Primary Topic: Coastal hydrodynamics
Secondary Topic: Coasts and climate change

STATISTICAL WAVE PROPERTIES IN SHALLOW WATER USING CNOIDAL THEORY

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1. Introduction

Various properties of wind-generated waves in coastal regions are significantly different from those in deep water regions. The differences are largely due to the influence of bathymetry, which is more pronounced in shallower water.

In general, deep water waves are considered a Gaussian random process with only minor discrepancies between the observed and theoretical probability density functions. The deviations from the Gaussian model are exhibited by that fact that high crests are observed more frequently than deep troughs (Holthuijsen 2007). In shallow water, these deviations are more pronounced due to the relative importance of non-linearity in these waves. Indeed irregularities in bathymetry, changes in wave height and wave steepness as the mean water depth decreases towards the shore affect wave properties and their probability distribution as a result. The steepening process near shore causes higher and sharper wave crests and shallower and flatter wave troughs. Under such conditions, the Gaussian model is no longer sufficient for describing wave behaviour as it underestimates the higher values and overestimates the lower values of the observed surface elevation. Hence, a non-Gaussian probability density function has to be applied for representing shallow water wave profiles (Ochi and Wang 1985).

Previous statistical analyses on the non-Gaussian characteristics of coastal waves include the results of Ochi et al. (1982) and Ochi and Wang (1985). In these works, wave records were obtained at a location along the CERC Field Research Facility at Duck North Carolina. These wave records were taken during the growth stage of a storm in the ARSLOE project. The results show that the skewness of the distribution modelling the free surface elevation was the dominant parameter affecting the degree of deviation from the Gaussian model. To account for the skewness, a non-Gaussian probability density function was used to more accurately represent the distribution of the free surface elevation near the shore. The Gram-Charlier probability density function showed good agreement with the histograms of the surface elevation obtained near the shore in both studies.

While the studies mentioned above are based on measurements, the present study embodies a numerical framework for estimating the coastal surface elevation distribution. As will be elaborated on in this paper, the combination of linear shoaling theory in deep water and non-linear cnoidal theory in shallow waters yields good agreement with the experimental results found in the above studies. In particular, with the numerical approach used in the present paper, the distribution of the free surface elevation is also found to be non-Gaussian and well represented by a Gram-Charlier distribution.

2. Background

In shoaling processes, wave energy is generally conserved while wave momentum may vary. The linear theory of wave shoaling imposes energy conservation to obtain the wave height of a shoaling wave. For the nonlinear case, momentum and energy balances are described using the KdV equation together with periodic cnoidal wave solutions.

Linear wave theory is generally limited to small-steepness, small amplitude surface gravity waves. This implies that $a/\lambda \ll 1$ and $a/h \ll 1$, respectively (Kundu and Cohen 2001). Here, a is the amplitude, λ is the wavelength and h is the depth. The solution to the linear problem is found by assuming the surface elevation η takes the form of a simple sinusoidal wave propagating in the positive x-direction

$$\eta(x,t) = a\cos(kx - \omega(k)t), \tag{1}$$

where k is the wave number and ω is given by the dispersion relation $\omega(k) = \sqrt{gk \tanh kh}$. The velocity potential is given by

$$\phi(x,z,t) = \frac{a\omega(k)}{k} \frac{\cosh(k(z+h))}{\sinh(kh)} \sin(kx - \omega(k)t). \tag{2}$$

In shoaling processes, the wave speed generally decreases, and as a consequence, the kinetic energy decreases. However, the total energy of a wave consists of both kinetic energy and potential energy which is conserved. A direct result of the decrease in the kinetic energy is then an increase in potential energy which is found to be directly proportional to the wave height. Consider first the energy per unit horizontal area

$$E = \frac{1}{\lambda} \int_0^\lambda \int_{-h}^0 \left[\frac{\rho}{2} \mid \nabla\phi \mid^2 + \rho g z \right] dz dx. \tag{3}$$

