DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **26**, Number **4**, April **2010**

doi:10.3934/dcds.2010.26.1121

pp. **1121–1139**

A KDV-TYPE BOUSSINESQ SYSTEM: FROM THE ENERGY LEVEL TO ANALYTIC SPACES

Jerry L. Bona

Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago
Chicago, IL 60607, USA

Zoran Grujić

Department of Mathematics
University of Virginia
Charlottesville, VA 22904, USA

Henrik Kalisch

Department of Mathematics
University of Bergen
5008 Bergen, Norway

Abstract. Considered here is the well-posedness of a KdV-type Boussinesq system modeling two-way propagation of small-amplitude long waves on the surface of an ideal fluid when the motion is sensibly two dimensional. Solutions are obtained in a range of Sobolev-type spaces, from the energy level to the analytic Gevrey spaces. In addition, a criterion for detecting the possibility of blow-up in finite time in terms of loss of analyticity is derived.

1. **Introduction.** Consideration is given to the following coupled systems of two nonlinear dispersive wave equations in one space dimension, namely,

$$\begin{aligned} w_t + \eta_x + w w_x + \eta_{xxx} &= 0, \\ \eta_t + w_x + (w\eta)_x + w_{xxx} &= 0, \end{aligned} \tag{1}$$

and its symmetric version (see [4]),

$$\begin{aligned} w_t + \eta_x + \tfrac{3}{2}\, w w_x + \tfrac{1}{2}\, \eta\eta_x + \eta_{xxx} &= 0, \\ \eta_t + w_x + \tfrac{1}{2}\, (w\eta)_x + w_{xxx} &= 0. \end{aligned} \tag{2}$$

These systems have been indicated as models for long waves of small amplitude at the surface of an ideal fluid in a long rectangular channel with a flat bottom. They are known to be a valid approximation of the full, two-dimensional Euler equations for fluid motion under the influence of gravity in suitably small amplitude, long wavelength regimes (see [1, 2, 4]). The dependent variable η is proportional to the deflection of the free surface from its rest position, whilst w represents the horizontal

2000 *Mathematics Subject Classification.* Primary: 35Q35, 35Q53, 35A07; Secondary: 76B07, 76B15.

Key words and phrases. Boussinesq systems, two-way propagation of water waves, well-posedness for nonlinear dispersive equations, analyticity, Gevrey-space analysis.

This research was partially supported by the US National Science Foundation and the Research Council of Norway. The authors very much appreciated the commentary of two referees whose suggestions considerably improved the exposition.

velocity of the fluid measured at a height of $\sqrt{\frac{2}{3}}h$, where h is the undisturbed depth of the fluid. The system (1) is a special case of a three-parameter family of long-wave systems derived in [1]. In [2], the linear and nonlinear well posedness of these systems was investigated. In general, it was found that whenever a system is linearly well posed, the same result holds for the nonlinear system. The well posedeness of (2) was studied in [4]. In these works, a local well-posedness theory was developed in the L^2-based Sobolev class H^s for $s > \frac{3}{4}$. Detailed numerical simulations of solutions to these systems may be found in [5] and [6].

As is apparent, these systems have a structure related to the Korteweg-de Vries equation (KdV-equation)

$$\eta_t + \eta_x + \eta\eta_x + \eta_{xxx} = 0. \tag{3}$$

This equation has been the subject of a great number of works over the last four decades. In particular, the last two decades have seen the development of a very satisfactory theory of existence and uniqueness for the initial-value problem for (3). The initial-value problem for (3) is known to be globally well posed in all Sobolev classes with nonnegative index [14, 30]. Indeed, there exist unique global solutions in the L^2-based Sobolev classes H^s down to $s \geq -\frac{3}{4}$ [16, 17, 15, 10]. It can also be shown that the initial-value problem is not well posed in a strong sense if $s < -\frac{3}{4}$. The property that fails for very low values of s is that the flow map from initial data to solution is not of class C^2 [15].

Considering the mature state of the theory for (3), it seems natural to develop corresponding theory for the systems (1) and (2). Here, we focus on existence of solutions in Sobolev spaces of nonnegative index. In particular, this includes local-in-time existence in the space of square-integrable functions on the real line, which is the energy space for the systems in view, as well as global-in-time existence for the symmetric version (2). Additionally, it will be proved that solutions are analytic in the spatial variable whenever the initial data are analytic in an appropriate sense and, for some $s > \frac{3}{2}$, the solutions remain bounded in the H^s-norm. Conversely, if a solution corresponding to analytic initial data loses analyticity in finite time, this will preclude the solution from being global in any Sobolev space H^s, $s > \frac{3}{2}$. In other words, loss of analyticity is a faithful indicator of a possible finite-time blow-up. The radius of analyticity is not only useful in the study of possible singularity formation, but also for proving exponential convergence of numerical approximations obtained via spectral methods. A study pertaining to the spectral projection method for approximating solutions of the KdV equation (3) has been conducted in [22, 23]. As already mentioned, numerical simulations of the systems appearing in (1) and (2) have appeared recently in [5] and [6].

Related theory for similar types of systems include the recent developments of Bona, Cohen and Wang [3] that develops local and global theory set in Sobolev classes H^s where s is only restricted by $s > -\frac{3}{4}$, as in the KdV-case. See also the papers [28] and [29] where related systems are considered. Issues of analyticity are not addressed in these works, and the range of systems considered in [3] does not include the present, physically relevant systems.

It is worth remarking that explicit examples in the KdV context may be constructed of complex-valued solutions that begin life analytic in a strip and, when restricted to the real axis, lose smoothness in finite time (see [11], [12] and [13] and the references therein).

To decouple the linear part of these systems, define new dependent variables

$$\begin{aligned} U &= \tfrac{1}{4}(w+\eta), \\ V &= \tfrac{1}{4}(w-\eta), \end{aligned}$$

so that the resulting systems for U and V are

$$\begin{aligned} U_t + U_x + U_{xxx} + 3UU_x + (UV)_x - VV_x = 0, \\ V_t - V_x - V_{xxx} - UU_x + (UV)_x + 3VV_x = 0 \end{aligned} \tag{4}$$

and

$$\begin{aligned} U_t + U_x + U_{xxx} + 3UU_x + (UV)_x + VV_x = 0, \\ V_t - V_x - V_{xxx} + UU_x + (UV)_x + 3VV_x = 0. \end{aligned} \tag{5}$$

Since w and η can be easily recovered from U and V, our development of a well posedness theory will be focused on (4) and (5), with initial data U_0 and V_0. The core of the argument used to show well posedness will be to obtain a contraction mapping in a conormal weighted space-time Sobolev space of Bourgain type. Banach spaces of this kind have been used extensively by many authors as a means to overcome the derivative loss inherent in dispersive equations with derivative nonlinearity. However, one problematic issue is that U and V are members of different Banach spaces, making the proof of the several required nonlinear estimates somewhat delicate. These difficulties are nevertheless overcome by carefully subdividing the domain of integration in Fourier space, as will be shown in Section 3.

As an overall guide to the rest of the paper, the main theorems to be proved are now stated. For $\sigma \geq 0$ and $s \geq 0$, define $G_{\sigma,s}$ to be the subspace of $L^2 = L^2(\mathbb{R})$ for which

$$\|u_0\|^2_{G_{\sigma,s}} = \int_{-\infty}^{\infty} (1+|\xi|)^{2s} e^{2\sigma(1+|\xi|)} |\widehat{u_0}(\xi)|^2 \, d\xi$$

is finite. Note that for $\sigma = 0$, the definition yields the usual Sobolev spaces, denoted by H^s. With the definition of $G_{\sigma,s}$ in hand, we are ready to state the theorems.

Theorem 1.1. *Let $\sigma \geq 0$ and $s \geq 0$. For initial data $U_0, V_0 \in G_{\sigma,s}$, there exists a positive time t_0 and a solution $U, V \in C([-t_0, t_0]; G_{\sigma,s})$ of the initial-value problems associated to (4) and (5).*

It will appear from the proof of Theorem 1.1 that the solution depends continuously on the initial data, but uniqueness holds only in a smaller class of functions. As mentioned before, the special case $\sigma = 0$ yields existence of a solution for initial data in H^s, and in particular when $s = 0$, we obtain existence in the energy class L^2. As the symmetric version (2) formally conserves the L^2-norm, it transpires that local existence of a solution immediately yields global-in-time existence.

The second main theorem to be proved states that the radius of analyticity σ when regarded as a function of time cannot decrease faster than algebraically.

Theorem 1.2. *Let $\sigma_0 > 0$ and $s > \frac{1}{2}$, and let $T \geq t_0$. Let $U_0, V_0 \in G_{\sigma_0,s}$, and suppose that the associated solution pair $U, V \in C([-4T, 4T];\, H^{s+1})$. Then*

$$U, V \in C([-T, T];\, G_{\sigma(T)/2,s}),$$

where $\sigma(T)$ is given by

$$\sigma(T) = \min\left\{\sigma_0, K(1+T)^{-12}\right\},$$

for some constant K.

This shows in particular that if the initial data are analytic in some strip about the real axis and if for some $s > \frac{3}{2}$, the Sobolev H^s-norm of the solution stays finite, then the solution will remain analytic for all time.

The proofs of these two theorems are developed in the following way. In the next section, notation and some basic properties of function spaces will be recorded. In Section 3, some linear estimates are recalled, laying the groundwork for the multilinear estimates which form the heart of the analysis. The multilinear estimates will be established in Section 4. Finally, a proof of Theorem 1.1 will given in Section 5, whilst Theorem 1.2 is established in Section 6.

