A Bound on Oscillations in an Unsteady Undular Bore

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Abstract

Using a model equation for the evolution of long waves at the surface of an incompressible fluid, the number of rapid oscillations of an undular bore is estimated. The estimate relies on the analysis of the Burgers-KdV equation in spaces of analytic functions. Special attention is paid to the effect of viscosity.

AMS Subject Classification (2000): 35Q53, 76B15, 76D33

Keywords: Undular bore, KdV-Burgers equation, Jensen's formula, counting oscillations, radius of analyticity

1 Introduction

The tidal bore is a well known phenomenon in fluid mechanics, having been observed in many rivers around the world. Two of the better known rivers where bores regularly appear are the Severn River in England [20] and the Qiantang River in China [18]. The usual circumstances in which a bore can appear develop when a tidal swell causes a difference in surface elevation in the mouth of a river or further upstream. In this case, long waves start to propagate upstream, and if the conditions are favorable, the main front steepens, and a nearly steady transition profile develops. Through field measurements and experiments [10] and [17], it has been found that bores appear in two types. If the ratio of the difference in surface elevation between the two uniform states to the undisturbed depth is greater than approximately 0.75, a so-called turbulent bore may be seen. If this ratio is smaller than 0.28, then the bore tends to exhibit undular character, in other words, the bore will feature oscillations in the downstream part. If the ratio is between 0.28 and 0.75, the bore will be turbulent, but also feature some oscillations.

Quite commonly, the bore is studied in the context of shallow-water theory. In this case, an analysis using conservation of mass and momentum shows that energy must be lost at the front of the bore. In the case of an undular bore, the energy is thought to be disseminated through an increasing number of oscillations behind the bore, while in the turbulent bore, the energy appears to dissipate through turbulent motion at the front of the bore. A turbulent bore featuring some undulations experiences both types of energy loss.

In the case of a purely undular bore, the transition from low to high surface elevation is rather gentle, so that a long-wave approximation is justified. Moreover, as the difference in

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amplitude is small, the assumption of small amplitude may also be used, so that an equation like the Korteweg-deVries (KdV) equation can be used. In dimensional variables, this equation takes the form

$$\eta_t + c_0\eta_x + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x + \frac{1}{6}c_0h_0^2\eta_{xxx} = 0, \tag{1.1}$$

where η is the deflection of the free surface, h_0 is the undisturbed depth of the river, $c_0 = \sqrt{gh_0}$ is the limiting long-wave speed, and g is the gravitational acceleration.

To recall the rationale behind using this equation, we first note that the flow is assumed in a channel or a river whose breadth is hardly varying. Thus it is reasonable to neglect transverse effects. It is also assumed that the river bed is essentially flat. Now if the surface is slowly varying, i.e. if waves are long compared to the depth of the water, and if the change in wave amplitude is also subtle, then the KdV equation is thought to be a reasonable model on short to intermediate time scales for waves that propagate mainly in the direction of increasing values of x.

There have been a number of studies aimed at understanding whether the use of the KdV equation in the description of an undular bore can account for the loss of energy exhibited in the shallow-water theory. Assuming a wavetrain of cnoidal waves behind the bore, Benjamin and Lighthill [1] found that the loss of energy in the front, and the energy contained in the dispersive tail do not agree, thus concluding that dissipation is required even in a purely undular bore. Using experimental results of Favre [10], Sturtevant [22] also found that when the undulations are assumed to have cnoidal shape, there is some excess energy which he attributed to the existence of a boundary layer behind the bore. According to his computations, up to 20% of the excess energy is dissipated by the boundary layer. In light of these findings, it seems expedient to include some form of dissipation directly into the model equation. Perhaps the most obvious way to include viscosity into the description is to consider the so-called KdV-Burgers equation

$$u_t + uu_x + u_{xxx} = \nu u_{xx}. \tag{1.2}$$

This equation appears in non-dimensional form, and in a frame of reference moving with the linear long wave speed c_0. The variable $u(x,t)$ now denotes the nondimensional deflection of the free surface, and the parameter ν represents viscous effects. A formal derivation of (1.2) can be found for instance in [14]. It is evident that the coefficients multiplying the nonlinear and dispersive terms in (1.1) have been scaled out, and appear in front of the dissipation term in (1.2). Thus the parameter ν is not the viscosity per se, but is inversely proportional to the Reynolds number. A dissipative Boussinesq system of similar character as (1.2) has also appeared recently in [7] and [8]. A more sophisticated approach is to include viscous effects only where they are strongest, namely in the boundary layer, thus paying heed to the findings of Sturtevant [22]. Such an analysis leads to a different model equation which features a nonlocal dissipative term, and a derivation of such an equation can be found in the book by Johnson [15]. Still other types of equations are found if one uses a Chezy law for bottom friction. This has been done for instance in the recent work of El, Grimshaw and Kamchatnov [9]. They have also made progress using asymptotic methods applied to a dissipative Boussinesq-type system to study the time development of an undular bore [8].

