Streamline calculations. Lecture note 1

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1 Application of streamlines

- Visualization and analyzation of vector fields (flow fields). Used in e.g.
  - Fluid dynamics
  - Aerodynamics
  - Magnetostatics (e.g. medical imaging)
  - Electrostatics
- Flow through porous media
  - Reservoir simulation (multiphase flow in porous media)
    - Visualization
    - Reformulation/simplification of the flow equations
    - History matching
    - Upscaling
  - Ground water flow
    - Contaminant transport (particle tracking)
    - Flow nets
- Flow based gridding

2 Streamlines and path lines

- General definition of a streamline $\boldsymbol{s}(\tau) = \boldsymbol{x}(\tau)$ says that the tangent to the streamline should be equal to the velocity at a given instant in time:

$$\frac{d\boldsymbol{x}(\tau)}{d\tau} = \boldsymbol{v}(\boldsymbol{x}, t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{1}$$

- Here, τ is a parameter that follows (parameterizes) the streamline, whereas t is the real (physical) time.

- Since the streamlines are independent of time, the streamlines describe the (direction of the) flow field at a given *instant* in time.
- The parameter τ measures the pseudo-time (measured in τ) needed for a particle to travel a given distance along the streamline at a given instant of the physical time t.
- Note: In reservoir simulation, both τ and the shape of the streamline are important, wheras in other applications, such as visualization, only the shape of the streamlines may be important.
- General definition of a path line $\boldsymbol{p}(t) = \boldsymbol{x}(t)$ says that the tangent to the pathline, at a given time t and position $\boldsymbol{x}$, should be equal to the velocity at the same time and space position:

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{v}(\boldsymbol{x}, t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{2}$$

- A pathline describes the geometry of the trajectory of a particle exposed to the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$.
- The parameter t of the path line gives the (physical) time needed for a particle to travel a given distance along its path line.
- Note: We use notation $\boldsymbol{x}(\cdot)$ for both path lines and streamlines, which makes it difficult to distinguish between a path line and a streamline. We have therefore introduced the additional notation $\boldsymbol{s}(\tau)$ for a streamline, and $\boldsymbol{p}(t)$ for a path line. Alternatively, Equations (1) and (2) could have been written

$$\frac{d\boldsymbol{s}(\tau)}{d\tau} = \boldsymbol{v}(\boldsymbol{s}), \qquad \boldsymbol{s}(0) = \boldsymbol{s}_0$$
$$\frac{d\boldsymbol{p}(t)}{dt} = \boldsymbol{v}(\boldsymbol{p}, t), \qquad \boldsymbol{p}(0) = \boldsymbol{p}_0.$$

  Since this notation did not seem to be less unambiguous, it was discarded.

Example

Compute the path lines $\boldsymbol{p}(t)$, and then the streamlines $\boldsymbol{s}(\tau)$, at $t = 1$ for

$$\boldsymbol{v} = \begin{bmatrix} t \\ 1 \end{bmatrix}, \tag{3}$$

such that $\boldsymbol{s}(0) = \boldsymbol{p}(0) = [0, 0]$. The solution is easily obtained by direct integration (or inspection) of the component equations of (1)

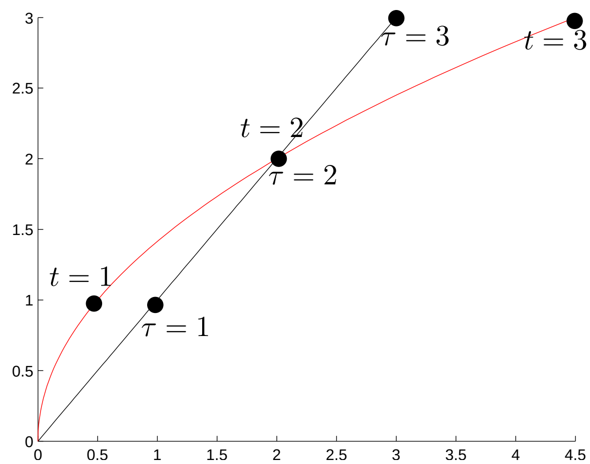

Figure 1: A path line in red, and a streamline in blue. Values of the parameters t and τ are indicated at some points.