2. **Functional setting.** The Fourier transform $\widehat{u_0}$ of a function u_0 belonging to the Schwartz class is taken to be

$$\widehat{u_0}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0(x) e^{-ix\xi}\, dx.$$

For a function $u(x,t)$ of two variables, the spatial Fourier transform is denoted by

$$\mathcal{F}_x u(\xi, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-ix\xi}\, dx,$$

whereas the notation $\hat{u}(\xi,\tau)$ designates the space-time Fourier transform

$$\hat{u}(\xi,\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x,t) e^{-ix\xi} e^{-it\tau}\, dx\, dt.$$

Define Fourier multiplier operators A and Λ by

$$\widehat{Au}(\xi,\tau) = (1 + |\xi|)\hat{u}(\xi,\tau)$$

and

$$\widehat{\Lambda u}(\xi,\tau) = (1 + |\tau|)\hat{u}(\xi,\tau).$$

The following notation is used to signify the L_x^p-L_t^q norms;

$$\|u\|_{L_p L_q} = \left\{ \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} |u(x,t)|^q\, dt \right|^{\frac{p}{q}} dx \right\}^{\frac{1}{p}}.$$

The usual modification applies for L^∞-norms. Sometimes, L^p-L^q norms are used in Fourier variables, and this will be indicated by using subscripts, *viz.* $L_\xi^p L_\tau^q$. A class of analytic functions suitable for our analysis is the analytic Gevrey class $G_{\sigma,s}$, introduced in the context of nonlinear evolution equations by Foias and Temam [19]. $G_{\sigma,s}$ may be defined as the domain of the operator $A^s e^{\sigma A}$ in L^2, and as was mentioned in the introduction, the analytic Gevrey norm of a function $u_0 = u_0(x)$ is defined by

$$\|u_0\|_{G_{\sigma,s}}^2 = \int_{-\infty}^{\infty} (1 + |\xi|)^{2s} e^{2\sigma(1+|\xi|)} |\widehat{u_0}(\xi)|^2\, d\xi.$$

It is straightforward to check that a function in $G_{\sigma,s}$ is the restriction to the real axis of a function which is analytic on a strip symmetric about the real axis of width 2σ. This strip, $\{z = x + iy : |y| < \sigma\}$, will be denoted by S_σ. If $s = 0$, we write simply G_σ for $G_{\sigma,0}$. The following proposition shows that if $u_0 \in G_\sigma$ and ε is such that $0 < \varepsilon < \sigma$, then u_0 and all of its derivatives are bounded on the smaller strip $S_{\sigma-\varepsilon}$.

Proposition 2.1. *Let ε with $0 < \varepsilon < \sigma$ and $n \in \mathbb{N}$ be given, and suppose that $u_0 \in G_\sigma$. Then there exists a constant c depending on ε and n, such that*

$$\sup_{x+iy \in S_{\sigma-\varepsilon}} |\partial_x^n u_0(x+iy)| \leq c\|u_0\|_{G_\sigma}.$$

Proof. This is a direct consequence of the Sobolev embedding theorem and the inequality

$$\|u_0\|_{G_{\sigma-\varepsilon,n+1}} \leq c_{n,\varepsilon}\|u_0\|_{G_\sigma} \tag{6}$$

which holds for $n \in \mathbb{N}$. The inequality (6) follows from the relation

$$\sup_{\xi\in\mathbb{R}} \left\{ e^{-\varepsilon(1+|\xi|)}(1+|\xi|)^{n+1} \right\} = c_{n,\varepsilon},$$

for a constant $c_{n,\varepsilon}$. □

The space $C([-T,T];G_{\sigma,s})$ of continuous functions on the interval $[-T,T]$ with values in $G_{\sigma,s}$ is denoted by $\mathcal{C}_{T,\sigma,s}$. This space is a Banach space when equipped with the norm

$$|u|_{\mathcal{C}_{T,\sigma,s}} = \sup_{-T\leq t\leq T} \|u(\cdot,t)\|_{G_{\sigma,s}}.$$

To efficiently exploit the dispersive effects inherent in (1), we consider a space that is a blend between the analytic Gevrey space and a space of Bourgain-type. More precisely, for $\sigma \geq 0$, $s \in \mathbb{R}$, and $b \in [-1,1]$, define $X^+_{\sigma,s,b}$ to be the Banach space equipped with the norm

$$\|v\|^2_{X^+_{\sigma,s,b}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1+|\tau+\xi-\xi^3|)^{2b}(1+|\xi|)^{2s}e^{2\sigma(1+|\xi|)}|\hat{v}(\xi,\tau)|^2\, d\xi d\tau$$

and define $X^-_{\sigma,s,b}$ using the norm

$$\|v\|^2_{X^-_{\sigma,s,b}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1+|\tau-\xi+\xi^3|)^{2b}(1+|\xi|)^{2s}e^{2\sigma(1+|\xi|)}|\hat{v}(\xi,\tau)|^2\, d\xi d\tau.$$

For $\sigma = 0$, $X^\pm_{\sigma,s,b}$ coincides with the spaces $X^\pm_{s,b}$ introduced by Bourgain, and Kenig, Ponce and Vega, whose norms are

$$\|v\|^2_{X^\pm_{s,b}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1+|\tau\pm\xi\mp\xi^3|)^{2b}(1+|\xi|)^{2s}|\hat{v}(\xi,\tau)|^2\, d\xi d\tau.$$

These spaces are effective because the Fourier weight is well adapted to the linear part of the evolution equations. To see how this works, let $W(t)^\pm$ be the solution groups associated with the homogeneous linear problems

$$\left.\begin{array}{c} u_t + u_x + u_{xxx} = 0, \\ u(x,0) = u_0(x), \end{array}\right\}$$

and

$$\left.\begin{array}{c} v_t - v_x - v_{xxx} = 0, \\ v(x,0) = v_0(x), \end{array}\right\}$$

respectively. These groups have the representations

$$u(x,t) = W^+(t)u_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}e^{-it\xi}e^{it\xi^3}\,\hat{u}_0(\xi)\, d\xi$$

and

$$v(x,t) = W^-(t)v_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}e^{it\xi}e^{-it\xi^3}\,\hat{v}_0(\xi)\, d\xi.$$

It follows immediately that

$$\left\|W^{\pm}(t)f\right\|_{X^{\pm}_{\sigma,s,b}} = \|A^s e^{\sigma A}\Lambda^b f\|_{L_2L_2}. \tag{7}$$

When $b > \frac{1}{2}$, the spaces $X^{\pm}_{\sigma,s,b}$ are included in the space of continuous functions with values in $G_{\sigma,s}$.

Proposition 2.2. *If $b > \frac{1}{2}$, then the spaces $X^{\pm}_{\sigma,s,b}$ are embedded in $C([-T,T];G_{\sigma,s})$.*

Proof. It follows directly from (7) and the Sobolev embedding theorem that the inequalities

$$\sup_{t\in[-T,T]} \|v(\cdot,t)\|_{G_{\sigma,s}} \le c\ \|v\|_{X^{\pm}_{\sigma,s,b}}$$

and

$$\sup_{t\in[-T,T]} \|v(\cdot,t)\|_{H^s} \le c\ \|v\|_{X^{\pm}_{s,b}}$$

hold for $b > \frac{1}{2}$. This proves the proposition. □

The product spaces denoted impressionistically by $Y = X^+ \times X^-$ have the usual product norm $\|u\|_{X^+_{\sigma,s,b}} + \|v\|_{X^-_{\sigma,s,b}}$. The linear matrix operator $W(t)$ is

$$W(t) = \begin{pmatrix} W^+(t) & 0 \\ 0 & W^-(t) \end{pmatrix},$$

so that for $\begin{pmatrix} u \\ v \end{pmatrix} \in Y$, $W(t)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} W^+(t)u \\ W^-(t)v \end{pmatrix}$. Finally, for any function space Z, Z^2 will denote the product space $Z \times Z$.

3. **Linear estimates.** Since the analysis is based on boundedness in $Y_{\sigma,s,b}$ of an integral operator given by a variation-of-constants formula, certain estimates of the solutions of the corresponding linear problem are needed. These estimates are addressed now. Let ψ be an infinitely differentiable cut-off function such that $0 \le \psi \le 1$ everywhere and

$$\psi(t) = \begin{cases} 0, & |t| \ge 2, \\ 1, & |t| \le 1, \end{cases}$$

and, for $T > 0$, let $\psi_T(t) = \psi(t/T)$.

Lemma 3.1. *Let $T > 0, \sigma \ge 0, s \ge 0, b > \frac{1}{2}$ and choose b' such that $b - 1 < b' < 0$. Then there is a constant c such that*

$$\left\|\psi_T(t)\, W^{\pm}(t)u_0\right\|_{X^{\pm}_{\sigma,s,b}} \le c \max\left\{T^{\frac{1}{2}}, T^{\frac{1-2b}{2}}\right\} \|u_0\|_{G_{\sigma,s}} \tag{8}$$

and

$$\left\|\psi_T(t) \int_0^t W^{\pm}(t-s)u(s)\, ds\right\|_{X^{\pm}_{\sigma,s,b}} \le c \max\left\{T, T^{1-b+b'}\right\} \|u\|_{X^{\pm}_{\sigma,s,b'}} \tag{9}$$

Proof. The proof of (8) is immediate from the definition of the $X_{\sigma,s,b}$-spaces and the linearity of the operator $e^{\sigma A}$. For the proof of (9), one follows the proof of Lemma 2.1 in [20] step by step, separating the cases $T \le 1$ and $T > 1$. □

The second kind of linear estimates needed are Kato-type smoothing inequalities and maximal function inequalities. For a suitable function f, define F_ρ^+ and F_ρ^- via their Fourier transforms according to

$$\widehat{F}_\rho^+(\xi,\tau) = \frac{|f(\xi,\tau)|}{\big(1+|\tau+\xi-\xi^3|\big)^\rho}$$

and

$$\widehat{F}_\rho^-(\xi,\tau) = \frac{|f(\xi,\tau)|}{\big(1+|\tau-\xi+\xi^3|\big)^\rho}.$$

Lemma 3.2. [Bourgain] *Let $\rho > \frac{1}{4}$ be given. Then there is a constant c, depending on ρ, such that*

$$\|A^{\frac{1}{2}} F_\rho^\pm\|_{L_4L_2} \le c\|f\|_{L^2_\xi L^2_\tau}. \tag{10}$$

For the proof of this lemma, the reader is referred to [14].