Substituting the solution of the velocity potential (2), and computing the integrals gives the expression

$$E = \frac{1}{8} \rho g H^2 \tag{4}$$

for the total energy. Now, the phase speed c is defined as $c = \frac{\omega}{k} = \sqrt{\frac{g}{k} \tanh kh}$, and so the group velocity (the velocity with which the overall envelope shape of the wave propagates) is

$$c_g = \frac{d\omega}{dk} = \frac{c}{2} \left[1 + \frac{2kh}{\sinh(2kh)} \right] \tag{5}$$

Conservation of the energy flux Ec_g then implies that the wave height H at a current depth is solely determined by the wave height at the offshore depth and the respective group velocities at each depth. Namely $H = H_0 \sqrt{\frac{c_{g0}}{c_g}}$, where the subscript '0' denotes the offshore depth (Sorensen 1993). To close the problem, conservation of the wave period T can be used in combination with the dispersion relation, leading to the equation $\frac{2\pi}{T} - gk \tanh(kh) = 0$, which can be solved for k numerically, so that H may be determined at the shoaling depth.

When waves become too steep or the local depth becomes too shallow, the assumptions of linear theory are no longer satisfied and a new, higher-order framework is required. The Korteweg-de Vries equation is one example of such a framework and has been used with its cnoidal solution to describe wave behaviour during shoaling processes. Previous studies on the shoaling of non-linear cnoidal waves were presented by Ostrovskiy and Pelinovskiy (1970) and Svendsen and Brink-Kjær (1972), Svendsen and Buhr Hansen (1977).

The Korteweg-de Vries (KdV) equation is a weakly non-linear dispersive model equation given in dimensional variables by

$$\eta_t + c_0\eta_x + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x + \frac{c_0 h_0^2}{6}\eta_{xxx} = 0 \tag{6}$$

where c_0 denotes the shallow water approximation of the phase speed and h_0 denotes the local water depth. The KdV equation has an exact travelling wave solution given by

$$\eta(x,t) = f_2 + (f_1 - f_2)\mathrm{cn}^2\left(\sqrt{\frac{3(f_1 - f_3)}{4h_0^3}}(x - ct); m\right), \tag{7}$$

where f_1 is the wave crest, f_2 is the wave trough, m is the elliptic parameter, cn is the Jacobian elliptic function and $f_3 = f_1 - \frac{1}{m}(f_1 - f_2)$. The wave speed c and wavelength λ can be defined as

$$c = c_0\left(1 + \frac{f_1 + f_2 + f_3}{2h_0}\right) \quad \text{and} \quad \lambda = K(m)\sqrt{\frac{16h_0^3}{3(f_1 - f_3)}} \tag{8}$$

where $K(m)$ is the complete elliptic integral of the first kind. It has been shown (Ali and Kalisch 2010, Ali and Kalisch 2012, Ali and Kalisch 2014) that the energy balance in the KdV equation is given by

$$\frac{\partial}{\partial t}E + \frac{\partial}{\partial x}q_E = 0 \tag{9}$$

to the second order, where

$$E = c_0^2\left(\frac{1}{h_0\eta^2} + \frac{1}{4h_0^2}\eta^3 + \frac{h_0}{6}\eta\eta_{xx} + \frac{h_0}{6}\eta_x^2\right) \tag{10}$$

and

$$q_E = c_0^3\left(\frac{1}{h_0}\eta^2 + \frac{5}{4h_0^2}\eta^3 + \frac{h_0}{2}\eta\eta_{xx}\right). \tag{11}$$

The wave height of a shoaling wave can now be determined by imposing preservation of wave frequency, conservation of mass and conservation of energy. Thus, if the wave motion at a certain water depth h_A is given, the wave height at water depth h was found by Khorsand and Kalisch (2014) to be given by the following equations:

$$\frac{c_A}{\lambda_A} = \frac{c}{\lambda}, \qquad \int_0^T q_{E_A}dt = \int_0^T q_E dt, \qquad \int_0^\lambda \eta_A dx = \int_0^\lambda \eta dx.$$

Using the stationary solution of the KdV equation (7) with wave speed and wavelength given in (8) and also utilizing the energy flux (11), a system of three non-linear equations that can be solved for f_1, f_2 and f_3 and the height of a wave at depth h can be determined. For more details on the numerical procedure see Paulsen and Kalisch (2022).