Beginning already with the work of Peregrine [19], recent studies of undular bores have focused on the time-dependent problem, and in particular on the initial onset of the undular bore and the nature of the developing oscillations. Since the energy at the bore is mainly lost to oscillations, a complete understanding of the physical principles underlying bore-type phenomena and connections with properties of mathematical models such as the ones just mentioned must certainly be based on an understanding of these initial oscillations. On the basis of an analysis using the non-dissipative equation (1.1), Sturtevant expects that new undulations will be generated continuously, and a long time after the bore has been generated, there will be an infinite train of oscillations. On the other hand, Peregrine [19] states that he believes the profile to approach a near-steady state, an outcome that appears to be closer to what is observed in actual river bores. For the equation (1.2), it has been proven by Bona, Rajopadhye and Schonbek [3, 4] that stationary profiles of the KdV-Burgers equation exist, and that they are stable if the Reynolds number is not too big. Thus at least for the case of the simple mathematical model (1.2), a steady profile is possible. If however the initial wave form is far from the steady profile, then it is not clear how the undulations will develop. Given the presence of the dissipative term νu_{xx}, one might expect that any strong oscillations will be damped immediately. However, according to the results of [3, 4, 9], if the parameter ν is small enough, the solution might still have a growing number of oscillations.

In the present article, we are using a method based on rigorous estimates of the solutions of the KdV-Burgers equation in spaces of analytic functions. Using these estimates, we make a first attempt at giving an upper bound on the number of undulations that emerge from an arbitrary initial wave profile. What we are able to show is that given a length scale L, the number of rapid oscillations of u on an interval of length L is proportional to L, thus also giving a lower bound on the average wavelength of these oscillations. This result follows essentially from the analyticity of solutions of (1.2) in the space variable. This analyticity imposes on the solution a certain rigidity which can be exploited to estimate the number of critical points of the function. This in turn leads to an upper bound on the number of oscillations of the surface. The paper is organized as follows. In the next section, the initial-value problem is stated, and some notation established. In Section 3, estimates on the solution are obtained. Finally, in Section 4, a bound on the number of rapid oscillations is found.

2 Setting of the problem

We study the equation (1.2) with initial data given by $u(x,0) = u_0(x)$. It is assumed that the disturbance is localized enough so that boundary effects can be ignored. The equation will therefore be posed on an infinite interval, and it will be required that the solution approaches two different surface elevations upstream and downstream of the bore, namely $\lim_{x\to-\infty} u(x,t) = \rho_1$ for all t, and $\lim_{x\to\infty} u(x,t) = \rho_2$ for all t. The geometry of the problem is schematized in Figure 1. To aid in the analysis of the initial-value problem, it is convenient to use an analytic function $h(x)$, with the same boundary values as u, and to investigate the equation satisfied by $v = u - h$. As will become clear in a moment, the function $h(x)$ must be chosen in such a way that it can be continued to an analytic function $f(z)$ in some strip $\{z = x + iy : |y| < \sigma\}$ around the real axis. To make the computations as explicit as possible we take the special form

$$h(x) = \rho_2 + \frac{\rho_1 - \rho_2}{2}\left[1 - \tanh\left(\frac{x}{\lambda}\right)\right]. \tag{2.1}$$

This function is analytic in the strip $\{z = x + iy : |y| < \pi/2\lambda\}$, and we have $\sup_x |h'(x)| = \frac{\rho_1-\rho_2}{2\lambda}$. The equation for v is

$$v_t + vv_x + v_{xxx} = \nu v_{xx} - (vh)_x + F, \tag{2.2}$$

where $F = \nu h_{xx} - h_{xxx} - hh_x$. For (2.2), we consider the initial-value problem, with initial data

$$v(x,0) = u_0(x) - h(x). \tag{2.3}$$

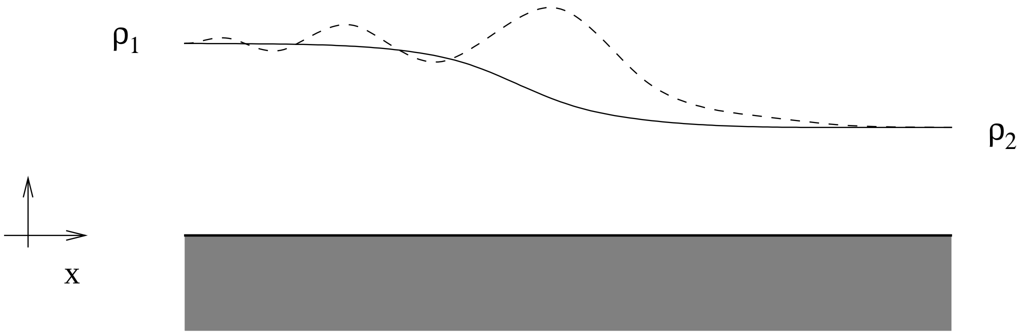

Figure 1: The solid curve shows the function $h(x)$. The dashed curve represents a possible surface profile, and the grey shaded area represents the river bed.

