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On persistence of spatial analyticity for the dispersion-generalized periodic KdV equation

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A B S T R A C T

Persistence of spatial analyticity is studied for periodic solutions of the dispersion-generalized KdV equation $u_t - |D_x|^\alpha u_x + uu_x = 0$ for $\alpha \geq 2$. For a class of analytic initial data with a uniform radius of analyticity $\sigma_0 > 0$, we obtain an asymptotic lower bound $\sigma(t) \geq ct^{-p}$ on the uniform radius of analyticity $\sigma(t)$ at time t, as $t \to \infty$, where $p = \max(1, 4/\alpha)$. The proof relies on bilinear estimates in Bourgain spaces and an approximate conservation law.

1. Introduction

Consider the Cauchy problem for the dispersion-generalized KdV equation posed on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$,

$$\begin{cases} \partial_t u - \partial_x |D_x|^\alpha u + u\partial_x u = 0, & u(x,t)\colon \mathbb{T}\times\mathbb{R} \to \mathbb{R}, \\ u(x,0) = \phi(x), \end{cases} \tag{1}$$

for a given number $\alpha \geq 2$. Here $|D_x|^\alpha$ is the Fourier multiplier given by $\widehat{|D_x|^\alpha \phi}(k) = |k|^\alpha \widehat{\phi}(k)$, where $\widehat{\phi}\colon \mathbb{Z} \to \mathbb{C}$ is the Fourier transform of $\phi\colon \mathbb{T} \to \mathbb{R}$. For $\alpha = 2$ one recovers the Cauchy problem for the periodic KdV equation, which has been extensively studied.

In this work, we are interested in the persistence of spatial analyticity for the solutions of this Cauchy problem, given initial data in a class of analytic functions. This is motivated naturally by observing that

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many special solutions of (1) such as for instance solitary and cnoidal waves in the case $\alpha = 2$ are analytic in a strip about the real axis.

Concerning the Cauchy problem (1) in the case $\alpha = 2$ and with initial data which are members of a space of analytic functions, it was established in [1–3] that the radius of analyticity is locally constant, and [1] also contains an argument proving that the radius of analyticity will be non-zero for all time under certain conditions. In [4], exponential lower bounds on the decrease of the radius of analyticity were established; however, as also mentioned in that work, exponential decay of the radius of analyticity is not an optimal result, and could lead to the perception of non-analyticity in a short time. This scenario would be particularly problematic if the analyticity properties of solutions of (1) are used to analyze numerical schemes or physical implications of the equations.

In the body of this paper, we will prove algebraic lower bounds on the radius of analyticity. These estimates will be useful in the analysis of convergence of infinite-order numerical methods, such as Fourier-spectral methods [5,6,4] since such a numerical treatment in general requires periodic boundary conditions. In addition, as shown in [5], accurate information about the analyticity of the solutions can aid in a more judicious choice of the number of modes to be used in the numerical approximation, and thereby lead to improved performance.

On the other hand, if more general boundary conditions are to be incorporated, one may look to the application of inertial manifold methods, such as explained in [7]. If these methods are to be used for the approximation of solutions of dissipative models such as the Kuramoto–Sivashinsky equation [8] or Navier–Stokes equations [9], it is important to have good estimates on the determining modes of the attractor, such as provided for instance in [10,11]. Quantitative information about the radius of analyticity of solutions may also be used in the context of the Navier–Stokes equations, or dispersive–dissipative models such as the KdV–Burgers equation (see [12,13]) to obtain information about qualitative properties of the solutions which are of physical relevance.

By the Paley–Wiener Theorem, the radius of analyticity of a function can be related to decay properties of its Fourier transform. It is therefore natural to take data for (1) in the Gevrey space $G^{\sigma,s}(\mathbb{T})$, defined by the norm

$$\|\phi\|_{G^{\sigma,s}(\mathbb{T})} = \left(\sum_{k\in\mathbb{Z}} e^{2\sigma|k|} \langle k\rangle^{2s} |\widehat{\phi}(k)|^2 \right)^{1/2} \qquad (\sigma \geq 0,\ s \in \mathbb{R})$$

where $\langle k\rangle = (1 + |k|^2)^{1/2}$. When $\sigma = 0$, this space reduces to the Sobolev space $H^s(\mathbb{T})$, with norm

$$\|\phi\|_{H^s(\mathbb{T})} = \left(\sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} |\widehat{\phi}(k)|^2 \right)^{1/2},$$

while for $\sigma > 0$, any function in $G^{\sigma,s}(\mathbb{T})$ has a radius of analyticity of at least σ at each point $x \in \mathbb{T}$. Indeed, we recall the following (see, e.g., [14]).

Paley–Wiener Theorem. *Let $\sigma > 0$ and $s \in \mathbb{R}$. A 2π-periodic function $\phi(x)$ belongs to $G^{\sigma,s}(\mathbb{T})$ if and only if it is the restriction to the real line of a function $\Phi(x + iy)$ which is 2π-periodic in x, holomorphic in the strip $\{x + iy\colon |y| < \sigma\}$, and satisfies $\sup_{|y|<\sigma} \|\Phi(\cdot + iy)\|_{H^s(\mathbb{T})} < \infty$.*

The spaces $G^{\sigma,s}(\mathbb{T})$ were introduced by Foias and Temam [9] in the study of spatial analyticity of periodic solutions to the Navier–Stokes equations, and various refinements of their method have since been applied to prove lower bounds on the radius of spatial analyticity for a number of nonlinear evolution equations [15–20,2,21,3,22–26]. The method we apply here relies heavily on Bourgain's Fourier restriction norm method [27] and further developments of that method from [28,29].

