Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
C. R. Mecanique 332 (2004) 525–530
COMPTES RENDUS MECANIQUE

A new model for large amplitude long internal waves

Walter Craig [a], Philippe Guyenne [a,*], Henrik Kalisch [b]

[a] *Department of Mathematics, McMaster University, 1280 Main Street West, Hamilton, Ontario L8S 4K1, Canada*
[b] *Centre for Mathematical Sciences, Lund University, Lund 221 00, Sweden*

Received 2 December 2003; accepted after revision 18 February 2004

Available online 13 May 2004

Presented by Yves Meyer

Abstract

We derive a new model for the description of large amplitude internal waves in a two-fluid system. The displacement of the interface between the two fluids is assumed to be of small slope, but no smallness assumption is made on the wave amplitude. The derivation of the model is based on the perturbation theory for Hamiltonian systems. In the case of a single fluid layer, the model reduces to the classical shallow water regime for surface water waves. ***To cite this article: W. Craig et al., C. R. Mecanique 332 (2004).***

Résumé

Un nouveau modèle pour les ondes internes longues de grande amplitude. Nous établissons un nouveau modèle pour la description des ondes internes de grande amplitude dans un système à deux fluides. On suppose que le déplacement de l'interface entre les deux fluides est de faible pente, mais on ne fait aucune hypothèse de faible amplitude pour les ondes. L'écriture du modèle est basée sur la théorie des perturbations pour les systèmes hamiltoniens. Dans le cas d'une seule couche de fluide, le modèle se réduit au régime classique en eau peu profonde pour les ondes de surface. ***Pour citer cet article : W. Craig et al., C. R. Mecanique 332 (2004).***

Keywords: Fluid mechanics; Internal waves; Hamiltonian systems; Dirichlet–Neumann operators

Mots-clés : Mécanique des fluides ; Ondes internes ; Systèmes hamiltoniens ; Opérateurs Dirichlet–Neumann

Version française abrégée

Nous considérons le problème des ondes internes se propageant à l'interface $\{y = \eta(x,t)\}$ entre deux fluides parfaits de profondeur finie. Les fluides inférieur et supérieur ont pour densités respectives ρ et ρ_1 avec $\rho > \rho_1$. Le domaine fluide est séparé en deux régions définies par $S(\eta) = \{(x, y)\colon x \in \mathbb{R},\ -h < y < \eta\}$ et $S_1(\eta) = \{(x, y)\colon x \in$

* Corresponding author. Tel.: +46-905-525-9140 x27364, fax: +46-905-522-0935.
E-mail addresses: craig@math.mcmaster.ca (W. Craig), guyenne@math.mcmaster.ca (P. Guyenne), kalisch@maths.lth.se (H. Kalisch).

1631-0721/$ – see front matter
doi:10.1016/j.crme.2004.02.026

$\mathbb{R},\ \eta < y < h_1\}$ où h et h_1 sont respectivement les profondeurs des couches inférieure et supérieure. Dans chacune des deux régions, l'écoulement est potentiel, décrit par les Éqs. (1) où φ, φ_1 désignent les potentiels des vitesses respectifs. Deux conditions cinématiques (2) ainsi que la condition de Bernoulli (3) sont imposées sur l'interface, où g représente l'accélération de gravité. On impose également les conditions de Neumann aux frontières inférieure et supérieure du domaine. Cela correspond à la situation où l'écoulement est confiné entre des parois solides horizontales.

L'hamiltonien (4) du système peut s'exprimer directement en fonction des variables canoniques $\eta(x)$ et $\xi(x) = \rho\Phi(x) - \rho_1\Phi_1(x)$ [1] en introduisant les opérateurs Dirichlet–Neumann $G(\eta)$ et $G_1(\eta)$ pour les régions $S(\eta)$ et $S_1(\eta)$ respectivement [2]. Les quantités $\Phi(x)$ et $\Phi_1(x)$ désignent les valeurs des potentiels des vitesses φ et φ_1 évaluées sur l'interface. Cela conduit à l'expression (5) pour l'hamiltonien et l'évolution de l'interface est alors déterminée par les équations canoniques (6). Dans notre analyse, nous utilisons la propriété que les opérateurs Dirichlet–Neumann $G(\eta)$ et $G_1(\eta)$ peuvent s'écrire sous la forme d'une série de Taylor convergente dont les termes sont homogènes en η [3]. Ceux-ci peuvent être déterminés de façon explicite à l'aide d'une formule de récurrence [4].