The well posedness of the initial-value problem (2.2), (2.3) has been studied in great detail in [4]. There, it has been shown that under reasonable assumptions on the initial data, a unique solutions exists. In particular, for smooth initial data $v(x,0)$ which are square-integrable on the real line, a solution exists for all time. Here, we focus on analyticity properties of these solutions. Extensive use will be made of the L^2-norm, defined by

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2 \, dx.$$

This norm can also be expressed via the inner product by $\|f\|^2 = (f,f)$, where $(f,g) = \int_{-\infty}^{\infty} f(x)g(x)\,dx$. Here it should be noted that all functions appearing are real when restricted to the real axis. The Fourier transform of a function f is defined by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ix\xi} \, dx,$$

whenever the integral converges. The inverse Fourier transform is defined by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi)e^{ix\xi} \, dx.$$

Of course, for functions $f \in L^2(\mathbb{R})$, a limiting procedure has to be used to define $\hat{f}$. With the help of the Fourier transform, we may define two operators A and $e^{\sigma A}$ that will be of crucial importance in the following development. For $\sigma > 0$, let

$$\widehat{\{Af\}}(\xi) = |\xi|\hat{f}(\xi),$$

and

$$\widehat{\{e^{\sigma A}f\}}(\xi) = e^{\sigma|\xi|}\hat{f}(\xi).$$

The quantity $\|e^{\sigma A}f\|$ can be used as a norm, and it is straightforward to check that a function for which $\|e^{\sigma A}f\|$ is finite is the restriction to the real axis of a function analytic on a symmetric strip of width 2σ. The strip $\{z = x+iy : \, |y| < \sigma\}$ will be denoted by S_σ, and the upper and lower boundaries of S_σ will be denoted by Γ_σ and $\Gamma_{-\sigma}$, respectively. The following result gives a quantitative relation between $\|e^{\sigma A}f\|$ and a path-integral over Γ_σ and $\Gamma_{-\sigma}$.

Proposition 1 *Let $f \in L^2(\mathbb{R})$ be a function for which $\|e^{\sigma A}f\|$ is finite. Then f can be continued analytically to the strip S_σ. Moreover, the following inequalities hold.*

$$\|e^{\sigma A}f\|^2 \leq \int_{\Gamma_\sigma} |f|^2 \, dx + \int_{\Gamma_{-\sigma}} |f|^2 \, dx \;\leq\; 2\,\|e^{\sigma A}f\|^2.$$

Proof: This is essentially the Paley-Wiener theorem. The proof of the first part, i.e. that f can be continued analytically to S_σ can be found in Katznelsen [16], page 174. To obtain the estimate, one makes use of the identity $\widehat{\{f(x-ia)\}}(\xi) = e^{a\xi}\hat{f}(\xi)$, which holds pointwise so long as $|a| < \sigma$, and still holds in the L^2 sense for $|a| = \sigma$. The estimate then follows by taking the limit as $a \to \pm\sigma$, and using Plancherel's formula and the dominated convergence theorem. □

According to Proposition 1, the sum of the two integral expressions in the center of the above inequality may be used as an equivalent norm, denoted by

$$\|f\|_\sigma^2 = \int_{\Gamma_\sigma} |f|^2\, dx + \int_{\Gamma_{-\sigma}} |f|^2\, dx.$$

The number σ is called radius of analyticity.

3 Estimates on the solution

The main thrust in this section will be in the direction of obtaining analyticity in the spatial variable. The first step is an priori estimate for square-integrable solutions. Namely, it will be shown that the L^2-norm of a solution can be controlled for all time.

