IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **37** (2024) 025013 (32pp) https://doi.org/10.1088/1361-6544/ad1a47

Convergence of mechanical balance laws for water waves: from KdV to Euler

Samer Israwi[1,2], Henrik Kalisch[3] and Bashar Khorbatly[3,*]

[1] Arts, Sciences, and Technology University in Lebanon, CRAMS: Center for Research in Applied Mathematics and Statistics, Faculty of Sciences, Beirut, Lebanon

[2] Department of Mathematics, Faculty of Sciences 1, Lebanese University, Beirut, Lebanon

[3] Department of Mathematics, University of Bergen, Postbox 7800, 5020 Bergen, Norway

E-mail: bashar-elkhorbatly@hotmail.com

Received 23 March 2023; revised 21 November 2023
Accepted for publication 3 January 2024
Published 16 January 2024

Recommended by Dr Tamara Grava

Abstract

This article takes into account the Korteweg–de Vries (KdV) equation as an approximate model of long waves of small amplitude at the free surface with inviscid fluid. It is demonstrated that the mechanical balance quantities, as defined by the solution of the KdV equation, rigorously approximate those in the Euler system within the L^∞ space. Furthermore, these approximations are estimated in relation to the parameter ε characterizing the long-wave behavior.

Keywords: KdV equation, Euler equations, mechanical balance laws, surface waves

Mathematics Subject Classification numbers: 35Q35, 35Q53, 76B15, 76M45

1. Introduction

In this paper, we consider approximations of physical balance laws associated with the Korteweg–de Vries equation (KdV) from a mathematical point of view. The KdV equation is written in non-dimensional form as

$$\eta_t + \eta_x + \varepsilon\frac{3}{2}\eta\eta_x + \mu\frac{1}{6}\eta_{xxx} = 0\,, \tag{1.1}$$

[*] Author to whom any correspondence should be addressed.

where ε and μ are parameters representing the relative influence of nonlinearity and dispersion, respectively. As will be shown below, these parameters quantify the amplitude and wavenumber of a typical wave to be described by the equation. In fact, the KdV equation is known to be a good model for waves at the free surface of an incompressible and inviscid fluid if transverse effects can be neglected and if the relations $\mu \ll 1$ and $\varepsilon = \mathcal{O}(\mu)$ hold. This approximation is rigorously justified using the theory developed in [7, 12, 14, 15, 26, 28, 29] and others.

Even before it was shown that the KdV equation is well posed in a mathematical sense, it was well known that the KdV equation features an infinite number of formally conserved integrals (indeed the conservation can be made rigorous by following the work of [8]). If the equation is given in the non-dimensional form (1.1), the first three conserved integrals are

$$\int_{-\infty}^{\infty} \eta \, dx, \qquad \int_{-\infty}^{\infty} \eta^2 \, dx, \quad \text{and} \quad \int_{-\infty}^{\infty} \left(\tfrac{\mu}{3\varepsilon}\eta_x^2 - \eta^3\right) dx. \tag{1.2}$$

The first integral is found to be invariant with respect to time t as soon as it is recognized that the KdV equation can be written in the form

$$\frac{\partial}{\partial t}(\eta) + \frac{\partial}{\partial x}\left(\eta + \varepsilon\frac{3}{4}\eta^2 + \mu\frac{1}{6}\eta_{xx}\right) = 0, \tag{1.3}$$

where the quantity appearing under the time derivative is interpreted as excess mass density, and the term appearing under the spatial derivative is the mass flux through a cross section of unit width due to the passage of a surface wave (see [17] for more details). The second and third integral are sometimes called momentum and energy, but this terminology may be misleading since these integrals are not readily interpreted as approximations of the physical momentum and energy appearing in the context of the Euler equations. Indeed, the authors of [1] already state clearly that they do not believe these integrals to be approximations of the physical momentum and density, and further doubt was cast on this interpretation in more recent work [2, 19, 20, 23].

Recently, the problem was considered in [4], and asymptotic expressions for physically motivated fluxes and densities were found. For example, following the procedure laid out in [4] gives the expression for momentum density as

$$\mathcal{I} = \eta + \varepsilon\frac{3}{4}\eta^2 + \mu\frac{1}{6}\eta_{xx}. \tag{1.4}$$

Since the analysis in [4] was based on a formal asymptotic analysis, the question of whether these physical identities can be made mathematically rigorous have so far remained open (note however that in the special case of the momentum density $\mathcal{I}$ defined above in (1.4) it was shown in [16] that this expression converges to the corresponding quantity in the full Euler equations if the parameters μ and ε tend to zero. In the present paper, similar convergence results will be proved also for the momentum flux, as well as the energy density and energy flux.

In the present article we will give a firm mathematical proof for convergence of all physical densities and fluxes. The main results to be proved thus state that the mechanical densities and fluxes found asymptotically in [4] converge to the corresponding quantities defined in terms of a solutions of the governing Euler equation for a perfect fluid if μ and ε tend to zero. Denoting the original (dimensional) variables with a tilde, we introduce a scaling to make the small amplitude and long wavelength relative to the undisturbed depth explicit. Thus we define new variables (without a tilde) by

$$\tilde{x} = \lambda x, \quad \tilde{z} = h_0(z-1), \quad \tilde{\eta} = a\eta, \quad \tilde{t} = \frac{\lambda}{c_0}t, \quad \tilde{\phi} = \frac{a\lambda g}{c_0}\phi.$$

Here a is a dominant amplitude of the waves; λ is a typical wavelength; h is the undisturbed depth; g is the gravitational acceleration, and $c_0 = \sqrt{gh_0}$ is the limiting long-wave speed. Then the free surface Bernoulli's dimensionless formulation of the water waves problem reads

$$\begin{cases} \mu\partial_x^2\varphi + \partial_z^2\varphi = 0 & \text{in} \quad \Omega_t, \\ \partial_t\varphi + \frac{\varepsilon}{2}\left(\partial_x\varphi\right)^2 + \frac{\varepsilon}{2\mu}\left(\partial_z\varphi\right)^2 = -\eta & \text{at} \quad z = 1+\varepsilon\eta, \\ \partial_z\varphi = 0, & \text{at} \quad z = 0, \\ \partial_t\eta - \frac{1}{\mu}\left(-\mu\varepsilon\partial_x\eta\partial_x\varphi + \partial_z\varphi\right) = 0 & \text{at} \quad z = 1+\varepsilon\eta, \end{cases} \tag{1.5}$$

where

$$\Omega_t = \left\{(x,z), 0 < z < 1 + \varepsilon\eta(x,t)\right\},$$

is the fluid domain limited by the free surface $\{z = 1+\eta(x,t)\}$, and the flat bottom $\{z = 0\}$, and where $\varphi(x,z,t)$, defined on Ω_t is the velocity potential associated to the flow (that is, the two-dimensional velocity field $\mathbf{v}$ is given by $\mathbf{v} = (\partial_x\varphi, \partial_z\varphi)^T$). As is well known, the existence of the velocity potential is guaranteed by the assumption of irrotational flow. Finally, as mentioned above, ε and μ are the dimensionless parameters defined as

$$\varepsilon = \frac{a}{h}, \qquad \mu = \frac{h^2}{\lambda^2}.$$

Making assumptions on the respective size of ε and μ, one is led to derive (simpler) asymptotic models from (1.5). Sometimes the Stokes number

$$S = \frac{\varepsilon}{\mu},$$

is introduced in order to quantify the applicability of the equation to a particular regime of surface waves. Let us assume for the time being that the Stokes number is equal to unity, so that we can work with a single small parameter ε. The equations above are formally equivalent to the Zakharov–Craig–Sulem equations. They are written in terms of the Dirichlet–Neumann operator as

$$\begin{cases} \eta_t - \frac{1}{\varepsilon}\mathcal{G}_\varepsilon\left[\varepsilon\eta\right]\psi = 0, \\ \psi_t + \eta + \frac{\varepsilon}{2}\psi_x^2 - \dfrac{\left[\mathcal{G}_\varepsilon\left[\varepsilon\eta\right]\psi + \varepsilon^2\eta_x\psi_x\right]^2}{2\left(1+\varepsilon^3\eta_x^2\right)} = 0. \end{cases} \tag{1.6}$$

It is shown in [26] that if a number of assumptions are met, a solution (ψ,ζ) exists on a time interval $[0, T/\varepsilon]$ with a certain regularity, and with a bound on Sobolev norm. Given a solution of this system, one may then reconstruct the potential φ by solving the Laplace equation. More precisely, in terms of the trace of the velocity potential at the free surface defined as

$$\psi = \varphi_{|z=1+\varepsilon\eta},$$

the Dirichlet–Neumann operator $\mathcal{G}_\varepsilon[\varepsilon\eta]\cdot$ is given by

$$\mathcal{G}_\varepsilon\left[\varepsilon\zeta\right]\psi = -\varepsilon^2\eta_x\left(\partial_x\varphi\right)_{|z=1+\varepsilon\eta} + \left(\partial_z\varphi\right)_{|z=1+\varepsilon\eta}, \tag{1.7}$$

with φ solving the boundary value problem

$$\begin{cases} \varepsilon\partial_x^2\varphi + \partial_z^2\varphi = 0, \\ \partial_z\varphi_{|z=0} = 0, \\ \varphi_{|z=1+\varepsilon\eta} = \psi. \end{cases} \tag{1.8}$$

1.1. Statement of the results

We start in section 2 by developing an approximate potential function φ^{app}, laying the groundwork for subsequent proof of error estimates between the Euler system (1.5) and the KdV equation (1.1).

In section 3, the discrepancy between vertical integrals of the space and time derivatives of the complete $\varphi = \varphi^{\text{Euler}}$ and approximate potentials φ^{app} is assessed.

In section 4, a linkage between the Euler solutions $(\eta^{\text{Euler}}, \varphi^{\text{Euler}})$ and the KdV solutions constructed, η^{KdV} and $w^{\text{KdV}} = \eta^{\text{KdV}} - \frac{1}{4}\varepsilon(\eta^{\text{KdV}})^2 + \frac{1}{3}\varepsilon\eta^{\text{KdV}}_{xx}$, through an approximation of the horizontal velocity at the flattened bottom $\varphi^{\text{app}}_{|z=0}$ and some observations on Boussinesq-type systems.

In section 5, approximate expressions for the velocity field and pressure in the entire fluid column are introduced in the context of the KdV equation. These variables will thus be expressed not only in terms of x and t, but also in terms of z. The goal of this section is to devise formulas for these variables, demonstrating their convergence to the relevant quantities defined within the full context of the Euler equations as the small parameter ε tends to zero. Here and throughout the rest of this paper we denote by C any constant depending on $h_{\min}^{-1}$, $\varepsilon_{\max}, |\eta_0|_{H^{s+N+1}}, |\psi_{0,x}|_{H^{s+N}}$ with $N \geqslant 8$.