Nous nous intéressons ici à la propagation des ondes internes ayant des longueurs d'ondes λ grandes par rapport à la profondeur des deux couches de fluides. Cependant, nous ne supposons pas que leur amplitude a est faible par rapport à h ou h_1. L'approche est basée sur la théorie des perturbations pour les systèmes hamiltoniens. On définit $\varepsilon^2 \simeq (h/\lambda)^2 \simeq (h_1/\lambda)^2 \simeq (a/\lambda)^2 \ll 1$ comme petit paramètre dans le problème et on introduit le changement d'échelle $x' = \varepsilon x$, $\eta' = \eta$, $\xi' = \varepsilon\xi$. On obtient alors au premier ordre pour l'hamiltonien

$$H = \frac{1}{2}\int_{\mathbb{R}} \left[R_0(\eta)u^2 + g(\rho - \rho_1)\eta^2\right] \mathrm{d}x + \mathrm{O}\left(\varepsilon^2\right)$$

où $R_0(\eta) = (h+\eta)(h_1-\eta)/(\rho_1(h+\eta) + \rho(h_1-\eta))$ et $u = \partial_x\xi$. Les équations du mouvement correspondantes s'écrivent

$$\partial_t\eta = -\partial_x\delta_u H = -\partial_x(R_0 u), \qquad \partial_t u = -\partial_x\delta_\eta H = -\partial_x\left[\frac{1}{2}(\partial_\eta R_0)u^2 + g(\rho - \rho_1)\eta\right]$$

Le facteur $R_0(\eta)$ est non-singulier dans tout le domaine $-h < \eta < h_1$, s'annulant aux deux bords $\eta = -h$ et $\eta = h_1$. Ce système d'équations est un système de lois de conservation hyperboliques seulement dans une certaine région de l'espace des phases (η, u). Dans le cas $\rho_1 = 0$ (une seule couche de fluide), les variables canoniques sont $\eta(x)$ et $\xi(x) = \rho\Phi(x)$, et les équations du mouvement se réduisent aux équations classiques en eau peu profonde pour les ondes de surface. Les termes à l'ordre suivant s'obtiennent de manière tout aussi directe.

1. Introduction

We consider the problem of internal waves propagating at the interface between two ideal fluids of finite depth. The fluid motion is described by potential flow in each fluid layer. In realistic situations, internal waves are observed which have large amplitudes, comparable to the depth of one or both layers. The small parameter characterizing these solutions is the steepness of the interface. We derive model equations describing wave motion in the regime of small slopes but with no restriction on the wave amplitudes. Our work was motivated by the paper of Choi and Camassa [5] on the shallow water regime for internal waves. The derivation of the model is performed through a systematic perturbation of the Hamiltonian representing the system. This is made possible by introducing the Dirichlet–Neumann operators for both layers in the Hamiltonian. At leading order, we obtain a pair of coupled equations with rational dependence on the interface elevation. In the case of a single layer, these equations reduce to the classical shallow water equations for surface water waves. We also derive the next order terms in the model equations, which are unusual nonlinear dispersive evolution equations.

2. Equations of motion

The goal is to describe two-dimensional wave motion of an interface $\{y = \eta(x,t)\}$ between two immiscible ideal fluids of finite depth. The densities of the lower and upper fluids are respectively ρ and ρ_1 with $\rho > \rho_1$. The fluid domain consists of two regions defined by $S(\eta) = \{(x,y)\colon\ x \in \mathbb{R},\ -h < y < \eta\}$ and $S_1(\eta) = \{(x,y)\colon\ x \in \mathbb{R},\ \eta < y < h_1\}$. In each region, the evolution is given by potential flow, so that