In [27], Bourgain introduced the $X^{s,b}$ spaces and used them to prove local well-posedness for the KdV equation (the case $\alpha = 2$ of (1)) with data in $H^s(\mathbb{T})$ for $s \geq 0$; since the L^2 norm is conserved in the evolution, this result then implies global well-posedness. Bourgain's method was pushed further by Kenig et al. [28], who obtained local well-posedness for the KdV equation in $H^s(\mathbb{T})$ for $s \geq -1/2$, a result that was extended globally in time by Colliander et al. [29]. The exponent $s = -1/2$ turns out to be critical for the regularity of the initial data if one requires that the flow map be smooth (see [30]) or locally uniformly continuous (see [31]), but can be improved if one only requires continuity (see [32,33]).

The well-posedness theory in $H^s(\mathbb{T})$ for (1) in the case $\alpha = 2$ is thus completely understood. To some extent this theory has also been extended to the higher dispersion case $\alpha > 2$, which appears naturally in several applications (see [34] and the references therein). Gorsky and Himonas [35] extended the result from [28] to all even integers $\alpha \geq 2$, showing that $s \geq -1/2$ remains sufficient for local well-posedness in $H^s(\mathbb{T})$. This was improved to $s \geq -\alpha/4$ by Hirayama [36] and also by Li and Shi [37], who removed the assumption that α be an even integer. Hirayama also showed that $s = -\alpha/4$ is sharp in the sense that the bilinear estimate in the $X^{s,b}$ spaces fails for $s < -\alpha/4$. Using more complicated spaces, Kato [38] proved that for $\alpha = 4$ one has local well-posedness for $s \geq -3/2$, and that this is sharp if one requires smoothness of the flow map. Recently, Kato's result was extended to all even integers $\alpha \geq 2$ by Yan et al. [39], with the sharp exponent $s = 1/2 - \alpha/2$.

In this paper we study the well-posedness of (1) in the Gevrey space $G^{\sigma,s}(\mathbb{T})$. We have the following main result.

Theorem 1. *Given $\alpha \geq 2, \sigma_0 > 0, s \geq -\alpha/4$, and a real-valued $\phi \in G^{\sigma_0,s}(\mathbb{T})$, then for any $T > 0$ the solution to (1) satisfies*

$$u \in C\left([-T,T]; G^{\sigma(T),s}(\mathbb{T})\right)$$

where

$$\sigma(T) = \min\left\{\sigma_0, \frac{c}{T^p}\right\}, \quad p = \begin{cases} \dfrac{4}{\alpha} & \text{if } 2 \leq \alpha \leq 4, \\ 1 & \text{if } \alpha > 4, \end{cases}$$

and $c > 0$ is a constant depending on ϕ, α, σ_0 and s.

In view of the Paley–Wiener theorem, this result implies that $u(\cdot,t)$ has radius of analyticity at least $\sigma(|t|)$ for every $t \in \mathbb{R}$.

Concerning the local problem, we mention that local well-posedness of (1) in $G^{\sigma,s}(\mathbb{T})$ has been proved by Li [3] for $s \geq 1$ and $\alpha = 2$, and by Gorsky et al. [22] for $s \geq -\alpha/4$ with $\alpha \geq 2$ an even integer.

We have no reason to believe that the lower bounds obtained here are sharp, but they seem to be the best possible with the method used, which we now briefly outline. The first step is to prove the following local-in-time result, where the radius of analyticity remains constant.

Theorem 2. *Given $\sigma > 0$ and $s \geq -\alpha/4$, then for any $\phi \in G^{\sigma,s}(\mathbb{T})$ there exists a time $\delta = \delta(\|\phi\|_{G^{\sigma,s}(\mathbb{T})}) > 0$ and a unique solution $u \in C([-\delta,\delta]; G^{\sigma,s}(\mathbb{T}))$ of the Cauchy problem (1) on $\mathbb{T} \times (-\delta,\delta)$. Moreover, $\delta = c_0(1 + \|\phi\|_{G^{\sigma,s}(\mathbb{T})})^{-a}$ for some constants $a, c_0 > 0$ depending only on s and α.*

The second step is to prove an approximate conservation law for the $G^{\sigma,0}(\mathbb{T})$ norm of the solution, which involves $\sigma > 0$ as a small parameter and which reduces to the exact conservation law in $L^2(\mathbb{T})$ in the limit $\sigma \to 0$.

Theorem 3. *Given $\sigma > 0$ and $\phi \in G^{\sigma,0}(\mathbb{T})$, let $u \in C([-\delta,\delta]; G^{\sigma,0}(\mathbb{T}))$ be the local solution of (1) obtained in Theorem 2 (with $s = 0$). Setting*

$$N(t) = \|u(\cdot,t)\|^2_{G^{\sigma,0}(\mathbb{T})},$$

we then have

$$\sup_{|t|\le\delta} N(t) \le N(0) + C\sigma^{\min(1,\alpha/4)} N(0)^3,$$

where the constant $C > 0$ depends only on α.

Applying the last two theorems repeatedly, then by taking $\sigma > 0$ small enough we can cover any time interval $[-T, T]$ and obtain the main result, Theorem 1.

The method used here for proving lower bounds on the radius of analyticity was introduced in [25] in the context of the 1D Dirac–Klein–Gordon equations. It was applied to the non-periodic KdV equation in [26], where the rate $t^{-(4/3+\varepsilon)}$ was obtained, improving an earlier result of Bona et al. [18], who obtained t^{-12} by a different approach. Note that the rate $t^{-(4/3+\varepsilon)}$ for the KdV equation in the non-periodic case is much better than the rate t^{-2} obtained here in the periodic case (take $\alpha = 2$ in Theorem 1). The explanation for this is that the dispersion for the KdV equation on the circle is weaker than on the real line, as is also reflected in the different behavior concerning H^s well-posedness in the two cases (see [28,29]).