In section 6, we establish the main results of this paper, where the mechanical laws in the Euler equations converge to the mechanical laws (mass, momentum and energy) defined in terms of the function of the solution of the KdV equation for a perfect fluid as the physical parameter ε approaches zero. In other words, as long as the assumption of corollary 1 is satisfied and for any constant C depending on $h_{\min}^{-1}$, $\varepsilon_{\max}, |\eta_0|_{H^{s+N+1}}, |\psi_{0,x}|_{H^{s+N}}$ with $N \geqslant 8$ and for all $t \in [0, T/\varepsilon]$ the following results are true:

(i) Theorems 4 and 5, state that the mass balance law of the water wave equations can be approximated in the long wave limit by appropriate balance quantities defined by solutions of the KdV equation:

$$\left|\mathcal{M}^{\text{Euler}} - \mathcal{M}^{\text{KdV}}\right|_{L^\infty} + \left|\mathcal{Q_M}^{\text{Euler}} - \mathcal{Q_M}^{\text{KdV}}\right|_{L^\infty} \leqslant \varepsilon^2 (1+t)\, C\,, \tag{1.9}$$

where the mass density and mass flux for the full Euler and KdV equations are given by: $\mathcal{M}^{\text{Euler}} = \int_0^{1+\varepsilon\eta^{\text{Euler}}} dz$, $\mathcal{M}^{\text{KdV}} = 1 + \varepsilon\eta^{\text{KdV}}$ and

$$\mathcal{Q_M}^{\text{Euler}} = \int_0^{1+\varepsilon\eta^{\text{Euler}}} \varphi_x^{\text{Euler}}\, dz, \qquad \mathcal{Q_M}^{\text{KdV}} = \eta^{\text{KdV}} + \frac{3\varepsilon}{4}\left(\eta^{\text{KdV}}\right)^2 + \frac{\varepsilon}{6}\eta^{\text{KdV}}_{xx}\,.$$

(ii) Theorem 7 states that the momentum balance law of the water wave equations can be approximated in the long wave limit by appropriate balance quantities defined by solutions of the KdV equation:

$$\left|\mathcal{I}^{\text{Euler}} - \mathcal{I}^{\text{KdV}}\right|_{L^\infty} + \left|\mathcal{Q_I}^{\text{Euler}} - \mathcal{Q_I}^{\text{KdV}}\right|_{L^\infty} \leqslant \varepsilon^2 (1+t)\, C\,, \tag{1.10}$$

where the momentum density and momentum flux for the full Euler and KdV equations are given by: $\mathcal{I}^{\text{Euler}} = \mathcal{Q_M}^{\text{Euler}}, \mathcal{I}^{\text{KdV}} = \mathcal{Q_M}^{\text{KdV}}$, and

$$\mathcal{Q_I}^{\text{Euler}} = \int_0^{1+\varepsilon\eta^{\text{Euler}}} \left(\varepsilon\left(\varphi_x^{\text{Euler}}\right)^2 + (P')^{\text{Euler}} - \frac{1}{\varepsilon}(z-1)\right) dz,$$
$$\mathcal{Q_I}^{\text{KdV}} = \frac{1}{2\varepsilon} + \eta^{\text{KdV}} + \frac{3\varepsilon}{2}\left(\eta^{\text{KdV}}\right)^2 + \frac{\varepsilon}{3}\eta^{\text{KdV}}_{xx}\,,$$

with $(P')^{\text{Euler}}$ is the dynamic pressure of the fluid (see (5.12)).

(iii) Theorems 8 and 9, state that the energy balance law of the water wave equations can be approximated in the long wave limit by appropriate balance quantities defined by solutions of the KdV equation:

$$\left|\mathcal{E}^{\text{Euler}}-\mathcal{E}^{\text{KdV}}\right|_{L^\infty}+\left|\mathcal{Q}_{\mathcal{E}}{}^{\text{Euler}}-\mathcal{Q}_{\mathcal{E}}{}^{\text{KdV}}\right|_{L^\infty}\leqslant\varepsilon^2\left(1+t\right)C\,, \tag{1.11}$$

where the energy density and energy flux for the full Euler and KdV equations are given by:

$$\mathcal{E}^{\text{Euler}}=\int_0^{1+\varepsilon\eta^{\text{Euler}}}\left(\frac{\varepsilon}{2}\left(\varphi_x^{\text{Euler}}\right)^2+\frac{1}{2}\left(\varphi_z^{\text{Euler}}\right)^2+\frac{1}{\varepsilon}z\right)dz,$$

$$\mathcal{Q}_{\mathcal{E}}{}^{\text{Euler}}=\int_0^{1+\varepsilon\eta^{\text{Euler}}}\left(\frac{\varepsilon^2}{2}\left(\varphi_x^{\text{Euler}}\right)^2+\frac{\varepsilon}{2}\left(\varphi_z^{\text{Euler}}\right)^2+\varepsilon\left(P'\right)^{\text{Euler}}+1\right)\varphi_x^{\text{Euler}}\,dz,$$

$$\mathcal{E}^{\text{KdV}}=\frac{1}{2\varepsilon}+\eta^{\text{KdV}}+\varepsilon\left(\eta^{\text{KdV}}\right)^2,\qquad \mathcal{Q}_{\mathcal{E}}{}^{\text{KdV}}=\eta^{\text{KdV}}+\frac{7\varepsilon}{4}\left(\eta^{\text{KdV}}\right)^2+\frac{\varepsilon}{6}\eta_{xx}^{\text{KdV}}\,.$$

1.2. Notation

We denote by $C(\lambda_1,\lambda_2,\dots)$ a constant depending on the parameters λ_1, $\lambda_2\dots$. To study the regularity properties of the solution of the transformed problem, we introduce the following functional spaces on the flat strip $\mathcal{S}$. In particular, the Banach space $H^{s,k}=\bigcap_{j=0}^k H^j\big((-1,0);H^{s-j}(\mathbb{R})\big)$ endowed with the norm $\|u\|_{H^{s,k}}=\sum_{j=0}^k\|\Lambda^{s-j}\partial_z^j u\|_2$ are introduced to control functions that are differentiated s times in x and z, with at most k derivatives in z where Λ^s is the pseudo-differential operator $\Lambda^s=(1-\partial_x)^{s/2}$. We write $L^\infty H^s=L^\infty\big((-1,0);H^s(\mathbb{R})\big)$ endowed with the canonical norm $\operatorname{supess}_{z\in(-1,0)}|u(\cdot,z)|_{H^s(\mathbb{R})}$.

2. The approximate potential φ^{app}

This section is devoted to constructing an approximate potential function which will be useful later for proving error estimates between the Euler system and the KdV equation. The main task is to obtain an asymptotic expansion in the small parameter ε of the potential φ in terms of various combinations of derivatives of the trace ψ. Thus we look for an asymptotic expansion of φ of the form

$$\varphi^{\text{app}}=\sum_{j=0}^N\varepsilon^j\varphi_j\,. \tag{2.1}$$

Indeed, substituting this expression into the boundary-value problem (1.8) one can remove the residual up to the order $\mathcal{O}(\varepsilon^{N+1})$ provided that

$$\forall j=0,\dots,N,\qquad \partial_z^2\varphi_j=-\partial_x^2\varphi_{j-1} \tag{2.2}$$

(with the convention that $\varphi_{-1}=0$), together with the boundary conditions

$$j=0\qquad\begin{cases}\varphi_{j|_{z=1+\varepsilon\eta}}=\psi,\\ \partial_z\varphi_j=0,\end{cases} \tag{2.3}$$

and

$$\forall j=1,\dots,N,\qquad\begin{cases}\varphi_{j|_{z=1+\varepsilon\eta}}=0,\\ \partial_z\varphi_j=0,\end{cases} \tag{2.4}$$

The differential equation (2.2) can be solved by integrating twice. In view of the boundary conditions (2.3) and (2.4), one finds

$$\varphi_0 = \psi, \tag{2.5}$$

$$\begin{aligned}\varphi_1 &= -\frac{1}{2}\left(z^2-1\right)\psi_{xx} + \varepsilon\eta\psi_{xx} + \frac{\varepsilon^2}{2}\eta^2\psi_{xx} \\ &= -\frac{1}{2}z^2\psi_{xx} + \frac{1}{2}\left(1+\varepsilon\eta\right)^2\psi_{xx}.\end{aligned} \tag{2.6}$$

In our derivation it is essential to find φ_2, therefore by proceeding in the same way as above (see [22, 24] for more details of the calculations) one obtains

$$\varphi_2 = \frac{1}{24}z^4\psi_{xxxx} - \frac{1}{4}z^2\psi_{xxxx} + \frac{5}{24}\psi_{xxxx} + \mathcal{O}(\varepsilon). \tag{2.7}$$

In the case $N = 2$, φ^{app} satisfies the elliptic problem

$$\begin{cases}\nabla\cdot I_\varepsilon\nabla\varphi^{app} = \varepsilon^3 R \ \text{ in } \ \Omega_t, \\ \varphi^{app}_{|z=1+\varepsilon\eta} = \psi, \qquad \partial_z\varphi^{app}_{|z=0} = 0,\end{cases} \tag{2.8}$$

where following [26], we use the matrix

$$I_\varepsilon = \begin{pmatrix}\varepsilon & 0\\ 0 & 1\end{pmatrix} \tag{2.9}$$

and the remainder term R is a polynomial in z with coefficients given in terms of ε and various derivatives of ψ and η.

We proceed to provide error estimates for φ^{app}. In order to avoid confusion in what follows, we denote the velocity potential associated to the Euler system by φ^{Euler}. We assume that the flow depth is always bounded from below, an assumption which is known as the non-cavitation condition:

$$\exists h_{min} > 0, \qquad \forall x\in\mathbb{R}, \qquad 1+\varepsilon\eta =: h \geqslant h_{min}. \tag{2.10}$$

For the sake of notational convenience, we consider the difference $u = \varphi^{\mathrm{Euler}} - \varphi^{\mathrm{app}}$. First of all, by construction, u satisfies (1.8) with homogeneous boundary conditions up to the same error term $\varepsilon^3 R$ as in (2.8):

$$\begin{cases}\nabla\cdot I_\varepsilon\nabla u = \varepsilon^3 R \ \text{ in } \ \Omega_t, \\ u_{|z=1+\varepsilon\eta} = 0, \qquad \partial_z u_{|z=0} = 0.\end{cases} \tag{2.11}$$

Based on [26], it is convenient to transform the above problem on Ω_t into an elliptic boundary value problem on a flat strip $\mathcal{S}$ defined as $\mathcal{S} = \mathbb{R}\times(-1,0)$. If condition (2.10) is met, this transformation is effected by introducing the diffeomorphism

$$\begin{aligned}\Sigma: \quad &\mathcal{S}\to\Omega_t \\ &(x,\hat{z}) \longmapsto \Sigma(x,\hat{z}) = \left(x, (1+\varepsilon\zeta)\hat{z} + 1 + \varepsilon\eta\right).\end{aligned} \tag{2.12}$$

Then, the composite functions $\mathsf{u} = u\circ\Sigma$ and $\mathsf{R} = R\circ\Sigma$ satisfy the following elliptic boundary-value problem on the fixed domain $\mathcal{S}$:

$$\begin{cases}\nabla\cdot P(\Sigma)\nabla\mathsf{u} = \varepsilon^3\mathsf{R} \quad \text{in } \mathbb{R}\times(-1,0), \\ \mathsf{u}_{|\hat{z}=0} = 0, \qquad \mathbf{e}_{\hat{z}}\cdot P(\Sigma)\nabla\mathsf{u}_{|\hat{z}=-1} = 0,\end{cases} \tag{2.13}$$

and the matrix P is given by

$$P(\Sigma) = J_\Sigma^{-1} I_\varepsilon \left(J_\Sigma^{-1}\right)^T |\det J_\Sigma| = \begin{pmatrix} \varepsilon(1+\varepsilon\eta) & -\varepsilon^2(1+\hat{z})\eta_x \\ -\varepsilon^2(1+\hat{z})\eta_x & \frac{1+\varepsilon^3(1+\hat{z})^2\eta_x^2}{1+\varepsilon\eta} \end{pmatrix},$$

where I_ε is defined in (2.9) and J_Σ is the Jacobian matrix of Σ.