$$\begin{aligned} &\mathbf{u} = \nabla\varphi, \quad \Delta\varphi = 0 \quad \text{in } S(\eta) \\ &\mathbf{u}_1 = \nabla\varphi_1, \quad \Delta\varphi_1 = 0 \quad \text{in } S_1(\eta) \end{aligned} \tag{1}$$

where $\mathbf{u}$, $\mathbf{u}_1$ denote the fluid velocities and φ, φ_1 the corresponding velocity potentials. Three conditions are imposed on the interface $\{y = \eta(x,t)\}$, namely two kinematic conditions

$$\partial_t\eta = \partial_y\varphi - \partial_x\eta\partial_x\varphi, \qquad \partial_t\eta = \partial_y\varphi_1 - \partial_x\eta\partial_x\varphi_1 \tag{2}$$

and the Bernoulli condition

$$\rho\left(\partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2 + g\eta\right) = \rho_1\left(\partial_t\varphi_1 + \frac{1}{2}|\nabla\varphi_1|^2 + g\eta\right) \tag{3}$$

where g is the acceleration due to gravity. The effects of surface tension are neglected. In addition, Neumann boundary conditions are imposed on the bottom and top boundaries of the fluid domain, so that $\partial_y\varphi = 0$ at $y = -h$ and $\partial_y\varphi_1 = 0$ at $y = h_1$. This corresponds to the configuration where the fluid domain is confined between horizontal rigid lids.

3. Hamiltonian formulation and Dirichlet–Neumann operators

The Hamiltonian of the system is given by

$$H = \frac{1}{2}\int_{\mathbb{R}}\int_{-h}^{\eta} \rho|\nabla\varphi|^2\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}}\int_{\eta}^{h_1} \rho_1|\nabla\varphi_1|^2\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}} g(\rho-\rho_1)\eta^2\,\mathrm{d}x \tag{4}$$

The first two terms in (4) represent the kinetic energies in the two fluid regions, while the last term represents the potential energy (Lamb [6]). Letting $\Phi(x) = \varphi(x,\eta(x))$ and $\Phi_1(x) = \varphi_1(x,\eta(x))$ be the boundary values of the two velocity potentials on the interface $\{y = \eta(x)\}$, and following Benjamin and Bridges [1], we introduce the choice of canonical variables $\eta(x)$ and $\xi(x) = \rho\Phi(x) - \rho_1\Phi_1(x)$. In order to express the Hamiltonian in an explicit form in terms of the canonical variables, we introduce the Dirichlet–Neumann operators $G(\eta)$ and $G_1(\eta)$ for the two fluid regions $S(\eta)$ and $S_1(\eta)$ respectively. Let N be the unit exterior normal to the lower fluid region $S(\eta)$ along the interface. The Dirichlet–Neumann operator for the domain $S(\eta)$ returns the exterior normal derivative of $\varphi(x)$ from its boundary values $\Phi(x)$, so that $G(\eta)\Phi(x) = (1+(\partial_x\eta)^2)^{1/2}\,\nabla\varphi\cdot N$. There is a similar definition for the domain $S_1(\eta)$ in terms of the exterior normal derivative of $\varphi_1(x)$, namely $G_1(\eta)\Phi_1(x) = -(1+(\partial_x\eta)^2)^{1/2}\,\nabla\varphi_1\cdot(-N)$. These operators are linear in $\Phi(x)$ and $\Phi_1(x)$, however they are nonlinear with explicit nonlocal behavior on $\eta(x)$. As shown in Craig and Groves [2], the Hamiltonian can be rewritten as

$$H = \frac{1}{2}\int_{\mathbb{R}} \xi G(\eta)B^{-1}G_1(\eta)\xi\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} g(\rho-\rho_1)\eta^2\,\mathrm{d}x \tag{5}$$

where $B(\eta) = \rho G_1(\eta) + \rho_1 G(\eta)$, and the evolution equations (2), (3) for the interface are equivalent to Hamilton's canonical equations