Lemma 3.3. [Kenig-Ponce-Vega] *Let s and ρ be given. There is a constant c, depending on s and ρ, such that the following inequalities hold.*

(i) If $\rho > \frac{1}{2}$, then

$$\|AF_\rho^\pm\|_{L_\infty L_2} \le c\|f\|_{L^2_\xi L^2_\tau}. \tag{11}$$

(ii) If $\rho > \frac{1}{2}$ and $s > 3\rho$, then

$$\|A^{-s}F_\rho^\pm\|_{L_2L_\infty} \le c\|f\|_{L^2_\xi L^2_\tau}. \tag{12}$$

(iii) If $\rho > \frac{1}{2}$ and $s > \frac{1}{4}$, then

$$\|A^{-s}F_\rho^\pm\|_{L_4L_\infty} \le c\|f\|_{L^2_\xi L^2_\tau}. \tag{13}$$

(iv) If $\rho > \frac{1}{2}$ and $s > \frac{1}{2}$, then

$$\|A^{-s}F_\rho^\pm\|_{L_\infty L_\infty} \le c\|f\|_{L^2_\xi L^2_\tau}. \tag{14}$$

The inequality (11) was proved in [26]. The estimates (12) and (14) were established in [21], and (13) can be proved analogously using an estimate appearing in [25]. Finally, note that for any $\rho \ge 0$, the trivial estimate

$$\|F_\rho^\pm\|_{L_2L_2} \le c\|f\|_{L^2_\xi L^2_\tau} \tag{15}$$

holds.

4. **Bilinear estimates.**

The purpose of this section is the proof of several useful bilinear estimates. The first theorem contains the estimates to be used in the proof of local existence of solutions of (1) and (2).

Theorem 4.1. *Let $s \ge 0$, $\sigma \ge 0$, $\frac{1}{2} < b < \frac{3}{4}$, and $b-1 < b' < -\frac{1}{4}$. Then there exists a constant c depending only on s, b, and b' such that*

$$\|\partial_x(uv)\|_{X^+_{\sigma,s,b'}} \le c\,\|u\|_{X^+_{\sigma,s,b}}\,\|v\|_{X^+_{\sigma,s,b}}\,,$$

$$\|\partial_x(uv)\|_{X^-_{\sigma,s,b'}} \le c\,\|u\|_{X^-_{\sigma,s,b}}\,\|v\|_{X^-_{\sigma,s,b}}\,,$$

$$\|\partial_x(uv)\|_{X^+_{\sigma,s,b'}} \le c\,\|u\|_{X^+_{\sigma,s,b}}\,\|v\|_{X^-_{\sigma,s,b}}\,,$$

$$\|\partial_x(uv)\|_{X^+_{\sigma,s,b'}} \le c\,\|u\|_{X^-_{\sigma,s,b}}\,\|v\|_{X^-_{\sigma,s,b}}\,,$$

$$\|\partial_x(uv)\|_{X^-_{\sigma,s,b'}} \le c\, \|u\|_{X^+_{\sigma,s,b}}\, \|v\|_{X^-_{\sigma,s,b}}\,,$$

$$\|\partial_x(uv)\|_{X^-_{\sigma,s,b'}} \le c\, \|u\|_{X^+_{\sigma,s,b}}\, \|v\|_{X^+_{\sigma,s,b}}\,.$$

Proof. In the proof, we focus on the case $s = 0$ and $\sigma = 0$ and comment on the situations wherein $s > 0$ or $\sigma > 0$ at the end. The first two estimates can be proved as in [27]. Of the remaining four inequalities, the fourth and the sixth are most demanding. Since the proofs are virtually the same, only the latter estimate will be proved. First note that the sixth inequality can be written more explicitly as

$$\left\| \left(1 + |\tau - \xi + \xi^3|\right)^{b'} (i\xi)\ \widehat{uv}(\xi,\tau)\ \right\|_{L^2_\xi L^2_\tau} \le c\, \|u\|_{X^+_{\sigma,s,b}}\, \|v\|_{X^+_{\sigma,s,b}}\,.$$

If the functions f and g are defined by

$$f(\xi,\tau) = (1 + |\tau + \xi - \xi^3|)^b \hat{u}(\xi,\tau)$$

and

$$g(\xi,\tau) = (1 + |\tau + \xi - \xi^3|)^b \hat{v}(\xi,\tau),$$

respectively, then the sought for bilinear estimate is equivalent to

$$\Bigg\| \frac{\xi}{\left(1+|\tau-\xi+\xi^3|\right)^{-b'}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(\xi_1,\tau_1)}{\left(1+|\tau_1+\xi_1-\xi_1^3|\right)^b} \times \frac{g(\xi-\xi_1,\tau-\tau_1)}{\left(1+|\tau-\tau_1+\xi-\xi_1-(\xi-\xi_1)^3|\right)^b}\, d\xi_1 d\tau_1 \Bigg\|_{L^2_\xi L^2_\tau} \le c \|f\|_{L^2_\xi L^2_\tau} \|g\|_{L^2_\xi L^2_\tau}.$$

This inequality will be obtained by duality. Letting $d\mu = d\xi_1 d\tau_1 d\xi d\tau$, and supposing $h(\xi,\tau)$ to be an arbitrary element of the unit ball of $L^2(\mathbb{R}^2)$, we need to estimate the quantity

$$\sup_{\|h\|_{L^2_\xi L^2_\tau}\le 1} \int_{\mathbb{R}^4} \frac{h(\xi,\tau)\ |\xi|}{\left(1+|\tau-\xi+\xi^3|\right)^{-b'}}\ \frac{f(\xi_1,\tau_1)}{\left(1+|\tau_1+\xi_1-\xi_1^3|\right)^b} \times \frac{g(\xi-\xi_1,\tau-\tau_1)}{\left(1+|\tau-\tau_1+\xi-\xi_1-(\xi-\xi_1)^3|\right)^b}\ d\mu. \quad (16)$$

After changing variables, the domain of integration is split into the two subdomains where $|\xi|^2 + |\xi_1|^2 \ge 2$ and $|\xi|^2 + |\xi_1|^2 < 2$ (a choice whose genesis will be obvious in a moment). Thus the integral in (16) can be written as the sum $I_1 + I_2$, where

$$I_1 = \int_{|\xi|^2+|\xi_1|^2\ge 2} \frac{h(\xi+\xi_1,\tau+\tau_1)\ |\xi+\xi_1|}{\left(1+|\tau+\tau_1-(\xi+\xi_1)+(\xi+\xi_1)^3|\right)^{-b'}} \times \frac{f(\xi_1,\tau_1)}{\left(1+|\tau_1+\xi_1-\xi_1^3|\right)^b} \frac{g(\xi,\tau)}{\left(1+|\tau+\xi-\xi^3|\right)^b}\ d\mu$$

and

$$I_2 = \int_{|\xi|^2+|\xi_1|^2< 2} \frac{h(\xi+\xi_1,\tau+\tau_1)\ |\xi+\xi_1|}{\left(1+|\tau+\tau_1-(\xi+\xi_1)+(\xi+\xi_1)^3|\right)^{-b'}} \times \frac{f(\xi_1,\tau_1)}{\left(1+|\tau_1+\xi_1-\xi_1^3|\right)^b} \frac{g(\xi,\tau)}{\left(1+|\tau+\xi-\xi^3|\right)^b}\ d\mu.$$

Focusing on I_1, observe the relation

$$(\tau+\xi-\xi^3)+(\tau_1+\xi_1-\xi_1^3)-\big[(\tau+\tau_1)-(\xi+\xi_1)+(\xi+\xi_1)^3\big]$$
$$=-(\xi+\xi_1)(2\xi^2+\xi\xi_1+2\xi_1^2-2)$$
$$=-(\xi+\xi_1)(\xi^2+\xi\xi_1+\xi_1^2+\xi^2+\xi_1^2-2).$$