3. Regularity estimates

On the strip $\mathcal{S}$, it is possible to define mixed Sobolov spaces in a straightforward fashion.

Definition 1. For $s \in \mathbb{R}$, we define L^2H^s as the space of functions

$$L^2H^s = \left\{ f \in L^2(\mathcal{S}) \,:\, \|f\|_{L^2H^s} = \left(\int_{-1}^{0} |f(\cdot,\hat{z})|_{H^s}^2 d\hat{z} \right)^{1/2} < \infty \right\}.$$

Here we use shorthand notation by denoting the generic norm associated to a function space over $\mathbb{R}$ by $|\cdot|$, and use the double bar notation $||\cdot||$ for functions defined on the flat strip $\mathcal{S}$. We shall use intensively the following main estimates on $\mathcal{S}$.

Lemma 1 ([26]). *Let $s \geqslant 0$ and $t_0 > 1/2$. Assume that $\mathsf{R}, \partial_t \mathsf{R} \in H^{s,0}$ and $\mathsf{u} \in H^{s+1,1}$ be a solution to the boundary value problem (2.13) such that assumption (2.10) is satisfied. Then there exists a constant C depending on h_{min}^{-1}, ε_{max}, $|\eta|_{H^{s+1\vee t_0+1}}$ and $|\partial_t\eta|_{H^{s+1\vee t_0+1}}$ such that the estimates*

$$\|\Lambda^s \nabla \mathsf{u}\|_{L^2(\mathcal{S})} \leqslant \varepsilon^3 C\left(h_{min}^{-1}, \varepsilon_{max}, |\eta|_{H^{s+1\vee t_0+1}}\right) \|\Lambda^s \mathsf{R}\|_{L^2(\mathcal{S})}, \tag{3.1}$$

$$\|\Lambda^s \nabla \partial_t \mathsf{u}\|_{L^2(\mathcal{S})} \leqslant \varepsilon^3 C\left(h_{min}^{-1}, \varepsilon_{max}, |\eta|_{H^{s+1\vee t_0+1}}, |\partial_t\eta|_{H^{s+1\vee t_0+1}}\right) \|\Lambda^s \partial_t \mathsf{R}\|_{L^2(\mathcal{S})}, \tag{3.2}$$

holds. Here ε_{max} is an upper bound of the parameter ε and we have used the notation $\nabla = (\partial_x, \partial_z)^T$.

Proof. The proof of the first estimate (3.1) is exactly the same proof of [lemma 3.43, p 83, [26]]. In order to prove the second estimate stated in the lemma, we differentiate (2.13) with respect to t to get

$$\begin{cases} \nabla\cdot P(\Sigma)\nabla\partial_t \mathsf{u} + \nabla\cdot\partial_t[P(\Sigma)]\nabla\mathsf{u} = \varepsilon^3\partial_t\mathsf{R} & \text{in } \mathbb{R}\times(-1,0), \\ \partial_t\mathsf{u}_{|\hat{z}=0} = 0, \quad \mathbf{e}_{\hat{z}}\cdot P(\Sigma)\nabla\partial_t\mathsf{u}_{|\hat{z}=-1} + \mathbf{e}_{\hat{z}}\cdot\partial_t[P(\Sigma)]\nabla\mathsf{u}_{|\hat{z}=-1} = 0. \end{cases}$$

Note now that $v = \partial_t \mathsf{u}$ can be decomposed into $v = v_1 + v_2$ so that the above problem is now v_1 and v_2 solving

$$\begin{cases} \nabla\cdot P(\Sigma)\nabla v_1 = \varepsilon^3\partial_t\mathsf{R} & \text{in } \mathbb{R}\times(-1,0), \\ v_{1|\hat{z}=0} = 0, \quad \mathbf{e}_{\hat{z}}\cdot P(\Sigma)\nabla v_{1|\hat{z}=-1} = 0. \end{cases} \tag{3.3}$$

$$\begin{cases} \nabla\cdot P(\Sigma)\nabla v_2 = -\nabla\cdot\partial_t[P(\Sigma)]\nabla\mathsf{u} & \text{in } \mathbb{R}\times(-1,0), \\ v_{2|\hat{z}=0} = 0, \quad \mathbf{e}_{\hat{z}}\cdot P(\Sigma)\nabla v_{2|\hat{z}=-1} = -\mathbf{e}_{\hat{z}}\cdot\partial_t[P(\Sigma)]\nabla\mathsf{u}_{|\hat{z}=-1}. \end{cases} \tag{3.4}$$

As for (3.1), since v_1 is solves the BVP (3.3), from [lemma 3.43, p 83, [26]] it holds that $\|\Lambda^s\nabla v_1\|_{L^2} \leqslant \varepsilon^3 C(h_{min}^{-1}, \varepsilon_{max}, |\eta|_{H^{s+1\vee t_0+1}})\|\Lambda^s\partial_t\mathsf{R}\|_{L^2}$. To obtain similar estimate on v_2, we will use first [lemma 2.38, p 50, [26]] to bound from above $\|\Lambda^s\nabla v_2\|_{L^2} \leqslant C\left(h_{min}^{-1},, \varepsilon_{max}, |\eta|_{H^{s+1\vee t_0+1}}\right)\|\Lambda^s\partial_t[P(\Sigma)]\nabla\mathsf{u}\|_{L^2}$. Now, it is not hard to check that $\|\Lambda^s\partial_t[P(\Sigma)]\nabla \mathsf{u}\|_{L^2} \leqslant \varepsilon C\big(h_{min}^{-1}, \varepsilon_{max}, |\partial_t\eta|_{H^{s+1\vee t_0+1}}\big)\|\Lambda^s\nabla\mathsf{u}\|_{L^2}$. Then using again (3.1), the desired estimate holds. □

We shall also need the following continuity results of the Dirichlet-Neumann operator $\mathcal{G}_\varepsilon[\varepsilon\eta]$.

Lemma 2 ([26]). *Let $\eta \in H^{s+1/2}(\mathbb{R}) \cap H^{t_0+2}(\mathbb{R})$ satisfying (2.10) with $s \geqslant 0$, $t_0 > 1/2$. Then, the following two mappings $\mathcal{G}_\varepsilon[\varepsilon\eta]\colon \dot{H}^{s+1/2}(\mathbb{R}) \to H^{s-1/2}(\mathbb{R})$ and $\nu[\varepsilon\eta]\colon \dot{H}^{s+1/2}(\mathbb{R}) \to H^{s-1/2}(\mathbb{R})$ defined by*

$$\psi \mapsto \mathcal{G}_\varepsilon[\varepsilon\eta]\psi\ , \qquad \psi \mapsto \nu[\varepsilon\eta]\psi = \frac{\left[\mathcal{G}_\varepsilon[\varepsilon\eta]\psi + \varepsilon^2\eta_x\psi_x\right]^2}{2\left(1+\varepsilon^3\eta_x^2\right)}$$

are continuous. In particular, one has

$$\left|\mathcal{G}_\varepsilon[\varepsilon\eta]\psi\right|_{H^s(\mathbb{R})} \leqslant \sqrt{\varepsilon}C\left(h_{min}^{-1}, \varepsilon_{\max}, |\eta|_{H^{t_0+2}}\right)\left(|\psi_x|_{H^s(\mathbb{R})} + |\eta|_{H^{s+1}(\mathbb{R})}|\psi_x|_{H^{t_0+1}(\mathbb{R})}\right), \tag{3.5}$$

$$\left|\nu[\varepsilon\eta]\psi\right|_{H^s(\mathbb{R})} \leqslant \sqrt{\varepsilon}C\left(h_{min}^{-1}, \varepsilon_{\max}, |\eta|_{H^{t_0+2}}\right)\left(|\psi_x|_{H^s(\mathbb{R})} + |\eta|_{H^{s+1}(\mathbb{R})}|\psi_x|_{H^{t_0+1}(\mathbb{R})}\right). \tag{3.6}$$

$\dot{H}^{s+1}(\mathbb{R}) = \{f \in L^2_{loc}(\mathbb{R}),\quad f_x \in H^s(\mathbb{R})\}$ stands for the Beppo-Levi spaces [13] endowed with the (semi) norm $|f|_{\dot{H}^{s+1}(\mathbb{R})} = |f_x|_{H^s(\mathbb{R})}$.

Proof. More general version of the first inequality (3.5) was proved in [theorem 3.15, p 67, [26]]. Consequently, one may deduce the desired estimate directly. The second inequality follows using (3.5) combined with the following estimate on $f/(1+g)$ for any $f \in H^s(\mathbb{R})$ and $g \in H^s \cap H^{t_0}(\mathbb{R})$ (see [proposition B.4, [26]] for the proof):

$$\left|\frac{f}{1+g}\right|_{H^s} \leqslant C\left(h_{min}^{-1}, |g|_{H^{t_0}}\right)\left(|f|_{H^s} + |f|_{H^{t_0}}|g|_{H^s}\right). \tag{3.7}$$

□

The main goal of this section is to estimate the difference between of vertical integrals the space and time derivatives of the full and approximate potentials. These estimates will be required later on to estimate the energy density and flux, as well as the pressure forces. To do so, let us introduce the following inequalities that we shall use intensively in our analysis.

First of all, we shall use the product estimate:

$$\|fg\|_{L^2H^s} \leqslant \|f\|_{L^\infty H^s}\|g\|_{L^2H^s} \qquad \text{for } f \in L^\infty H^s \text{ and } g \in L^2H^s\ , \tag{3.8}$$

we refer the reader to corollary $B.5(1)$ in [26] for detailed proof. Moreover, we need the following essential Poincaré inequality which reads:

$$\|\mathsf{u}\|_{L^2(\mathcal{S})} \leqslant 2\|\partial_{\hat{z}}\mathsf{u}\|_{L^2(\mathcal{S})} \leqslant 2\|\nabla\mathsf{u}\|_{L^2(\mathcal{S})}\ . \tag{3.9}$$

Indeed, for any $(x,\hat{z}) \in \mathcal{S}$, by Cauchy-Schwarz inequality we have

$$|\mathsf{u}|^2 = 2\int_z^0 \mathsf{u}\partial_{\hat{z}}\mathsf{u}\, d\hat{z} \leqslant 2\|\mathsf{u}\|_{L^2(-1,0)}\|\partial_{\hat{z}}\mathsf{u}\|_{L^2(-1,0)}\ .$$

Integrating the latter on $(-1,0)$ in $\hat{z}$, we get $\|\mathsf{u}\|^2_{L^2(-1,0)} \leqslant 4\|\partial_{\hat{z}}\mathsf{u}\|^2_{L^2(-1,0)}$. Integrating now on $\mathbb{R}$ in x, the desired inequality (3.9) holds. Also, it is not hard to check that by Cauchy-Schwarz inequality and Fubini's theorem we have

$$\left|\int_{-1}^0 f\, d\hat{z}\right|_{H^s} \leqslant \|f\|_{L^2H^s}\ . \tag{3.10}$$

Now we turn on stating and proving the main results of this section in three distinct propositions.

