $C$ and $L$ we see, for $3 \leq L^2 \leq 5$, that $z = 0, M.L = 6$, contradicting (vi) of Lemma 3.1. If $L^2 = 2$ by the Hodge index theorem and (vi) of Lemma 3.1 we have that $z = 0, M.L = 6$. By (12), (13) we deduce that $L \sim 2\tilde{H} - G_i - G_j$ for $i \neq j$ and by (iii) of Lemma 3.1 we have that there are two points $P_1, P_2 \in C$ such that $A \cong L_i - P_1 - P_2$. If $L^2 = 1$ by the Hodge index theorem and (vi) of Lemma 3.1 we have that $z = 1, M.L = 5$. By (12), (13) we deduce that either $L \sim \tilde{H}$ or $L \sim 2\tilde{H} - G_1 - G_2 - G_3$. By (iii) of Lemma 3.1 we have that $A \cong L_i$, giving the contradiction $h^0(A) = 3$. If $L^2 = 0$ by (iv) of Lemma 3.1 we have that $M.L = 6$ contradicting (vi) of Lemma 3.1. Finally when $L^2 = 6$ the Hodge index theorem applied to $C$ and $L$ implies that $C \equiv 2L$ and $z = 0$. Therefore $L \sim M \sim -K_X$ whence the exact sequence (11) splits since $\text{Ext}^1(O_X(-K_X), O_X(-K_X)) = 0$ and we get $E \cong O_X(-K_X)^{12}$. Therefore $E(K_X)$
is globally generated and so is \((-K_X) \otimes A^{-1}\) by (10). Moreover again by (10) we get that \((-K_X) \otimes A^{-1}\) is a \(g^1_0\). Also such a \(g^1_0\) cannot coincide with the other type \((2\bar{H} - G_i - G_j) \otimes P_1 - P_2\), for otherwise we would have that \((-K_X) \otimes A^{-1} \sim (2\bar{H} - G_i - G_j) \otimes P_1 - P_2\), whence \(A \cong (\bar{H} - G_k) \otimes P_1 + P_2\) would have two base points. This proves (f).

(g) We have that \(X \cong \mathbb{P}(O_E \oplus O_E(-1))\) where \(E \subset \mathbb{P}^5\) is a smooth elliptic normal curve. Let \(C_0\) be a section and \(f\) be a fiber so that \(C_0^2 = -6\) and the intersection form is even. Moreover \(C \equiv 2C_0 + 12f\), \(C^2 = 24\) and \(k = 4, 5\). From (ii) of Lemma 3.1 and the Hodge index theorem applied to \(C\) and \(L\) we deduce, if \(L^2 \geq 2\), that \(k = 5\), \(L^2 = 2\), \(z = 0\) and \(M.L = 5\). On the other hand if \(L^2 = 0\) we have that \(M.L = k\) and \(A \cong L^0(C)\) by (iv) of Lemma 3.1. Let \(L \equiv aC_0 + bf\) so that \(M \equiv (2 - a)C_0 + (12 - b)f\) and \(L^2 = 2a(b - 3a)\). Moreover, by (v) of Lemma 3.1 we have \(a = f.L \geq 0\). If \(L^2 = 2\) we get \(a = 1, b = 4\) giving the contradiction \(M.L = 6\). Therefore \(L^2 = 0\) whence either \(a = 0\) or \(b = 3a\). In the second case we get \(k = M.L = 6a\), a contradiction. Therefore \(a = 0\) and \(k = M.L = 2b\), that is \(k = 4, b = 2\) and \(L \equiv 2f\) as desired. This proves (g).

(h) We have that \(X \cong \mathbb{P}(O_{\mathbb{P}^2} \oplus O_{\mathbb{P}^2}(-2))\). Let \(C_0\) be a section and \(f\) be a fiber so that \(C_0^2 = -2\) and the intersection form is even. Moreover \(C \equiv 4C_0 + 8f\), \(C^2 = 32\) and \(k = 4\). From (ii) of Lemma 3.1 and the Hodge index theorem applied to \(C\) and \(L\) we have a contradiction if \(L^2 \geq 2\). Hence \(L^2 = 0\), \(M.L = 4\) and \(A \cong L^0(C)\) by (iv) of Lemma 3.1. Then we get that either \(L \sim f\) or \(L \sim C_0 + f\). Since \(C_0, C = 0\), this proves (h).

(i) We retain the notation used in (h) except that now \(k = 6\). From (ii) of Lemma 3.1 and the Hodge index theorem applied to \(C\) and \(L\) we deduce, if \(L^2 \geq 2\), that \(L^2 = 2\), \(z = 0\), \(M.L = 6\) and \(C \equiv 4L\), whence \(L \sim C_0 + 2f \sim \bar{H}\). By (iii) of Lemma 3.1 we have that there are two points \(P_1, P_2 \subset C\) such that \(A \cong \bar{H}(C) - P_1 - P_2\). When \(L^2 = 0\) we get \(M.L = 6\) by (iv) of Lemma 3.1, contradicting (vi) of Lemma 3.1. This proves (i).

(j) Again we use the notation in (i). Suppose there is an effective divisor \(Z \subset C\) such that \(f|_C + Z\) is a complete base-point free \(g^1_0\) on \(C\). By Riemann–Roch we get that

\[
h^0((2C_0 + 3f)_C - Z) = h^0(K_C - f|_C - Z) = 3
\]

and the exact sequence

\[
0 \longrightarrow O_X(-2C_0 - 5f) \longrightarrow J_{Z/X}(2C_0 + 3f) \longrightarrow J_{Z/C}(2C_0 + 3f) \longrightarrow 0
\]

gives that also \(h^0(J_{Z/X}(2C_0 + 3f)) = 3\), whence, since \(h^0(2C_0 + 3f) = 6\), that \(Z\) does not impose independent conditions to \([2C_0 + 3f]\). Now let \(Z' \subset Z\) be an effective divisor of degree 3 and set \(Z' + P = Z\). By the exact sequence

\[
0 \longrightarrow O_X(-2C_0 - 5f) \longrightarrow J_{Z'/X}(2C_0 + 3f) \longrightarrow J_{Z'/C}(2C_0 + 3f) \longrightarrow 0
\]

and Riemann–Roch we have

\[
h^0(J_{Z'/X}(2C_0 + 3f)) = h^0(J_{Z'/C}(2C_0 + 3f)) = h^1(f|_C + Z - P)
= h^0(f|_C + Z - P) + 1 = 3.
\]

Therefore \(Z\) is in special position with respect to \(2C_0 + 3f \sim \mathcal{L} + K_X\), where \(\mathcal{L} \sim 4C_0 + 7f\). By [32], [17], [7], [27] there is a rank 2 vector bundle \(\mathcal{E}\) on \(X\) sitting in an
exact sequence

\[ 0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E} \rightarrow \mathcal{J}_{Z/X} \otimes \mathcal{L} \rightarrow 0 \]  
(14)

with \( c_1(\mathcal{E}) = \mathcal{L} \) and \( c_2(\mathcal{E}) = 4 \) so that \( \Delta(\mathcal{E}) = \mathcal{L}^2 - 16 = 8 > 0 \). Therefore \( \mathcal{E} \) is Bogomolov unstable and ([6], [32]) there are two line bundles \( A, B \) on \( X \) and a zero-dimensional subscheme \( W \subset X \) sitting in an exact sequence

\[ 0 \rightarrow A \rightarrow \mathcal{E} \rightarrow \mathcal{J}_{W/X} \otimes B \rightarrow 0. \]  
(15)

Moreover \( \mathcal{L} \sim A + B, A, B + \text{length}(W) = 4, (A - B)^2 = 8 + 4\text{length}(W) \) and \( A - B \) lies in the positive cone of \( M \).