Noticing that $\xi^2+\xi\xi_1+\xi_1^2\geq|\xi||\xi_1|$, and recalling that $|\xi|^2+|\xi_1|^2\geq 2$ in the present case, it is clear that

$$\big|\tau+\xi-\xi^3+(\tau_1+\xi_1-\xi_1^3)-\big[(\tau+\tau_1)-(\xi+\xi_1)+(\xi+\xi_1)^3\big]\big|$$
$$\geq|\xi_1||\xi+\xi_1||\xi| \quad (17)$$

holds. This implies that one of the cases

$(a)\quad |\tau+\xi-\xi^3|\geq\frac{1}{3}|\xi+\xi_1||\xi||\xi_1|,$
$(b)\quad |\tau_1+\xi_1-\xi_1^3|\geq\frac{1}{3}|\xi+\xi_1||\xi||\xi_1|$ or
$(c)\quad |\tau+\tau_1-(\xi+\xi_1)+(\xi+\xi_1)^3|\geq\frac{1}{3}|\xi+\xi_1||\xi||\xi_1|$

always occurs. In case (a), I_1 is bounded by

$$\int_{|\xi|^2+|\xi_1|^2\geq 2}\frac{|h(\xi+\xi_1,\tau+\tau_1)|\ |\xi+\xi_1|^{1-b}}{\big(1+|\tau+\tau_1-(\xi+\xi_1)+(\xi+\xi_1)^3|\big)^{-b'}}$$
$$\times\frac{|f(\xi_1,\tau_1)||\xi_1|^{-b}}{\big(1+|\tau_1+\xi_1-\xi_1^3|\big)^{b}}\ |g(\xi,\tau)||\xi|^{-b}d\mu$$
$$\leq\ 2\int_{|\xi|^2+|\xi_1|^2\geq 2}\frac{|h(\xi+\xi_1,\tau+\tau_1)|\ (1+|\xi+\xi_1|)^{1-b}}{\big(1+|\tau+\tau_1-(\xi+\xi_1)+(\xi+\xi_1)^3|\big)^{-b'}}$$
$$\times\frac{|f(\xi_1,\tau_1)|(1+|\xi_1|)^{-b}}{\big(1+|\tau_1+\xi_1-\xi_1^3|\big)^{b}}\ |g(\xi,\tau)|(1+|\xi|)^{-b}d\mu.$$

Changing to the original variables, it transpires that

$$\begin{aligned}I_1\ &\leq\ 2\int_{\mathbb{R}^4}\frac{|h(\xi,\tau)|\ (1+|\xi|)^{1-b}}{\big(1+|\tau-\xi+\xi^3|\big)^{-b'}}\ \frac{|f(\xi_1,\tau_1)|(1+|\xi_1|)^{-b}}{\big(1+|\tau_1+\xi_1-\xi_1^3|\big)^{b}}\\ &\qquad\times|g(\xi-\xi_1,\tau-\tau_1)|(1+|\xi-\xi_1|)^{-b}\ d\mu\\ &\leq\ 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A^{1-b}H^-_{-b'}(x,t)\ A^{-b}F^+_b(x,t)\ A^{-b}G_0(x,t)\ dx\,dt\\ &\leq\ 2\,\|A^{1-b}H^-_{-b'}\|_{L_4L_2}\ \|A^{-b}F^+_b\|_{L_4L_\infty}\ \|A^{-b}G^+_0\|_{L_2L_2}\\ &\leq\ c\ \|h\|_{L_2L_2}\|f\|_{L_2L_2}\ \|g\|_{L_2L_2}.\end{aligned}$$

where Lemmas 3.2 and 3.3 have been used in the last step. It appears immediately that case (b) is similar to case (a). In case (c), we follow a similar reasoning as in case (a) to see that I_1 is bounded by

$$2\int_{\mathbb{R}^4}|h(\xi+\xi_1,\tau+\tau_1)|\ (1+|\xi+\xi_1|)^{1+b'}\ \frac{|f(\xi_1,\tau_1)|\ (1+|\xi_1|)^{b'}}{\big(1+|\tau_1+\xi_1-\xi_1^3|\big)^{b}}\ \frac{|g(\xi,\tau)|(1+|\xi|)^{b'}}{\big(1+|\tau+\xi-\xi^3|\big)^{b}}\ d\mu.$$

Changing back to the original variables, it is then required to estimate the integral

$$2\int_{\mathbb{R}^4} |h(\xi,\tau)|\ (1+|\xi|)^{1+b'}\ \frac{|f(\xi_1,\tau_1)|\ (1+|\xi_1|)^{b'}}{\big(1+|\tau_1+\xi_1-\xi_1^3|\big)^b} \times \frac{|g(\xi-\xi_1,\tau-\tau_1)|(1+|\xi-\xi_1|)^{b'}}{\big(1+|\tau-\tau_1+\xi-\xi_1-(\xi-\xi_1)^3|\big)^b}\ d\mu. \quad (18)$$

Split the domain of integration into two further subregions, namely the regions where $|\xi_1| > |\xi-\xi_1|$ and $|\xi_1| \le |\xi-\xi_1|$, respectively. In the region where $|\xi_1| > |\xi-\xi_1|$, the integral just displayed is dominated by

$$2\int_{\mathbb{R}^4} |h(\xi,\tau)|\ \frac{|f(\xi_1,\tau_1)||\xi_1|^{1+b'+b'}}{\big(1+|\tau_1+\xi_1-\xi_1^3|\big)^b}\ \frac{|g(\xi-\xi_1,\tau-\tau_1)||\xi-\xi_1|^{b'}}{\big(1+|\tau-\tau_1+\xi-\xi_1-(\xi-\xi_1)^3|\big)^b}\ d\mu.$$

This integral can be further bounded by

$$2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_0(x,t)\ A^{1+2b'}F_b(x,t)\ A^{b'}G_b(x,t)\ dxdt \le c\ \|H_0\|_{L_2L_2}\ \|A^{1+2b'}F_b\|_{L_4L_2}\ \|A^{b'}G_b\|_{L_4L_\infty} \le c\ \|h\|_{L_2L_2}\|f\|_{L_2L_2}\ \|g\|_{L_2L_2}.$$

Since the last two factors in the integral (18) have identical structure, the analysis in the region where $|\xi_1| \le |\xi-\xi_1|$ is the same.

With the integral I_1 appropriately bounded, we turn to the estimation of I_2. It is clear that I_2 can be dominated by

$$c\int_{\mathbb{R}^4} \frac{h(\xi,\tau)}{\big(1+|\tau-\xi+\xi^3|\big)^{-b'}}\ \frac{f(\xi_1,\tau_1)}{\big(1+|\tau_1+\xi_1-\xi_1^3|\big)^b} \times \frac{g(\xi-\xi_1,\tau-\tau_1)}{\big(1+|\tau-\tau_1+\xi-\xi_1-(\xi-\xi_1)^3|\big)^b}\ d\mu. \quad (19)$$

Split the domain of integration into the two subregions $D_1 = \{|\xi_1| > |\xi-\xi_1|\}$ and $D_2 = \{|\xi_1| \le |\xi-\xi_1|\}$ as in case (c) above. Again using Lemmas 3.2 and 3.3 along with the trivial estimate (15), the integrand appearing in (19), when integrated over D_1 and D_2, respectively, has the upper bounds

$$\|H_{-b'}\|_{L_2L_2}\|A^{\frac12}F_b\|_{L_4L_2}\ \|A^{-\frac12}G_b\|_{L_4L_\infty}\ \le c\ \|h\|_{L_2L_2}\|f\|_{L_2L_2}\ \|g\|_{L_2L_2},$$

and

$$\|H_{-b'}\|_{L_2L_2}\|A^{-\frac12}G_b\|_{L_4L_\infty}\ \|A^{\frac12}F_b\|_{L_4L_2}\ \le c\ \|h\|_{L_2L_2}\|f\|_{L_2L_2}\ \|g\|_{L_2L_2}.$$

It thus transpires that

$$\sup_{\|h\|_{L^2_\xi L^2_\tau}\le 1}\int_{\mathbb{R}^4} \frac{h(\xi,\tau)\ |\xi|}{\big(1+|\tau-\xi+\xi^3|\big)^{-b'}}\ \frac{f(\xi_1,\tau_1)}{\big(1+|\tau_1+\xi_1-\xi_1^3|\big)^b} \times \frac{g(\xi-\xi_1,\tau-\tau_1)}{\big(1+|\tau-\tau_1+\xi-\xi_1-(\xi-\xi_1)^3|\big)^b}\ d\mu \le c\ \|f\|_{L_2L_2}\ \|g\|_{L_2L_2}$$

as advertised.

In the case when s or σ are nonzero, define the functions f and g as

$$f(\xi,\tau) = (1+|\tau+\xi-\xi^3|)^b(1+|\xi|)^s e^{\sigma(1+|\xi|)}\hat{u}(\xi,\tau)$$

and

$$g(\xi,\tau) = (1+|\tau+\xi-\xi^3|)^b(1+|\xi|)^s e^{\sigma(1+|\xi|)}\hat{v}(\xi,\tau),$$

then use the inequality $e^{(1+|\xi|)} \le e^{(1+|\xi_1|)}e^{(1+|\xi-\xi_1|)}$ and observe that positive values of s actually aid the procedure of the proof. □

Next, a bilinear estimate to be used in the proof of large time estimates in Section 6 is stated and proved.