Proposition 1. *Suppose that the assumption of lemma 1 holds. Let* $\eta \in H^{s+7}(\mathbb{R}) \cap H^{t_0+2}(\mathbb{R})$ *satisfying* (2.10) *and* $\psi_x \in H^{s+6}(\mathbb{R}) \cap H^{t_0+1}(\mathbb{R})$ *with* $s \geqslant 0$, $t_0 > 1/2$. *Then the following estimate holds*

$$\left| \int_0^{1+\varepsilon\eta} \varphi_t^{\mathrm{Euler}} - \varphi_t^{\mathrm{app}} dz \right|_{H^s} \leqslant \varepsilon^3 C \left(h_{min}^{-1}, \varepsilon_{\max}, |\eta|_{H^{s+7\vee t_0+2}}, |\psi_x|_{H^{s+6\vee t_0+1}} \right) . \tag{3.11}$$

Proof. Recall that $\partial_t \mathsf{u} := \varphi_t^{\mathrm{Euler}} - \varphi_t^{\mathrm{app}}$, by (3.10), the product estimate (3.8) and Poincaré inequality (3.9), one may observe that

$$\begin{aligned} \left| \int_0^{1+\varepsilon\eta} \partial_t \mathsf{u}\, dz \right|_{H^s(\mathbb{R})} &= \left| \int_{-1}^{0} \left[h\partial_t \mathsf{u} - \varepsilon\eta_t \left(1+\hat{z}\right) \partial_{\hat{z}} \mathsf{u} \right] \circ \Sigma \, d\hat{z} \right|_{H^s(\mathbb{R})} \\ &\leqslant \left(\varepsilon |\eta|_{H^s(\mathbb{R})} + 1 \right) \|\partial_t \mathsf{u}\|_{L^2 H^s} + 2\varepsilon |\eta_t|_{H^s} \|\Lambda^s \partial_{\hat{z}} \mathsf{u}\|_{L^2(\mathcal{S})} \\ &\leqslant 2 \left(\varepsilon |\eta|_{H^s(\mathbb{R})} + 1 \right) \|\Lambda^s \partial_t \nabla \mathsf{u}\|_{L^2(\mathcal{S})} + 2\varepsilon |\eta_t|_{H^s(\mathbb{R})} \|\Lambda^s \nabla \mathsf{u}\|_{L^2(\mathcal{S})} . \end{aligned} \tag{3.12}$$

The first equation of (1.6) combined with (3.5) yields

$$|\eta_t|_{H^s(\mathbb{R})} \leqslant \frac{1}{\sqrt{\varepsilon}} C \left(h_{min}^{-1}, \varepsilon_{\max}, |\eta|_{H^{t_0+2}} \right) \left(|\psi_x|_{H^s(\mathbb{R})} + |\eta|_{H^{s+1}(\mathbb{R})} |\psi_x|_{H^{t_0+1}(\mathbb{R})} \right) . \tag{3.13}$$

Now, combining the latter inequality with (3.1) and (3.2) in (3.12), yields

$$\begin{aligned} \left| \int_0^{1+\varepsilon\eta} \partial_t \mathsf{u}\, dz \right|_{H^s(\mathbb{R})} &\leqslant \varepsilon^3 C \left(h_{min}^{-1}, \varepsilon_{max}, |\eta|_{H^{s+2\vee t_0+2}}, |\psi_x|_{H^{s+1\vee t_0+1}} \right) \\ &\quad \times \left(\|\Lambda^s \partial_t \mathsf{R}\|_{L^2(\mathcal{S})} + \|\Lambda^s \mathsf{R}\|_{L^2(\mathcal{S})} \right) . \end{aligned} \tag{3.14}$$

It remains to estimate $|\mathsf{R}|_{H^s(\mathbb{R})}$ and $|\partial_t \mathsf{R}|_{H^s(\mathbb{R})}$. To do so, remark that by definitions (2.8)–(2.5)–(2.6)–(2.7), we have $R(x,z) = \partial_x(\eta_x h)\partial_x^2\psi + \frac{1}{2}(h+1)\eta\partial_x^4\psi + (\frac{1}{24}z^4 - \frac{1}{4}z^2 + \frac{5}{24})\partial_x^6\psi$ in Ω_t. Now we recall that η and ψ are independent from the vertical variable, then for all $(x,\hat{z}) \in \mathcal{S}$, we have

$$\begin{aligned} \mathsf{R}(x,\hat{z}) = R \circ \Sigma(x,z) &= \partial_x (\eta_x h) \partial_x^2 \psi + \frac{1}{2} (h+1) \eta \partial_x^4 \psi \\ &\quad + \left(\frac{1}{24} h^4 \left(\hat{z}^4 + 1 \right) - \frac{1}{4} h^2 \left(\hat{z} + 1 \right)^2 + \frac{5}{24} \right) \partial_x^6 \psi . \end{aligned}$$

Therefore, it is not hard to check that

$$|\mathsf{R}|_{H^s(\mathbb{R})} \leqslant C(|\eta|_{H^{s+2}}) |\partial_x^5 \psi_x|_{H^s} \quad \text{and} \quad |\partial_t \mathsf{R}|_{H^s(\mathbb{R})} \leqslant C(|\eta|_{H^{s+1}}, |\partial_t \eta|_{H^{s+2}}) |\partial_t \partial_x^5 \psi_x|_{H^s} . \tag{3.15}$$

It is worth noticing that as for estimate (3.13), one may control $|\partial_t \eta|_{H^{s+2}}$. Moreover, by the continuity properties of the operator $\nu[\varepsilon\eta]\psi$ provided in (3.6) combined with the second equation of the Zakharov-Craig-Sulem system (1.6), one may observe that

$$|\partial_t \psi_x|_{H^{s+5}} \leqslant C \left(h_{min}^{-1}, \varepsilon_{\max}, |\eta|_{H^{s+7\vee t_0+2}}, |\psi_x|_{H^{s+6\vee t_0+1}} \right) . \tag{3.16}$$

Gathering the information provided by the above estimates (3.13)–(3.15)–(3.16) in (3.14) the proof is complete. □

Proposition 2. *Suppose that the assumption of lemma 1 holds. Let* $\eta \in H^{s+3}(\mathbb{R}) \cap H^{t_0+1}(\mathbb{R})$ *satisfying* (2.10) *and* $\psi_x \in H^{s+6}(\mathbb{R})$ *with* $s \geqslant 0$, $t_0 > 1/2$. *Then the following estimates hold*

$$\Big| \int_0^{1+\varepsilon\eta} \left(\varphi_x^{\text{Euler}}\right)^2 - \left(\varphi_x^{\text{app}}\right)^2 \, dz \Big|_{H^s} \leqslant \varepsilon^3 C\left(h_{min}^{-1}, \varepsilon_{max}, |\eta|_{H^{s+3\vee t_0+1}}, |\psi_x|_{H^{s+6}}\right) , \tag{3.17}$$

$$\Big| \int_0^{1+\varepsilon\eta} \left(\varphi_z^{\text{Euler}}\right)^2 - \left(\varphi_z^{\text{app}}\right)^2 \, dz \Big|_{H^s} \leqslant \varepsilon^3 C\left(h_{min}^{-1}, \varepsilon_{max}, |\eta|_{H^{s+2\vee t_0+1}}, |\psi_x|_{H^{s+5}}\right) . \tag{3.18}$$

Proof. We start by using the transformation Σ, for any function $f(t,x,z)$, such that it holds

$$\begin{aligned}\int_0^{1+\varepsilon\eta} (\partial_x f)^2 \, dz &= \int_{-1}^{0} \frac{1}{h} \left(h\partial_x f + (\hat{z}h_x + \varepsilon\eta_x)\,\partial_{\hat{z}} f\right)^2 \, d\hat{z} \\ &= \int_{-1}^{0} h\,(\partial_x f)^2 + 2\,(\hat{z}h_x + \varepsilon\eta_x)\,\partial_x f \partial_{\hat{z}} f + \frac{1}{h}\,(\hat{z}h_x + \varepsilon\zeta_x)^2\,(\partial_{\hat{z}} f)^2 \, d\hat{z} \, .\end{aligned} \tag{3.19}$$

To estimate (3.17), we use the above identity to write

$$\begin{aligned}\int_0^{1+\varepsilon\eta} \left(\varphi_x^{\text{Euler}}\right)^2 - \left(\varphi_x^{\text{app}}\right)^2 \, dz = &\int_{-1}^{0} h \left[\left(\varphi_x^{\text{Euler}} + \varphi_x^{\text{app}}\right)\left(\varphi_x^{\text{Euler}} - \varphi_x^{\text{app}}\right)\right] \circ \Sigma \, d\hat{z} \\ &+ 2\int_{-1}^{0} (\hat{z}+1)\,\varepsilon\eta_x \left[\partial_x\varphi^{\text{Euler}}\partial_{\hat{z}}\varphi^{\text{Euler}} - \partial_x\varphi^{\text{app}}\partial_{\hat{z}}\varphi^{\text{app}}\right] \circ \Sigma \, d\hat{z} \\ &+ \int_{-1}^{0} \frac{1}{h}\,(\hat{z}+1)^2 \varepsilon^2\eta_x^2 \left[\left(\partial_{\hat{z}}\varphi^{\text{Euler}}\right)^2 - \left(\partial_{\hat{z}}\varphi^{\text{app}}\right)^2\right] \circ \Sigma \, d\hat{z} =: \mathbf{I} + \mathbf{II} + \mathbf{III} \, .\end{aligned}$$

To control $\mathbf{I}$, by (3.10), the product estimate (3.8), the Poincaré inequality (3.9), the fact that $|\mathsf{u}_x|_{L^\infty H^s} \leqslant \text{supess}_{z\in(-1,0)} |\mathsf{u}|_{H^{s+1,1}}$, then it holds that

$$\begin{aligned}|\mathbf{I}|_{H^s} &\leqslant \|h\,(\mathsf{u}_x + 2\varphi_x^{\text{app}})\,\mathsf{u}_x\|_{L^2H^s} \leqslant \|h\,(\mathsf{u}_x + 2\varphi_x^{\text{app}})\,\|_{L^\infty H^s}\|\mathsf{u}_x\|_{L^2H^s} \\ &\leqslant (|h-1|_{H^s} + 1)\,\|\mathsf{u}_x + 2\varphi_x^{\text{app}}\|_{L^\infty H^s}\|\mathsf{u}\|_{L^2H^{s+1}} \\ &\leqslant (|h-1|_{H^s} + 1)\,(\|\mathsf{u}_x\|_{L^\infty H^s} + 2\,(|\eta|_{H^{s+1}} + 1)\,|\psi_{xxxxx}|_{H^s})\,\|\Lambda^{s+1}\mathsf{u}\|_{L^2(\mathcal{S})} \\ &\leqslant (|h-1|_{H^s} + 1)\,(\|\mathsf{u}\|_{H^{s+1,1}} + 2\,(|\eta|_{H^{s+1}} + 1)\,|\psi_{xxxxx}|_{H^s})\,\|\Lambda^{s+1}\nabla\mathsf{u}\|_{L^2(\mathcal{S})} \, .\end{aligned} \tag{3.20}$$

Now using estimate (3.1) for $s+1$ instead of s, it holds that $|\mathbf{I}|_{H^s} \leqslant \varepsilon^3 C\big(h_{min}^{-1}, \varepsilon_{max}, |\eta|_{H^{s+2\vee t_0+1}},$ $|\Lambda^{s+1}\mathsf{R}|_{L^2}, |\psi_x|_{H^{s+4}}\big)$. We recall that from (3.15), we have $|\mathsf{R}|_{H^{s+1}(\mathbb{R})} \leqslant C\big(|\eta|_{H^{s+3}}\big)|\psi_x|_{H^{s+6}}$ and then (3.17) holds.