Proposition 2 *Suppose v is a smooth solution of* (2.2) *and* (2.3) *on* $\mathbb{R} \times (0,T]$. *Suppose the function $h(x)$ is chosen as in* (2.1). *Then the L^2-norm of v can be estimated according to*

$$\|v(\cdot,t)\|^2 \leq \gamma(t), \tag{3.1}$$

where

$$\gamma(t) = e^{(\frac{\rho_1-\rho_2}{2\lambda}+1)t}\|v(\cdot,0)\|^2 + \, te^{(\frac{\rho_1-\rho_2}{2\lambda}+1)t}\, \|F\|^2. \tag{3.2}$$

Proof: Multiply (1.2) by v, and integrate in x to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} v^2\, dx = \nu \int_{-\infty}^{\infty} vv_{xx}\, dx + \int_{-\infty}^{\infty} hvv_x\, dx + \int_{-\infty}^{\infty} vF\, dx.$$

Note that the nonlinear term and the term containing the third derivative integrate out. Two integrations by parts yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} v^2\, dx + \nu\int_{-\infty}^{\infty} v_x^2\, dx \quad &= \quad -\frac{1}{2}\int_{-\infty}^{\infty} v^2 h_x\, dx + \int_{-\infty}^{\infty} vF\, dx \\ &\leq \quad \frac{1}{2}\sup_x |h_x|\, \|v\|^2 + \|v\|\|F\|.\end{aligned}$$

Noticing that the second term on the left is nonnegative, and using the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$, there appears

$$\frac{d}{dt}\|v(\cdot,t)\|^2 \quad \leq \quad \sup_x |h_x(x)|\, \|v\|^2 + \|v\|^2 + \|F\|^2.$$

Finally, using Gronwall's lemma, it follows that

$$\|v(\cdot,t)\|^2 \, \leq e^{(\sup_x |h_x|+1)t}\|v(\cdot,0)\|^2 \, + \, te^{(\sup_x |h_x|+1)t}\, \|F\|^2.$$

Noting that $\sup_x |h_x| = (\rho_1 - \rho_2)/2\lambda$ finishes the proof. □

Proposition 2 establishes a bound on the L^2-norm of a smooth solution. The next order of business is to establish analyticity of a solution $v(x,t)$ as a function of x. Indeed, it will be shown that any solution of (2.2) must be real-analytic, and can be extended analytically to a strip S_σ about the real axis. Moreover, $v(x,t)$ must satisfy a certain bound as stated in the following theorem.

Theorem 1 *Let v be a solution of* (2.2) *and* (2.3)*, and let $T > 0$ be fixed. Let $\gamma(T)$ be defined as in* (3.2)*, let $\kappa > 2$, and define σ_T by*

$$\sigma_T = \min\left\{ \frac{\pi}{4\lambda} \,,\ \frac{1}{2}\,\frac{\nu^3}{\nu^2(1+H^2) + 2\kappa^2\gamma^2(T) + \nu^3(1+C)} \right\}, \tag{3.3}$$

where $H = \sup_{z\in S_{\pi/4\lambda}} |h(z)|$, and $C = \left\| e^{\frac{\pi}{4\lambda}A}(hh_x + \nu h_{xx} + h_{xxx}) \right\|^2$. Then there exists a $t_0 > 0$, such that for any t with $t_0 \le t \le T$, $v(\cdot,t)$ is analytic in S_{σ_T}, and satisfies the bound

$$\|e^{\sigma_T A} v(\cdot,t)\|^2 \le \kappa\gamma(T). \tag{3.4}$$

Before the proof can be given, a couple of lemmas are needed.

Lemma 1 *Let f be a real-analytic function, such that $\|e^{\sigma A} f\|$ and $\|e^{\sigma A} A f\|$ are finite. Then the following inequality holds.*

$$\left(e^{\sigma A}(f f_x), e^{\sigma A} f\right) \le \sup_{x\in\mathbb{R}} |e^{\sigma A} f(x)|\, \|e^{\sigma A} f\|\, \|A e^{\sigma A} f\|.$$

Lemma 2 *Let f be a real-analytic function, such that $\|e^{\sigma A} f\|$ and $\|e^{\sigma A} A f\|$ are finite. Let h be defined as in* (2.1)*. Then the following inequality holds for $\sigma < \pi/2\lambda$.*

$$\left(e^{\sigma A}(hf)_x, e^{\sigma A} f\right) \le\ 2 \sup_{z\in\Gamma_\sigma\cup\Gamma_{-\sigma}} \{|h(z)|\}\ \|e^{\sigma A} f\|\, \|A e^{\sigma A} f\|.$$

The proof of these estimates is standard, and can be found for instance in [11, 12]. For the first lemma, use Parseval's formula to rewrite the inner product in terms of Fourier transforms. Then use the triangle inequality on the exponential factors. For the second lemma, use Cauchy-Schwarz, and then Proposition 1 and the equivalent norm $\|\cdot\|_\sigma$.

Proof of Theorem 1: The proof will be obtained by providing an a priori bound on the quantity $\|e^{\sigma_T A} v(\cdot,t)\|$. To this end, a standard Galerkin procedure is employed. Notice that the space of functions for which the norm $\|e^{\frac{\pi}{4\lambda}A} f\|$ is finite is a separable Hilbert space, and therefore has a countable orthogonal basis. Moreover, this space is also dense in any larger space of functions for which the norm $\|e^{\sigma A} f\|$ with $\sigma < \pi/4$ is finite. The solution v is now approximated by a sequence of smooth functions which are analytic as functions of x, and are also continuous as mappings $t \mapsto \|e^{\frac{\pi}{4\lambda}A} v(\cdot,t)\|$.