The remainder of this paper is devoted to the proof of Theorem 1. In the next section we introduce the necessary function spaces. In Section 3 we prove the local well-posedness, Theorem 2. The almost conservation law, Theorem 3, is proved in Section 4. Finally, the main result is proved in Section 5.

To prove Theorem 2 we use a modification of the contraction argument used to prove the corresponding result in the Sobolev spaces $H^s(\mathbb{T})$, which is based on appropriate bilinear estimates in $X^{s,b}$ norms. The key observation is that these estimates continue to hold when the norm is "Gevrey-modified" by inserting the Fourier multiplier $e^{\sigma|D_x|}$, since the triangle inequality $|\xi| \le |\xi-\eta| + |\eta|$ (applied in Fourier space) implies

$$e^{\sigma|\xi|} \le e^{\sigma|\xi-\eta|} e^{\sigma|\eta|}. \tag{2}$$

This observation goes back to Bourgain [1] in the context of the Kadomtsev–Petviashvili equation. In the same paper, Bourgain also gave a simple argument to show that if, for some s, one has local well-posedness in H^s and moreover the H^s norm is conserved in the evolution, then spatial analyticity persists globally in time, but no lower bound on the radius of analyticity is obtained. In fact, this argument can easily be extended to give an exponential lower bound $\sigma(t) \ge \sigma_0 e^{-a|t|}$, but this is of course much weaker than the algebraic bounds obtained in Theorem 1.

As in earlier works on the periodic problem, we reduce from the start to the case of initial data with mean zero, that is,

$$\int_{\mathbb{T}} \phi\, dx = 0.$$

Indeed, if one has proved Theorem 1 for this case, then it follows that the theorem holds also without this assumption. This can be seen from the invariance of (1) under the Galilean transformation

$$u(x,t) \longrightarrow v(x,t) := u(x+ct,t) - c, \qquad \phi(x) \longrightarrow \psi(x) := \phi(x) - c,$$

and from the identity

$$\|u(\cdot,t)\|^2_{G^{\sigma,s}(\mathbb{T})} = 2\pi c^2 + \|v(\cdot,t)\|^2_{G^{\sigma,s}(\mathbb{T})}.$$

Henceforth we therefore assume $\int_{\mathbb{T}} \phi\, dx = 0$. Then the solution $u(\cdot,t)$ will also have mean zero for every t, as is to be expected, since $\int_{\mathbb{T}} u(x,t)\, dx$ is a conserved quantity.

2. Notation and function spaces

The Fourier transform on $\mathbb{T}$ is defined by

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} e^{-ikx} f(x)\, dx \quad (k \in \mathbb{Z}),$$

so the inversion formula reads

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} e^{ikx} \widehat{f}(k)$$

and we have Parseval's identity

$$\int_{\mathbb{T}} f(x)\overline{g(x)}\, dx = \sum_{k\in\mathbb{Z}} \widehat{f}(k)\overline{\widehat{g}(k)}.$$

Set $D_x = -i\partial_x$. Given a function $m\colon \mathbb{R} \to \mathbb{C}$, let $m(D_x)$ be the Fourier multiplier with symbol $m(k)$, that is,

$$\left(m(D_x)f\right)(x) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} e^{ikx} m(k)\widehat{f}(k).$$

Note that if $\overline{m(-k)} = m(k)$ for all $k \in \mathbb{Z}$, then $m(D_x)f$ is real-valued whenever f is. For example, this holds for $e^{\sigma|D_x|}$ and $|D_x|^\alpha$, and also for the unitary group $S(t)$ associated to the linear part of (1), given by

$$\widehat{S(t)f}(k) = e^{itk|k|^\alpha} \widehat{f}(k).$$

To prove local existence for (1), we rely on a contraction argument in certain spaces of functions on the space–time $\mathbb{T} \times \mathbb{R}$. First of all, for given $s, b \in \mathbb{R}$ we will use the space $X^{s,b} = X^{s,b}(\mathbb{T} \times \mathbb{R})$, introduced by Bourgain [27], with norm

$$\|u\|_{X^{s,b}} = \left(\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \langle k\rangle^{2s} \langle \tau - k|k|^\alpha\rangle^{2b} |\widetilde{u}(k,\tau)|^2 \, d\tau \right)^{1/2}.$$

Here we use the notation

$$\langle\cdot\rangle = \left(1 + |\cdot|^2\right)^{1/2},$$

and

$$\widetilde{u}(k,\tau) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{T}} e^{-i(kx+\tau t)} u(x,t)\, dx\, dt \quad (k \in \mathbb{Z}, \tau \in \mathbb{R})$$

is the space–time Fourier transform, whose inversion reads

$$u(x,t) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{i(kx+\tau t)} \widetilde{u}(k,\tau)\, d\tau. \tag{3}$$

We remark that the Banach space $X^{s,b}$ can either be realized directly as a subspace of the tempered distributions on $\mathbb{R}_x \times \mathbb{R}_t$ with periodicity in x, by means of the Fourier transform, or it can be defined as the completion of a suitable subspace of $C^\infty(\mathbb{T} \times \mathbb{R})$ with respect to the above norm, for example, the space of all u given by (3) with $\widetilde{u}\colon \mathbb{Z} \times \mathbb{R} \to \mathbb{C}$ some smooth and compactly supported function.

The study of well-posedness of the periodic KdV equation in [27,28] was based on iteration in the space $X^{s,1/2}(\mathbb{T})$. As observed in [29], however, this space is deficient in the sense that it does not embed into $C(\mathbb{R}; H^s(\mathbb{T}))$. Following [29], we fix this problem by adding an extra term to the norm, defining

$$\|u\|_{Y^s} = \|u\|_{X^{s,1/2}} + \left(\sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} \left(\int_{\mathbb{R}} |\widetilde{u}(k,\tau)|\, d\tau \right)^2 \right)^{1/2}.$$

The additional term in the norm ensures that the desired embedding holds.