Theorem 4.2. *Let $\sigma > 0, s \ge 0, b > \frac{1}{2}$ and $b' \le -\frac{3}{8}$. Then there exists a constant c depending only on s, b, and b' such that*

$$\|\partial_x(uv)\|_{X^+_{\sigma,s,b'}} \le c\,\|u\|_{X^+_{s,b}}\|v\|_{X^+_{s,b}} + c\,\sigma^{\frac{1}{4}}\ \|u\|_{X^+_{\sigma,s,b}}\ \|v\|_{X^+_{\sigma,s,b}}\,,$$

$$\|\partial_x(uv)\|_{X^-_{\sigma,s,b'}} \le c\,\|u\|_{X^-_{s,b}}\|v\|_{X^-_{s,b}} + c\,\sigma^{\frac{1}{4}}\ \|u\|_{X^-_{\sigma,s,b}}\ \|v\|_{X^-_{\sigma,s,b}}\,,$$

$$\|\partial_x(uv)\|_{X^+_{\sigma,s,b'}} \le c\,\|u\|_{X^-_{s,b}}\|v\|_{X^+_{s,b}} + c\,\sigma^{\frac{1}{4}}\ \|u\|_{X^-_{\sigma,s,b}}\ \|v\|_{X^+_{\sigma,s,b}}\,,$$

$$\|\partial_x(uv)\|_{X^+_{\sigma,s,b'}} \le c\,\|u\|_{X^-_{s,b}}\|v\|_{X^-_{s,b}} + c\,\sigma^{\frac{1}{4}}\ \|u\|_{X^-_{\sigma,s,b}}\ \|v\|_{X^-_{\sigma,s,b}}\,,$$

$$\|\partial_x(uv)\|_{X^-_{\sigma,s,b'}} \le c\,\|u\|_{X^-_{s,b}}\|v\|_{X^+_{s,b}} + c\,\sigma^{\frac{1}{4}}\ \|u\|_{X^-_{\sigma,s,b}}\ \|v\|_{X^+_{\sigma,s,b}}\,,$$

$$\|\partial_x(uv)\|_{X^-_{\sigma,s,b'}} \le c\,\|u\|_{X^+_{s,b}}\|v\|_{X^+_{s,b}} + c\,\sigma^{\frac{1}{4}}\ \|u\|_{X^+_{\sigma,s,b}}\ \|v\|_{X^+_{\sigma,s,b}}\,.$$

Proof. The first two cases have essentially been proved in [8]. The proofs of the third and fifths case are symmetric, as are the proofs of the fourth and sixth cases. We prove only the fifth case which is somewhat more straightforward than the fourth, because no change of variables is necessary. Only the case $s = 0$ is treated; the case $s > 0$ is straightforwardly reduced to the case $s = 0$. The elementary inequality

$$e^{\sigma(1+|\xi|)} \le e + \sigma^{\frac{1}{4}}(1+|\xi|)^{\frac{1}{4}}e^{\sigma(1+|\xi|)} \tag{20}$$

will play a role at a certain point in the proof of the theorem.

As in the proof of Theorem 1, the estimate may be rewritten in the form

$$\left\|\left(1+|\tau-\xi+\xi^3|\right)^{b'} e^{\sigma(1+|\xi|)}|\xi|\ \widehat{uv}(\xi,\tau)\ \right\|_{L^2_\xi L^2_\tau} \le c\,\|u\|_{X^-_{s,b}}\|v\|_{X^+_{s,b}} + c\,\sigma^{\frac{1}{4}}\ \|u\|_{X^-_{\sigma,s,b}}\ \|v\|_{X^+_{\sigma,s,b}}\,.$$

Setting

$$f(\xi,\tau) = (1+|\tau-\xi+\xi^3|)^b e^{\sigma(1+|\xi|)}\hat{u}(\xi,\tau)$$

and

$$g(\xi,\tau) = (1+|\tau+\xi-\xi^3|)^b e^{\sigma(1+|\xi|)}\hat{v}(\xi,\tau),$$

it is required to bound appropriately the quantity

$$\int_{\mathbb{R}^4} \frac{h(\xi,\tau)\ |\xi|\ e^{\sigma(1+|\xi|)}}{\left(1+|\tau-\xi+\xi^3|\right)^{-b'}}\,\frac{f(\xi_1,\tau_1)\ e^{-\sigma(1+|\xi_1|)}}{\left(1+|\tau_1-\xi_1+\xi_1^3|\right)^{b}} \times \frac{g(\xi-\xi_1,\tau-\tau_1)\ e^{-\sigma(1+|\xi-\xi_1|)}}{\left(1+|\tau-\tau_1+\xi-\xi_1-(\xi-\xi_1)^3|\right)^{b}}\ d\mu \tag{21}$$

uniformly for h belonging to the unit ball B in $L^2(\mathbb{R}^2)$. As before, the notation $d\mu = d\xi_1 d\tau_1 d\xi d\tau$ has been employed. Using the inequality (20), the integral (21) is bounded by the sum of the two terms

$$I = e \int_{\mathbb{R}^4} \frac{|h(\xi,\tau)||\xi|}{\big(1+|\tau-\xi+\xi^3|\big)^{-b'}} \frac{|f(\xi_1,\tau_1)|\; e^{-\sigma(1+|\xi_1|)}}{\big(1+|\tau_1-\xi_1+\xi_1^3|\big)^{b}} \times \frac{|g(\xi-\xi_1,\tau-\tau_1)|\; e^{-\sigma(1+|\xi-\xi_1|)}}{\big(1+|\tau-\tau_1+\xi-\xi_1-(\xi-\xi_1)^3|\big)^{b}}\, d\mu$$

and

$$J = \sigma^{\frac14} \int_{\mathbb{R}^4} \frac{|h(\xi,\tau)|\; |\xi|(1+|\xi|)^{\frac14}\; e^{\sigma(1+|\xi|)}}{\big(1+|\tau-\xi+\xi^3|\big)^{-b'}} \frac{|f(\xi_1,\tau_1)|\; e^{-\sigma(1+|\xi_1|)}}{\big(1+|\tau_1-\xi_1+\xi_1^3|\big)^{b}} \times \frac{|g(\xi-\xi_1,\tau-\tau_1)|\; e^{-\sigma(1+|\xi-\xi_1|)}}{\big(1+|\tau-\tau_1+\xi-\xi_1-(\xi-\xi_1)^3|\big)^{b}}\, d\mu.$$

To analyze I, proceed as in the proof of Theorem 4.1 to split the domain of integration into the region where $|\xi|^2+|\xi_1|^2<2$ and the region where $|\xi|^2+|\xi_1|^2\ge 2$. In the first region, the proof is the same as the corresponding part of the proof of Theorem 1. In the second region, use the inequality

$$\big|\tau-\xi+\xi^3+(\tau_1-\xi_1+\xi_1^3)-\big[(\tau-\tau_1)+(\xi-\xi_1)-(\xi-\xi_1)^3\big]\big| \ge |\xi_1||\xi-\xi_1||\xi|,$$

which is derived in a similar way as (17), to conclude that one of the cases

$$\begin{array}{ll} (a) & |\tau-\xi+\xi^3| \ge \frac13|\xi+\xi_1||\xi||\xi_1|, \\ (b) & |\tau_1-\xi_1+\xi_1^3| \ge \frac13|\xi+\xi_1||\xi||\xi_1| \text{ or} \\ (c) & |\tau-\tau_1+(\xi-\xi_1)-(\xi-\xi_1)^3| \ge \frac13|\xi-\xi_1||\xi||\xi_1| \end{array} \tag{22}$$

always occurs. After noticing that the factors $e^{-\sigma(1+|\xi_1|)}$ and $e^{-\sigma(1+|\xi-\xi_1|)}$ are always less than 1, the estimation of I then proceeds along the lines of the proof of Theorem 4.1.

Attention is next turned to the task of estimating J. Because of the relation $e^{(1+|\xi|)} \le e^{(1+|\xi_1|)}e^{(1+|\xi-\xi_1|)}$, it is plain that J can be dominated by

$$\sigma^{\frac14} \int_{\mathbb{R}^4} \frac{|h(\xi,\tau)|\; (1+|\xi|)^{\frac54}}{\big(1+|\tau-\xi+\xi^3|\big)^{-b'}} \frac{|f(\xi_1,\tau_1)|}{\big(1+|\tau_1-\xi_1+\xi_1^3|\big)^{b}} \times \frac{|g(\xi-\xi_1,\tau-\tau_1)|}{\big(1+|\tau-\tau_1+\xi-\xi_1-(\xi-\xi_1)^3|\big)^{b}}\, d\mu.$$

Again splitting the domain of integration into the region where $|\xi|^2+|\xi_1|^2\ge 2$ and the region where $|\xi|^2+|\xi_1|^2<2$ leads to the need to estimate the integrals

$$J_1 = \sigma^{\frac14} \int_{|\xi|^2+|\xi_1|^2\ge 2} \frac{|h(\xi,\tau)|\; (1+|\xi|)^{\frac54}}{\big(1+|\tau-\xi+\xi^3|\big)^{-b'}} \frac{|f(\xi_1,\tau_1)|}{\big(1+|\tau_1-\xi_1+\xi_1^3|\big)^{b}} \times \frac{|g(\xi-\xi_1,\tau-\tau_1)|}{\big(1+|\tau-\tau_1+\xi-\xi_1-(\xi-\xi_1)^3|\big)^{b}}\, d\mu$$

and

$$J_2 = \sigma^{\frac{1}{4}} \int_{|\xi|^2+|\xi_1|^2<2} \frac{|h(\xi,\tau)|\ (1+|\xi|)^{\frac{5}{4}}}{\big(1+|\tau-\xi+\xi^3|\big)^{-b'}} \ \frac{|f(\xi_1,\tau_1)|}{\big(1+|\tau_1-\xi_1+\xi_1^3|\big)^{b}} \\ \times \frac{|g(\xi-\xi_1,\tau-\tau_1)|}{\big(1+|\tau-\tau_1+\xi-\xi_1-(\xi-\xi_1)^3|\big)^{b}}\ d\mu.$$

As before, it is shown that for J_1, one of the cases (a), (b) or (c) in (22) must occur. In case (a), J_1 is bounded by

$$2\,\sigma^{\frac{1}{4}} \int_{\mathbb{R}^4} |h(\xi,\tau)|\ |\xi|^{\frac{5}{4}+b'}\ \frac{\big(1+|\xi_1|\big)^{b'}|f(\xi_1,\tau_1)|}{\big(1+|\tau_1+\xi_1-\xi_1^3|\big)^{b}} \\ \times \frac{\big(1+|\xi-\xi_1|\big)^{b'}|g(\xi-\xi_1,\tau-\tau_1)|}{\big(1+|\tau-\tau_1+\xi-\xi_1-(\xi-\xi_1)^3|\big)^{b}}\ d\mu.$$