To control the cross term $\mathbf{II}$, remark that

$$\partial_x\varphi^{\text{Euler}}\partial_{\hat{z}}\varphi^{\text{Euler}} - \partial_x\varphi^{\text{app}}\partial_{\hat{z}}\varphi^{\text{app}} = \partial_x\mathsf{u}\partial_{\hat{z}}\mathsf{u} - \partial_x\mathsf{u}\partial_{\hat{z}}\varphi^{\text{app}} + \partial_{\hat{z}}\mathsf{u}\partial_x\varphi^{\text{app}}. \tag{3.21}$$

Consequently, as in (3.20), it holds that $|\mathbf{II}|_{H^s} \leqslant \varepsilon^3 C\big(h_{min}^{-1}, \varepsilon_{max}, |\eta|_{H^{s+3\vee t_0+1}}, |\psi_x|_{H^{s+6}}\big)$. The control of $|\mathbf{III}|_{H^s} \leqslant \varepsilon^3 C\big(h_{min}^{-1}, \varepsilon_{max}, |\eta|_{H^{s+2\vee t_0+1}}, |\psi_x|_{H^{s+5}}\big)$ follows similarly using in addition the below estimate :

$$\Big\| \frac{f}{1+\varepsilon\eta} \Big\|_{L^2H^s} \leqslant C\left(h_{min}^{-1}, \varepsilon_{max}, |\eta|_{H^s}\right) \|f\|_{L^2H^s}. \tag{3.22}$$

We refer the reader to corollary $B.6$ in [26] for the proof of the latter estimate. As a result, estimate (3.17) holds. To estimate (3.18), as for $\mathbf{III}$, using (3.22) the desired estimate holds. □

Proposition 3. *Suppose that the assumption of lemma 1 holds. Let $\eta \in H^{s+3}(\mathbb{R}) \cap H^{t_0+1}(\mathbb{R})$ satisfying (2.10) and $\psi_x \in H^{s+6}(\mathbb{R})$ with $s \geqslant 0$, $t_0 > 1/2$. Then the following estimates hold:*

$$\Big| \int_0^{1+\varepsilon\zeta} \left(\varphi_x^{\text{Euler}} - \varphi_x^{\text{app}}\right) dz \Big|_{H^s} \leqslant \varepsilon^3 C\left(h_{min}^{-1}, \varepsilon_{max}, |\eta|_{H^{s+3\vee t_0+1}}, |\psi_x|_{H^{s+6}}\right) , \quad (3.23)$$

$$\Big| \int_0^{1+\varepsilon\zeta} \left(\varphi_z^{\text{Euler}} - \varphi_z^{\text{app}}\right) dz \Big|_{H^s} \leqslant \varepsilon^3 C\left(h_{min}^{-1}, \varepsilon_{max}, |\eta|_{H^{s+2\vee t_0+1}}, |\psi_x|_{H^{s+5}}\right) . \quad (3.24)$$

Proof. Recall that $\mathsf{u} = (\varphi^{\text{Euler}} - \varphi^{\text{app}}) \circ \Sigma$ and $\mathsf{R} = R \circ \Sigma$. Changing variables in the integral in (3.23), we get

$$\int_0^{1+\varepsilon\eta} \partial_x u\, dz = \int_{-1}^{0} h\partial_x \mathsf{u} + (\hat{z}+1)\,\varepsilon\eta_x \partial_{\hat{z}} \mathsf{u}\, d\hat{z} = I + II .$$

As in Propsition 2, by (3.10), the product estimate (3.8) and (3.9), it holds that

$$|I|_{H^s} \leqslant (|h-1|_{H^s} + 1)\, \|\mathsf{u}_x\|_{L^2 H^{s+1}} \leqslant C(|\eta|_{H^s})\, \|\Lambda^{s+1} \nabla \mathsf{u}\|_{L^2(\mathcal{S})} ,$$

$$\text{and} \qquad |II|_{H^s} \leqslant \varepsilon |\eta_x|_{H^s} \|\Lambda^s \partial_{\hat{z}} \mathsf{u}\|_{L^2(\mathcal{S})} \leqslant C(|\eta|_{H^{s+1}})\, \|\Lambda^s \nabla \mathsf{u}\|_{L^2(\mathcal{S})} .$$

Now using the first estimate of lemma 1 for s and $s+1$ combined with (3.15), estimate (3.23) holds. The proof of the second estimate (3.24) follows similarly as follows.

$$\Big| \int_0^{1+\varepsilon\eta} \partial_z u\, dz \Big|_{H^s} = \Big| \int_{-1}^{0} \partial_{\hat{z}} \mathsf{u}\, d\hat{z} \Big|_{H^s} \leqslant \|\partial_{\hat{z}} \mathsf{u}\|_{L^2 H^s} \leqslant 2 \|\Lambda^s \nabla \mathsf{u}\|_{L^2(\mathcal{S})} .$$

□

Proposition 4. *Suppose that the assumption of lemma 1 holds. Let $\eta \in H^{s+3}(\mathbb{R}) \cap H^{t_0+1}(\mathbb{R})$ satisfying (2.10) and $\psi_x \in H^{s+6}(\mathbb{R})$ with $s \geqslant 0$, $t_0 > 1/2$. Then the following estimates hold*

$$\Big| \int_0^{1+\varepsilon\eta} \left(\varphi_x^{\text{Euler}}\right)^3 - \left(\varphi_x^{\text{app}}\right)^3 dz \Big|_{H^s} \leqslant \varepsilon^3 C\left(h_{min}^{-1}, \varepsilon_{max}, |\eta|_{H^{s+3\vee t_0+1}}, |\psi_x|_{H^{s+6}}\right) , \quad (3.25)$$

$$\Big| \int_0^{1+\varepsilon\eta} \left(\varphi_z^{\text{Euler}}\right)^3 - \left(\varphi_z^{\text{app}}\right)^3 dz \Big|_{H^s} \leqslant \varepsilon^3 C\left(h_{min}^{-1}, \varepsilon_{max}, |\eta|_{H^{s+2\vee t_0+1}}, |\psi_x|_{H^{s+5}}\right) . \quad (3.26)$$

Proof. For the first estimate, remark that

$$\begin{aligned} \int_0^{1+\varepsilon\eta} \left(\varphi_x^{\text{Euler}}\right)^3 - \left(\varphi_x^{\text{app}}\right)^3 dz &= \int_0^{1+\varepsilon\eta} \left(\left(\varphi_x^{\text{Euler}}\right)^2 - \left(\varphi_x^{\text{app}}\right)^2\right) u_x\, dz \\ &\quad + \int_0^{1+\varepsilon\eta} \varphi_x^{\text{app}} \left(\left(\varphi_x^{\text{Euler}}\right)^2 - \left(\varphi_x^{\text{app}}\right)^2\right) dz + \int_0^{1+\varepsilon\eta} \left(\varphi_x^{\text{app}}\right)^2 u_x\, dz \\ &= J_1 + J_2 + J_3 . \end{aligned}$$

To control $J_2 + J_3$, first remark that by definitions (2.8)–(2.5)–(2.6)–(2.7), it holds that

$$\Big| \sup_{z \in (0, 1+\varepsilon\eta)} \varphi_x^{\text{app}} \Big|_{H^s} \leqslant C\left(h_{min}^{-1}, \varepsilon_{max}, |\eta|_{H^{s+2}}, |\psi_x|_{H^{s+5}}\right) .$$

Consequently, using the latter inequality combined with the estimates (3.17)–(3.23), we get that $|J_2+J_3|_{H^s} \leqslant \varepsilon^3 C\big(h_{min}^{-1}, \varepsilon_{max}, |\eta|_{H^{s+2\vee t_0+1}}, |\psi_x|_{H^{s+5}}\big)$. To control J_1, we start by using the transformation Σ. In view of (3.19) and (3.21), it is not hard to check that

$$\begin{aligned} J_1 =& \int_{-1}^{0} h\left(\mathsf{u}_x + 2\varphi_x^{\text{app}}\right)\mathsf{u}_x^2\, d\hat{z} + 2\int_{-1}^{0} (\hat{z}+1)\,\varepsilon\eta_x\left(\mathsf{u}_x\mathsf{u}_{\hat{z}} - \mathsf{u}_x\varphi_{\hat{z}}^{\text{app}} + \mathsf{u}_{\hat{z}}\varphi_x^{\text{app}}\right)\mathsf{u}_x\, d\hat{z} \\ &+ \int_{-1}^{0} \frac{1}{h}(\hat{z}+1)^2\varepsilon^2\eta_x^2\left(\mathsf{u}_{\hat{z}} + 2\varphi_{\hat{z}}^{\text{app}}\right)\mathsf{u}_{\hat{z}}\mathsf{u}_x\, d\hat{z} + \int_{-1}^{0} \varepsilon(\hat{z}+1)\,\eta_x\left(\mathsf{u}_x + 2\varphi_x^{\text{app}}\right)\mathsf{u}_x\mathsf{u}_{\hat{z}}\, d\hat{z} \\ &+ 2\int_{-1}^{0} \frac{1}{h}(\hat{z}+1)^2\varepsilon^2\eta_x^2\left(\mathsf{u}_x\mathsf{u}_{\hat{z}} - \mathsf{u}_x\varphi_{\hat{z}}^{\text{app}} + \mathsf{u}_{\hat{z}}\varphi_x^{\text{app}}\right)\mathsf{u}_{\hat{z}}\, d\hat{z} + \int_{-1}^{0} \frac{1}{h^2}(\hat{z}+1)^3\varepsilon^3\eta_x^3\left(\mathsf{u}_{\hat{z}} + 2\varphi_{\hat{z}}^{\text{app}}\right)\mathsf{u}_{\hat{z}}^2\, d\hat{z}\,. \end{aligned}$$

Controlling of the above integrals follows same lines of the proof of proposition 2, in particular, we refer to (3.20) for similar procedure. To estimate (3.26), remark that

$$\left(\varphi_z^{\text{Euler}}\right)^3 - \left(\varphi_z^{\text{app}}\right)^3 = \left[\left(\varphi_z^{\text{Euler}}\right)^2 - \left(\varphi_z^{\text{app}}\right)^2\right] u_z + \left(\varphi_z^{\text{app}}\right)^2 u_z + \varphi_z^{\text{app}}\left[\left(\varphi_z^{\text{Euler}}\right)^2 - \left(\varphi_z^{\text{app}}\right)^2\right].$$