Sea surface elevation is typically modelled by a Gaussian distribution. An alternative model is provided by the two-parameter Gram-Charlier distribution, obtained by perturbing a standard Gaussian density by Hermite polynomials. It is formally defined by

$$f(z) = \left(1 + \frac{\sqrt{\beta_1}}{3!}H_3(z) + \frac{\beta_2 - 3}{4!}H_4(z)\right)p(z) \tag{12}$$

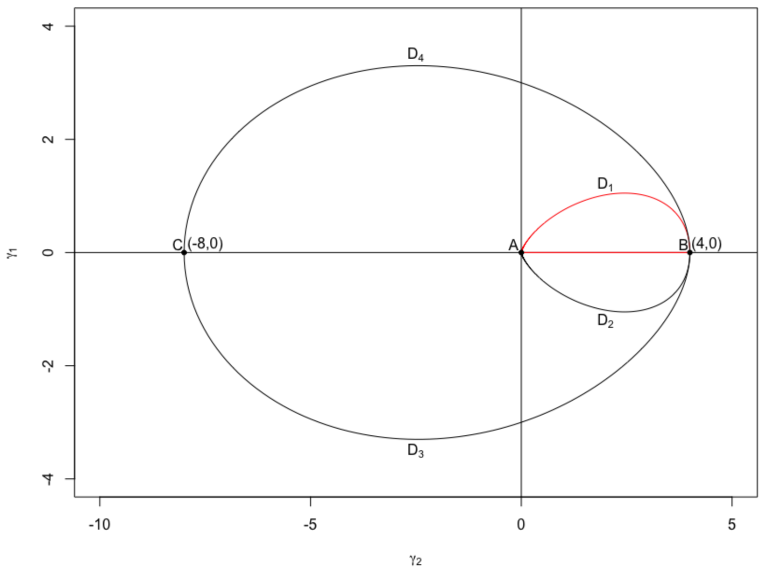

Figure 1: The red lines define the admissibility region of the parameters of the Gram-Charlier density.

where $p(z)$ is the standard normal distribution,

$$H_n(z) = (-1)^n \frac{d^n p}{dz^n} \frac{1}{p(z)}$$

is the *n*th-order Hermite polynomial, and β_1 and β_2 are a skewness and a kurtosis parameter, respectively. When $\beta_1 = 0$ and $\beta_2 = 3$, $f(z)$ reduces to a standard Normal distribution.

The function (12) is however a proper probability density distribution only when the parameters $\gamma_1 = \sqrt{\beta_1}$ and $\gamma_2 = (\beta_2 - 3)$ lie within a specific admissibility region, such as found by Barton and Dennis (1952). The region of admissibility is displayed in Figure 1 This restriction complicates maximum likelihood estimation of the parameters which can be obtained by a maximization algorithm with nonlinear constraints. We obtained parameter estimates by exploiting a simple grid search algorithm that explores the likelihood surface within the admissibility region.

3. Application

For a given sea state with significant wave height $H_s = 1, 2, 3\text{m}$ and peak period $T_p = 8, 10, 12\text{s}$, Rayleigh-distributed wave heights were randomly sampled. The non-linear transfer function implemented by Paulsen and Kalisch (2022) was then readily applied to each sample with their corresponding frequency $f = 1/T$ to acquire the local wave heights, wave lengths, modulus m and root solutions f_2 in shallow water.