$$\partial_t\eta = \delta_\xi H, \qquad \partial_t\xi = -\delta_\eta H \tag{6}$$

Our analysis involves the expansion of the Hamiltonian in a Taylor series in the variables η and ξ. The potential energy is clearly quadratic in η, and the effort lies in performing the Taylor expansion of the operator $G(\eta)B^{-1}G_1(\eta)$ appearing in the expression for the total kinetic energy. As discussed in Coifman and Meyer [3], the operators $G(\eta)$ and $G_1(\eta)$ depend analytically on the variable $\eta \in \mathrm{Lip}(\mathbb{R})$, and therefore one can write $G(\eta) = \sum_{j \geqslant 0} G^{(j)}(\eta)$ where the Taylor polynomials $G^{(j)}$ are homogeneous of degree j in η. Given $G(\eta)$, the operator $G_1(\eta)$ can be determined through the relation $G_1(\eta, h_1) = G(-\eta, h_1)$. When $\eta \in \mathrm{Lip}(\mathbb{R})$, Craig and Sulem [4] showed that explicit expressions for the $G^{(j)}$ can be computed using a recursion formula. The first two terms of $G(\eta)$ are $G^{(0)} = D\tanh(hD)$, $G^{(1)} = D\eta D - G^{(0)}\eta G^{(0)}$, where $D = -\mathrm{i}\partial_x$ and $G^{(0)}$ represent Fourier multiplier operators.

4. Scaling regime of small steepness

We focus on the regime in which the typical wavelength λ of the internal waves is long compared to the depths h and h_1 of the two layers. However the typical wave amplitude a is not assumed to be small compared to h or h_1 unlike the classical Boussinesq regime. In the framework of Hamiltonian perturbation theory, we take the small parameter to be $\varepsilon^2 \simeq (h/\lambda)^2 \simeq (h_1/\lambda)^2 \simeq (a/\lambda)^2 \ll 1$ characterizing steepness, and we introduce the scaling $x' = \varepsilon x$, $\eta' = \eta$, $\xi' = \varepsilon\xi$. The procedure follows that used in Craig and Groves [7] for the derivation of long wave limits of the surface water wave problem. Expanding $G^{(0)} = D\tanh(hD) = \varepsilon D' \tanh(\varepsilon h D') = \varepsilon^2 h D'^2 - \frac{1}{3}\varepsilon^4 h^3 D'^4 + \cdots$ together with higher-order contributions, and grouping terms in powers of ε in the Hamiltonian, one finds up to order $\mathrm{O}(1)$

$$H = \frac{1}{2}\int_{\mathbb{R}} \left[R_0(\eta)u^2 + g(\rho - \rho_1)\eta^2\right] \mathrm{d}x + \mathrm{O}\big(\varepsilon^2\big) \tag{7}$$

where $R_0(\eta) = (h + \eta)(h_1 - \eta)/(\rho_1(h + \eta) + \rho(h_1 - \eta))$ and $u = \partial_x \xi$. For convenience, we have dropped the primes in (7). The corresponding approximate equations of motion are given by

$$\partial_t \eta = -\partial_x \delta_u H = -\partial_x (R_0 u), \qquad \partial_t u = -\partial_x \delta_\eta H = -\partial_x \left[\frac{1}{2}(\partial_\eta R_0)u^2 + g(\rho - \rho_1)\eta\right] \tag{8}$$

Note that the factor $R_0(\eta)$ is nonsingular in the whole domain $-h < \eta < h_1$, vanishing at both endpoints $\eta = -h$ and $\eta = h_1$. In the case $\rho_1 = 0$, the canonical variables are $\eta(x)$ and $\xi(x) = \rho\Phi(x)$, and the equations of motion (8) reduce to

$$\partial_t \eta = -\frac{1}{\rho}\partial_x\big((h + \eta)u\big), \qquad \partial_t u = -\frac{1}{\rho}u\partial_x u - g\rho\partial_x \eta \tag{9}$$

which are the classical shallow water equations for surface water waves. The system (8) is a hyperbolic conservation law in (η, u) in a certain region of state space. The integrand of the Hamiltonian H in (7) depends upon (η, u) but not on their derivatives. In this situation, Eqs. (8) take the form

$$\partial_t \begin{pmatrix} \eta \\ u \end{pmatrix} + \begin{pmatrix} \partial_{\eta u} H & \partial_u^2 H \\ \partial_\eta^2 H & \partial_{\eta u} H \end{pmatrix} \partial_x \begin{pmatrix} \eta \\ u \end{pmatrix} = \partial_t \begin{pmatrix} \eta \\ u \end{pmatrix} + A(\eta, u)\partial_x \begin{pmatrix} \eta \\ u \end{pmatrix} = 0 \tag{10}$$