Lemma 1. *Let $s \in \mathbb{R}$. For any $u \in Y^s$,*

$$\sup_{t\in\mathbb{R}} \|u(\cdot,t)\|_{H^s(\mathbb{T})} \le \|u\|_{Y^s}\,.$$

Moreover, $Y^s \hookrightarrow BC(\mathbb{R}; H^s(\mathbb{T}))$, where the latter denotes the Banach space of bounded and continuous maps $t \mapsto u(\cdot,t)$ from $\mathbb{R}$ into $H^s(\mathbb{T})$, with the sup norm.

Proof. By density we may assume that $\widetilde{u}\colon \mathbb{Z}\times\mathbb{R} \to \mathbb{C}$ is bounded and compactly supported. By Fourier inversion in time,

$$\widehat{u}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{it\tau} \widetilde{u}(k,\tau)\, d\tau, \tag{4}$$

hence

$$|\widehat{u}(k,t)| \le \int_{\mathbb{R}} |\widetilde{u}(k,\tau)|\, d\tau,$$

and it follows that

$$\|u(\cdot,t)\|_{H^s(\mathbb{T})}^2 = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} |\widehat{u}(k,t)|^2 \le \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} \left(\int_{\mathbb{R}} |\widetilde{u}(k,\tau)|\, d\tau\right)^2. \tag{5}$$

This proves the desired inequality. Moreover, it is straightforward to check that the map $t \mapsto u(\cdot,t)$ belongs to $C(\mathbb{R}; H^s(\mathbb{T}))$, by using (4).

By a density argument, the same properties then extend to any $u \in Y^s$. Indeed, for $j \in \mathbb{N}$ choose u_j such that $\widetilde{u}_j$ is bounded and compactly supported, and such that $u_j \to u$ in Y^s. Then $u_j \in C(\mathbb{R}; H^s(\mathbb{T}))$ and (5) applied to $u_j - u_k$ shows that u_j is Cauchy in this space. □

To run an iteration argument for (1) in Y^s one needs estimates in that space for the solution of the Cauchy problem

$$\partial_t u - \partial_x |D_x|^\alpha u = F(x,t), \quad u(x,0) = f(x), \tag{6}$$

for given $F(x,t)$ and $f(x)$ periodic in x. Notice that on the space–time Fourier transform side, the left equation becomes

$$i(\tau - k|k|^\alpha)\widetilde{u}(k,\tau) = \widetilde{F}(k,\tau),$$

so heuristically, estimating u in Y^s means estimating the inverse Fourier transform of $\frac{\widetilde{F}(k,\tau)}{i(\tau-k|k|^\alpha)}$ in Y^s. Of course, this idea breaks down when $\tau - k|k|^\alpha$ vanishes, but it turns out that by truncating in time (which is unproblematic if one is studying local existence) one can in effect replace the singular denominator $\tau - k|k|^\alpha$ by the inhomogeneous version $\langle \tau - k|k|^\alpha\rangle$. Thus, the heuristic is that estimating u in Y^s requires estimating the inverse Fourier transform of $\frac{\widetilde{F}(k,\tau)}{\langle\tau-k|k|^\alpha\rangle}$ in Y^s, and this leads to the introduction of the norm, again following [29],

$$\|u\|_{Z^s} = \|u\|_{X^{s,-1/2}} + \left(\sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} \left(\int_{\mathbb{R}} \frac{|\widetilde{u}(k,\tau)|}{\langle \tau - k|k|^\alpha\rangle}\, d\tau\right)^2\right)^{1/2},$$

in which the nonlinear part of the equation in (1) will have to be estimated.

To deal with the time truncation, we define the restriction of the spaces Y^s and Z^s to a time interval I in the standard way, and we denote them Y_I^s and Z_I^s, respectively. That is, Y_I^s is defined by the norm

$$\|u\|_{Y_I^s} = \inf\{\|v\|_{Y^s} : v \in Y^s,\ v = u \text{ on } \mathbb{T}\times \mathrm{Int}(I)\},$$

where $\mathrm{Int}(I)$ is the interior of I. In other words, Y_I^s is the quotient space $Y^s/\mathcal{M}$, where $\mathcal{M}$ is the closed subspace consisting of $u \in Y^s$ such that $u = 0$ on $\mathbb{T} \times \mathrm{Int}(I)$ (in the sense of distributions). Thus, Y_I^s is a Banach space with the above norm (see [40, Section 5.1, Exercise 12]). The restriction Z_I^s of Z^s is similarly defined.

We are now ready to state two lemmas from [29, Section 7.3], which provide estimates for the solution

$$u(\cdot,t) = S(t)f + \int_0^t S(t-t')F(\cdot,t')\,dt'$$

of the Cauchy problem (6).

Lemma 2 *([29]).* *There exists $C > 0$ such that for any $s \in \mathbb{R}$ and any time interval I which contains $t = 0$ and has length $|I| \le 1$, we have the estimate*

$$\|S(t)f\|_{Y_I^s} \le C\,\|f\|_{H^s(\mathbb{T})}\,.$$

Lemma 3 *([29]).* *There exists $C > 0$ such that for any $s \in \mathbb{R}$ and any time interval I which contains $t = 0$ and has length $|I| \le 1$, we have the estimate*

$$\left\|\int_0^t S(t-t')F(t')\,dt'\right\|_{Y_I^s} \le C\,\|F\|_{Z_I^s}\,.$$

We will also need the following property of the $X^{s,b}$ spaces.