The rest of the proof follows as above. □

Proposition 5. *Suppose that the assumption of lemma 1 holds. Let* $\eta \in H^{s+7}(\mathbb{R}) \cap H^{t_0+2}(\mathbb{R})$ *satisfying (2.10) and* $\psi_x \in H^{s+6}(\mathbb{R}) \cap H^{t_0+1}(\mathbb{R})$ *with* $s \geqslant 0$, $t_0 > 1/2$. *Then the following estimate holds*

$$\left|\int_0^{1+\varepsilon\eta} \varphi_t^{\text{Euler}}\varphi_x^{\text{Euler}} - \varphi_t^{\text{app}}\varphi_x^{\text{app}}\, dz\right|_{H^s} \leqslant \varepsilon^3 C\left(h_{min}^{-1}, \varepsilon_{\max}, |\eta|_{H^{s+7\vee t_0+2}}, |\psi_x|_{H^{s+6\vee t_0+1}}\right), \tag{3.27}$$

$$\left|\int_0^{1+\varepsilon\eta} \left(\varphi_z^{\text{Euler}}\right)^2\varphi_x^{\text{Euler}} - \left(\varphi_z^{\text{app}}\right)^2\varphi_x^{\text{app}}\, dz\right|_{H^s} \leqslant \varepsilon^3 C\left(h_{min}^{-1}, \varepsilon_{max}, |\eta|_{H^{s+2\vee t_0+1}}, |\psi_x|_{H^{s+5}}\right). \tag{3.28}$$

Proof. To estimate (3.27), remark that

$$\begin{aligned} \int_0^{1+\varepsilon\eta} \varphi_t^{\text{Euler}}\varphi_x^{\text{Euler}} - \varphi_t^{\text{app}}\varphi_x^{\text{app}}\, dz =& \int_0^{1+\varepsilon\eta} \left(\varphi_t^{\text{Euler}} - \varphi_t^{\text{app}}\right) u_x\, dz \\ &+ \int_0^{1+\varepsilon\eta} \varphi_x^{\text{app}}\left(\varphi_t^{\text{Euler}} - \varphi_t^{\text{app}}\right) dz + \int_0^{1+\varepsilon\eta} \varphi_t^{\text{app}} u_x\, dz \\ =& \ \mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3\,. \end{aligned}$$

To control $\mathbf{J}_2 + \mathbf{J}_3$, first remark that by definitions (2.8)–(2.5)–(2.6)–(2.7), it holds that

$$\Big|\sup_{z\in(0,1+\varepsilon\eta)} \varphi_t^{\text{app}}\Big|_{H^s} \leqslant C\left(h_{min}^{-1}, \varepsilon_{max}, |\eta_t|_{H^{s+2}}, |\partial_t\psi_x|_{H^{s+5}}\right).$$

Consequently, using the latter inequality combined with the estimates (3.11), (3.23), (3.12) and (3.16), we get that $|\mathbf{J}_2+\mathbf{J}_3|_{H^s} \leqslant \varepsilon^3 C\big(h_{min}^{-1}, \varepsilon_{\max}, |\eta|_{H^{s+7\vee t_0+2}}, |\psi_x|_{H^{s+6\vee t_0+1}}\big)$. To control $\mathbf{J}_1$, we start by using the transformation Σ. In view of (3.19) and (3.21), it is not hard to check that

$$\begin{aligned} \mathbf{J}_1 =& \int_{-1}^{0} h\mathsf{u}_x\mathsf{u}_t\, d\hat{z} + \varepsilon\int_{-1}^{0} \eta_x(\hat{z}+1)\,\mathsf{u}_{\hat{z}}\mathsf{u}_t\, d\hat{z} - \varepsilon\int_{-1}^{0} \eta_t(\hat{z}+1)\,\mathsf{u}_{\hat{z}}\mathsf{u}_x\, d\hat{z} - \varepsilon^2\int_{-1}^{0} \frac{1}{h}\eta_x(\hat{z}+1)\,\mathsf{u}_{\hat{z}}^2\, d\hat{z} \\ =& \ \mathbf{J}_{11} + \mathbf{J}_{12} + \mathbf{J}_{13} + \mathbf{J}_{14}. \end{aligned}$$

By (3.10), the product estimates (3.8)–(3.22) and Poincaré inequality (3.9), one may observe that

$$|\mathbf{J}_{11}+\mathbf{J}_{12}|_{H^s} \leqslant \left(\varepsilon|\eta|_{H^s(\mathbb{R})}+1\right)\|\mathsf{u}_x\|_{L^\infty H^s}\|\partial_t\mathsf{u}\|_{L^2H^s}+\varepsilon|\eta_x|_{H^s}\|\mathsf{u}_{\hat{z}}\|_{L^\infty H^s}\|\partial_t\mathsf{u}\|_{L^2H^s}$$
$$\leqslant 2\left(\varepsilon|\eta|_{H^s(\mathbb{R})}+1+\varepsilon|\eta_x|_{H^s}\right)\|\mathsf{u}\|_{H^{s+1,1}}\|\Lambda^s\partial_t\nabla\mathsf{u}\|_{L^2(\mathcal{S})}$$

and $$|\mathbf{J}_{13}+\mathbf{J}_{14}|_{H^s} \leqslant 2\left(\varepsilon|\eta_t|_{H^s(\mathbb{R})}+\varepsilon^2C|\eta_x|_{H^s}\right)\|\mathsf{u}\|_{H^{s+1,1}}\|\Lambda^{s+1}\nabla\mathsf{u}\|_{L^2(\mathcal{S})}\,.$$

Now as in the proof of proposition 1, estimate (3.27) holds. To estimate (3.28), remark that

$$\left(\varphi_z^{\text{Euler}}\right)^2\varphi_x^{\text{Euler}}-(\varphi_z^{\text{app}})^2\varphi_x^{\text{app}}=\left[\left(\varphi_z^{\text{Euler}}\right)^2-(\varphi_z^{\text{app}})^2\right]u_x+(\varphi_z^{\text{app}})^2u_x+\varphi_x^{\text{app}}\left[\left(\varphi_z^{\text{Euler}}\right)^2-(\varphi_z^{\text{app}})^2\right]\,.$$

The rest of the proof follows as in the proof of proposition 2. □

4. Mathematical justification of the KdV equation

The next step is to connect the approximated velocity potential to the KdV equation. Completing this step turns out to be dependent on an approximation of the horizontal velocity at the bottom and some observations on Boussinesq-type systems. For the sake of readability, we introduce the following simplified notation.

Definition 2. We denote by $\mathcal{O}(\varepsilon^n)$, with $n\in\mathbb{N}$ any family of functions $\{g^\varepsilon\}_{\varepsilon\in(0,1)}$ such that $\dfrac{1}{\varepsilon^n}g^\varepsilon$ remains bounded in $L^\infty([0,\frac{T}{\varepsilon}]\times\mathbb{R})$ or in $L^\infty([0,\frac{T}{\varepsilon}],H^s(\mathbb{R}))$ for some $s>1/2$ and for all ε small enough.

4.1. The intermediate system

This section is dedicated to revealing the fully symmetric Boussinesq system (4.6) as a bridge between the Euler system and the KdV equation. Also we will present the mathematical justification of the system (4.6).

It will be convenient in the following to use the velocity potential evaluated at the bottom. By using the above approximations of φ (2.1), it becomes clear that to order $\mathcal{O}(\varepsilon^2)$, this quantity is represented by

$$f := \varphi^{\text{app}}{}_{|_{z=0}} = \psi+\frac{1}{2}\varepsilon\psi_{xx}\,. \tag{4.1}$$

It is to hard to check that at the free surface we have $\partial_x\varphi^{app}=\psi_x+\mathcal{O}(\varepsilon^2)$ and $\partial_z\varphi^{app}=-\varepsilon\psi_{xx}-\varepsilon^2\eta\psi_{xx}-\frac{1}{3}\varepsilon^2\psi_{xxxx}+\mathcal{O}(\varepsilon^3)$. Thus, in view of the function *f*, the second boundary condition in (1.5) becomes

$$\eta_t+f_{xx}+\varepsilon\eta f_{xx}+\varepsilon\eta_x f_x-\frac{1}{6}\varepsilon f_{xxxx}=\mathcal{O}\left(\varepsilon^2\right), \tag{4.2}$$

where f_x represents the horizontal velocity at the bottom (note that in the present inviscid theory, the velocity component tangential to the bottom is unrestricted). Furthermore, the third boundary condition in (1.5) becomes

$$\eta+f_t-\frac{\varepsilon}{2}f_{xxt}+\frac{\varepsilon}{2}{f_x}^2=\mathcal{O}\left(\varepsilon^2\right). \tag{4.3}$$

Differentiating (4.3) with respect to x, using (4.2) as the first equation, and denoting $w = f_x$ yields the following approximation:

$$\begin{aligned} \eta_t + w_x + \varepsilon(\eta w)_x - \frac{1}{6}\varepsilon w_{xxx} &= \mathcal{O}\left(\varepsilon^2\right), \\ w_t + \eta_x + \varepsilon w w_x - \frac{1}{2}\varepsilon w_{xxt} &= \mathcal{O}\left(\varepsilon^2\right). \end{aligned} \tag{4.4}$$

The last system can be approximated using the fact that $\eta_x = -w_t + \mathcal{O}(\varepsilon)$:

$$\begin{aligned} \eta_t + \left(1 - \frac{1}{6}\varepsilon\partial_x^2\right) w_x + \varepsilon(\eta w)_x &= \mathcal{O}\left(\varepsilon^2\right), \\ \left(1 - \frac{2}{3}\varepsilon\partial_x^2\right) w_t + \left(1 - \frac{1}{6}\varepsilon\partial_x^2\right)\eta_x + \varepsilon w w_x &= \mathcal{O}\left(\varepsilon^2\right). \end{aligned} \tag{4.5}$$

Let us now define two new unknowns, (ξ, v), where ξ is an approximation of the free surface elevation η and v is an approximation of the horizontal velocity on the bottom w. The unknowns (ξ, v) are solutions of the system (4.5), but with the error terms removed. To put it another way, we look at the fully symmetric Boussinesq equations given by

$$\begin{cases} \xi_t + \left(1 - \frac{1}{6}\varepsilon\partial_x^2\right) v_x + \varepsilon(\xi v)_x = 0, \\ \left(1 - \frac{2}{3}\varepsilon\partial_x^2\right) v_t + \left(1 - \frac{1}{6}\varepsilon\partial_x^2\right)\xi_x + \varepsilon v v_x = 0. \end{cases} \tag{4.6}$$

This system acts as an intermediate system between the Euler system and the KdV equation. To make this more rigorous, we define the concept of consistency between the water-wave problem and the fully symmetric Boussinesq system (4.6) in the following sense:

Definition 3 (Consistency). Let $\varepsilon \ll 1$. We say that the water-wave equations (1.6) are consistent at order $\mathcal{O}(\varepsilon^2)$ with the fully symmetric Boussinesq equations (4.6) if there exists $n \in \mathbb{N}$ and $T > 0$ such that for all $s \geqslant 0$, $t_0 > 1/2$,

- There exists a solution $(\eta, \psi_x) \in C([0, T/\varepsilon]; (H^{s+n})^2)$ to the water-wave equations (1.6).
- Defining $w = \psi_x + \frac{1}{2}\varepsilon\psi_{xxx}$, one has

$$\begin{cases} \eta_t + \left(1 - \frac{1}{6}\varepsilon\partial_x^2\right) w_x + \varepsilon(\eta w)_x = \varepsilon^2 R_1, \\ \left(1 - \frac{2}{3}\varepsilon\partial_x^2\right) w_t + \left(1 - \frac{1}{6}\varepsilon\partial_x^2\right)\eta_x + \varepsilon w w_x = \varepsilon^2 R_2, \end{cases} \tag{4.7}$$

  with $|R_1|_{H^s} + |R_2|_{H^s} \leqslant C\left(h_{\min}^{-1}, \varepsilon_{\max}, |\eta|_{H^{s+7\vee t_0+2}}, |\psi_x|_{H^{s+6\vee t_0+1}}\right)$ on $[0, T/\varepsilon]$.