At this point, the domain of integration is again split into two further subregions, $|\xi_1| > |\xi-\xi_1|$ and $|\xi_1| \le |\xi-\xi_1|$. In the region where $|\xi_1| > |\xi-\xi_1|$, the quantity in the just displayed integral is dominated by

$$2\,\sigma^{\frac{1}{4}} \int_{\mathbb{R}^4} |h(\xi,\tau)|(1+|\xi|)^{b'+\frac{3}{8}}\ \frac{\big(1+|\xi_1|\big)^{\frac{7}{8}+b'}|f(\xi_1,\tau_1)|}{\big(1+|\tau_1+\xi_1-\xi_1^3|\big)^{b}} \\ \times \frac{\big(1+|\xi-\xi_1|\big)^{b'}|g(\xi-\xi_1,\tau-\tau_1)|}{\big(1+|\tau-\tau_1+\xi-\xi_1-(\xi-\xi_1)^3|\big)^{b}}\ d\mu.$$

The latter integral can be further bounded above by

$$2\,\sigma^{\frac{1}{4}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A^{b'+\frac{3}{8}} H_0^-(x,t)\ A^{\frac{7}{8}+b'} F_b^-(x,t)\ A^{b'} G_b^+(x,t)\,dx\,dt \\ \le c\ \|A^{b'+\frac{3}{8}} H_0^-\|_{L_2L_2}\ \|A^{\frac{7}{8}+b'} F_b^-\|_{L_4L_2}\ \|A^{b'} G_b^+\|_{L_4L_\infty} \\ \le c\ \|h\|_{L_2L_2}\|f\|_{L_2L_2}\ \|g\|_{L_2L_2}.$$

Since the last two factors in the integral have identical structure, the analysis in the region where $|\xi_1| \le |\xi-\xi_1|$ is the same. In case (b), J_1 is dominated by

$$2\,\sigma^{\frac{1}{4}} \int_{\mathbb{R}^4} \frac{|h(\xi,\tau)|\ (1+|\xi|)^{\frac{5}{4}-b}}{\big(1+|\tau-\xi+\xi^3|\big)^{-b'}}\ \frac{|f(\xi_1,\tau_1)|}{\big(1+|\xi_1|\big)^{b}} \\ \times \frac{|g(\xi-\xi_1,\tau-\tau_1)|}{\big(1+|\xi-\xi_1|\big)^{b}\big(1+|\tau-\tau_1+\xi-\xi_1-(\xi-\xi_1)^3|\big)^{b}}\ d\mu.$$

We split the domain of integration into the same two subregions as before. In the region where $|\xi_1| > |\xi-\xi_1|$, J_1 is dominated by

$$2\,\sigma^{\frac{1}{4}} \int_{\mathbb{R}^4} \frac{|h(\xi,\tau)|\ (1+|\xi|)^{\frac{5}{4}-2b}}{\big(1+|\tau-\xi+\xi^3|\big)^{-b'}}\ |f(\xi_1,\tau_1)| \\ \times \frac{|g(\xi-\xi_1,\tau-\tau_1)|}{\big(1+|\xi-\xi_1|\big)^{b}\big(1+|\tau-\tau_1+\xi-\xi_1-(\xi-\xi_1)^3|\big)^{b}}\ d\mu,$$

and the latter can be bounded by

$$2\,\sigma^{\frac{1}{4}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A^{\frac{5}{4}-2b} H^{-}_{-b'}(x,t)\ ,\ F_0(x,t)\ A^{-b} G^{+}_{b}(x,t)\ dx\,dt$$
$$\leq c\ \sigma^{\frac{1}{4}} \|A^{\frac{5}{4}-2b} H^{-}_{-b'}\|_{L_4L_2}\ \|F_0\|_{L_2L_2}\ \|A^{-b}G^{+}_{b}\|_{L_4L_\infty}$$
$$\leq c\,\sigma^{\frac{1}{4}}\ \|h\|_{L_2L_2}\|f\|_{L_2L_2}\ \|g\|_{L_2L_2}.$$

In the region where $|\xi_1| \leq |\xi - \xi_1|$, the integral is dominated by

$$2\,\sigma^{\frac{1}{4}} \int_{\mathbb{R}^4} \frac{|h(\xi,\tau)|\ (1+|\xi|)^{1-b}}{\big(1+|\tau-\xi+\xi^3|\big)^{-b'}}\ \frac{|f(\xi_1,\tau_1)|}{\big(1+|\xi_1|\big)^{b}}$$
$$\times \frac{(1+|\xi-\xi_1|)^{\frac{1}{4}}|g(\xi-\xi_1,\tau-\tau_1)|}{\big(1+|\xi-\xi_1|\big)^{b}\big(1+|\tau-\tau_1+\xi-\xi_1-(\xi-\xi_1)^3|\big)^{b}}\ d\mu,$$

which in turn is bounded above, *viz.*

$$2\,\sigma^{\frac{1}{4}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A^{1-b} H^{-}_{-b'}(x,t)\ A^{-b}F_0(x,t)\ A^{\frac{1}{4}-b} G^{+}_{b}(x,t)\ dx\,dt$$
$$\leq\ c\,\sigma^{\frac{1}{4}}\ \|A^{1-b} H^{-}_{-b'}\|_{L_4L_2}\ \|A^{-b}F^{-}\|_{L_2L_2}\ \|A^{\frac{1}{4}-b}G^{+}_{b}\|_{L_4L_\infty}$$
$$\leq c\,\sigma^{\frac{1}{4}}\ \|h\|_{L_2L_2}\|f\|_{L_2L_2}\ \|g\|_{L_2L_2}.$$

The proof in case (c) in (22) is similar to the proof in case (b). Finally, the estimation of J_2 is carried out as in the corresponding case in the proof of Theorem 4.1. □

5. **Local-in-time existence and global-in-time existence for the symmetric version in L^2.** With the inequalities provided in the previous section, local-in-time existence and uniqueness of solutions in $Y_{\sigma,s,b}$ to the initial-value problems can be proved for any $s \geq 0$ and $\sigma \geq 0$ with the help of the contraction-mapping theorem. Consider the integral operator

$$\Gamma \begin{pmatrix} u \\ v \end{pmatrix} = \psi(t) \begin{pmatrix} W^{+}(t)U_0 \\ W^{-}(t)V_0 \end{pmatrix}$$
$$- \psi_{t_0}(t) \int_0^t \begin{pmatrix} W^{+}(t-t')\partial_x\big(\frac{3}{2}u^2+uv-\frac{1}{2}v^2\big)(t') \\ W^{-}(t-t')\partial_x\big(\frac{3}{2}v^2+uv-\frac{1}{2}u^2\big)(t') \end{pmatrix} dt'. \quad (23)$$

Let $r = \|U_0\|_{G_{\sigma,s}} + \|V_0\|_{G_{\sigma,s}}$. It will be proved that t_0 can be chosen so that Γ is a contraction in the ball $B(2cr) \subset Y_{\sigma,s,b}$ of radius $2cr$ centered at 0.

Lemma 5.1. *There exists a positive time $t_0 \leq 1$, such that the operator Γ as defined in (23) is a contraction in the ball $B(2cr)$.*

Proof. We consider the system (4) – the proof in the case of the symmetric version (5) is the same.

First, it is proved that Γ is a mapping on $B(2cr)$. Using (8), (9) and the nonlinear estimates, it is seen that

$$\begin{aligned}
\left\|\Gamma\begin{pmatrix}u\\v\end{pmatrix}\right\|_{Y_{\sigma,s,b}} &\le \left\|\psi(t)W^+(t)U_0\right\|_{X^+_{\sigma,s,b}} + \left\|\psi(t)W^-(t)V_0\right\|_{X^-_{\sigma,s,b}} \\
&\quad + c\left\|\psi_{t_0}(t)\int_0^t W^+(t-t')\partial_x\left(\tfrac{3}{2}u^2+uv-\tfrac{1}{2}v^2\right)(t')\,dt'\right\|_{X^+_{\sigma,s,b}} \\
&\quad + c\left\|\psi_{t_0}(t)\int_0^t W^-(t-t')\partial_x\left(\tfrac{3}{2}v^2+uv-\tfrac{1}{2}u^2\right)(t')\,dt'\right\|_{X^-_{\sigma,s,b}} \\
&\le c\left\|U_0\right\|_{G_{\sigma,s}} + c\left\|V_0\right\|_{G_{\sigma,s}} \\
&\quad + c\,t_0^{1-b+b'}\ \left\|\partial_x\left(\tfrac{3}{2}u^2+uv-\tfrac{1}{2}v^2\right)\right\|_{X^+_{\sigma,s,b'}} \\
&\quad + c\,t_0^{1-b+b'}\ \left\|\partial_x\left(\tfrac{3}{2}v^2+uv-\tfrac{1}{2}u^2\right)\right\|_{X^-_{\sigma,s,b'}} \\
&\le cr\ +\ c\,t_0^{1-b+b'}\ \left(\|u\|^2_{X^+_{\sigma,s,b}} + \|v\|^2_{X^-_{\sigma,s,b}}\right) \\
&\le cr + c\ t_0^{1-b+b'}(2cr)^2 \le 2cr
\end{aligned}$$

for $t_0 = c_{b,b'}r^{-\frac{1}{1-b+b'}}$. With the same choice of t_0, the same set of inequalities yield the contraction inequality

$$\left\|\Gamma\begin{pmatrix}u_1\\v_1\end{pmatrix} - \Gamma\begin{pmatrix}u_2\\v_2\end{pmatrix}\right\|_{Y_{\sigma,s,b}} \le \frac{1}{2}\ \left\|\begin{pmatrix}u_1\\v_1\end{pmatrix} - \begin{pmatrix}u_2\\v_2\end{pmatrix}\right\|_{Y_{\sigma,s,b}}.$$

□

It follows that Γ has a unique fixed point

$$\Gamma\begin{pmatrix}U\\V\end{pmatrix} = \begin{pmatrix}U\\V\end{pmatrix} \in B(2cr).$$

The functions U and V provide a solution of the initial-value problem associated to (4), and it is necessarily unique in the class $Y_{\sigma,s,b}$. Continuous dependence follows immediately since the mapping Γ depends continuously on U_0 and V_0 (cf. [27, 21]).