Lemma 4. *Let η be a smooth, compactly supported function of time t. Let $s \in \mathbb{R}$. If $0 \le b \le 1/2$, there is a constant C depending only on η and b, such that for any $\delta \in (0,1]$ we have the estimate*

$$\|\eta(t/\delta)u\|_{X^{s,b}} \le C\delta^{1/2-b}\sqrt{\log(2/\delta)}\,\|u\|_{X^{s,1/2}}\,. \tag{7}$$

Proof. The left side of the claimed estimate equals

$$\left\|\langle k\rangle^s\langle\tau - k|k|^\alpha\rangle^b\int_{\mathbb{R}}\delta\widehat{\eta}(\delta\lambda)\widetilde{u}(k,\tau-\lambda)\,d\lambda\right\|_{L_\tau^2 l_k^2}.$$

Since $\langle\tau - k|k|^\alpha\rangle^b \le C(\langle\tau-\lambda-k|k|^\alpha\rangle^b + \langle\lambda\rangle^b)$, we reduce to proving

$$A, B \le C\delta^{1/2-b}\sqrt{\log(2/\delta)}\,\|F(k,\tau)\|_{L_\tau^2 l_k^2}\,,$$

where

$$A = \left\|\int_{\mathbb{R}}\frac{\delta\widehat{\eta}(\delta\lambda)}{\langle\tau-\lambda-k|k|^\alpha\rangle^{1/2-b}}F(k,\tau-\lambda)\,d\lambda\right\|_{L_\tau^2 l_k^2},$$

$$B = \left\|\int_{\mathbb{R}}\frac{\delta\langle\lambda\rangle^b\widehat{\eta}(\delta\lambda)}{\langle\tau-\lambda-k|k|^\alpha\rangle^{1/2}}F(k,\tau-\lambda)\,d\lambda\right\|_{L_\tau^2 l_k^2},$$

and

$$F(k,\tau) = \langle k\rangle^s\langle\tau-k|k|^\alpha\rangle^{1/2}\widetilde{u}(k,\tau).$$

We partition the λ-integration according to whether $\langle\tau-\lambda-k|k|^\alpha\rangle \ge 1/\delta$ or $\langle\tau-\lambda-k|k|^\alpha\rangle < 1/\delta$, and correspondingly we write $A \le A' + A''$ and $B \le B' + B''$ by the triangle inequality.

In the case $\langle\tau-\lambda-k|k|^\alpha\rangle \ge 1/\delta$ we estimate

$$A' \le \delta^{1/2-b}\left\|\int_{\mathbb{R}}\delta\widehat{\eta}(\delta\lambda)F(k,\tau-\lambda)\,d\lambda\right\|_{L_\tau^2 l_k^2} \le \delta^{1/2-b}\left(\int_{\mathbb{R}}\delta|\widehat{\eta}(\delta\lambda)|\,d\lambda\right)\|F\|_{L^2 l^2}\,,$$

and the last integral is independent of δ, so the desired estimate holds, without the logarithmic factor. Similarly,

$$B' \le \delta^{1/2} \left(\int_{\mathbb{R}} \delta \langle \lambda \rangle^b |\widehat{\eta}(\delta\lambda)| \, d\lambda \right) \|F\|_{L^2 l^2} \le C\delta^{1/2-b} \|F\|_{L^2 l^2} .$$

In the case $\langle \tau - \lambda - k|k|^\alpha \rangle < 1/\delta$ we apply Cauchy–Schwarz to the λ-integral, obtaining

$$A'' \le \left\| \left(\iint_{\langle \tau-\lambda-k|k|^\alpha \rangle < 1/\delta} \frac{\delta^2 |\widehat{\eta}(\delta\lambda)|^2}{\langle \tau - \lambda - k|k|^\alpha \rangle^{1-2b}} \, d\tau \, d\lambda \right)^{1/2} \|F(k,\cdot)\|_{L^2} \right\|_{l^2_k} .$$

Performing the τ-integration, we dominate this by

$$\left(\int_{\mathbb{R}} \delta^{-2b} \log(2/\delta) \delta^2 |\widehat{\eta}(\delta\lambda)|^2 \, d\lambda \right)^{1/2} \|F\|_{L^2 l^2} \le C\delta^{1/2-b} \sqrt{\log(2/\delta)} \, \|F\|_{L^2 l^2} ,$$

where the logarithmic factor can be removed unless $b = 0$. Similarly,

$$B'' \le C \left(\int_{\mathbb{R}} \log(2/\delta) \delta^2 \langle \lambda \rangle^{2b} |\widehat{\eta}(\delta\lambda)|^2 \, d\lambda \right)^{1/2} \|F\|_{L^2 l^2} \le C\delta^{1/2-b} \sqrt{\log(2/\delta)} \, \|F\|_{L^2 l^2} ,$$

completing the proof of the lemma. $\square$

Finally, we define the "Gevrey-modified" versions of the spaces defined above. For $s \in \mathbb{R}$ and $\sigma > 0$, we introduce the norms

$$\|u\|_{Y^{\sigma,s}} = \left\| e^{\sigma|D_x|} u \right\|_{Y^s},$$
$$\|u\|_{Z^{\sigma,s}} = \left\| e^{\sigma|D_x|} u \right\|_{Z^s}.$$

The spaces $Y^{\sigma,s}$ and $Z^{\sigma,s}$ are defined as the completions, with respect to these norms, of the space of all u given by (3) with $\widetilde{u} \colon \mathbb{Z} \times \mathbb{R} \to \mathbb{C}$ some smooth and compactly supported function. The time-restricted spaces $Y_I^{\sigma,s}$ and $Z_I^{\sigma,s}$, for a time interval I, are then defined by the standard procedure, as described above.

The key properties of the spaces Y^s and Z^s and their time-restrictions now carry over to the Gevrey-modified spaces, by the substitution $f \mapsto e^{\sigma|D_x|} f$. Thus, by Lemma 1 we have

$$Y_I^{\sigma,s} \hookrightarrow BC(I; G^{\sigma,s}(\mathbb{T})),$$

and the analogues of Lemmas 2 and 3 hold in $Y_I^{\sigma,s}$ and $Z_I^{\sigma,s}$, as is seen by simply replacing (f, F) by $(e^{\sigma|D_x|} f, e^{\sigma|D_x|} F)$.