To compute the surface elevation η in both deep and shallow water, the parameter m was used as a switch. Using each m_i to calculate $K(m_i)$ which is the complete elliptic integral of first kind, the Jacobian elliptic function cn was computed for each m_i. Now, m gives periodic waves for $0 \leq m < 1$. For the case $m = 0$, the cnoidal solution given in (7) reduces to the linear solution given in (1). The surface elevation of each individual wave was then computed at 100 uniformly spaced grid points x_i so that $-\frac{\lambda_i}{2} \leq x_i \leq \frac{\lambda_i}{2}$, using either the linear or non-linear solution depending on the nature of the wave.

Table 1: Estimated values of γ_1, γ_2 and σ for simulated sea states with $T = 8s$.

$H_{s,0}$	H_s	γ_1	γ_2	σ	β_1	β_2
1m	1.05m	0.517	0.552	0.249	0.267	3.552
2m	2.05m	0.879	1.379	0.512	0.773	4.379
3m	3.18m	0.983	1.793	0.731	0.966	4.793

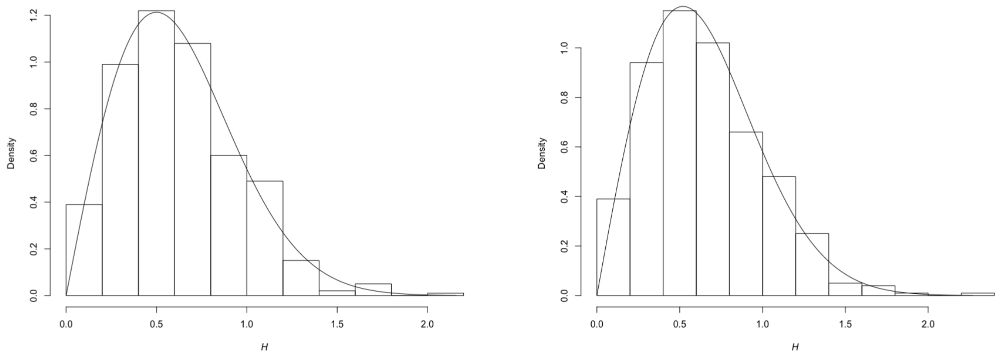

Figure 2: Distribution of wave height at 70m depth (left) and at 5m depth (right), given a period $T = 8s$. The continuous line shows the fit of the Rayleigh distribution.

A statistical analysis of both the wave height and the surface elevation was then carried out.

Deep water and coastal depth were set equal to 70m and 5m, respectively. Table 1 shows the estimated values of the skewness (γ_1) and kurtosis (γ_2) parameters of the Gram-Charlier distribution as well as the standard deviation (σ) of the normal distribution $p(z)$. The parameter $H_{s,0}$ defines the significant wave height in deep water whereas H_s denotes the shallow water significant wave height.

Figure 2 shows that the Rayleigh distribution nicely fits wave height at both 70m and 5m depth (Kolmogorov-Smirnov test p = 0.77). Figure 3 displays individual wave profiles as function of wave length λ. An increase in wave height and decrease in wavelength can be observed while the frequency remains constant in each case. This is due to the group velocity changing with water depth. A decrease in the group velocity is analogous to a decrease in the wave-energy transport velocity and must be compensated for. Since wave energy is conserved, a decrease in the kinetic energy leads accordingly to an increase in the potential energy and thus an increase in wave height.

Figure 4 shows the obtained histograms of the free surface elevation. The Gaussian distribution (solid line) fits the data well in deep water. Regarding the surface elevation in shallow water, the results vary depending on the significant wave height. As can be observed, sea states with a smaller significant wave height are in general better approximated by a Gram-Charlier density. As the significant wave height increases, the surface elevation data becomes excessively skewed which can possibly be explained by the non-linearity of the waves. Recall that the modulus $m \in [0, 1)$ gives periodic waves. For $m = 0$, the solution to the problem is given in terms of (1). When the non-linear terms are more dominant however, the parameter m increases and causes a surface deformation in the form of sharper crests and flatter troughs which can be seen in the histograms. In general, the non-linear terms seem more dominant in