and (2):

$$\boldsymbol{s}(\tau) = \begin{bmatrix} \tau \\ \tau \end{bmatrix} \qquad \wedge \qquad \boldsymbol{p}(t) = \begin{bmatrix} \frac{t^2}{2} \\ t \end{bmatrix}. \tag{4}$$

See Figure 1

- If $\boldsymbol{v}$ is independent of t, streamlines and path lines coincide.
- We can think of a streamline as a path line for a steady flow field.
- Alternative definitions of streamline:
  - Cross product

$$\boldsymbol{v} \times d\boldsymbol{s} = 0 \tag{5}$$

  - The system of equations (1) or (5) can be written in component form as

$$\frac{dx}{d\tau} = v_x, \quad x(0) = x_0, \tag{6}$$

$$\frac{dy}{d\tau} = v_y, \quad y(0) = y_0, \tag{7}$$

$$\frac{dz}{d\tau} = v_z, \quad z(0) = z_0. \tag{8}$$

  - Elimination of time parameter. In 2D:

$$\frac{dx}{v_x} = \frac{dy}{v_y} \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{v_y}{v_x} \tag{9}$$

    In 3D:

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z} \tag{10}$$

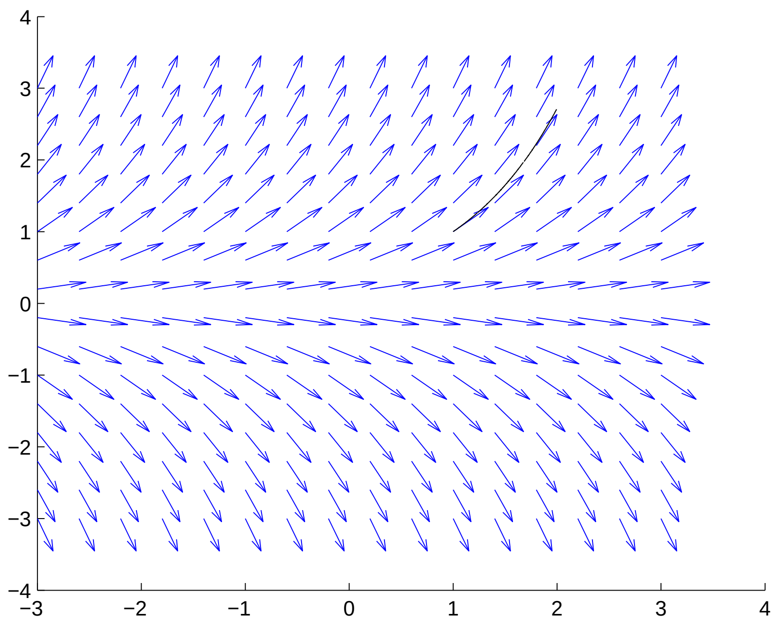

Figure 2: Field plot for $\boldsymbol{v} = [1, y]$. A streamline $\boldsymbol{s}(\tau)$ is started from $\boldsymbol{s}(0) = \boldsymbol{s}_0 = [1, 1]$ and traced to the point $\boldsymbol{s}(1)$.

2.1 Integration

- We start by analytical methods.
- Simplest case: Equations are separable: Example: $\boldsymbol{v} = [1, y]$, $\boldsymbol{x}_0 = [1, 1]$. Then $\boldsymbol{s}(\tau) = [\tau + 1, e^{\tau}]$, or $y = e^{x-1}$. See Figure 2.
- Divergence of a vector field:

$$\nabla \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}. \tag{11}$$

- Curl of a vector field:

$$\nabla \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \boldsymbol{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \boldsymbol{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \boldsymbol{k}$$

- In 2D, if $\boldsymbol{v} = [v_x(x, y), v_y(x, y), 0]$, then

$$\nabla \times \boldsymbol{v} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \boldsymbol{k} \tag{12}$$

2.1.1 Potential flow

- Suppose we have both divergence free and irrotational flow in 2D

$$\nabla \cdot \boldsymbol{v} = 0 \qquad \wedge \qquad \nabla \times \boldsymbol{v} = 0 \tag{13}$$