3. Bilinear estimate and local existence

The following bilinear estimate will be a key ingredient in our proof of the main theorem.

Lemma 5 *([29,36,37]). Given $s \ge -\alpha/4$ there exists a constant $C > 0$ such that*

$$\|\partial_x(uv)\|_{Z^s} \le C \|u\|_{Y^s} \|v\|_{Y^s}$$

for all $u, v \in Y^s$ with zero mean in x for every t, that is, $\widehat{u}(0,t) = \widehat{v}(0,t) = 0$ for every t.

A closer examination of the proof of Lemma 5 reveals that a slightly sharper estimate can be obtained. Indeed it transpires that

$$\|\partial_x(uv)\|_{Z^s} \le C \|u\|_{X^{s,b}} \|v\|_{X^{s,1/2}} + C \|u\|_{X^{s,1/2}} \|v\|_{X^{s,b}} \tag{8}$$

for some $b < 1/2$. The fact that $b < 1/2$ in this estimate enables us, by a time localization, to use Lemma 4 and deduce the following key estimate.

Lemma 6. *Under the hypotheses of Lemma 5, there is a constant C such that for any time interval I, we have the estimate*

$$\|\partial_x(uv)\|_{Z_I^s} \le C|I|^\varepsilon \|u\|_{Y_I^s} \|v\|_{Y_I^s},$$

for some $\varepsilon > 0$ independent of s.

Proof. Without loss of generality we may assume that $I = [-\delta, \delta]$. Choose a smooth cut-off function $\eta \in C_c^\infty(\mathbb{R})$ such that $\eta(t) = 1$ for $|t| \le 1$. Let $u', v' \in Y^s$ denote extensions of u and v outside of $\mathbb{T} \times I$. That is, on $\mathbb{T} \times \mathrm{Int}(I)$ we have $u' = u$ and $v' = v$. Then choosing $0 < \varepsilon < 1/2 - b$, with $b < 1/2$ as in (8), we have

$$\begin{aligned}\|\partial_x(uv)\|_{Z_I^s} &\le \|\partial_x(\eta(t/\delta)u' \cdot \eta(t/\delta)v')\|_{Z^s} \\ &\le C\|\eta(t/\delta)u'\|_{X^{s,b}} \|\eta(t/\delta)v'\|_{X^{s,1/2}} + C\|\eta(t/\delta)u'\|_{X^{s,1/2}} \|\eta(t/\delta)v'\|_{X^{s,b}} \\ &\le C\delta^\varepsilon \|u'\|_{X^{s,1/2}} \|v'\|_{X^{s,1/2}},\end{aligned}$$

where in the second step we used (8) and in the last step we used Lemma 4. Taking the infimum over all extensions $u', v' \in Y^s$, we then get the desired estimate. □

The analogous estimate in the σ-modified spaces $Y_I^{\sigma,s}$ and $Z_I^{\sigma,s}$ follows immediately, by the inequality (2). Thus, we obtain:

Corollary 1. *Under the hypotheses of Lemma 5, there is a constant C, such that for any time interval I and any $\sigma > 0$, we have the estimate*

$$\|\partial_x(uv)\|_{Z_I^{\sigma,s}} \le C|I|^\varepsilon \|u\|_{Y_I^{\sigma,s}} \|v\|_{Y_I^{\sigma,s}},$$

for some $\varepsilon > 0$ independent of σ.

Then by a standard contraction argument applied to the integral formulation

$$u(\cdot, t) = S(t)\phi - \int_0^t S(t-t')\frac{1}{2}\partial_x\left(u^2\right)(\cdot, t')\,dt',$$

and using also the Gevrey-modified versions of Lemmas 2 and 3, one obtains the following local existence theorem (a slightly more detailed statement of Theorem 2 from the Introduction):

Theorem 4. *Given $\sigma > 0$ and $s \ge -\alpha/4$, then for any $\phi \in G^{\sigma,s}(\mathbb{T})$ there exists a time $\delta = \delta(\|\phi\|_{G^{\sigma,s}(\mathbb{T})}) > 0$ and a solution*

$$u \in C\left([-\delta, \delta]; G^{\sigma,s}(\mathbb{T})\right)$$

of the Cauchy problem (1) on $\mathbb{T} \times (-\delta, \delta)$. Moreover,

$$\delta = \frac{c_0}{\left(1 + \|\phi\|_{G^{\sigma,s}(\mathbb{T})}\right)^a}$$

for some constants $a, c_0 > 0$ depending only on s and α, and

$$\|u\|_{Y^{\sigma,s}_{[-\delta,\delta]}} \le C\|\phi\|_{G^{\sigma,s}(\mathbb{T})}$$

for some constant $C > 0$ depending only on s and α.

We remark that since the solution is smooth, it is of course unique.

The idea is now to apply the local result repeatedly on consecutive short time intervals to cover any large time interval $[0, T]$. The crucial point is then to be able to control the growth of $\|u(\cdot, t)\|_{G^{\sigma,0}(\mathbb{T})}$ in this iteration process, and to achieve this we will make use of an approximate conservation law, which we prove next.