Eigenvalues of the matrix $A(\eta, u)$ are $\mu_\pm(\eta, u) = \partial_{\eta u} H \pm \sqrt{(\partial_\eta^2 H)(\partial_u^2 H)}$ where $\partial_u^2 H = R_0(\eta)$ and

$$\partial_{\eta u} H = \frac{\rho(h_1 - \eta)^2 - \rho_1(h + \eta)^2}{(\rho_1(h + \eta) + \rho(h_1 - \eta))^2}u, \qquad \partial_\eta^2 H = g(\rho - \rho_1) - u^2 \frac{\rho\rho_1(h + h_1)^2}{(\rho_1(h + \eta) + \rho(h_1 - \eta))^3} \tag{11}$$

Because $\partial_u^2 H > 0$ for $-h < \eta < h_1$, the two eigenvalues $\mu_\pm$ for system (8) are real and distinct in the region of phase space determined by $\partial_\eta^2 H > 0 \Leftrightarrow u^2 < g(\rho - \rho_1)(\rho_1(h + \eta) + \rho(h_1 - \eta))^3/(\rho\rho_1(h + h_1)^2)$, which is

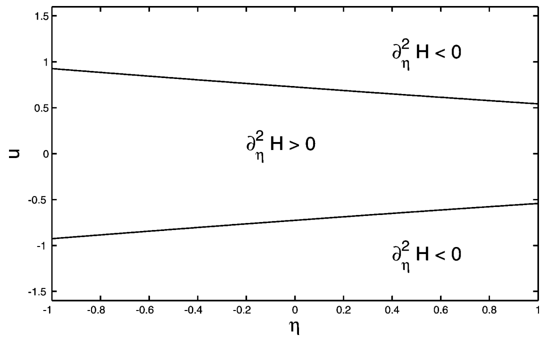

Fig. 1. Regions of phase space (η, u) where $\partial_\eta^2 H < 0$ and $\partial_\eta^2 H > 0$. The parameters are $h = h_1 = 1$, $\rho = 1$ and $\rho_1 = 0.7$.

Fig. 1. Des régions de l'espace des phases (η, u) où $\partial_\eta^2 H < 0$ et $\partial_\eta^2 H > 0$. Les paramètres sont $h = h_1 = 1$, $\rho = 1$, et $\rho_1 = 0{,}7$.

pictured in Fig. 1. When $\partial_\eta^2 H < 0$ (complex eigenvalues for A), system (8) is elliptic and the initial value problem is ill-posed; physically, solutions develop rapid oscillations and are unstable. The curves $\partial_\eta^2 H = 0$ correspond to the transition regime.

The solution of system (8) forms shocks or other discontinuities at which time they are no longer in the region of validity of the above scaling regime. This implies that one should examine the equations at higher order. The next approximation can be derived in a straightforward manner. Retaining terms up to order $O(\varepsilon^2)$, one gets

$$\begin{aligned} H = \frac{1}{2}\int_{\mathbb{R}} & R_0(\eta)u^2 + g(\rho - \rho_1)\eta^2 \\ & + \varepsilon^2\big[R_1(\eta)(\partial_x u)^2 + \big(\partial_x R_2(\eta)\big)\partial_x\big(u^2\big) + R_3(\eta)(\partial_x\eta)^2u^2\big]\,\mathrm{d}x + O\big(\varepsilon^4\big). \end{aligned} \tag{12}$$