Remark 5.2. Multiplying the equations in (5) by U and V, respectively, integrating over the real line, integrating by parts and summing the result reveals that the total energy, $\int\left(U^2+V^2\right)\,dx$, of the symmetric system is conserved. This combined with the local-in-time well-posedness yields global-in-time existence for the symmetric version in L^2.

Remark 5.3 Using the estimates of Section 4 combined with techniques used in [26], see [3], it is also possible to obtain existence of solutions in Sobolev classes of negative index.

Since the main focus of this paper is on analytic solutions, we do not stop here to provide an existence result in negative Sobolev classes. However, it would be interesting to find the critical Sobolev index below which well posedness becomes impossible, and to compare it to the situation for the initial-value problem associated to the KdV equation (3). For the related, but simpler systems studied in [3], well-posedness in $H^s \times H^s$ is known for $s > -\frac{3}{4}$. Presumably this result is sharp in the same sense it is for (3).

6. **Large time estimates on the radius of analyticity.** This section is devoted to the proof of Theorem 1.2 which is concerned with large time lower bounds on the uniform radius of spatial analyticity of solutions emanating from analytic initial data. Recall that Theorem 1.2 states that if initial data (U_0, V_0) are in an analytic Gevrey space $G^2_{\sigma_0,s}$ and for some $s > \frac{1}{2}$, the H^{s+1}-Sobolev norm of the solution remains bounded, then the solution will remain analytic and the radius of analyticity will decrease at worst at a rate of T^{-12}. This *algebraic* decrease of the radius of analyticity in time is a major improvement over previous results for other evolution equations which provided exponential decrease [24, 7]. The theorem is proved by first deriving *a priori* bounds in $G^2_{\sigma(T),s}$ on a suitable sequence $\{(U^n, V^n)\}$ of approximations of (U, V) on the time interval $[0, T]$ where $T > 0$ is arbitrary.

One important tool to be used in the proof is the following result relating boundedness of a Bourgain-type norm to boundedness of a Sobolev norm.

Lemma 6.1. *Let $s > \frac{1}{2}$ and $-1 < b < 1$. Let $t_0 > 0$ be the local existence time provided by Lemma 5.1, and suppose that (U, V) is a solution to (4) or (5) in $C([-2T, 2T]; H^{s+1})^2$. Then there exists a constant $c = c(s, b, t_0)$ such that*

$$\|\psi_T(t)U(\cdot,t)\|_{X^+_{s,b}} + \|\psi_T(t)V(\cdot,t)\|_{X^-_{s,b}} \le cT^{\frac{1}{2}}(1 + \alpha_T(U, V)),$$

for all $T \ge t_0$, where

$$\alpha_T(U, V) \equiv \sup_{t\in[-2T,2T]} (\|U\|_{s+1} + \|V\|_{s+1})^2.$$

The proof is analogous to the proof of Lemma 4 in [8].

In what follows, the system (4) is examined in detail. An analysis of the symmetric version follows exactly along the same lines.

The sequence of approximations $\{(U^n, V^n)\}_{n=1}^{\infty}$ is defined by

$$\begin{aligned} U^n_t + U^n_x + U^n_{xxx} &= -3(\eta_n * \psi_T U^n)(\eta_n * \psi_T U^n)_x \\ &\quad - ((\eta_n * \psi_T U^n)(\eta_n * \psi_T V^n))_x \\ &\quad + (\eta_n * \psi_T V^n)(\eta_n * \psi_T V^n)_x \\ &\equiv \big[NL(U)\big]_n, \end{aligned} \tag{24}$$

$$\begin{aligned} V^n_t - V^n_x - V^n_{xxx} &= -3(\eta_n * \psi_T V^n)(\eta_n * \psi_T V^n)_x \\ &\quad - ((\eta_n * \psi_T U^n)(\eta_n * \psi_T V^n))_x \\ &\quad + (\eta_n * \psi_T U^n)(\eta_n * \psi_T U^n)_x \\ &\equiv \big[NL(V)\big]_n, \end{aligned} \tag{25}$$

supplemented with the initial conditions $U^n(x, 0) = U_0(x), V^n(x, 0) = V_0(x)$ for $n \in \mathbb{N}$ and $T > 0$, where η_n is defined via its Fourier transform to be such that

$$\widehat{\eta_n}(\xi) = \begin{cases} 0, |\xi| \ge 2n \\ 1, |\xi| \le n \end{cases}$$

and $\widehat{\eta_n}$ is smooth and monotone on $(-2n, -n)$ and $(n, 2n)$.

Remark 6.2. (i) If $(U_0, V_0) \in (H^r)^2$ and $(U, V) \in C([-2T, 2T]; H^r)^2$ for some $r \ge 0$, standard $X^{\pm}_{r,s}$-estimates applied to $(U^n - U, V^n - V)$ show that for n large enough, (U^n, V^n) exists on the same time interval and $\{(U^n, V^n)\}$ converges to (U, V) in $C([-2T, 2T]; H^r)^2$.

(ii) If $(U_0, V_0) \in G^2_{\sigma_0,s}$, the proof given in the previous section yields local-in-time well-posedness for (24) - (25) in $Y_{\sigma_0,s,b}$ with the same interval of existence and the same bounds.

(iii) If $(U_0, V_0) \in G^2_{\sigma_0,s}$ and $(U, V) \in C([-4T, 4T]; H^{s+1})^2$ then for n large enough $(\psi_T U^n, \psi_T V^n)$ belongs to $Y_{\sigma,s,b}$ for all $\sigma > 0$ (this is a consequence of the spatial convolution with η_n which is an entire function of exponential type; for details see p. 199 in [9]).

Proposition 6.3. Let $\sigma_0 > 0$ and $s > \frac{1}{2}$ be given, and let b be such that $\frac{1}{2} < b < \frac{3}{4}$ and b' be such that $b - 1 < b' < -\frac{3}{8}$, Let $T \geq t_0$ where t_0 is the local existence time in $Y_{\sigma_0,s,b}$. Suppose that $(U_0, V_0) \in G^2_{\sigma_0,s}$ and $(U, V) \in C([-4T, 4T]; H^{s+1})^2$. Then there exists a suitable constant K depending on s, b, $\|(U_0, V_0)\|_{G^2_{\sigma_0,s}}$ and $\alpha_T(U, V)$, such that $\{(\psi_T U^n, \psi_T V^n)\}$ is bounded in $Y_{\sigma(T),s,b}$, where

$$\sigma(T) = \min\left\{\sigma_0, K(1 + T)^{-12}\right\}.$$

Proof. Applying the linear estimates from Section 3, the bilinear estimates from Section 4, and Lemma 6.1 to

$$\begin{aligned}
\psi_T(t)U^n &= \psi_T(t)W^+(t)U_0 + \psi_T(t)\int_0^t W^+(t-s)\big[NL(U)\big]_n(s)\,ds \\
\psi_T(t)V^n &= \psi_T(t)W^-(t)U_0 + \psi_T(t)\int_0^t W^-(t-s)\big[NL(V)\big]_n(s)\,ds
\end{aligned}$$

yields

$$\begin{aligned}
&\|\psi_T U^n\|_{X^+_{\sigma,s,b}} + \|\psi_T V^n\|_{X^-_{\sigma,s,b}} \\
&\quad\leq c\Big(T^{\frac{1}{2}} + t_0^{\frac{1-2b}{2}}\Big)\Big(\|U_0\|_{G_{\sigma_0,s}} + \|V_0\|_{G_{\sigma_0,s}}\Big) \\
&\quad + c\Big(T + T^{1-b+b'}\Big)\Big(\|\psi_T U^n\|_{X^+_{s,b}} + \|\psi_T V^n\|_{X^-_{s,b}}\Big)^2 \\
&\quad + c\Big(T + T^{1-b+b'}\Big)\sigma^{\frac{1}{4}}\Big(\|\psi_T U^n\|_{X^+_{\sigma,s,b}} + \|\psi_T V^n\|_{X^-_{\sigma,s,b}}\Big)^2 \\
&\quad\leq c\Big(T^{\frac{1}{2}} + t_0^{\frac{1-2b}{2}}\Big)\Big(\|U_0\|_{G_{\sigma_0,s}} + \|V_0\|_{G_{\sigma_0,s}}\Big) \\
&\quad + c\Big(T + T^{1-b+b'}\Big)\Big(T^{\frac{1}{2}}(1 + \alpha_T(U, V))\Big)^2 \\
&\quad + c\Big(T + T^{1-b+b'}\Big)\sigma^{\frac{1}{4}}\Big(\|\psi_T U^n\|_{X^+_{\sigma,s,b}} + \|\psi_T V^n\|_{X^-_{\sigma,s,b}}\Big)^2
\end{aligned}$$

for n large enough and a constant c.