4. Approximate conservation law

Here we prove Theorem 3. We start by recalling that

$$\int_{\mathbb{T}} u^2(x,t)\,dx$$

is conserved for a smooth solution u of (1), since

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{T}} u^2\,dx\right) = \int_{\mathbb{T}} u\partial_t u\,dx = \int_{\mathbb{T}} u\partial_x |D_x|^\alpha u\,dx - \int_{\mathbb{T}} u^2\partial_x u\,dx =: A - B,$$

where A and B both vanish. Indeed, by Parseval's identity we first calculate

$$A = \int_{\mathbb{T}} u\partial_x|D_x|^\alpha u\,dx = \sum_{k\in\mathbb{Z}} \widehat{u}(k,t)(-ik)|k|^\alpha \overline{\widehat{u}(k,t)} = -i\sum_{k\in\mathbb{Z}} k|k|^\alpha |\widehat{u}(k,t)|^2,$$

which shows that A is pure imaginary. On the other hand, A must be real, since u is real-valued, hence so is $|D_x|^\alpha u$. Thus, A equals zero. Next, we observe that

$$B = \int_{\mathbb{T}} u^2\partial_x u\,dx = \int_{\mathbb{T}} \partial_x(u^3/3)\,dx = 0.$$

To obtain the approximate conservation law, we apply the same calculations to the σ-modified field

$$U := e^{\sigma|D_x|}u.$$

Assuming u satisfies (1), then U satisfies

$$\partial_t U - \partial_x|D_x|^\alpha U + U\partial_x U = F,$$

where

$$F = U\partial_x U - e^{\sigma|D_x|}(u\partial_x u) = \frac{1}{2}\partial_x\left((e^{\sigma|D_x|}u)\cdot(e^{\sigma|D_x|}u) - e^{\sigma|D_x|}(u\cdot u)\right).$$

Setting

$$N(t) = \int_{\mathbb{T}} U^2(x,t)\,dx = \|u(\cdot,t)\|^2_{G^{\sigma,0}(\mathbb{T})},$$

and proceeding as above, we then obtain

$$N'(t) = \frac{d}{dt}\int_{\mathbb{T}} U^2\,dx = \int_{\mathbb{T}} 2U\partial_t U\,dx = \int_{\mathbb{T}} 2UF\,dx,$$

hence

$$N(t) \le N(0) + \left|\int_0^t \int_{\mathbb{T}} 2UF(x,t')\,dx\,dt'\right|. \tag{9}$$

To estimate the last term we need the following.

Lemma 7 ([29]). *There exists a constant $C>0$ such that for any time interval J,*

$$\left|\int_{\mathbb{R}}\int_{\mathbb{T}} \chi_J(t)u(x,t)v(x,t)\,dx\,dt\right| \le C\,\|u\|_{Y^0}\,\|v\|_{Z^0}$$

for all $u\in Y^0$ and $v\in Z^0$, where χ_J denotes the characteristic function of J.

This immediately implies the time-restricted version:

Corollary 2. *There exists a constant $C > 0$ such that for any time intervals J and I with $J \subset I$, we have*

$$\left| \int_{\mathbb{R}} \int_{\mathbb{T}} \chi_J(t) u(x,t) v(x,t)\, dx\, dt \right| \le C \|u\|_{Y_I^0} \|v\|_{Z_I^0}.$$

Proof. Let $u' \in Y^s$ and $v' \in Z^s$ denote extensions of u and v, respectively, outside of $\mathbb{T} \times I$. That is, on $\mathbb{T} \times \mathrm{Int}(I)$ we have $u' = u$ and $v' = v$. Then by Lemma 7,

$$\left| \int_{\mathbb{R}} \int_{\mathbb{T}} \chi_J(t) u(x,t) v(x,t)\, dx\, dt \right| \le C \|u'\|_{Y^0} \|v'\|_{Z^0},$$

and taking the infimum over the extensions we get the desired estimate. □

Thus, for a time interval I containing $t = 0$, we get from (9) and the last corollary that

$$\sup_{t \in I} N(t) \le N(0) + C \|U\|_{Y_I^0} \|F\|_{Z_I^0}, \tag{10}$$

and to estimate the norm of F, we apply the following symbol estimate, which quantifies the lack of commutation when we apply the multiplier $e^{\sigma|D_x|}$ to a product.

Lemma 8. *For $\sigma > 0, \theta \in [0,1]$ and $k_1, k_2 \in \mathbb{Z}$,*

$$e^{\sigma|k_1|} e^{\sigma|k_2|} - e^{\sigma|k_1+k_2|} \le [2\sigma \min(|k_1|,|k_2|)]^\theta e^{\sigma|k_1|} e^{\sigma|k_2|}.$$

Proof. If k_1 and k_2 have the same sign, the left side of the inequality vanishes. Now assume that k_1 and k_2 have opposite signs. By symmetry we may assume $k_1 \ge 0$ and $k_2 \le 0$. If $|k_2| \le |k_1|$, then $k_1 + k_2 \ge 0$ and

$$\begin{aligned} e^{\sigma|k_1|} e^{\sigma|k_2|} - e^{\sigma|k_1+k_2|} &= \left(e^{-2\sigma k_2} - 1\right) e^{\sigma(k_1+k_2)} \\ &\le (2\sigma|k_2|)^\theta e^{-2\sigma k_2} e^{\sigma(k_1+k_2)} \\ &= (2\sigma|k_2|)^\theta e^{\sigma|k_1|} e^{\sigma|k_2|}, \end{aligned}$$

where we used the fact that $e^x - 1 \le e^x$ and $e^x - 1 \le x e^x$ for $x \ge 0$, hence also

$$e^x - 1 \le x^\theta e^x \quad \text{for } x \ge 0 \text{ and } \theta \in [0,1].$$