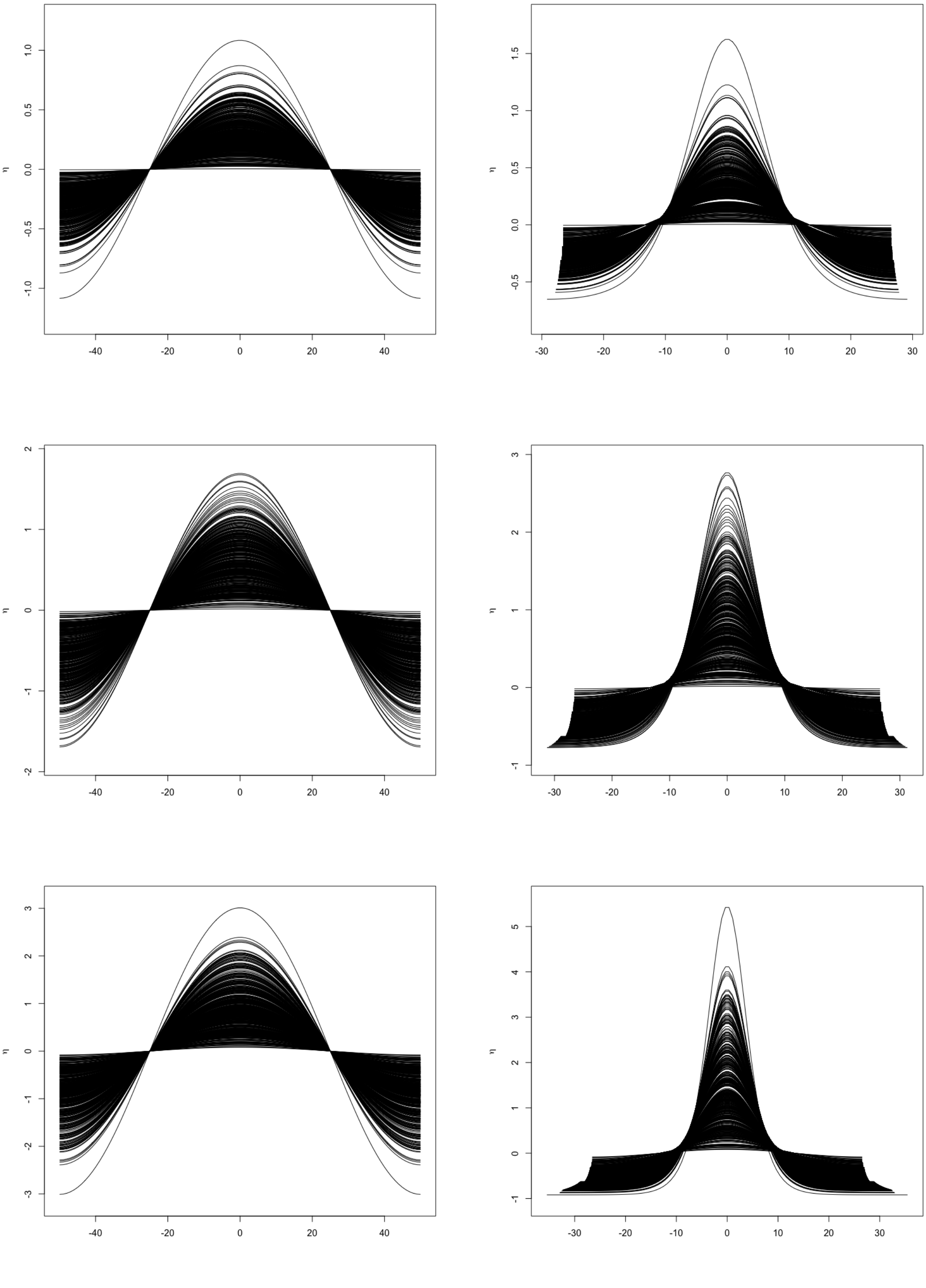

Figure 3: Surface elevation η at 70m and at 5m depth for waves with $T = 8s$ over each of their respective wavelengths λ. Left: $H_{s,0} = 1, 2, 3\text{m}$. Right: $H_s = 1.05, 2, 2.05\text{m}$