To prove (16) and (17) let \( h \) be the nef divisors and (15) twisted by \( -A > 0 \). By (16) with \( M \geq 0 \) we get

\[ A^2 + B^2 \geq 16. \]  
(18)

Moreover \( \mathcal{L} \) lies in the positive cone of \( X \), whence, by [4, VIII.1], \( (A - B).\mathcal{L} > 0 \), that is

\[ A^2 > B^2. \]  
(19)

Now if \( A^2 \leq 8 \) we deduce by (19) that \( B^2 \leq 6 \), contradicting (18). Therefore

\[ A^2 \geq 10. \]  
(20)

Suppose that \( A \sim aC_0 + a_1f \) so that \( B \sim (4 - a)C_0 + (7 - a_1)f \). Intersecting \( A \) with the nef divisors \( f, C_0 + 2f \) and using (17), we see that \( a \geq 0, a_1 \geq 0 \), whence \( A \geq 0 \) and in fact \( A > 0 \) by (20). Also \( a > 0 \), for otherwise \( A^2 = 0 \). Now the exact sequences (14) and (15) twisted by \(-A\) give

\[ h^0(\mathcal{J}_{Z/X}(B)) \geq h^0(\mathcal{E}(-A)) \geq 1 \]  
(21)

whence also \( B > 0 \). The nefness of \( C_0 + 2f \) then implies \( 7 - a_1 = B.(C_0 + 2f) \geq 0 \), whence \( a_1 \leq 7 \), while the nefness of \( f \) implies that \( 4 - a = B.f \geq 0 \), whence \( a \leq 4 \).

By (16) with \( M = C_0 + 2f \) we get \( 2a_1 - 7 = (A - B).(C_0 + 2f) \geq 0 \), whence \( a_1 \geq 4 \).

Finally by (20) we have that \( a(a_1 - a) \geq 5 \). Therefore we have proved that

\[ 1 \leq a \leq 4, \quad 4 \leq a_1 \leq 7, \quad a(a_1 - a) \geq 5. \]  
(22)

If \( a = 1, 2 \) we get that \( A^2 + B^2 \leq 12 \), contradicting (18). Recall now that \( C_0 \cap C = \emptyset \) since \( C_0, C = 0 \). When \( a = 3 \) we have \( A^2 + B^2 = 4a_1 - 6 \) whence \( a_1 = 6, 7 \) by (18).
When $a_1 = 7$ we have $B \sim C_0$, whence $B = C_0$. By (21) we deduce the contradiction $Z \subset C_0 \cap C = \emptyset$. When $a_1 = 6$ we have $B \sim C_0 + f$, whence $B = C_0 \cup F$ for some ruling $F$. As above we have that $Z \cap C_0 = \emptyset$, whence $Z \subset F \cap C$. Since $F.C = 4$ we have that $Z = F \cap C$, whence $Z \sim f|_C$ and therefore $f|_C + Z \sim 2f|_C$ is a complete base-point free $g_1^2$ on $C$. This is of course a contradiction since on $X$ we have that $2f|_C$ is a complete base-point free $g_1^3$ on $C$. Finally when $a_1 = 4$ we have $B \sim (7 - a_1)f$ whence $a_1 \leq 6$ as $B > 0$. By (22) we get $a_1 = 6$ whence $B \sim f$, therefore again $B = F$ for some ruling $F$. Hence $Z \subset F \cap C$, giving the same contradiction above. This proves (j).

\[\textbf{Remark 3.5.}\] Let $C$ be a smooth tetragonal curve of genus 7 such that $\dim W^1_1(C) = 0$ and $\dim W^1_5(C) = 1$ (as in the case $C \sim -2K_X$ on $X = \Sigma_3$). By [1] $W^1_6(C)$ has an irreducible component of dimension at least 3 and whose general element $A$ is a complete $g_1^3$ on $C$. Moreover $A$ is base-point free since $\dim W^1_1(C) = 0$ and $\dim W^1_5(C) = 1$. Also the same holds for $K_C - A$ thus proving that, for these curves, there is a family of dimension at least 3 of complete base-point free $g_1^3$'s whose residual is also base-point free.

\section{Some results on Enriques surfaces}

We will use the following well-known

\[\textbf{Definition 4.1.}\] Let $L$ be a line bundle on an Enriques surface $S$ such that $L^2 > 0$. Following [12] we define

$$\phi(L) = \inf\{|F\cdot L| : F \in \text{Pic} \ S, F^2 = 0, F \neq 0\}.$$ 

This function has two important properties:

(i) $\phi(L)^2 \leq L^2$ ([12, Corollary 2.7.1]);

(ii) If $L$ is nef, then there exists a genus one pencil $|2E|$ such that $E \cdot L = \phi(L)$ (by [11, 2.11] or by [12, Corollary 2.7.1, Proposition 2.7.1 and Theorem 3.2.1]).

We will often use the

\[\textbf{Definition 4.2.}\] Let $S$ be an Enriques surface. A nodal curve on $S$ is a smooth rational curve contained in $S$.

We will now briefly recall some results on line bundles on Enriques surfaces, proved in [24] and [22], that we will often use.

\[\textbf{Lemma 4.3.}\] [22, Lemma 2.2] Let $L > 0$ and $\Delta > 0$ be divisors on an Enriques surface $S$ with $L^2 \geq 0$, $\Delta^2 = -2$ and $k := -\Delta \cdot L > 0$. Then there exists an $A > 0$ such that $A^2 = L^2$, $A \cdot \Delta = k$ and $L \sim A + k\Delta$. Moreover if $L$ is primitive then so is $A$. 
Lemma 4.4. [22, Lemma 2.3] Let $S$ be an Enriques surface and let $L$ be a line bundle on $S$ such that $L > 0, L^2 > 0$. Let $F > 0$ be a divisor on $S$ such that $F^2 = 0$ and $\phi(L) = |F_L|$. Then

(a) $F.L > 0$;
(b) if $\alpha > 0$ is such that $(L - \alpha F)^2 \geq 0$, then $L - \alpha F > 0$.

Lemma 4.5. [24, Lemma 2.1] Let $X$ be a smooth surface and let $A > 0$ and $B > 0$ be divisors on $X$ such that $A^2 \geq 0$ and $B^2 \geq 0$. Then $A.B \geq 0$ with equality if and only if there exists a primitive divisor $F > 0$ and integers $a \geq 1, b \geq 1$ such that $F^2 = 0$ and $A \equiv aF, B \equiv bF$.

Definition 4.6. An effective line bundle $L$ on a K3 or Enriques surface is said to be quasi-nef if $L^2 \geq 0$ and $L.D \geq -1$ for every $D$ such that $\Delta > 0$ and $\Delta^2 = -2$.

Theorem 4.7. [24, Corollary 2.5] An effective line bundle $L$ on a K3 or Enriques surface is quasi-nef if and only if $L^2 \geq 0$ and either $h^1(L) = 0$ or $L \equiv nE$ for some $n \geq 2$ and some primitive and nef divisor $E > 0$ with $E^2 = 0$.

Theorem 4.8. [22, Corollary 1] Let $|L|$ be a base-component free linear system on an Enriques surface $S$ such that $L^2 > 0$ and let $C \in |L|$ be a general curve. Then

$$\text{gon}(C) = 2\phi(L)$$

unless $L$ is of one of the following types:

(a) $L^2 = \phi(L)^2$ with $\phi(L) \geq 2$ and even. In these cases $\text{gon}(C) = 2\phi(L) - 2$.
(b) $L^2 = \phi(L)^2 + \phi(L) - 2$ with $\phi(L) \geq 3$, $L \not\equiv 2D$ for $D$ such that $D^2 = 10$, $\phi(D) = 3$. In these cases $\text{gon}(C) = 2\phi(L) - 1$ except for $\phi(L) = 3, 4$ when $\text{gon}(C) = 2\phi(L) - 2$.
(c) $(L^2, \phi(L)) = (30, 5), (22, 4), (20, 4), (14, 3), (12, 3) \text{ and } (6, 2)$. In these cases $\text{gon}(C) = \lfloor \frac{L^2}{2} \rfloor + 2 = 2\phi(L) - 1$.

5 Tetragonal curves on Enriques surfaces and on surfaces of degree $g - 1$ in $\mathbb{P}^{g-1}$

Let $C$ be a smooth irreducible tetragonal curve of genus $g \geq 6$ and let $M$ be a line bundle on $C$ such that $H^1(M) = 0$ and $\mu_{M,\omega_C}$ is surjective. To have the surjectivity of the Gaussian map $\Phi_{M,\omega_C}$, it is necessary, by Proposition 2.18(ii), that $h^0(2K_C - M - b_{2,A}A) = 0$ for every $g_A^1$ on $C$. On the other hand when $h^0(2K_C - M) = 1$ we need that $b_{2,A} \geq 1$ for every $g_A^1$ on $C$, that is (see 2.15) $b_2(C) \geq 1$, because in this case, by Proposition 2.18(ii), $h^0(2K_C - M - b_{2,A}A) = \text{cork} \Phi_{M,\omega_C}$ is independent of $A$. As we have seen in 2.15, in the canonical embedding, $C = Y_A \cap Z_A \subset \mathbb{P}^{g-1}$ where $Y_A$ is a surface of degree $g - 1 + b_{2,A}$ by Lemma 2.16. Moreover $b_{2,A} = 0$ if and only if $C$ is a quadric section of $Y_A$. Therefore saying that $b_2(C) \geq 1$ is equivalent to saying that $C$, in
its canonical embedding, can never be a quadric section of a surface $Y_A$ of degree $g - 1$ in $\mathbb{P}^{g-1}$.