A priori estimates are then obtained on the approximating functions. Since this procedure and the ensuing limiting process are standard, we will write the estimates formally in terms of v with the understanding that they are actually obtained for each member of the sequence v_n. Since they hold for each member of the sequence, they also hold for the limit v. The a priori estimates are obtained by means of a differential inequality involving the operator $e^{\sigma(\tau)A}$ where the radius of analyticity is allowed to depend on τ. The differential inequality is obtained in the following way. After applying the operator $e^{2\tau A}$ to the equation (2.2), take the inner product of v and every term in the equation. Using the relation

$$\frac{d}{d\tau}(e^{\tau A} v, e^{\tau A} v) = 2\,(A e^{\tau A} v, e^{\tau A} v) + 2\,(e^{\tau A} v_\tau, e^{\tau A} v),$$

for the shifted function $v(x, t-t_0+\tau)$, there appears the inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}\|e^{\tau A}v\|^2 - \|Ae^{\tau A}v\|\|e^{\tau A}v\| \quad &\leq \quad -\left(e^{\tau A}(vv_x), e^{\tau A}v\right) - \left(e^{\tau A}(hv)_x, e^{\tau A}v\right) \\ &\quad + \nu\left(e^{\tau A}v_{xx}, e^{\tau A}v\right) + \left(e^{\tau A}F, e^{\tau A}v\right).\end{aligned}$$

The term containing the third derivative is not present because the third derivative is skew-adjoint. Using the estimates in Lemma 2 and Lemma 3, the Cauchy-Schwarz inequality, and the basic inequality $\sup_x |f(x)| \leq \frac{1}{2}\|f\|^{\frac{1}{2}}\|Af\|^{\frac{1}{2}}$, there appears

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}\|e^{\tau A}v\|^2 + \nu\|Ae^{\tau A}v\|^2 \quad &\leq \quad \|e^{\tau A}v\|\|Ae^{\tau A}v\| + \|e^{\tau A}v\|^{\frac{3}{2}}\|Ae^{\tau A}v\|^{\frac{3}{2}} \\ &\quad + \sup_{z\in\Gamma_\tau\cup\Gamma_{-\tau}}|h(z)|\|e^{\tau A}v\|\|Ae^{\tau A}v\| + \|e^{\tau A}F\|\|e^{\tau A}v\|.\end{aligned}$$

Observe that $\sup_{z\in\Gamma_\tau\cup\Gamma_{-\tau}}|h(z)|$ is finite as long as $\tau < \pi/2\lambda$. We will take $\tau < \pi/4\lambda$ to be safely inside the domain of analyticity of h. Next, using Young's inequality as before, and in the form $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$, where $\frac{1}{p}+\frac{1}{q}=1$ and $q=4$ on the nonlinear term, there appears the inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}\|e^{\tau A}v\|^2 + \nu\|Ae^{\tau A}v\|^2 \quad &\leq \quad \frac{\nu}{2}\|Ae^{\tau A}v\|^2 + \frac{1}{2\nu}\left(1+H^2\right)\|e^{\tau A}v\|^2 \\ &\quad + \frac{\nu}{2}\|Ae^{\tau A}v\|^2 + \frac{27}{32}\frac{1}{\nu^3}\|e^{\tau A}v\|^6 \\ &\quad + \frac{1}{2}\|e^{\tau A}F\|^2 + \frac{1}{2}\|e^{\tau A}v\|^2,\end{aligned}$$

where $H = \sup_{z\in S_{\pi/4\lambda}}|h(z)|$. Finally, we obtain the differential inequality

$$\frac{d}{d\tau}\|e^{tA}v\|^2 \leq c_1\,\|e^{\tau A}v\|^2 + c_2\,\|e^{\tau A}v\|^6 + C, \tag{3.5}$$

where the constants c_1 and c_2 are given by

$$c_1 = \frac{1}{\nu}(1+H^2)+1, \qquad \text{and} \qquad c_2 = \frac{2}{\nu^3}.$$

Recall that C was defined by

$$C = \left\|e^{\frac{\pi}{4\lambda}A}F\right\|^2 = \left\|e^{\frac{\pi}{4\lambda}A}(hh_x + \nu h_{xx} + h_{xxx})\right\|^2,$$

and that inequality (3.5) is valid as long as $\tau < \pi/4\lambda$. To determine the radius of analyticity at some t, first define

$$t_0 = \min\left\{\frac{\pi}{4\lambda}\ ,\ \frac{(\kappa-1)\gamma(T)}{c_1\,\kappa\gamma(T) + c_2\,\kappa^3\gamma^3(T) + C}\right\}. \tag{3.6}$$