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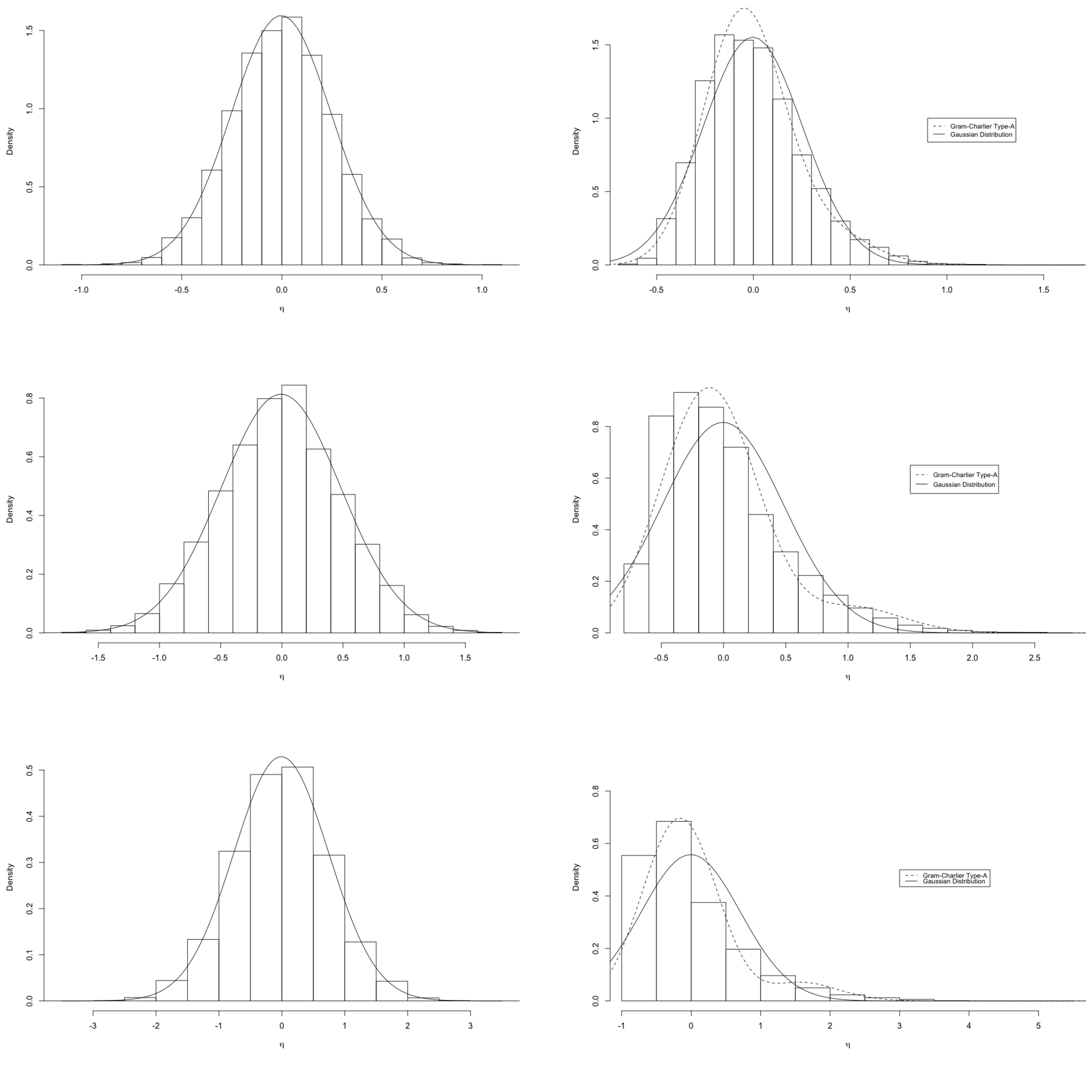
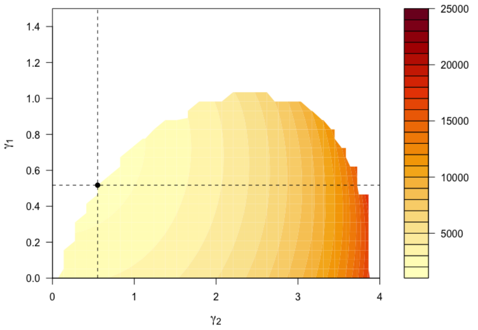
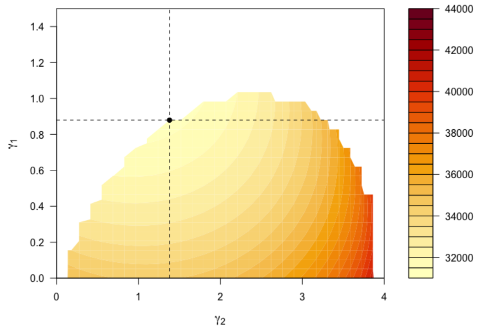