- Define an *analytic* function $F(z)$ (complex potential) as

$$F(z) = \phi(x,y) + i\psi(x,y) \tag{14}$$

From the Cauchy-Riemann equations, we know that

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{15}$$

Define $\boldsymbol{v} = \nabla\phi$, or

$$v_x = \frac{\partial \phi}{\partial x} \qquad v_y = \frac{\partial \phi}{\partial y}, \tag{16}$$

then $\phi(x,y)$ is harmonic, and

$$\nabla \cdot \boldsymbol{v} = \nabla^2\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial x}\frac{\partial \psi}{\partial y} + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0, \tag{17}$$

and

$$|\nabla \times \boldsymbol{v}| = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} = 0. \tag{18}$$

- We also have,

$$\nabla\phi \cdot \nabla\psi = \frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y} = 0. \tag{19}$$

which implies that level curves of ϕ and ψ are orthogonal.

- The function ϕ is called a potential function, and ψ is called the stream function.

- The function ψ must be the harmonic conjugate of ϕ.

- Since the velocity $\boldsymbol{v}$ is orthogonal to the level surfaces of $\phi(x,y)$ ($\boldsymbol{v} = \nabla\phi$), the function ψ must describe the streamlines.

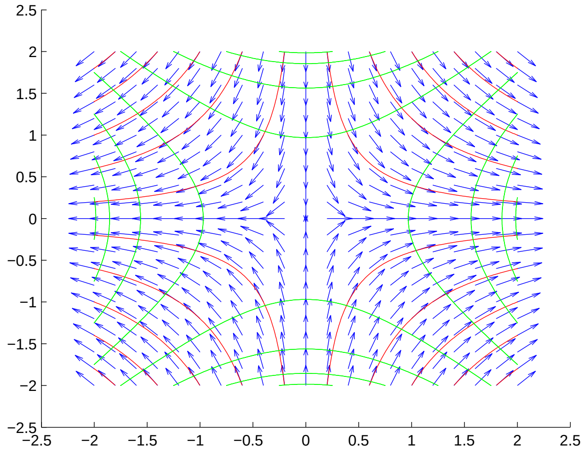

Figure 3: Field lines (blue arrows) for $\boldsymbol{v} = [-2x, 2y]$, streamlines (red curves), and equipotential curves (green lines).

Example

Let

$$\boldsymbol{v} = \begin{bmatrix} 2x \\ -2y \end{bmatrix} \tag{20}$$

We have

$$\frac{\partial \phi}{\partial x} = 2x \qquad \Rightarrow \qquad \phi(x, y) = \int 2x dx = x^2 + k(y), \tag{21}$$

and

$$\frac{\partial \phi}{\partial y} = h'(y) = -2y \quad \Rightarrow \quad h(y) = -y^2 \quad \Rightarrow \quad \phi(x, y) = x^2 - y^2 \tag{22}$$

The function ψ is the harmonic conjugate of ϕ. It is easy to see that

$$\psi(x, y) = 2xy \tag{23}$$

2.1.2 Stream function in 2D

- Assume only $\nabla \cdot \boldsymbol{v} = 0$. Then

$$\boldsymbol{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \frac{\partial \psi}{\partial y} \\ -\frac{\partial \psi}{\partial x} \end{bmatrix} \tag{24}$$

where $\psi(x, y)$ is called the stream function.

- A streamline can be defined by rewriting Equations (6) and (7) as

$$v_y - v_x \frac{dy}{dx} = 0. \tag{25}$$

We then have

$$\frac{d}{dx}\psi(x, y(x)) = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = -v_y + v_x \frac{dy}{dx} = 0. \tag{26}$$

Thus,

$$\psi(x, y) = const \tag{27}$$

represents a streamline.