The reason for this choice will become more transparent in a moment, as will the fact, that t_0 can also be chosen smaller if necessary to allow for $t_0 < T$. On the other hand, t_0 decreases as a function of T, so that a large enough choice of T will also suffice. Pick a time t in the interval $[t_0, T]$, and observe that $\|v(\cdot, t-t_0)\|^2 \leq \gamma(T)$ by Proposition 2. Since $\|e^{\tau A}v(\cdot, t-t_0+\tau)\|$ is a continuous function of τ, it is apparent that the inequality

$$\|e^{\tau A}v(\cdot, t-t_0+\tau)\|^2 \leq \kappa\,\gamma(T) \tag{3.7}$$

will hold for $\kappa > 2$ and small enough τ. Thus for those τ, the differential inequality (3.5) becomes

$$\frac{d}{d\tau}\|e^{\tau A}v(\cdot,t-t_0+\tau)\|^2 \quad \leq \quad c_1\,\kappa\gamma(T)+c_2\,\kappa^3\gamma^3(T)+C.$$

Integrating this relation yields

$$\|\, e^{\tau A}v(\cdot,t-t_0+\tau)\|^2 \leq \Big(c_1\,\kappa\gamma(T)+c_2\,\kappa^3\gamma^3(T)+C\Big)\,\tau\ \ +\gamma(T).$$

Now this estimate shows that (3.7) holds for $0 \leq \tau \leq t_0$ if t_0 is defined as in (3.6). But since the operator $e^{\tau A}$ is used in (3.7), t_0 represents the radius of analyticity of $v(\cdot, t-t_0+\tau)$ at $\tau = t_0$, or in other words of $v(\cdot,t)$. Hence $\|e^{t_0 A}v(\cdot,t)\|$ is finite, and what's more,

$$\|e^{t_0 A}v(\cdot,t)\|^2 \leq \kappa\gamma(T)$$

by (3.7), so that we may define $\sigma(t) = t_0$. Then assuming that $\gamma(T) \geq 1$ and using the expressions for c_1 and c_2, it transpires that

$$\begin{aligned}\frac{(\kappa-1)\gamma(T)}{c_1\,\kappa\gamma(T)+c_2\,\kappa^3\gamma^3(T)+C} \quad &\geq \quad \frac{\kappa-1}{\kappa}\frac{1}{c_1\ +c_2\,\kappa^2\gamma^2(T)+C} \\ &\geq \quad \frac{1}{2}\,\frac{1}{\frac{1}{\nu}(1+H^2)+1+\frac{2}{\nu^3}\kappa^2\gamma^2(T)+C} \\ &= \quad \frac{1}{2}\,\frac{\nu^3}{\nu^2(1+H^2)+2\kappa^2\gamma^2(T)+\nu^3(1+C)}.\end{aligned}$$

Thus defining σ_T as in (3.3), it appears immediately that

$$\|e^{\sigma_T A}v(\cdot,t)\|^2 \leq \|e^{t_0 A}v(\cdot,t)\|^2 \leq \kappa\gamma(T),$$

and (3.4) is satisfied. Since same procedure as above can be applied to any t with $t_0 \leq t \leq T$, the theorem is proved. □

A few remarks are in order. First, we note that this theorem does not supply a bound for the initial time $0 < t < t_0$. However, an argument similar to the one used in the proof just given can be used to establish bounds for $0 < t < t_0$. If the initial data are not analytic, then smaller radii of analyticity are obtained for $0 < t < t_0$. The initial growth of $\sigma(t)$ can be optimized by using for the operator $e^{\sqrt{\tau}A}$ in the proof. We have chosen not to provide the details here in order to keep the estimates in the next section relatively simple. Next, it should be remarked that the radius of analyticity σ_T is limited above by $\frac{\pi}{4\lambda}$ due to the choice of h. Furthermore, the definition of σ_T appearing in (3.3) is not sharp, but is chosen to make the expression as explicit as possible. A final remark concerns the asymptotic behavior of σ_T. The expression for σ_T yields the following asymptotic estimates for $T \sim 1$:

$$\begin{aligned}\sigma_T &\sim \frac{\nu^3}{4\kappa\gamma^2(T)} \qquad \text{for} \qquad \nu \to 0, \\ \sigma_T &\sim \tfrac{1/2}{1+C} \qquad\qquad \text{for} \qquad \nu \to \infty.\end{aligned}$$

Notice that in the second estimate, the radius of analyticity does not become infinite, but is limited by the choice of h. Note also that the KdV equation has only been proved to be valid on short to intermediate time scales [5, 6], so that $T \sim 1$ is an appropriate choice.