The present section we will be devoted to proving that tetragonal curves of genus $g \geq 7$, lying on an Enriques surface and general in their linear system, in their canonical embedding, can never be a quadric section of a surface $Y_A$ of degree $g - 1$ in $\mathbb{P}^{g-1}$. The latter fact will be then used to prove surjectivity of Gaussian maps for such curves in our main theorem.

We start by observing that we cannot do better in genus 6. Let $C$ be a smooth irreducible tetragonal curve of genus 6 and let $A$ be a $g^1_3$ on $C$. Now $K_C - A$ is a $g^2_9$ and has a base point if and only if $C$ is isomorphic to a plane quintic. Therefore if $C$ is not isomorphic to a plane quintic, then it has complete base-point free $g^2_9$ and either $C$ is bielliptic or the $g^2_9$ is birational. In the latter case the image of $C$ by the $g^2_9$ cannot have points of multiplicity higher than 2, therefore $C$ does lie on $X = \Sigma_4$ and is linearly equivalent to $-2K_X$.

Hence we can restrict our attention to curves of genus $g \geq 7$.

We will henceforth let $S$ be an Enriques surface. Consider a base-point free line bundle $L$ on $S$ with $L^2 \geq 12$ and let $C \in |L|$ be a general curve. By Theorem 4.8 we have that $C$ is not trigonal and moreover $C$ is tetragonal if and only if $\phi(L) = 2$.

Now assume that $\phi(L) = 2$. We have

**Theorem 5.1.** Let $L$ be a base-point free line bundle on an Enriques surface with $L^2 \geq 12$ and $\phi(L) = 2$. Then $b_2(C) \geq 1$ for a general curve $C \in |L|$.

The proof of this theorem will be essentially divided in two parts, namely a careful study of the cases $L^2 = 12, 14$ and 16 and an application of previous results for $L^2 \geq 18$. In both parts we will employ the following

**General remark 5.2.** Let $C$ be a tetragonal curve of genus $g$ and let $A$ be a $g^1_3$ on $C$ such that $b_{2,A} = 0$. Then, by 2.15 and Lemma 2.16, in its canonical embedding, $C$ is a quadric section of a surface $Y_A \subset \mathbb{P}^{g-1}$ of degree $g - 1$ whence, by Remark 3.3, $C$ is contained in a surface $X$ that is either $\Sigma_{10-g}$, or $\text{Bl}_V C_{g-1}$, or a smooth quadric in $\mathbb{P}^3$ or $\text{Bl}_V Q$ where $Q$ is a quadric cone in $\mathbb{P}^3$, or $\mathbb{P}^2$. Also $C$ is either bielliptic (in the case of $C_{g-1}$) or linearly equivalent to $-2K_X$.

We start with the cases of genus 7, 8 and 9.

### 5.1 Curves of genus 7

We will need the ensuing

**Lemma 5.3.** Let $L$ be a base-point free line bundle on an Enriques surface with $L^2 = 12$ and $\phi(L) = 2$. Let $|2E|$ be a genus one pencil such that $E.L = 2$. Then there exists a primitive divisor $E_1$ such that $E_1 > 0$, $E_1^2 = 0$, $E + E_1$ is nef, $h^0(E_1) = h^0(E_1 + K_S) = 1$ and one of the following cases occurs:

(i) $\phi(L - 2E) = 1$ and $L \sim 3E + 2E_1$, $E.E_1 = 1$;
(ii) $\phi(L - 2E) = 2$ and $L \sim 3E + E_1$, $E.E_1 = 2$.

Moreover, in Case (ii), for any smooth curve $C \in |L|$, we have that $h^0((E_1)|_C) = h^0((E_1 + K_S)|_C) = 2$. 
Proof. We have \((L - 3E)^2 = 0\), \(E.(L - 3E) = 2\) and by Lemma 4.4 we can write \(L \sim 3E + E_1'\) with \(E_1' \neq 0\), \((E_1')^2 = 0\) and \(E.E_1' = 2\). Also \(1 \leq \phi(L - 2E) \leq \sqrt{(L - 2E)^2} = 2\).

If \(\phi(L - 2E) = 2\) we set \(E_1 = E_1'\). Then certainly \(E_1\) is primitive and we have \(L \sim 3E + E_1, E.E_1 = 2\), as in (ii).

If \(\phi(L - 2E) = 0\) set \(F = 0\) be a divisor such that \(F^2 = 0\) and \(F.(E + E_1') = 1\) (\(F\) exists by Lemma 4.4). Then necessarily \(F.E = 1, F.E_1' = 0\) therefore \(E_1' = 2F\) by Lemma 4.5 and we can set \(E_1 = F\). Replacing, if necessary, \(E\) with \(E + K_S\), we have that \(E_1\) is primitive and \(L \sim 3E + 2E_1, E.E_1 = 1\), as in (i).

Since \(E_1\) is primitive, to see, in both Cases (i) and (ii), that \(h^0(E_1) = h^0(E_1 + K_S) = 1\), by [24, Corollary 2.5], we just need to show that \(E_1\) is quasi-nef. Let \(\Delta \geq 0\) be a divisor such that \(\Delta^2 = -2\) and \(k := -E_1.\Delta \geq 1\). By Lemma 4.3 we can write \(E_1 \sim A + k\Delta\) for some \(A > 0\) primitive with \(A^2 = 0, A.\Delta = k\). Now \(0 \leq L.\Delta = 3E.\Delta + E_1'.\Delta \leq 3E.\Delta - 1\) gives \(E.\Delta \geq 1\). From \(2 \geq E.E_1 = E.A + kE.\Delta\) we get that either \(k = 1\) or \(k = 2, E.\Delta = 1\) and \(E.A = 0\). In the latter case we have that \(E \equiv A\) by Lemma 4.5 and this is a contradiction since \(A.\Delta = 2\).

Therefore we have proved that \(E_1\) is quasi-nef and if \(E_1.\Delta \leq -1\) then \(E_1.\Delta = -1, E.\Delta \geq 1\). This of course implies that \(E + E_1\) is nef.

Suppose now that we are in Case (ii), let \(F \equiv E_1\) and let \(C \in |L|\) be a smooth curve. From the exact sequence

\[
0 \rightarrow F - L \rightarrow F \rightarrow F|_C \rightarrow 0
\]

and the fact just proved that \(h^0(F) = 1, h^1(F) = 0\), we see that \(h^0(F|_C) = 1 + h^1(F - L) = 2\) since \(F - L \equiv -3E\).

The above lemma allows to exclude quickly the bielliptic case.

Remark 5.4. Let \(L\) be a base-point free line bundle on an Enriques surface \(S\) with \(L^2 = 12\) and \(\phi(L) = 2\). Let \(|2E|\) be a genus one pencil such that \(E.L = 2\). Let \(C\) be a general curve in \(|L|\). If \(b_2(C) = 0\) we can certainly say that \(C\) is not bielliptic since if \(A\) is a complete base-point free \(g^1_4\) on \(C\) we have, by Proposition 3.4(g), that \(A \sim (f_1 + f_2)|C\) therefore \(|K_C - A| = |(f_1 + \cdots + f_4)|C|\) is not birational. On the other hand on the Enriques surface \(S\), if we pick \(A = (2E)|_C\), using the notation of Lemma 5.3, we have that either \(K_C - A \sim (E + E_1 + K_S)|_C\) or \(K_C - A \sim (E + 2E_1 + K_S)|_C\). Since the linear systems \(|E + E_1 + K_S|\) and \(|E + 2E_1 + K_S|\) define a map whose general fiber is finite by [12, Theorem 4.6.3 and Theorem 4.5.1], we get that \(|K_C - A|\) is birational for general \(C\) since \(|L|\) is birational by [12, Theorem 4.6.3 and Proposition 4.7.1].

According to the two cases in Lemma 5.3 we will have two propositions.