The corresponding equations of motion read

$$\begin{aligned} \partial_t\eta &= -\partial_x(R_0u) - \varepsilon^2\partial_x\big[-\partial_x(R_1\partial_x u) - \partial_x^2(R_2)u + R_3(\partial_x\eta)^2u\big] \\ \partial_t u &= -\partial_x\left[\frac{1}{2}(\partial_\eta R_0)u^2 + g(\rho - \rho_1)\eta\right] \\ &\quad - \varepsilon^2\partial_x\left[\frac{1}{2}(\partial_\eta R_1)(\partial_x u)^2 - \frac{1}{2}(\partial_\eta R_2)\partial_x^2\big(u^2\big) + \frac{1}{2}(\partial_\eta R_3)(\partial_x\eta)^2u^2 - \partial_x\big(R_3(\partial_x\eta)u^2\big)\right] \end{aligned} \tag{13}$$

where

$$R_1(\eta) = -\frac{1}{3}\frac{(h+\eta)^2(h_1-\eta)^2(\rho_1(h_1-\eta) + \rho(h+\eta))}{(\rho_1(h+\eta) + \rho(h_1-\eta))^2} \tag{14}$$

$$\partial_x R_2(\eta) = -\frac{1}{3}\rho\rho_1(h+h_1)(h+\eta)(h_1-\eta)\frac{(h_1-\eta)^2 - (h+\eta)^2}{(\rho_1(h+\eta) + \rho(h_1-\eta))^3}\partial_x\eta \tag{15}$$

$$R_3(\eta) = -\frac{1}{3}\rho\rho_1(h+h_1)^2\frac{\rho_1(h+\eta)^3 + \rho(h_1-\eta)^3}{(\rho_1(h+\eta) + \rho(h_1-\eta))^4} \tag{16}$$

These are novel evolution equations which exhibit nonlinear variations in wave speed and in their coefficients of dispersion. Using a different formulation and a different method, Choi and Camassa [5] also derived model equations with rational coefficients which have some similarities with (13), for large amplitude long internal waves in the configuration of two finite layers. The three-term expansion of (13) in small amplitudes (η, u), when additionally one specializes to the case of uni-directional wave motions, bears some resemblance to the extended

Korteweg–de Vries equation. There is a well-known singularity of the small amplitude/long wave limit in two-layer flows, having to do with the vanishing of the coefficient of nonlinearity when $\rho/h^2 = \rho_1/h_1^2$. Our rational coefficients for the nonlinearity include this case, in which the first Taylor coefficient of the nonlinear term vanishes. It would be of interest to understand the character of solutions to equations (13), including in particular the class of traveling wave solutions. Such solutions would be characterized as critical points of the Lagrangian $H - cI$, which is a dynamical system with two degrees of freedom. Here, $I(\eta, u) = \int \eta u \, \mathrm{d}x$ is the momentum integral and c the traveling wave speed. It would also be of interest to compare these solutions with those of the Choi–Camassa equations as well as the extended Korteweg–de Vries equation. Finally, we would like to remark that the present methods are not restricted to two dimensions and can be easily extended to three dimensions. These directions of inquiry will be addressed in the near future.

Acknowledgements

W.C. acknowledges support from the Canada Research Chairs Program, the NSERC through grant #238452-01 and the NSF through grant #DMS-0070218. P.G. acknowledges support from the NSERC and a SHARCNET Postdoctoral Fellowship.

References

[1] T.B. Benjamin, T.J. Bridges, Reappraisal of the Kelvin–Helmholtz problem. I. Hamiltonian structure, J. Fluid Mech. 333 (1997) 301–325.
[2] W. Craig, M.D. Groves, Normal forms for wave motion in fluid interfaces, Wave Motion 31 (2000) 21–41.
[3] R. Coifman, Y. Meyer, Nonlinear harmonic analysis and analytic dependence, in: Pseudodifferential Operators and Applications, Notre Dame, IN, 1984, American Mathematical Society, 1985, pp. 71–78.
[4] W. Craig, C. Sulem, Numerical simulation of gravity waves, J. Comput. Phys. 108 (1993) 73–83.
[5] W. Choi, R. Camassa, Fully nonlinear internal waves in a two-fluid system, J. Fluid Mech. 386 (1999) 1–36.
[6] H. Lamb, Hydrodynamics, sixth ed., Dover, New York, 1932, p. 738.
[7] W. Craig, M.D. Groves, Hamiltonian long-wave scaling limits of the water-wave problem, Wave Motion 19 (1994) 367–389.