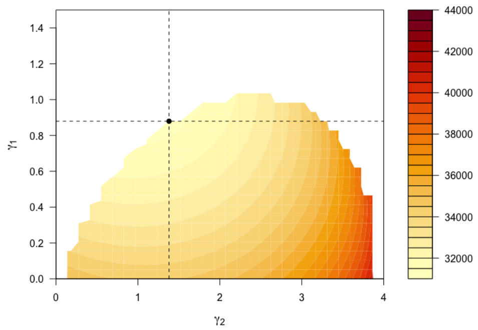
Figure 4: Histograms of surface elevation η at 70m and at 5m depth for waves with $T = 8s$. Left: $H_{s,0} = 1, 2, 3\text{m}$. Right: $H_s = 1.05, 2, 2.05\text{m}$



(a) $\gamma_1 = 0.517$ and $\gamma_2 = 0.552$



(b) $\gamma_1 = 0.879$ and $\gamma_2 = 1.379$



(c) $\gamma_1 = 0.983$ and $\gamma_2 = 1.793$

Figure 5: γ_1, γ_2 plane showing estimated parameters of the Gram-Charlier type-A expansion for waves with $T = 8s$. Color contours indicate the values oof the negative log-likelihood function.

the sea states with an original significant wave height $H_{s,0} = 3$m.

4. Discussion

Non-Gaussian behaviour of the free surface elevation in shallow water has been investigated for sea states consisting of waves with a single frequency. The wave heights obtained at 5m depth can still be considered Rayleigh distributed, but the surface elevation is not Gaussian distributed. As found in the field data collected by Ochi and Wang (1985), the distribution of the simulated surface elevation is decribed to a satisfactory degree by the Gram-Charlier series. However, the histograms of the surface elevation became excessively skewed for sea states with an offshore significant wave height of $H_{s,0} = 3$m or higher. A natural extension to these experiments would be the investigation of the limiting sea severity above which the Gram-Charlier series is no longer accurate in describing the distribution of the free surface elevation in shallow waters. A comparison between the Gram-Charlier series and the Tayfun distribution could then be carried out to identify which distribution is most accurate depending on the sea severity. It was also observed that the significant wave height did not change significantly after the non-linear transfer function was applied, even though the wave shape did undergo a noticeable change in the form of sharper crests and flatter troughs. Since the model used in this work does not take into account wave breaking. Wave breaking can be included in the present model in a way similar to the work of Bjørkavåg and Kalisch (2011), and further studies could involve the investigation of a region between the linear region and region dominated by non-linear effects where the waves have not yet reached breaking point but the significant wave height of the sea state undergoes a noticeable change during the shoaling process. In particular, it would be interesting to see whether waveheights follow the Gamma distribution observed in (Ochi and Wang, 1985) for locations in shallow water. However, this result likely relies on wave breaking dissipation, and may require more in-depth studies using a Boussinesq-type nearshore wave model such as the one put forward by Roeber and Cheung (2010).

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