Proposition 5.5. Let \(L\) be a base-point free line bundle on an Enriques surface with \(L^2 = 12\) and \(\phi(L) = 2\). Let \(|2E|\) be a genus one pencil such that \(E.L = 2\) and suppose that \(\phi(L - 2E) = 1\).

Then \(b_2(C) \geq 1\) for a general curve \(C \in |L|\).
Proof. We use the notation of Lemma 5.3. First we prove that either \((E + E_1)|_C\) or \((E + E_1 + K_S)|_C\) is a complete base-point free \(g^1_2\) on \(C\).

To this end note that since \((E + E_1)^2 = 2\) and \(E + E_1\) is nef by Lemma 5.3, we have by [12, Proposition 3.1.6 and Corollary 3.1.4] that either \(E + E_1\) or \(E + E_1 + K_S\) is base-component free with two base points. Let \(B \equiv E + E_1\) be the line bundle that is base-component free. As \(C\) is general in \(|L|\) we have that \(B|_C\) is base-point free. Now the exact sequence

\[
0 \rightarrow B - C \rightarrow B \rightarrow B|_C \rightarrow 0
\]

shows that also \(B|_C\) is a complete \(g^1_2\) since \(B - C \equiv -2E - E_1\) whence \(h^1(B - C) = 0\) because \(2E + E_1\) is nef by Lemma 5.3.

Now suppose that there exists a line bundle \(A\) that is a \(g^1_2\) on \(C\) and is such that \(b_{2,A} = 0\). By the general Remark 5.2 we know that \(C\) lies on a surface \(X\) (obtained by desingularizing \(Y_A\), if necessary) and either \(X = \Sigma_3\), \(C \sim -2K_X\) or \(X = B|_C \subset C_6\) and \(C\) is bielliptic. As \(C\) has a complete base-point free \(g^1_2\) the second case is excluded by Proposition 3.4(g) (or by Remark 5.4). When \(X \equiv \Sigma_3\) by Proposition 3.4(e) we know that there is a point \(P \in C\) such that either \(B|_C \sim \tilde{H}|_C - P\) or \(B|_C \sim (2\tilde{H} - G_1 - G_2 - G_3)|_C - P\).

If \(B|_C \sim \tilde{H}|_C - P\) then

\[
K_C \sim (3\tilde{H} - G_1 - G_2 - G_3)|_C \sim (L + K_S - B)|_C + \tilde{H}|_C - P
\]

whence

\[
(L + K_S - B)|_C - P \sim (2\tilde{H} - G_1 - G_2 - G_3)|_C \text{ is a } g^2_6 \text{ on } C.
\]

If \(B|_C \sim (2\tilde{H} - G_1 - G_2 - G_3)|_C - P\) then

\[
K_C \sim (3\tilde{H} - G_1 - G_2 - G_3)|_C \sim (L + K_S - B)|_C + (2\tilde{H} - G_1 - G_2 - G_3)|_C - P
\]

whence

\[
(L + K_S - B)|_C - P \sim \tilde{H}|_C \text{ is a } g^2_6 \text{ on } C.
\]

But using the Enriques surface \(S\) we have an exact sequence

\[
0 \rightarrow K_S - B \rightarrow L + K_S - B \rightarrow (L + K_S - B)|_C \rightarrow 0
\]

and \(L + K_S - B \equiv 2E + E_1\), \(h^1(K_S - B) = 0\) by Lemma 5.3, whence \((L + K_S - B)|_C\) is a base-point free \(g^2_7\) on \(C\), contradicting (23) and (24). \(\square\)

Now the other case.

Proposition 5.6. Let \(L\) be a base-point free line bundle on an Enriques surface \(S\) with \(L^2 = 12\) and \(\phi(L) = 2\). Let \(|2E|\) be a genus one pencil such that \(E\). \(L = 2\) and suppose that \(\phi(L - 2E) = 2\). Then the general curve in \(|L|\) possesses no \(g^2_6\) and satisfies \(b_2(C) \geq 1\).
The first part of the claim follows by the exact sequence (25).

Proof. The proof will be a variant of the method of [22, Section 4]. By Lemma 5.3 we have $L \sim 3E + E_1$ with $E > 0$, $E_1 > 0$ both primitive, $E^2 = E_1^2 = 0$, $E$ and $E + E_1$ are nef and $E.E_1 = 2$. Let $D = 2E + E_1$ so that $D^2 = 8$, $\phi(D) = 2$, $D.L = 10$ and $D$ is nef, whence base-point free by [12, Proposition 3.1.6, Proposition 3.1.4 and Theorem 4.4.1].

Now recall that by [12, Theorem 4.6.3 and Theorem 4.7.1] the linear system $|D|$ defines a birational morphism $\varphi_D : S \to \overline{S} \subset \mathbb{P}^4$ onto a surface $\overline{S}$ having some rational double points, corresponding to nodal curves $R \subset S$ such that $D.R = 0$, and two double lines, namely $\varphi_D(E)$ and $\varphi_D(E + K_S)$. More precisely by [20, Proposition 3.7] we see that if $Z \subset S$ is any zero-dimensional subscheme of length two not imposing independent conditions to $|D|$ then either $Z \subset E$ or $Z \subset E + K_S$ or any point $x \in \text{Supp}(Z)$ lies on some nodal curve contracted by $\varphi_D$. Observe that if $R \subset S$ is a nodal curve contracted by $\varphi_D$, then $0 = D.R = E.R + (E + E_1).R$ whence $E.R = E_1.R = 0$ by the nefness of $E$ and of $E + E_1$. This implies that $C.R = 0$, whence that $C \cap R = \emptyset$, for any $C \in |L|_sm$. Also, if $\overline{S}$ contains a line different from the two double lines, then this line is image of a nodal curve $\Gamma \subset S$ such that $D.\Gamma = 1$ whence, using again the nefness of $E + E_1$, we have that either $E.\Gamma = 0$, $E_1.\Gamma = 1$ or $E.\Gamma = 1$, $E_1.\Gamma = -1$. This implies that $C.\Gamma = 1, 2$ for any $C \in |L|_sm$. In particular, since $C.E = 2$, we find that for each line on $\overline{S}$ its inverse image in $S$ can contain at most two points of any $C \in |L|_sm$. Moreover $\overline{S}$ contains finitely many lines, namely the two lines $\varphi_D(E)$, $\varphi_D(E + K_S)$ and the images of the finitely many irreducible curves $\Gamma \subset S$ such that $D.\Gamma = 1$ (these are finitely many since if $D.\Gamma = 1$ we get $\Gamma^2 = -2$).

By Remark 5.4 we know that there is a proper closed subset $B \subset |L|_sm$ such that every element in $B$ is bielliptic and by Theorem 4.8 there is another proper closed subset $B_3 \subset |L|_sm$ such that every element in $B_3$ is trigonal or hyperelliptic and any element of $U := |L|_sm - (B \cup B_3)$ is tetragonal. We set $B_6^2$ for the closed subset of $|L|_sm$ whose elements correspond to curves having a $g_6^2$.

The goal will be to prove that the open subset $|L|_sm - (B \cup B_3 \cup B_6^2)$ is nonempty. We will therefore suppose that it is empty, so that every $C \in U$ has a linear series $A_C$ that is a $g_6^2$ on $C$.

Since $h^0(D(C - A_C)) = h^0(\omega_C - A_C - (E + K_S)_C) \geq h^1(A_C) - 2 \geq 1$, we see that there exists an effective divisor $T$ of degree 4 on $C$ such that $T \sim D(C - A_C)$.

Claim 5.7. For each $T$ as above we have $h^0(J_{T/S}(D)) = 3$ and $h^0(J_{T/S}(L)) = 4$.

Proof. The first part of the claim follows by the exact sequence (25)

$$0 \longrightarrow O_S(-E) \longrightarrow J_{T/S}(D) \longrightarrow J_{T/C}(D) \longrightarrow 0$$

since then $h^0(J_{T/S}(D)) = h^0(J_{T/C}(D)) = h^0(A_C) = 3$.

To see the second part of the claim consider the exact sequence

$$0 \longrightarrow O_S \longrightarrow J_{T/S} \otimes L \longrightarrow J_{T/C} \otimes L \longrightarrow 0$$

so that $h^0(J_{T/S} \otimes L) = 1 + h^0(L(C - T)) = 1 + h^0(A_C + E_C)$.