4 Bound on oscillations

In this final section, it will be our goal to obtain a bound on the number of spatial oscillations of $v(\cdot,t)$ at a given time t. The first step is to deduce from Theorem 1 a bound on the spatial derivative of v. The next few results are of a general nature, so that we use the notation f for a generic function. We will return to v in Theorem 4.

Lemma 3 *Let f be analytic in the strip S_σ. Then there exists a constant c, such that*

$$\sup_{x+iy\in S_{\sigma/2}} |f'(x+iy)| \leq c\,\|e^{\sigma A} f\|. \tag{4.1}$$

Proof: The Sobolev inequality implies that for fixed $|y| \leq \sigma$, there is a constant c_3 such that,

$$\sup_{x+iy\in S_{\sigma/2}} |f'(x+iy)| \leq c_3 \left(\|f(\cdot+iy)\| + \|A^2 f(\cdot+iy)\|\right).$$

Rewriting the right hand side using Parseval's identity, it appears that

$$\begin{aligned}\sup_{x+iy\in S_{\sigma/2}} |f'(x+iy)| &\leq c_3 \left(\|e^{y|\xi|}\,\hat{f}\| + \|e^{y|\xi|}|\xi|^2\,\hat{f}\|\right)\\ &\leq c_3 \left(\|e^{(\sigma/2)|\xi|}\,\hat{f}\| + \|e^{(\sigma/2)|\xi|}|\xi|^2\,\hat{f}\|\right).\end{aligned}$$

Thus to prove the lemma, we need to show that

$$\|e^{(\sigma/2)|\xi|}|\xi^2|\hat{f}(\xi)\| \leq c_4\,\|e^{\sigma|\xi|}\hat{f}(\xi)\|,$$

for some constant c_4. But this will be achieved by setting

$$c_4 = \max_{\xi\in\mathbb{R}} \left\{e^{-(\sigma/2)|\xi|}|\xi|^2\right\}.$$

□

The bound on oscillations will be achieved by making use of Jensen's formula, which we state in the following form:

Theorem 2 *Let f be analytic in an open disk $\{z : |z-z_0| < r\}$ of radius r about a point $z_0 \in \mathbb{C}$. Assume that $f(z_0) \neq 0$. Let $\alpha_1, \alpha_2, \ldots, \alpha_N$ be the zeros of f in the closed disk $\{z : |z-z_0| \leq r/2\}$ of radius $r/2$ about z_0. Then the following identity holds.*

$$|f(z_0)| \prod_{n=1}^{N} \frac{r/2}{|\alpha_n|} = exp\left\{\frac{1}{\pi r}\int_{|z-z_0|=r/2} \log|f(z-z_0)|dz\right\}. \tag{4.2}$$

This identity is classical, and can be found in most standard texts on complex analysis, such as [21]. An immediate consequence of Jensen's formula is the following bound on the number of zeros of the function f, contained in the smaller disk $\{z : |z-z_0| < r/4\}$.

Corollary 1 *Let $z_0 \in \mathbb{C}$ and $r > 0$. Let f be analytic in $\{z : |z-z_0| < r\}$, and suppose that $f(z_0) \neq 0$. Then*

$$\text{card}\,\{z : |z-z_0| \leq r/4,\ f(z) = 0\} \leq \frac{1}{\log 2} \log \frac{\max_{|z-z_0|=r/2}|f(z)|}{|f(z_0)|}$$

The proof is straightforward if it is observed that the product in formula (4.2) is greater than 2^N. Taking the logarithm and estimating the integral then yields the required inequality for $N = \text{card}\,\{z : |z-z_0| \leq r/4,\ f'(z) = 0\}$. For a full proof, the reader may consult [21] or [12]. Finally, with this estimate on the number of zeros of an analytic function in a subdisk, the following theorem can be proved.

Theorem 3 *Let $L > 0$, $r > 0$, and let f be analytic in $\{z = x + iy : |y| < r\}$. Then for any $\varepsilon > 0$, the interval $[0, L] \in \mathbb{R}$ is given by the union $I_\varepsilon \cup R_\varepsilon$, where I_ε is a union of at most $2\frac{L}{r}$ open intervals, and*

(i) $|f'(x)| < \varepsilon$, *for all* $x \in I_\varepsilon$,

(ii) $\mathrm{card}\{x \in R_\varepsilon : f'(x) = 0\} \leq \dfrac{2}{\log 2} \dfrac{L}{r} \log \dfrac{\max_{Im(z) \leq r/2} |f'(z)|}{\varepsilon}$.