Recall that $\{(\psi_{t_0} U^n, \psi_{t_0} V^n)\}$ is bounded in $Y_{\sigma_0,s,b}$ by the local-in-time theory, and denote the bound by M_{t_0}, so that

$$\|\psi_{t_0} U^n\|_{X^+_{\sigma_0,s,b}} + \|\psi_{t_0} v^n\|_{X^-_{\sigma_0,s,b}} \leq M_{t_0}.$$

Define T-dependent variables z, a and d by

$$\begin{aligned}
z(T) &= \|\psi_T U^n\|_{X^+_{\sigma,s,b}} + \|\psi_T V^n\|_{X^-_{\sigma,s,b}} \\
a(T) &= M_{t_0} + c\Big(T^{\frac{1}{2}} + t_0^{\frac{1-2b}{2}}\Big)\Big(\|U_0\|_{G_{\sigma_0,s}} + \|V_0\|_{G_{\sigma_0,s,b}}\Big) \\
&\quad + c\Big(T + T^{1-b+b'}\Big)\Big(T^{\frac{1}{2}}(1 + \alpha_T(U, V))\Big)^2 \\
d(T) &= c\Big(T + T^{1-b+b'}\Big).
\end{aligned}$$

Then the bound above implies

$$z \le a + d\sigma^{\frac{1}{4}} z^2. \tag{26}$$

Consider the inequality (26) for arbitrary T', where $t_0 \le T' \le T$ and define

$$\sigma(T') = \frac{\delta^4}{d^4(2a)^4}$$

for some $\delta > 0$ to be chosen presently. Setting $y(T') = \dfrac{z(T')}{2a(T')}$, the inequality (26) becomes

$$y(1 - \delta y) \le \frac{1}{2}.$$

Choosing δ suitably small yields the following dichotomy; either $y \le m^*$ or $y \ge M^*$ for some constants m^*, M^*, where $\frac{1}{2} < m^* < 1 < M^*$. Since $z(t_0) \le M_{t_0} < a(t_0)$, it follows that $y(t_0) < 1/2 < m^*$. Finally, note that $\|\psi_{T'} u^n(T')\|_{X^+_{\sigma(T'),s,b}} + \|\psi_{T'} v^n(T')\|_{X^-_{\sigma(T'),s,b}}$ is a continuous function of T' on $[t_0, T]$ (see p. 201 in [9] for a continuity argument in a similar setting). It follows that $y(T') \le m^* < 1$ for all T' with $t_0 \le T' \le T$. In particular, $y(T) < 1$, which is to say that $z(T) \le 2a(T)$, a bounded quantity. □

The proof of Theorem 1.2 is based on a compactness argument that was originally presented in [8] (see also [9]). Proposition 6.3 gives boundedness of the sequence $\{(\psi_T u^n, \psi_T v^n)\}$ in $Y_{\sigma(T),s,b}$. Since $b > \frac{1}{2}$, this implies boundedness of the sequence $\{(U^n, V^n)\}$ in $G^2_{\sigma(T),s}$, uniformly on $[-T, T]$. Recalling that the analytic Gevrey norm G_σ is equivalent to the classical Hardy $\mathcal{H}^2$-norm on the strip S_σ for any $\sigma > 0$, all the spatial derivatives of (U^n, V^n) are, via the Cauchy integral formulas, bounded uniformly on the compact subsets of, say, $S_{\sigma(T)/2}$. The system (24)-(25) satisfied by the approximations then yields boundedness of all the temporal derivatives and consequently there obtains uniform boundedness of all spatio-temporal derivatives on any compact subset of $S_{\sigma(T)/2} \times (-T, T)$. The Arzela-Ascoli theorem then implies locally uniform convergence on $S_{\sigma(T)/2} \times (-T, T)$. This allows us to pass to the limit in the approximations (24) and (25). The limit is an analytic function on $S_{\sigma(T)/2}$ inheriting uniform bounds for all t in $[-T, T]$. This, together with the local-in-time well-posedness, implies the result.

Remark 6.4. While the estimates of the width of the strip of analyticity is surely not sharp, the fact that it shrinks to zero as the solution looses regularity is shown explicitly in the examples worked out by Bona and Weissler [13].

REFERENCES

[1] J. L. Bona, M. Chen and J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I. Derivation and linear theory*, J. Nonlinear Sci., **12** (2002), 283–318.

[2] J. L. Bona, M. Chen and J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II. The nonlinear theory*, Nonlinearity, **17** (2004), 925–952.

[3] J. L. Bona, J. Cohen and G. Wang, *Global well-posedness for a system of KdV equations with coupled quadratic nonlinearities*, to appear.

[4] J. L. Bona, T. Colin and D. Lannes, *Long-wave approximations of water waves*, Arch. Rational Mech. Anal., **178** (2005), 373–410.

[5] J. L. Bona, V. A. Dougalis and D. Mitsotakis, *Numerical solution of the KdV-KdV systems of Boussinesq equations I. Numerical schemes and generalized solitary waves*, Math. Comput. Simulation, **74** (2007), 214–228.

[6] J. L. Bona, V. A. Dougalis and D. Mitsotakis, *Numerical solution of Boussinesq systems of KdV-KdV type. II. Evolution of radiating solitary waves*, Nonlinearity, **21** (2008), 2825–2848.

[7] J. L. Bona and Z. Grujić, *Spatial analyticity for nonlinear waves*, Math. Models & Methods in Appl. Sci., **13** (2003), 1–15.

[8] J. L. Bona, Z. Grujić and H. Kalisch, *Algebraic lower bounds for the uniform radius of spatial analyticity for the generalized KdV equation*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire, **22** (2005), 783–797.

[9] J. L. Bona, Z. Grujić and H. Kalisch, *Global solutions of the derivative Schrödinger equation in a class of functions analytic in a strip*, J. Diff. Eq., **229** (2006), 186–203.

[10] J. L. Bona, Shu-Ming Sun and Bing-Yu Zhang, *A nonhomogeneous boundary-value problem for the Korteweg-de Vries equation posed on a finite domain*, Comm. Partial Differential Equations, **28** (2003), 1391–1436.

[11] J. L. Bona and F. B. Weissler, *Similarity solutions of the generalized Korteweg-de Vries equation*, Math. Proc. Cambridge Philos. Soc., **127** (1999), 323–351.

[12] J. L. Bona and F. B. Weissler, *Blow-up of spatially periodic complex-valued solutions of nonlinear dispersive equations*, Indiana Univ. Math. J., **50** (2001), 759–782.

[13] J. L. Bona and F. B. Weissler, *Pole dynamics of interacting solitons and blow-up of complex-valued solutions of KdV*, Nonlinearity, **22** (2009), 311–349.

[14] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations*, Geom. & Functional Anal., **3** (1993), 107–156, 209–262.

[15] M. Christ, J. Colliander and T. Tao, *Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations*, American J. Math., **125** (2003), 1235–1293.

[16] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Global well-posedness for KdV in Sobolev spaces of negative index*, EJDE, **2001** (2001), 1–7.

[17] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$*, J. American Math. Soc., **16** (2003), 705–749.

[18] J. Colliander, G. Staffilani and H. Takaoka, *Global wellposedness for KdV below L^2*, Math. Res. Lett., **6** (1999), 755–778.

[19] C. Foias and R. Temam, *Gevrey class regularity for the solutions of the Navier-Stokes equations*, J. Functional Anal., **87** (1989), 359–369.

[20] J. Ginibre, Y. Tsutsumi and G. Velo, *On the Cauchy problem for the Zakharov system*, J. Functional Anal., **151** (1997), 384–436.

[21] Z. Grujić and H. Kalisch, *Local well-posedness of the generalized Korteweg-de Vries equation in spaces of analytic functions*, Diff. Integral Eq., **15** (2002), 1325–1334.

[22] H. Kalisch, *Rapid convergence of a Galerkin projection of the KdV equation*, C. R. Math. Acad. Sci. Paris, **341** (2005), 457–460.

[23] H. Kalisch and X. Raynaud, *On the rate of convergence of a collocation projection of the KdV equation*, M2AN Math. Model. Numer. Anal., **41** (2007), 95–110.

[24] T. Kato and K. Masuda, *Nonlinear evolution equations and analyticity I*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire, **3** (1986), 455–467.

[25] C. E. Kenig, G. Ponce and L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J., **40** (1991), 33–69.

[26] C. E. Kenig, G. Ponce and L. Vega, *On the Cauchy problem for the Korteweg-deVries equation in Sobolev spaces of negative indices*, Duke Math. J., **71** (1993), 1–20.

[27] C. E. Kenig, G. Ponce and L. Vega, *A bilinear estimate with applications to the KdV equation*, J. American Math. Soc., **9** (1996), 573–603.

[28] J. D. Wright and A. Scheel, *Solitary waves and their linear stability in weakly coupled KdV equations*, Z. Angew. Math. Phys., **58** (2007), 535–570.

[29] J. Wu and J.-M. Yuan, *Local well-posedness and local (in space) regularity results for the complex KdV equation*, Proc. Royal Soc. Edinburgh Sect. A, **137** (2007), 203–223.

[30] Y. Zhou, *Uniqueness of weak solutions of the KdV equation*, Internat. Math. Res. Notices, **6** (1997), 271–283.

Received March 2009; revised May 2009.

E-mail address: `bona@math.uic.edu;zg7c@virginia.edu;henrik.kalisch@math.uib.no`