We will prove that $h^0(A_C + E_C) = 3$. Now $h^0(A_C + E_C) \geq h^0(A_C) = 3$ and we need to exclude that $h^0(A_C + E_C) \geq 4$. 

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Assume henceforth that \( h^0(A_C + E|_C) \geq 4 \). Since \( \deg(A_C + E|_C) = 8 \) and \( \text{Cliff}(C) = 2 \), if \( h^0(A_C + E|_C) \geq 4 \), we must have \( h^0(A_C + E|_C) = 4 \), therefore \( h^0(J_{Z/S} \otimes L) = 5 \). Since \( h^0(L) = 7 \) we see that there is a zero-dimensional subscheme \( Z \subset T \) such that \( \text{length}(Z) = 3 \) and \( h^0(J_{Z/S} \otimes L) = 5 \). We claim that there is a proper subscheme \( Z' \subset Z \) such that \( \text{length}(Z') = 2 \) and \( h^0(J_{Z'/S} \otimes L) \geq 6 \). In fact if for every proper subscheme \( Z' \subset Z \) with \( \text{length}(Z') = 2 \) we have \( h^0(J_{Z'/S} \otimes L) = 5 \) then \( Z \) is in special position with respect to \( L + K_S \) and, since \( L^2 = 4 \text{length}(Z) = 12 \), we deduce by [20, Proposition 3.7] that there is an effective divisor \( B \) such that \( Z \subset B \) and \( L.B \leq B^2 + 3 \leq 6 \). Since \( B.L \geq 3 \) we get that

\[
3 \leq L.B \leq B^2 + 3 \leq 6
\]  

(26)

whence \( 0 \leq B^2 \leq 2 \). Note that for any \( F > 0 \) with \( F^2 = 0 \) we have either \( F.L \geq 4 \) or \( F \equiv E \) (whence \( F.L = 2 \)). Now if \( B^2 = 2 \) we can write \( B \sim F_1 + F_2 \) with \( F_1 > 0 \), \( F_i^2 = 0 \) for \( i = 1 \) and \( F_1.F_2 = 1 \). By (26) we have \( L.F_1 + L.F_2 = L.B \leq 5 \), whence the contraction \( F_1 \equiv E \equiv F_2 \). Therefore \( B^2 = 0 \) and \( L.B = 3 \) by (26), again a contradiction.

We have therefore proved that there is a proper subscheme \( Z' \subset Z \subset C \) such that \( \text{length}(Z') = 2 \) and \( h^0(J_{Z'/S} \otimes L) \geq 6 \), whence \( h^0(J_{Z'/S} \otimes L) = 6 \) as \( L \) is base-point free and therefore \( Z' \) is not separated by the morphism \( \varphi_L : S \to \mathbb{P}^6 \). Now recall that by [12, Theorem 4.6.3, Proposition 4.7.1 and Corollary 1, p.283] \( \varphi_L \) is a birational morphism onto a surface having some rational double points, corresponding to nodal curves \( R \subset S \) such that \( L.R = 0 \), and two double lines, namely \( \varphi_L(E) \) and \( \varphi_L(E + K_S) \) and that \( \varphi_L \) is an isomorphism outside \( E, E + K_S \) and the nodal curves contracted. In particular we deduce that either \( Z' = C \cap E \) or \( Z' = C \cap (E + K_S) \). We claim that this implies that either \( T \sim (2E)_C \) or \( T \sim (2E + K_S)_C \).

To see the latter suppose for example that \( Z' = C \cap E \) and set \( W = T - Z' \) on \( C \). Then \( \text{length}(W) = 2 \) and \( 4 = h^0(L|_C - T) \equiv h^0(D|_C + Z' - T) \equiv h^0(D|_C - W) \), whence the exact sequence

\[
0 \to \mathcal{O}_S(-E) \to J_{W/S} \otimes D \to J_{W/C} \otimes D \to 0
\]

shows that \( h^0(J_{W/S} \otimes D) = 4 \). Therefore \( W \) is not separated by the morphism \( \varphi_D : S \to \mathbb{P}^4 \). As \( C \cap R = 0 \), for any nodal curve \( R \) contracted by \( \varphi_D \) we have that either \( W \sim E|_C \) or \( W \sim (E + K_S)|_C \), whence either \( T \sim (2E)_C \) or \( T \sim (2E + K_S)_C \).

Finally since we know that \( T \sim D|_C - A_C \) we deduce that either \( A_C \sim (E_1)|_C \) or \( A_C \sim (E_1 + K_S)|_C \), but this contradicts Lemma 5.3. 

**Continuation of the proof of Proposition 5.6.** Consider the following incidence subscheme of \( \text{Hilb}^4(S) \times \mathcal{U} \):

\[
\mathcal{J} = \{(T, C) : T \in \text{Hilb}^4(S), C \in \mathcal{U}, T \subset C \text{ and } h^0(D|_C - T) \geq 3\}
\]

together with its two projections \( \pi : \mathcal{J} \to \text{Hilb}^4(S) \) and \( p : \mathcal{J} \to \mathcal{U} \).

Our assumption that any \( C \in \mathcal{U} \) carries a \( g^2_1 \) implies, as we have seen, that \( p \) is surjective, whence we deduce that \( \mathcal{J} \) has an irreducible component \( \mathcal{J}_0 \) such that \( \dim \mathcal{J}_0 \geq \)
6. Since the fibers of $\pi$ have dimension at most $h^0(J_{T/S}(L)) - 1 = 3$ by Claim 5.7, we get that $\dim(\pi(3_0)) \geq 3$.

Using $\pi(3_0)$ we build up an incidence subscheme of $\pi(3_0) \times |D|$: \[ J = \{(T, D') : T \in \pi(3_0), D' \in |D|, T \subset D'\} \]

To show that this fact leads to a contradiction let us return to the morphism $\varphi_D : S \to \mathcal{S}$. By (25) and the definition of $\pi(3_0)$ we have that $h$ is surjective. Since the fibers of $h$ have dimension $h^0(J_{T/S}(D)) - 1 = 2$ by Claim 5.7, we find that $J$ has an irreducible component $J_0$ such that $\dim J_0 \geq 5$.

To show that this fact leads to a contradiction let us return to the morphism $\varphi_D : S \to \mathcal{S} \subset \mathbb{P}^4$.

A general hyperplane section $\overline{\mathcal{D}} = \mathcal{S} \cap H \subset \mathbb{P}^4$ is a curve of degree 8 with two nodes, whence of arithmetic genus 7. Consider, for $i = 2, 3$, the exact sequence

\[ 0 \to \mathcal{O}_{\mathcal{S}}(i - 1) \to \mathcal{O}_{\mathcal{S}}(i) \to \mathcal{O}_{\overline{\mathcal{D}}}(i) \to 0. \]

Using Riemann–Roch on $\overline{\mathcal{D}}$ we get

\[ h^0(\mathcal{O}_{\mathcal{S}}(3)) \leq h^0(\mathcal{O}_{\mathcal{S}}(2)) + h^0(\mathcal{O}_{\overline{\mathcal{D}}}(3)) \leq h^0(\mathcal{O}_{\mathcal{S}}(1)) + h^0(\mathcal{O}_{\overline{\mathcal{D}}}(2)) + h^0(\mathcal{O}_{\overline{\mathcal{D}}}(3)) = 33 \]

whence $h^0(J_{\overline{\mathcal{D}}/\mathbb{P}^4}(3)) \geq 2$ and therefore there is a plane $\overline{\mathcal{D}} \subset \mathbb{P}^4$ such that $\mathcal{S} \cup \overline{\mathcal{D}}$ is a complete intersection of two cubics in $\mathbb{P}^4$.