The consistency result mentioned above for the fully symmetric Boussinesq system is the following.

Proposition 6 (Consistency of (4.6)). *Let $\varepsilon \ll 1$. The water-wave problem (1.6) is consistent at order $\mathcal{O}(\varepsilon^2)$ with the fully symmetric Boussinesq system (4.6).*

Proof. In order to check that the two equations are satisfied up to order $\mathcal{O}(\varepsilon^2)$, we need an asymptotic expansion of ψ_x in terms of w which can be deduced from (4.1) as follows :

$$\psi_x = w - \frac{1}{2}\varepsilon w_{xx} + \varepsilon^2 r. \tag{4.8}$$

Now, substitute $\mathcal{G}[\varepsilon\eta]\psi$ in the first equation of (1.6) by its expression (1.7) with φ replaced by φ^{app}:

$$\mathcal{G}[\varepsilon\eta]\,\psi = -\varepsilon\psi_{xx} - \varepsilon^2(\eta\psi_x)_x - \frac{1}{3}\varepsilon^2\psi_{xxxx} + \varepsilon^3 R = -w_x + \frac{1}{6}\varepsilon^2 w_{xxx} - \varepsilon^2(\eta w)_x + \varepsilon^3(r_x + R)\ . \tag{4.9}$$

Also, take the derivative of the second equation of (1.6) and replace ψ_x by its expression (4.8). We denote by $\varepsilon^2 R_1$ and $\varepsilon^2 R_2$ the residual of the first and the second equations corresponding to the consistency result stated above, one therefore has that R_1 and R_2 depends on the remainder $\partial_x r$ and R the same error term as in (2.8). Now we use (4.8) and definition of w to give some control on r as follows :

$$|r_x|_{H^s} \leqslant |\partial_x^5\psi_x|_{H^s} \qquad \text{and} \qquad |\partial_t r_x|_{H^s} \leqslant |\partial_x^5\partial_t\psi_x|_{H^s}\ . \tag{4.10}$$

Taking advantage of the estimates (3.13), (3.15) and (3.16) in the proof of proposition 1 the proof is complete with n large enough (mainly greater than 7). □

We turn our attention now to the full justification (existence+stability+convergence) of the fully symmetric Boussinesq system (4.6). In other words, we state here that the solutions to the water waves equations exist on the relevant time scale associated to the Boussinesq regime under consideration and remain close to the approximation furnished by (4.6). Let us first remark that the model (4.6) is in fact the same as the system (5.34) of [26] if one takes the parameter values

$$\lambda = 0, \quad \theta = \frac{1}{2}, \quad \delta = 0, \quad \alpha = \frac{3}{2},$$

in the notation from [26].

The well-posedness result of system (4.6) has been proved in [27], note that according to the notation used in [27], $a = c = -\dfrac{1}{6}$, $b = 0$ and $d = \dfrac{2}{3}$, for our system (4.6).

Theorem 1 (Local existence [27]). *Let $t_0 > 1/2$ and $s \geqslant t_0 + 2$. Assume that $V_0 = (\xi_0, v_0)^T \in X^s(\mathbb{R}) = H^{s+3}(\mathbb{R}) \times H^{s+4}(\mathbb{R})$ satisfying condition (2.10). Then there exist $T > 0$ independent of ε such that system (4.6) has a unique solution $V = (\xi, v)^T \in C([0, T/\varepsilon]; X^s(\mathbb{R}))$. Moreover, we have the following size estimate*

$$\max_{t\in[0,T/\varepsilon]} |(\xi, v)|_{X^s} \leqslant C\left(h_{\min}^{-1}, \varepsilon_{\max}, |\xi_0|_{H^{s+3}}, |v_0|_{H^{s+4}}\right)\ . \tag{4.11}$$

Theorem 1 is complemented by the following result, which shows the solution's stability with respect to perturbations and is very useful for justifying asymptotic approximations to the exact solution. The solution $V = (\xi, v)^T$ and time T in the following statement are provided by theorem 1.

Theorem 2 (A stability property). *Suppose that the assumption of theorem 1 is satisfied and moreover assume that there exists $\widetilde{V} = (\widetilde{\xi}, \widetilde{v})^T \in C\left([0, T/\varepsilon], X^{s+1}(\mathbb{R})\right)$ such that*

$$\begin{cases} \widetilde{\xi}_t + \left(1 - \frac{1}{6}\varepsilon\partial_x^2\right)\widetilde{v}_x + \varepsilon\left(\widetilde{\xi}\widetilde{v}\right)_x = \widetilde{r}\,, \\ \left(1 - \frac{2}{3}\varepsilon\partial_x^2\right)\widetilde{v}_t + \left(1 - \frac{1}{6}\varepsilon\partial_x^2\right)\widetilde{\xi}_x + \varepsilon\widetilde{v}\widetilde{v}_x = \widetilde{R}\,, \end{cases}$$

with $\widetilde{F} = (\widetilde{r}, \widetilde{R})^T \in L^\infty\left([0, T/\varepsilon], H^{s+5}(\mathbb{R})\right)$. Then for all $t \in [0, T/\varepsilon]$, the error $\boldsymbol{V} = V - \widetilde{V} = (\xi, v)^T - (\widetilde{\xi}, \widetilde{v})^T$ with respect to V given by theorem 1 satisfies for all $0 \leqslant t \leqslant T/\varepsilon$ the following inequality

$$\left|\boldsymbol{V}\right|_{L^\infty([0,t],X^s(\mathbb{R}))} \leqslant \widetilde{C}\left(\left|\boldsymbol{V}_{|t=0}\right|_{X^{s+1}(\mathbb{R})} + t\left|\widetilde{F}\right|_{L^\infty([0,t],H^{s+5}(\mathbb{R}))}\right), \tag{4.12}$$

where the constant $\widetilde{C}$ is depending on $h_{\min}^{-1}$, $|V|_{L^\infty([0,T/\varepsilon],X^s(\mathbb{R}))}$ and $|\widetilde{V}|_{L^\infty([0,T/\varepsilon],X^{s+1}(\mathbb{R}))}$.

Proof. The proof is a direct and classical consequence of similar energy estimates evaluated in [27] consists on the evaluation of $\frac{1}{2}\frac{d}{dt}\left|\mathbf{V}\right|^2_{X^s(\mathbb{R})}$ combined with Grönwall's inequality. The key step of the proof that we shall omit, is subtracting the equations satisfied by V and $\widetilde{V}$ then proceeding as in [27]. □

As a conclusion, we are able now to provide the full justification of system (4.6) through the below theorem.

Theorem 3 (Full justification of (4.6)). *Let $\varepsilon \in (0,1)$, $t_0 > 1/2$, $s \geqslant t_0+2$ and $(\eta_0, \psi_{0,x})^T \in H^{s+N+1}(\mathbb{R}) \times H^{s+N}(\mathbb{R})$ satisfying condition (2.10) where N sufficiently large (mainly greater than 8). Then there exists $T>0$, independent of ε such that*

- *There exists a unique solution $(\eta, \psi_x) \in C([0,T/\varepsilon]; H^{s+N+1}(\mathbb{R}) \times H^{s+N}(\mathbb{R}))$ to the water-wave equations (1.6), and to which one associates through $\partial_x \varphi_{|z=0} = w^{\mathrm{Euler}} \in C([0,T/\varepsilon]; H^{s+N-6})$ the bottom horizontal velocity.*
- *There exists a unique solution $(\xi, v) \in C([0,T/\varepsilon]; X^{s+N-6}) \subset C([0,T/\varepsilon]; X^{s+N-11})$ to (4.6) with initial data $(\xi_0, v_0) = (\eta_0, \psi_{0,x} + \frac{1}{2}\varepsilon\psi_{0,xxx}) + \mathcal{O}(\varepsilon^2)$.*
- *The following error estimate holds, for all $0 \leqslant t \leqslant T/\varepsilon$,*

$$|\xi - \eta|_{W^{2,\infty}} \leqslant \varepsilon^2 (1+t)\, C\left(h_{\min}^{-1}, \varepsilon_{\max}, |\eta_0|_{H^{s+N+1}}, |w_0|_{H^{s+N-6}}, |\psi_{0,x}|_{H^{s+N}}\right)\,, \tag{4.13}$$

$$|v - w|_{W^{2,\infty}} \leqslant \varepsilon^2 (1+t)\, C\left(h_{\min}^{-1}, \varepsilon_{\max}, |\eta_0|_{H^{s+N+1}}, |w_0|_{H^{s+N-6}}, |\psi_{0,x}|_{H^{s+N}}\right)\,, \tag{4.14}$$

and

$$|v - w^{\mathrm{Euler}}|_{W^{2,\infty}} \leqslant \varepsilon^2 (1+t)\, C\left(h_{\min}^{-1}, \varepsilon_{\max}, |\eta_0|_{H^{s+N+1}}, |w_0|_{H^{s+N-6}}, |\psi_{0,x}|_{H^{s+N}}\right)\,. \tag{4.15}$$

Proof. In view of the large assumption made on N, we refer to [theorem 4.16, p 102, [26]] for the deduction of the existence and uniqueness of a solution to (1.6). To prove the regularity of w^{Euler}, we recall that $w = \partial_x \varphi^{\mathrm{app}}_{|_{z=0}}$ and remark that

$$\begin{aligned}|w^{\mathrm{Euler}} - w|_{H^{s+N-6}} \leqslant |u_{|_{z=0}}|_{H^{s+N-5}} = \Big|\int_0^{1+\varepsilon\eta} \partial_z u\, dz\Big|_{H^{s+N-5}} \\ \leqslant \varepsilon^3 C\left(h_{min}^{-1}, \varepsilon_{max}, |\eta|_{H^{s+N-3\vee t_0+1}}, |\psi_x|_{H^{s+N}}\right)\,.\end{aligned} \tag{4.16}$$