- Determining the streamfunction: If $\nabla \cdot \boldsymbol{v} = 0$, the streamfunction can be determined from Equation (24) by integration:

$$\frac{\partial \psi}{\partial x} = -v_y \qquad \Rightarrow \qquad \psi(x, y) = -\int v_y dx + h(y)$$

Which implies,

$$v_x = \frac{\partial \psi}{\partial y} = -\int \frac{\partial v_y}{\partial y} dx + h'(y)$$

$$\Rightarrow \qquad h(y) = \int v_x dy + \int \left(\int \frac{\partial v_y}{\partial y} dx \right) dy \tag{28}$$

$$\Rightarrow \qquad \psi(x, y) = \int v_x dy - \int v_y dx + \int \left(\int \frac{dv_y}{dy} dx \right) dy \tag{29}$$

- Note: it might seem that the last two terms on the right hand side will cancel, since by changing the order of integration, we have

$$\int \left(\int \frac{dv_y}{dy} dx \right) dy = \int \left(\int \frac{dv_y}{dy} dy \right) dx = \int v_y dx, \tag{30}$$

but in fact, they will give quite different information about the unknown integration constant.

Example

Let

$$\boldsymbol{v} = \begin{bmatrix} Ax + B \\ -Ay + C \end{bmatrix} \qquad \Rightarrow \qquad \nabla \cdot \boldsymbol{v} = 0. \tag{31}$$

Find the streamfunction passing through $\boldsymbol{x} = (1, 1)$. We have

$$\psi(x, y) = \int v_x dy - \int v_y dx + \int \left(\int \frac{dv_y}{dy} dx \right) dy =$$
$$(Ax + B)y - (-Ay + C)x - Axy = Axy + By - Cx \tag{32}$$

The streamline is given by

$$\psi(x,y) = const \tag{33}$$

$$\psi(1,1) = A + B - C \qquad \Rightarrow \qquad const = A + B - C \tag{34}$$

2.1.3 Stream functions in 3D

- We will solve

$$\frac{dx}{v_x} = \frac{dy}{v_y} \qquad \wedge \qquad \frac{dy}{v_y} = \frac{dz}{v_z} \qquad \wedge \qquad \frac{dx}{v_x} = \frac{dz}{v_z} \tag{35}$$

- The solution to these equation should be a streamline. It can be shown, that if $\nabla \cdot \boldsymbol{v} = 0$, this can be expressed as the intersection of two independent surfaces,

$$f(x,y,z) = \xi \qquad \wedge \qquad g(x,y,z) = \eta, \tag{36}$$

where ξ and η are constants, such that (since the streamlines are embedded in level curves of both f and g)

$$\boldsymbol{v} = \nabla f \times \nabla g \tag{37}$$

- I have not found (so far) any general way to determine the functions f and g.

- For cases below we can determine them:

Example 1

Let $\boldsymbol{v} = [z, z, -(x+y)]$. Then

$$\frac{dx}{v_x} = \frac{dy}{v_y} \qquad \Rightarrow \qquad \frac{dx}{z} = \frac{dy}{z} \qquad \Rightarrow \qquad y = x + C_1 \tag{38}$$

Choose the constant $C_1 = 0$. Then

$$\frac{dx}{v_x} = \frac{dz}{v_z} \qquad \Rightarrow \qquad \frac{dx}{z} = \frac{dz}{-2x} \qquad \Rightarrow \qquad -2xdx = zdz$$

$$\Rightarrow \qquad x^2 + \frac{z^2}{2} = C_2 \qquad \Rightarrow \qquad \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_2 \tag{39}$$

Then let

$$f(x,y,z) = x - y \qquad g(x,y,z) = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} \tag{40}$$

And we can check that $\boldsymbol{v} = \nabla f \times \nabla g$.

Example 2

Let $\boldsymbol{v} = [x, 1, 1]$. Then

$$\frac{dx}{v_x} = \frac{dy}{v_y} \qquad \Rightarrow \qquad \frac{dx}{x} = \frac{dy}{1} \qquad \Rightarrow \qquad x = C_3 e^y \tag{41}$$

Choose $C_3 = 1$. Then

$$\frac{dy}{v_y} = \frac{dz}{v_z} \qquad \Rightarrow \qquad \frac{dy}{1} = \frac{dz}{1} \qquad \Rightarrow \qquad y = z + C_4$$

Choose $C_4 = 0$. Then let

$$f(x, y, z) = x - e^y \qquad\qquad g(x, y, z) = y - z \tag{42}$$

But now

$$\nabla f \times \nabla g = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -e^y & 0 \\ 0 & 1 & -1 \end{vmatrix} = [e^y, 1, 1] \neq \boldsymbol{v} \tag{43}$$

References