Now $\pi(3_0)$ has three important properties. First of all we know that $T \subset C$ for some $C \in \mathcal{U}$ and $C \cap R = \emptyset$ for every nodal curve $R$ contracted by $\varphi_D$, therefore also $T \cap R = \emptyset$ for every nodal curve $R$ contracted by $\varphi_D$. Secondly, since $C.E = 2$, we get that $\deg(T \cap E) \leq 2$ and $\deg(T \cap (E + K_S)) \leq 2$. Thirdly the linear span $l_T := (\varphi_D(T)) \subset \mathbb{P}^4$ is a line by Claim 5.7. Moreover let us prove that we cannot have infinitely many elements $T \in \pi(3_0)$ such that $l_T$ is the same line. Suppose to the contrary that there is an infinite set $Z \subset \pi(3_0)$ and a line $l \subset \mathbb{P}^4$ such that $l_T = l$ for every $T \in Z$. If $l$ is not contained in $\mathcal{S}$ then it meets $\mathcal{S}$ in finitely many points, therefore there is a point $P \in l$ and an infinite set $V \subset S$ such that $\varphi_D(x) = P$ for every $x \in V$ and each $x \in V$ lies on some $T \in Z$. Now $V \subset \varphi_D^{-1}(P)$ therefore $\varphi_D^{-1}(P)$, being infinite, must be a nodal curve contracted by $\varphi_D$ (recall that $\varphi_D$ is 2 to 1 on $E$ and $E + K_S$) and this is absurd since for any $x \in V$ we have that $x \in T$ for some $T \in Z$ and we know that $T \cap R = \emptyset$ for every nodal curve $R$ contracted by $\varphi_D$. Therefore $l$ is contained in $\mathcal{S}$ and all $T \in Z$ lie in $\varphi_D^{-1}(l) \subset S$ and this is absurd since each $T$ is contained in some $C \in \mathcal{U}$ and we know that $\varphi_D^{-1}(l)$ can contain at most two points of any $C \in \mathcal{U}$.

Since $\dim(\pi(3_0)) \geq 3$ we have that there is a family of lines $l_T := \langle \varphi_D(T) \rangle$ of dimension at least 3 meeting $\mathcal{S}$ along $\varphi_D(T)$.

Now let $T \in \pi(3_0)$ be a general element. We cannot have that $\deg(\varphi_D(T)) \geq 4$, also $\varphi_D(T)$ is contained in $l_T \cup F_3$ for every cubic $F_3$ containing $\mathcal{S}$, that is $l_T$ is contained in $\mathcal{S} \cup \overline{\mathcal{D}}$, a contradiction since $\mathcal{S}$ contains finitely many lines and of course $\overline{\mathcal{D}} \cong \mathbb{P}^2$ contains a 2-dimensional family of lines.
There is no decomposition $D = A + B$ with $h^0(A) \geq 2$ and $h^0(B) \geq 2$.

**Proof.** Suppose such a decomposition exists. Then we get $A.D \geq 2\phi(D) = 4$ and similarly $B.D \geq 4$, whence $A.D = B.D = 4$, since $D^2 = 8$. Let $A \sim F_A + M_A$, $B \sim F_B + M_B$ be the decompositions into base-components and moving parts of $|A|$ and $|B|$. Then $h^0(M_A) \geq 2$ and $h^0(M_B) \geq 2$, whence, as above, $M_A.D = M_B.D = 4$.

Now by [12, Proposition 3.1.4] either $M_A \sim 2hE'$ for some genus one pencil $|2E'|$ or $M_A^2 > 0$. In both cases we can write $M_A \sim \sum_{i=1}^n F_i$ with $F_i > 0$, $F_i^2 = 0$ and $n \geq 2$, therefore $4 = M_A.D \geq n\phi(D) = 2n$. Hence $n = 2$ and $M_A \sim 2E$, since for any $F > 0$ with $F^2 = 0$ and $F.D = 2$ we must have $F \equiv E$. Similarly $M_B \sim 2E$ and
therefore $2E + E_1 = D \geq 4E$. But then $E_1 \geq 2E$ whence $h^0(E_1) \geq 2$, a contradiction by Lemma 5.3.

\begin{proof}
Claim 5.9. Let $D \sim \Delta + M$ for some $\Delta > 0$ and $M > 0$ with $M^2 \geq 6$. Then $M^2 = 6$, $\Delta^2 = -2$, $D.\Delta = 0$.

Proof. By Riemann–Roch we have that $h^0(M) \geq 4$, whence, by Claim 5.8, $h^0(\Delta) = 1$. Hence $\Delta^2 \leq 0$ by Riemann–Roch and $M^2 = (D-\Delta)^2 = 8 + \Delta^2 - 2D.\Delta \geq 6$, so that

$$2D.\Delta \leq 2 + \Delta^2.$$ 

If $\Delta^2 = 0$ we find the contradiction $2 \geq 2D.\Delta \geq 2\phi(D) = 4$. If $\Delta^2 \leq -2$, by the nefness of $D$, we find that $0 \geq 2D.\Delta \geq 0$, that is $M^2 = 6$, $\Delta^2 = -2$ and $D.\Delta = 0$. □

Now the reducible locus:

Claim 5.10. Let $W$ be an irreducible subvariety of $\{D' \in |D| : D' \text{ is reducible}\}$ such that $\dim W = 3$. Then there is a divisor $G_W > 0$ with $h^0(G_W) = 1$ and such that if $M \sim D - G_W$ then $|M|$ is base-component free and every curve $D' \in W$ is $D' = G_W + M'$ for some $M' \in |M|$. Moreover $M^2 = 6$, $G_W^2 = -2$ and $D.G_W = 0$.

Proof. Let $D'$ be an element of $W$. Since $D'$ is reducible we have that $D' = G + B$ with $G > 0$, $B > 0$ and, by Claim 5.8, we can assume that $h^0(G) = 1$. Since the divisor classes $G > 0$ such that $D - G > 0$ are finitely many, we see that $h^0(B) \geq 4$. Let $G'$ be the base component of $|B|$ and let $M$ be its moving part. Then also $h^0(M) \geq 4$ and $M^2 > 0$, for otherwise we have $M^2 = 0$ whence, by [12, Proposition 3.1.4], we get that $M \sim 2hE'$, with $|2E'|$ a genus one pencil and $h + 1 = h^0(M) \geq 4$, contradicting Claim 5.8, since then $D \sim 2E' + 2(h - 1)E' + G + G'$ and $h^0(2E') = 2$, $h^0(2(h - 1)E' + G + G') \geq 2$. Therefore $1 + \frac{M^2}{2} = h^0(M) \geq 4$, whence $M^2 \geq 6$ and of course $D \sim G + G' + M$ with $G + G' > 0$ and $h^0(G + G') = 1$ by Claim 5.8. By Claim 5.9 we have that $M^2 = 6$, $(G + G')^2 = -2$ and $D.(G + G') = 0$. Therefore $h^0(B) = h^0(M) = 4$.

Since the possible $G + G'$ are finitely many, we get that $\dim W = \dim |M| = 3$. Let $G_1, \ldots, G_n$ be the finite set of divisors $G > 0$ such that $D - G > 0$ and let $B_i = D - G_i$ for $i = 1, \ldots, n$. We have seen that for every $D' \in W$ there is an $i \in \{1, \ldots, n\}$ and a divisor $B' \in |B_i|$ so that $D' = G_i + B'$. Let $\phi_i : |B_i| \to |D|$ be the natural inclusion defined by $\phi_i(B) = B + G_i$. Then

$$W \subset \bigcup_{i=1}^{n} \text{Im } \phi_i$$

and since $\text{Im } \phi_i \cong |B_i|$ is a closed subset of $|D|$ and $W$ is irreducible, we deduce that there is some $G_W$ with $h^0(G_W) = 1$, $D' \sim G_W + M$ and every curve $D' \in W$ is $D' = G_W + M'$ for some $M' \in |M|$. Finally the remaining part follows by Claim 5.9. □
Conclusion of the proof of Proposition 5.6. Recall that \( \dim f_0(\mathcal{J}_0) = 3 \) and that a general element \( D_0 \in f_0(\mathcal{J}_0) \) is reducible. By Claim 5.10, there is a \( G > 0 \) with \( h^0(G) = 1 \) and such that if \( M \sim D - G \) then \( |M| \) is base-component free, \( M^2 = 6, \quad G^2 = -2, \quad D.G = 0 \) and every curve \( D' \in f_0(\mathcal{J}_0) \) is \( D' = G + M' \) for some \( M' \in |M| \). Moreover note that every irreducible component of \( G \) is a nodal curve contracted by \( \varphi_D \).

Therefore \( D_0 = \bigcup_{i=1}^n R_i \cup M_0 \) where the \( R_i \)'s are nodal curves contracted by \( \varphi_D \) and \( M_0 \) is general in \( |M| \). Now \( M_0 \) is a smooth irreducible curve by [12, Proposition 3.1.4 and Theorem 4.10.2] and \( \varphi_D(M_0) \) is a nondegenerate (since \( h^0(\sum_{i=1}^n R_i) = 1 \)) integral curve in \( \mathbb{P}^3 \). On the other hand we know that on \( D_0 \) there is a family of dimension at least \( 2 \) of divisors \( T \) such that \( (T, D_0) \in f_0^{-1}(D_0) \) and each \( T \) gets mapped to a line \( l_T \) by \( \varphi_D \). Since for each \( T \) we have that \( T \cap R_i = \emptyset \) for all \( i = 1, \ldots, n \), we deduce that all these \( T \)'s lie in \( M_0 \) and this gives a contradiction since then \( \varphi_D(M_0) \) would have a two dimensional family of lines \( l_T \) as above.