If $|k_2| \ge |k_1|$, then $k_1 + k_2 \le 0$ and

$$\begin{aligned} e^{\sigma|k_1|} e^{\sigma|k_2|} - e^{\sigma|k_1+k_2|} &= \left(e^{2\sigma k_1} - 1\right) e^{-\sigma(k_1+k_2)} \\ &\le (2\sigma|k_1|)^\theta e^{2\sigma k_1} e^{-\sigma(k_1+k_2)} \\ &= (2\sigma|k_1|)^\theta e^{\sigma|k_1|} e^{\sigma|k_2|}, \end{aligned}$$

completing the proof of the lemma. □

This lemma is combined with the fact that, from the triangle inequality,

$$\min(|k_1|,|k_2|) \le 2 \frac{(1+|k_1|)(1+|k_2|)}{(1+|k_1+k_2|)},$$

hence, for $\theta \in [0,1]$,

$$e^{\sigma|k_1|} e^{\sigma|k_2|} - e^{\sigma|k_1+k_2|} \le (4\sigma)^\theta \frac{(1+|k_1|)^\theta (1+|k_2|)^\theta}{(1+|k_1+k_2|)^\theta} e^{\sigma|k_1|} e^{\sigma|k_2|}.$$

This implies

$$\|2F\|_{Z_I^0} \le (4\sigma)^\theta \left\| \langle D_x \rangle^{-\theta} \partial_x \left(\langle D_x \rangle^\theta U \cdot \langle D_x \rangle^\theta U \right) \right\|_{Z_I^0},$$

so we can apply Lemma 5 with $s = -\theta$ to get

$$\|F\|_{Z_I^0} \le C\sigma^\theta \|U\|_{Y_I^0}^2 . \tag{11}$$

This requires $-\theta \ge -\alpha/4$, that is, $\theta \le \alpha/4$, in addition to $\theta \le 1$. Combining (10) and (11) we conclude that

$$\sup_{t\in I} N(t) \le N(0) + C\sigma^\theta \|U\|_{Y_I^0}^3 \quad \text{for } 0 \le \theta \le \min(1, \alpha/4).$$

But taking $I = [-\delta, \delta]$ as in Theorem 4 (with $s = 0$), we have

$$\|U\|_{Y_I^0} = \|u\|_{Y_I^{\sigma,0}} \le C \|\phi\|_{G^{\sigma,0}(\mathbb{T})} ,$$

hence we finally get the approximate conservation law, Theorem 3.

With this information in hand, we are now ready to iterate the local result to obtain the long-time result.

5. Proof of the main theorem

Here we show how Theorem 1 is proved using Theorems 2 and 3.

By the invariance of (1) under the reflection $u(x,t) \to u(-x,-t)$, it suffices to prove Theorem 1 for positive times. Moreover, we limit our attention to the case $s = 0$, since the general case can be reduced to this, as shown in [26]. Set

$$p = \frac{1}{\min(1, \alpha/4)} = \begin{cases} \dfrac{4}{\alpha} & \text{if } 2 \le \alpha \le 4, \\ 1 & \text{if } \alpha > 4. \end{cases}$$

Given $\sigma_0 > 0$ and $\phi \in G^{\sigma_0,0}(\mathbb{T})$, we want to prove that for large $T > 0$,

$$u(\cdot, t) \in G^{\sigma,0} \quad \text{for } \sigma = \frac{c}{T^p} \text{ and all } t \in [0, T],$$

where $c > 0$ depends on σ_0, ϕ and α.

Regarding σ as a parameter, define

$$N_\sigma(t) = \|u(\cdot, t)\|_{G^{\sigma,0}}^2 \quad \text{for } 0 < \sigma \le \sigma_0.$$

Suppose that for given $\sigma > 0$ and $t_0 > 0$ we have

$$\sup_{t\in[0,t_0]} N_\sigma(t) \le 2N_{\sigma_0}(0).$$

Then we can apply Theorem 4, with initial time $t = t_0$ and the time step

$$\delta = \frac{c_0}{(1 + 2N_{\sigma_0}(0))^a},$$

to extend the solution to $[t_0, t_0 + \delta]$. By Theorem 3, the approximate conservation law, we have

$$\sup_{t\in[t_0,t_0+\delta]} N_\sigma(t) \le N_\sigma(t_0) + C\sigma^{1/p} \left(2N_{\sigma_0}(0)\right)^3 .$$

In this way, we cover time intervals $[0, \delta], [\delta, 2\delta]$ etc., and obtain

$$\begin{aligned} N_\sigma(\delta) &\le N_\sigma(0) + C\sigma^{1/p} \left(2N_{\sigma_0}(0)\right)^3 , \\ N_\sigma(2\delta) &\le N_\sigma(\delta) + C\sigma^{1/p} \left(2N_{\sigma_0}(0)\right)^3 \le N_\sigma(0) + 2C\sigma^{1/p} \left(2N_{\sigma_0}(0)\right)^3 , \\ &\cdots \\ N_\sigma(n\delta) &\le N_\sigma(0) + nC\sigma^{1/p} \left(2N_{\sigma_0}(0)\right)^3 . \end{aligned}$$

This continues as long as

$$nC\sigma^{1/p}\left(2N_{\sigma_0}(0)\right)^3 \leq N_{\sigma_0}(0),$$

since then

$$N_\sigma(n\delta) \leq N_\sigma(0) + nC\sigma^{1/p}\left(2N_{\sigma_0}(0)\right)^3 \leq 2N_{\sigma_0}(0),$$

so we can take one more step.

Thus, the induction stops at the first integer n for which

$$nC\sigma^{1/p}\left(2N_{\sigma_0}(0)\right)^3 > N_{\sigma_0}(0).$$

and then we have reached a final time

$$T = n\delta,$$

so

$$\frac{T}{\delta}C\sigma^{1/p}2^3\left(N_{\sigma_0}(0)\right)^2 > 1.$$

Note that T will be arbitrarily large for $\sigma > 0$ small enough. Moreover,

$$\sigma^{1/p} > \frac{\delta}{8CT\left(N_{\sigma_0}(0)\right)^2} = \frac{c_0}{8CT\left(N_{\sigma_0}(0)\right)^2\left(1+2N_{\sigma_0}(0)\right)^a},$$

proving

$$\sigma \geq \frac{c}{T^p},$$

as claimed.

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