Proof: Let $x_1 = \inf\{x \in [0, L] : |f'(x)| \geq \varepsilon\}$. Then by Corollary 1, we have

$$\mathrm{card}\,\{x \in [x_1 - r/4, x_1 + r/4] : |f'(x)| = 0\} \leq \frac{1}{\log 2} \log \frac{\max_{|z - x_1| = r/2} |f'(z)|}{\varepsilon}.$$

Next, define $x_2 = \inf\{x \in [x_1 + r/4, L] : |f'(x)| \geq \varepsilon\}$, and by applying Corollary 1 again, we get exactly the same estimate on $[x_2 - r/4, x_2 + r/4]$. We can continue this process at most $N = L/(r/2)$ times, and finally obtain the lemma for $R_\varepsilon = \left(\bigcup_{i=1}^{N} [x_i - r/4, x_i + r/4]\right) \cap [0, L]$ and $I_\varepsilon = [0, L] - R_\varepsilon$. □

The main result of this paper now emerges. Note first that the function $v(\cdot, t)$ is analytic in the strip S_{σ_T} for $t_0 \leq t \leq T$, and that

$$\sup_{x+iy \in S_{\sigma_T/2}} |v_x(x + iy, t)| \leq c\,\kappa\,\gamma(T).$$

by Theorem 1 and Lemma 3. Moreover, the estimate in Theorem 3 applies to any interval of length L in the real line. The proof can be adapted by simply translating the function. Therefore the following result has been proved.

Theorem 4 *Let v be the solution of (2.2) and (2.3), and let $L > 0$, $T > 0$ and $\varepsilon > 0$ be given. Fix a time $t \in [t_0, T]$. Then any interval of length L contained in the real line, is given by the union $I_\varepsilon \cup R_\varepsilon$, where I_ε is a union of at most $2\frac{L}{\sigma_T}$ intervals, and*

(i) $|v_x(x, t)| < \varepsilon$, *for all* $x \in I_\varepsilon$,

(ii) $\mathrm{card}\{x \in R_\varepsilon : v_x(x, t) = 0\} \leq \dfrac{2}{\log 2} \dfrac{L}{\sigma_T} \log \left(\dfrac{c\,\kappa\,\gamma(T)}{\varepsilon} \right)$.

Observe that the collection of sets I_ε and R_ε depends both on the time t, and on the particular interval chosen. The important result is that the number of zeros of $v_x(x, t)$ is bounded by the same constant for all t with $t_0 < t < T$, and for all intervals of length L. Next, notice that the cardinality of the set appearing in (ii) in Theorem 4 is a number which is equal to twice the number of oscillations of $v(x, t)$ on R_ε at a given time t. On the other hand, the intervals that fall under (i) feature oscillations that are arbitrary slow (depending on ε), so that they can be essentially disregarded when counting rapid oscillations. Since the function $h(x)$ is monotone, and $v(x, t) = u(x, t) - h(x)$, we see that the number of rapid oscillations of $u(x, t)$ on a given interval of length L at a time t is linearly proportional to L. This finding has two immediate consequences. First, we may conclude that the average wavelength of oscillations of the free surface behind an undular bore is bounded below by a constant proportional to $1/L$. In addition, it appears that the waves cannot concentrate on a particular location, but must be spread out over a larger domain.

Noticing from (3.3) that σ_T gets smaller as T gets bigger, we see that the bound in (ii) in Theorem 4 gets worse as $T \to \infty$. On the other hand, this limit is not necessarily relevant, because the equation is only valid on intermediate time scales.

Recalling the asymptotic limits of ν found at the end of Section 3, we see that for small ν, the constant σ_T also tends to zero, so that the bound in (ii) in Theorem 4 is meaningless. This is of course expected, as $\nu \to 0$ approximates an inviscid theory. On the other hand, if

ν approaches infinity, the bound is limited by the choice of the function h, but dependence on T is weaker, so that in this case, the bound in (ii) is valid for larger times.

In conclusion, it is found that the presence of viscosity inhibits the development of oscillations behind an undular bore. This is of course what one would expect on physical grounds. A quantitative estimate is found in the form of an upper bound on the number of oscillations, but we do not expect this estimate to be sharp for several reasons. For one, there are a number of auxiliary parameters, such λ and ε which have a direct impact on the estimate (ii). Another drawback of our result is that it is gives a poor estimate for large times. However, this problem is mitigated by the fact that the equation itself is only valid up to intermediate time scales. This also shows that the question raised in the introduction about a possibly infinite number of oscillations having appeared after a long time is mostly of an academic nature. An analysis focusing on a steady profile affords a finer analysis with respect the effect of viscosity on the number of oscillations [3], but it is limited in that it neglects dynamical aspects of the problem, and therefore does not allow for the growth of new oscillations over time.

Acknowledgments: This work was supported in part by the Research Council of Norway. The authors also wish to thank two anonymous referees whose comments helped to significantly improve the exposition of the paper.

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