We have therefore proved that the general curve \( C \in |L| \) possesses no \( g^2_0 \).

To see that it satisfies \( b_2(C) \geq 1 \) suppose that there exists a line bundle \( A \) that is a \( g^1_4 \) on \( C \) and is such that \( b_2(A) = 0 \). By the general Remark 5.2 we know that \( C \) lies on a surface \( X \) (obtained by desingularizing \( Y_A \), if necessary) and either \( X = \Sigma_A, C \sim -2K_X \) or \( X = Bl_A C_0 \) and \( C \) is bielliptic. But this is clearly a contradiction since in both cases \( C \) carries \( g^2_0 \)'s.  

\[ \square \]

5.2 Curves of genus 8.

**Proposition 5.11.** Let \( L \) be a base-point free line bundle on an Enriques surface with \( L^2 = 14 \) and \( \phi(L) = 2 \). Then \( b_2(C) \geq 1 \) for a general curve \( C \in |L| \).

We will use the following

**Lemma 5.12.** Let \( L \) be a base-point free line bundle on an Enriques surface with \( L^2 = 14 \) and \( \phi(L) = 2 \). Let \( |2E| \) be a genus one pencil such that \( E.L = 2 \). Then there exists two primitive divisors \( E_1, E_2 \) such that \( E_i > 0, \quad E_i^2 = 0, \quad E.E_i = E_1.E_2 = 1 \) for \( i = 1, 2, \)

\[
L \sim 3E + E_1 + E_2
\]

and

(i) \( E + E_1 \) is nef;
(ii) either \( 2E + E_2 \) is nef or there exists a nodal curve \( \Gamma \) such that \( E_2 = E_1 + \Gamma, \quad E.\Gamma = 0, \quad E_1.\Gamma = 1, \quad E_2.\Gamma = -1 \). In particular \( 2E + E_2 \) is quasi-nef.

Moreover let \( C \in |L| \) be a general curve. Then

(iii) either \( (E + E_1)|_C \) or \( (E + E_1 + K_S)|_C \) is a complete base-point free \( g^1_4 \) on \( C \);
(iv) \( (2E + E_2)|_C \) and \( (2E + E_2 + K_S)|_C \) are complete base-point free \( g^2_0 \)'s on \( C \).

**Proof.** Using Lemma 4.4 and Lemma 4.5 we can write \( L \sim 3E + E_1 + E_2 \) with \( E_i > 0 \) primitive, \( E_i^2 = 0 \) and \( E.E_i = E_1.E_2 = 1, \quad i = 1, 2 \). We now claim that we can assume that \( E + E_1 \) is nef.

Suppose that there is a nodal curve \( \Gamma \) such that \( \Gamma.(E + E_1) < 0 \). Then \( E_1.\Gamma \leq -1 - E.\Gamma \leq -1 \) and \( k := -E_1.\Gamma \geq 1 + E.\Gamma \geq 1 \). By Lemma 4.3, we can write
$E_1 \sim A + k\Gamma$ with $A > 0$ primitive with $A^2 = 0$. If $E.\Gamma > 0$ we have that $k \geq 2$ giving the contradiction $1 = E.E_1 = E.A + kE.\Gamma \geq 2$. Therefore $E.\Gamma = 0$ and the nefness of $L$ implies that $E_2.\Gamma > 0$. From $1 = E_2.E_1 = E_2.A + kE_2.\Gamma \geq 1$ we deduce that $k = 1$ and $E_2.A = 0$ whence $E_2 \equiv A$ by Lemma 4.5 and therefore $E_1 \equiv E_2 + \Gamma$. Now if in addition we have that also $E + E_2$ is not nef then the same argument above shows that there is a nodal curve $\Gamma'$ such that $E_2 \equiv E_1 + \Gamma'$, giving the contradiction $\Gamma + \Gamma' \equiv 0$. Therefore either $E + E_1$ or $E + E_2$ is nef and (i) is proved.

Now let $\Delta > 0$ be such that $\Delta^2 = -2$, $\Delta.(2E + E_2) < 0$. Then $E_2.\Delta \leq -1 - 2E.\Delta \leq -1$ and $k := -E_2.\Delta \geq 1 + 2E.\Delta \geq 1$. By Lemma 4.3, we can write $E_2 \sim A + k\Delta$ with $A > 0$ primitive with $A^2 = 0$. If $E.\Delta > 0$ we have that $k \geq 3$ giving the contradiction $1 = E.E_2 = E.A + kE.\Delta \geq 3$. Therefore $E.\Delta = 0$ and the nefness of $L$ implies that $E_1.\Delta > 0$. From $1 = E_1.E_2 = E_1.A + kE_1.\Delta \geq 1$ we deduce that $k = 1$ and $E_1.A = 0$, whence $E_1 \equiv A$ by Lemma 4.5 and therefore $E_2 \equiv E_1 + \Delta$. Hence $2E + E_2$ is quasi-nef and if it is not nef then we can choose $\Delta$ to be a nodal curve. This proves (ii).

To see (iii) note that since $(E + E_1)^2 = 2$ and $E + E_1$ is nef by (i), we have by [12, Proposition 3.1.6 and Corollary 3.1.4] that either $E + E_1$ or $E + E_1 + KS$ is base-component free with two base points. Let $B \equiv E + E_1$ be the line bundle that is base-component free. As $C$ is general in $|L|$ we have that $B(C)$ is base-point free. Now the exact sequence

$$0 \to B - C \to B \to B(C) \to 0$$

shows that also $B(C)$ is a complete $g^1_6$ since $B - C = -2E - E_2$ whence $h^1(B - C) = 0$ by Theorem 4.7 because $2E + E_2$ is quasi-nef.

To see (iv) note that if $2E + E_2$ is nef then it is base-component free with two base points by [12, Proposition 3.1.6, Proposition 3.1.4 and Theorem 4.4.1] whence $(2E + E_2)(C)$ is base-point free, as $C$ is general. The same argument shows that $(2E + E_1)(C)$ and $(2E + E_1 + KS)(C)$ are base-point free by (i). Now if $2E + E_2$ is not nef then $2E + E_2 \equiv 2E + E_1 + \Gamma$ by (ii) whence again $(2E + E_2)(C)$ is base-point free, since $\Gamma.C = 0$. Now the exact sequence

$$0 \to -E - E_1 \to 2E + E_2 \to (2E + E_2)(C) \to 0$$

shows that also $(2E + E_2)(C)$ is a complete $g^2_6$ since $h^1(-E - E_1) = 0$ because $E + E_1$ is nef by (i). Similarly we can show the same for $(2E + E_2 + KS)(C)$. \hfill \Box

Before proving Proposition 5.11 we use the above lemma to deal with the case of $\Sigma_2$. This is used also in the proof of Proposition 4.17 in [22].

**Lemma 5.13.** Let $L$ be a base-point free line bundle on an Enriques surface with $L^2 = 14$ and $\phi(L) = 2$. Then the general curve $C \in |L|$ cannot be isomorphic to a curve linearly equivalent to $-2K_X$ on $X = \Sigma_2$.

**Proof.** By Lemma 5.12(iii) there is a line bundle $B$ such that $B \equiv E + E_1$ and $B(C)$ is a base-point free complete $g^1_6$ on $C$. By Proposition 3.4(c) there are two points $P_1, P_2 \in C$ such that $B(C) \sim (2H - G_1 - G_2)(C) - P_1 - P_2$. 

Now \[ K_C \sim (3H - G_1 - G_2)|_C \sim B|_C + P_1 + P_2 + \tilde{H}|_C \]
whence
\[ (L + K_S - B)|_C - P_1 - P_2 \sim \tilde{H}|_C \]
and this contradicts (28).

**Proof of Proposition 5.11.** Suppose that there exists a line bundle \( E \) that is a \( g^1_1 \) on \( C \) and is such that \( b_{2,A} = 0 \). By the general Remark 5.2 we know that \( C \) lies on a surface \( X \) obtained by desingularizing \( Y_A \) if necessary and either \( X = \Sigma_2 \), \( C \sim -2K_X \) or \( X = Bl V C_7 \) and \( C \) is bielliptic. The latter case is excluded since, by [22, Proposition 4.17], \( C \) has a unique \( g^1_1 \) while the first case was excluded in Lemma 5.13.

### 5.3 Curves of genus 9.

**Proposition 5.14.** Let \( L \) be a base-point free line bundle on an Enriques surface with \( L^2 = 16 \) and \( \varphi(L) = 2 \). Then \( b_2(C) \geq 1 \) for a general curve \( C \in |L| \).

We will use the following

**Lemma 5.15.** Let \( L \) be a base-point free line bundle on an Enriques surface with \( L^2 = 16 \) and \( \varphi(L) = 2 \). Let \( |2E| \) be a genus one pencil such that \( E.L = 2 \). Then there exists a divisor \( E_1 \) such that \( E_1 > 0, E_1^2 = 0 \) and \( E.E_1 = 2 \) and \( L \sim 4E + E_1 \).

Moreover if \( H^1(E_1 + K_S) \neq 0 \) there exists a divisor \( E_2 \) such that \( E_2 > 0, E_2^2 = 0 \) and \( E.E_2 = 2 \).

**Proof.** Since \( (L - 4E)^2 = 0 \) and \( E.(L - 4E) = 2 \), by Lemma 4.4 we can write \( L \sim 4E + E_1 \) with \( E_1 > 0, E_1^2 = 0 \) and \( E.E_1 = 2 \).

By Theorem 4.7 if \( H^1(E_1 + K_S) \neq 0 \) then either \( E_1 \equiv nE' \) for \( n \geq 2 \) and some genus one pencil \( |2E'| \) or \( E_1 \) is not quasi-nef. In the first case we have \( 2 = nE.E' \) whence \( n = 2, E.E' = 1 \) and we set \( E_2 = E' \). Also \( E + E_2 \) is nef in this case.

If \( E_1 \) is not quasi-nef there exists a \( \Delta > 0 \) such that \( \Delta^2 = -2, \Delta.E_1 \leq -2 \). By Lemma 4.3, we can write \( E_1 \sim A + k\Delta \) with \( A > 0, A^2 = 0, A.\Delta = k = -E_1.\Delta \geq 2 \). The nefness of \( L \) implies that \( E.\Delta > 0 \), whence from \( 2 = E.E_1 = E.A + kE.\Delta \geq 2 \) we deduce that \( k = 2, E.\Delta = 1 \) and \( E.A = 0 \). Hence \( A \equiv qE \) for some \( q \geq 1 \) by Lemma 4.5. Now \( 2 = A.\Delta = q \) and therefore \( E_1 \equiv 2E + 2\Delta \). We now set \( E_2 = E + \Delta \). Let us prove that \( E + E_2 = 2E + \Delta \) is nef. Let \( \Gamma \) be a nodal curve such that \( (2E + \Delta).\Gamma < 0 \). Since now \( L \equiv 6E + 2\Delta \) the nefness of \( L \) implies that \( E.\Gamma > 0 \). Now \( (2E + \Delta)^2 = 2 \) and \( (E + \Gamma)^2 \geq 0 \) whence \( (E + \Gamma).(2E + \Delta) \geq 1 \). But this is a contradiction since \( (E + \Gamma).(2E + \Delta) = 1 + \Gamma.(2E + \Delta) \leq 0 \).

Now that \( E + E_2 \) is nef we just observe that by [12, Proposition 3.1.6 and Corollary 3.1.4] either \( E + E_2 \) or \( E + E_2 + K_S \) is base-component free, whence to conclude we choose accordingly \( E_2 = E_2' \) or \( E_2 = E_2' + K_S \).
Proof of Proposition 5.14. We use the notation of Lemma 5.15. Suppose that there exists a line bundle $A$ that is a $g^1_4$ on $C$ and is such that $b_{2A} = 0$. By the general Remark 5.2 we know that $C$ lies on a surface $X$ (obtained by desingularizing $Y_A$, if necessary) and either $X = \Sigma_1$, $Bl_V Q$ and $C \sim -2K_X$ or $X = Bl_V C_8$ and $C$ is bielliptic. When $X = \Sigma_1$ or $Bl_V C_8$ we get that $C$ has a complete base-point free $g^2_2$ and this is excluded by [23, Proposition 3.5]. The bielliptic case can also be excluded in another way, since, by [22, Proposition 4.17], $C$ has a unique $g^1_4$. Therefore $C \sim -2K_X$ on $X = Bl_V Q$. By Proposition 3.4(h) we have that $C$ has a unique $g^1_4$, namely $f_C$. Hence $(2E)_C \sim f_C$ and we deduce that $h^0((4E)_C) = h^0(2f_C) = 4$. Now the exact sequence

$$0 \longrightarrow -E_1 \longrightarrow 4E \longrightarrow (4E)_C \longrightarrow 0$$

shows that $H^1(E_1 + K_S) \neq 0$, since $h^0(4E) = 3$. Therefore there exists a divisor $E_2$ as in Lemma 5.15.

Let us prove that $(2E + E_2)_C$ is a complete base-point free $g^2_2$ on $C$. To this end note that since $(2E + E_2)^2 = 4$ and $2E + E_2$ is base-component free with two base points by Lemma 5.15 and [12, Proposition 3.1.6, Proposition 3.1.4 and Theorem 4.4.1], we have that $(2E + E_2)_C$ is base-point free. Now the exact sequence

$$0 \longrightarrow 2E + E_2 - C \longrightarrow 2E + E_2 \longrightarrow (2E + E_2)_C \longrightarrow 0$$

shows that also $(2E + E_2)_C$ is a complete $g^2_2$ since $2E + E_2 - C \equiv -2E - E_2$ whence $h^1(2E + E_2 - C) = 0$ because $2E + E_2$ is nef.

Let $Z = E_2 \cap C$. Then $Z \subset C$ is an effective divisor such that $f_C + Z \sim (2E + E_2)_C$ is a complete base-point free $g^2_2$ on $C$, contradicting Proposition 3.4(j).

We can now complete the proof of Theorem 5.1.

Proof of Theorem 5.1. By Lemma 5.3, Propositions 5.5, 5.6, 5.11 and 5.14 we can assume that $L^2 \geq 18$. Let $C$ be a curve as in the theorem, let $g = \frac{L^2}{2} + 1 \geq 10$ be the genus of $C$ and suppose that $b_2(C) = 0$. By the general Remark 5.2 either $C$ is bielliptic or $g = 10$ and $C$ is isomorphic to a smooth plane sextic. Now by [22, Proposition 4.17] we have that $C$ has a unique $g^1_4$, therefore it cannot be bielliptic. On the other hand the case of $C$ isomorphic to a smooth plane sextic is excluded in [23, Proposition 3.1]. Therefore we have a contradiction in all cases and the theorem is proved.

6 Proof of the main theorem

We proceed with our main result.

Proof. Let $C$ be a curve as in the theorem and let $g = \frac{L^2}{2} + 1 \geq 3$ be its genus. Under the hypotheses (i) and (ii) the theorem follows immediately from Proposition 2.4, while if hypothesis (v) holds the theorem follows immediately from Corollary 2.13.

Now suppose we are under hypothesis (iii). By Theorem 4.8 and [22, Proposition 4.15] we have that $C$ is neither trigonal nor isomorphic to a smooth plane quintic, that is $\text{Cliff}(C) \geq 2$. Then the theorem follows by Proposition 2.11.
Finally suppose that hypothesis (iv) holds. Since $L^2 \geq 12$, by [15, Theorem 1.4] (or by Theorem 4.8) we get that $\text{Cliff}(C) \geq 2$. If $\text{Cliff}(C) \geq 3$ then (iv) follows by Proposition 2.11(ii). If $\text{Cliff}(C) = 2$ then, as is well-known, $C$ is either tetragonal or isomorphic to a smooth plane sextic. But the latter case was excluded in [23, Proposition 3.1]. Therefore $C$ is tetragonal and $\phi(L) = 2$ by Theorem 4.8. By Theorem 5.1 we have that $b_2(C) \geq 1$. Since $h^0(2K_C - M) = 1$ it follows that $h^0(2K_C - M - b_2A) = 0$ for every line bundle $A$ that is a $g^1_2$ on $C$. Therefore the theorem is a consequence of Proposition 2.18(i).

References


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