Here we did similar proof to that of (3.24) by using (3.1). Hence, the regularity of w holds. The second point follows directly from theorem 1. For the third point, proposition 6 implies that

$$\begin{cases} \eta_t + \left(1 - \frac{1}{6}\varepsilon\partial_x^2\right) w_x + \varepsilon\left(\eta w\right)_x = \varepsilon^2 R_1\,, \\ \left(1 - \frac{2}{3}\varepsilon\partial_x^2\right) w_t + \left(1 - \frac{1}{6}\varepsilon\partial_x^2\right)\eta_x + \varepsilon w w_x = \varepsilon^2 R_2\,, \end{cases}$$

with $|R_1|_{H^s} + |R_2|_{H^s} \leqslant C\left(h_{\min}^{-1}, \varepsilon_{\max}, |\eta|_{H^{s+7}}, |\psi_x|_{H^{s+6}}\right)$ on $[0,T/\varepsilon]$. Now by theorem 2, with $V = (\xi, v)$ and $\widetilde{V} = (\eta, w) \in C([0,T/\varepsilon]; X^{s+N-10})$ and $\widetilde{F} = \varepsilon^2(R_1, R_2)$, the stability property (4.12) implies that

$$
\begin{aligned}
|V-\widetilde{V}|_{L^\infty\left([0,t],X^{s+N-11}(\mathbb{R})\right)} &= \left|\xi-\eta\right|_{L^\infty\left([0,t],H^{s+N-8}(\mathbb{R})\right)} + \left|v-w\right|_{L^\infty\left([0,t],H^{s+N-7}(\mathbb{R})\right)} \\
&\leqslant \varepsilon^2\left(1+t\right)C\left(h_{\min}^{-1}, |\left(\eta,w\right)|_{X^{s+N-10}}, |\left(\xi,v\right)|_{X^{s+N-11}}, |\left(R_1,R_2\right)|_{H^{s+N-6}}\right) \\
&\leqslant \varepsilon^2\left(1+t\right)C\left(h_{\min}^{-1}, \varepsilon_{\max}, |\eta_0|_{H^{s+N+1}}, |\psi_{0,x}|_{H^{s+N}}\right) .
\end{aligned}
$$

For the latter inequality we used the size estimate (4.11) and that $|w_0|_{H^{s+N-6}} \leqslant 2|\psi_{0,x}|_{H^{s+N-4}}$. Finally, from the assumption made on N (mainly greater than 8) and the Sobolev embedding $H^{s-2}(\mathbb{R}) \subset L^\infty(\mathbb{R})$, the desired estimates (4.13) and (4.14) holds. The last estimate (4.15) follows from (4.14) and (4.16) combined with a triangle inequality. □

4.2. Full justification of the KdV equation

It's known that the KdV equation

$$
\eta_t^{\mathrm{KdV}} + \eta_x^{\mathrm{KdV}} + \frac{3}{2}\varepsilon\eta^{\mathrm{KdV}}\eta_x^{\mathrm{KdV}} + \frac{1}{6}\varepsilon\eta_{xxx}^{\mathrm{KdV}} = 0, \tag{4.17}
$$

can be derived from (4.6) by assuming a certain relation between the horizontal velocity and the surface deflection. We prove in this section that one can associate to the solutions of (4.17) a family of approximate solutions consistent with the fully symmetric Boussinesq equations (4.6) in the following sense:

Definition 4. Let $\varepsilon \ll 1$ and $T > 0$. We say that a family $(\eta^{\mathrm{KdV}}, w^{\mathrm{KdV}})$ is consistent at order $\mathcal{O}(\varepsilon^2)$ with the fully symmetric Boussinesq equations (4.6) if for all $s \geqslant 0$,

$$
\begin{cases}
\eta_t^{\mathrm{KdV}} + \left(1-\frac{1}{6}\varepsilon\partial_x^2\right)w_x^{\mathrm{KdV}} + \varepsilon\left(\eta^{\mathrm{KdV}}w^{\mathrm{KdV}}\right)_x = \varepsilon^2 r_1 , \\
\left(1-\frac{2}{3}\varepsilon\partial_x^2\right)w_t^{\mathrm{KdV}} + \left(1-\frac{1}{6}\varepsilon\partial_x^2\right)\eta_x^{\mathrm{KdV}} + \varepsilon w^{\mathrm{KdV}}w_x^{\mathrm{KdV}} = \varepsilon^2 r_2 ,
\end{cases}
$$

with $(r_1, r_2) \in L^\infty\left([0, T/\varepsilon]; H^s(\mathbb{R})^2\right)$.

The following proposition shows that there is one-parameter family of KdV equations of the form (4.17) which is consistent in the same sense of definition 4 with the fully symmetric Boussinesq equations (4.6).

Proposition 7. *Let $\varepsilon \ll 1$. Then there exists $n \in \mathbb{N}$ (mainly greater than 7), such that for all $s \geqslant 0$ and $T > 0$, and for all bounded families $\eta^{\mathrm{KdV}} \in C([0, \frac{T}{\varepsilon}]; H^{s+n}(\mathbb{R}))$ solving (4.17), the family $(\eta^{\mathrm{KdV}}, w^{\mathrm{KdV}})$, such that*

$$
w^{\mathrm{KdV}} := \eta^{\mathrm{KdV}} - \frac{1}{4}\varepsilon\left(\eta^{\mathrm{KdV}}\right)^2 + \frac{1}{3}\varepsilon\eta_{xx}^{\mathrm{KdV}}, \tag{4.18}
$$

is consistent of order $\mathcal{O}(\varepsilon^2)$ on $[0, \frac{T}{\varepsilon}]$ with the fully symmetric Boussinesq equations (4.6).

Remark 1. The KdV equation (4.17) is in fact globally well posed as shown in [8], but this is not needed for our present purposes.

Proof. Let n be large enough (mainly greater than 7), and suppose $\eta^{\mathrm{KdV}} \in C([0, \frac{T}{\varepsilon}]; H^{s+n}(\mathbb{R}))$ is a solution of the KdV equation (4.17). We seek a function $W(x,t)$ such that if $w^{\mathrm{KdV}} = \eta^{\mathrm{KdV}} + \varepsilon W$ and η^{KdV} solves (4.17), then the first equation of (4.6) is satisfied up to $\mathcal{O}(\varepsilon^2)$ term. This is equivalent to checking that

$$
\eta_t^{\mathrm{KdV}} + \eta_x^{\mathrm{KdV}} + \varepsilon W_x - \frac{1}{6}\varepsilon\eta_{xxx}^{\mathrm{KdV}} + 2\varepsilon\eta^{\mathrm{KdV}}\eta_x^{\mathrm{KdV}} = 0.
$$

Since η^{KdV} is a solution of the KdV equation (4.17), then replace $\eta_t^{\mathrm{KdV}} + \eta_x^{\mathrm{KdV}} = -\frac{3}{2}\varepsilon\eta^{\mathrm{KdV}}\eta_x^{\mathrm{KdV}} - \frac{1}{6}\varepsilon\eta_{xxx}^{\mathrm{KdV}}$. Therefore, it holds that

$$\varepsilon W_x = -\frac{1}{2}\varepsilon\eta\eta_x + \frac{1}{3}\varepsilon\eta_{xxx} .$$

If we integrate the last equation with respect to x, then one can choose

$$W = -\frac{1}{4}\left(\eta^{\mathrm{KdV}}\right)^2 + \frac{1}{3}\eta_{xx}^{\mathrm{KdV}}.$$

Consequently,

$$w^{\mathrm{KdV}} = \eta^{\mathrm{KdV}} - \frac{1}{4}\varepsilon\left(\eta^{\mathrm{KdV}}\right)^2 + \frac{1}{3}\varepsilon\eta_{xx}^{\mathrm{KdV}}.$$

Now, it is straightforward that the pair of functions $(\eta^{\mathrm{KdV}}, w^{\mathrm{KdV}})$ satisfies the first equation of (4.6) up to $\mathcal{O}(\varepsilon^2)$ containing $\partial_x^5\eta^{\mathrm{KdV}}$ as a highest derivative. On the other hand, for the second equation of (4.6), use the fact that $\eta_t^{\mathrm{KdV}} = -\eta_x^{\mathrm{KdV}} + \mathcal{O}(\varepsilon)$ so that this equation is also satisfied up to $\mathcal{O}(\varepsilon^2)$ terms containing $\partial_x^7\eta^{\mathrm{KdV}}$ as a highest derivative. □

A consequence of the following proposition is a stronger result: this consistent family $(\eta^{\mathrm{KdV}}, w^{\mathrm{KdV}})$ of solutions of the KdV equation (4.17) constructed, provides a good approximation of the exact solutions $(\underline{\xi}, \underline{v})$ of the fully symmetric Boussinesq equations (4.6) with same initial data in the sense that $(\underline{\xi}_0, \underline{v}_0) = (\eta_0^{\mathrm{KdV}}, w_0^{\mathrm{KdV}}) + \mathcal{O}(\varepsilon^2)$ for times of order ε^{-1}.

Proposition 8. *Suppose that the assumption of theorem 3 is satisfied, then it holds that:*

- *There is a unique family* $(\eta^{\mathrm{KdV}}, w^{\mathrm{KdV}}) \in C\big([0,T/\varepsilon]; H^{s+N+1}(\mathbb{R}) \times H^{s+N-1}(\mathbb{R})\big)$ *given by the resolution of* (4.17) *with initial condition* $\eta_0^{\mathrm{KdV}} = \eta_0$ *and formula* (4.18).
- *There exists a unique solution* $(\underline{\xi}, \underline{v}) \in C\big([0,T/\varepsilon]; H^{s+N-2}(\mathbb{R}) \times H^{s+N-1}(\mathbb{R}) = X^{s+N-5}\big)$ *to* (4.6) *with initial data* $(\underline{\xi}_0, \underline{v}_0) = (\eta_0, w_0^{\mathrm{KdV}} = \eta_0 - \frac{1}{4}\varepsilon\eta_0^2 + \frac{1}{3}\varepsilon\eta_{0,xx}) + \mathcal{O}(\varepsilon^2)$.
- *The following error estimate holds, for all* $0 \leqslant t \leqslant T/\varepsilon$,

$$|\underline{\xi} - \eta^{\mathrm{KdV}}|_{W^{2,\infty}} \leqslant \varepsilon^2(1+t)\, C\left(h_{\min}^{-1}, \varepsilon_{\max}, |\eta_0|_{H^{s+N+1}}\right) , \tag{4.19}$$

$$|\underline{v} - w^{\mathrm{KdV}}|_{W^{2,\infty}} \leqslant \varepsilon^2(1+t)\, C\left(h_{\min}^{-1}, \varepsilon_{\max}, |\eta_0|_{H^{s+N+1}}\right) . \tag{4.20}$$

Proof. The first point of this proposition is a direct consequence from the well posedness theory of the KdV equation (4.17) (see for instance [15, 18, 21]). The second point is deduced using theorem 1. For the third point, we know from proposition 7 that $(\eta^{\mathrm{KdV}}, w^{\mathrm{KdV}})$ is consistent with the fully symmetric Boussinesq equations (4.6) in the sense of definition 4, this implies that

$$\begin{cases} \eta_t^{\mathrm{KdV}} + \left(1 - \frac{1}{6}\varepsilon\partial_x^2\right) w_x^{\mathrm{KdV}} + \varepsilon\left(\eta^{\mathrm{KdV}} w^{\mathrm{KdV}}\right)_x = \varepsilon^2 r_1 , \\ \left(1 - \frac{2}{3}\varepsilon\partial_x^2\right) w_t^{\mathrm{KdV}} + \left(1 - \frac{1}{6}\varepsilon\partial_x^2\right)\eta_x^{\mathrm{KdV}} + \varepsilon w^{\mathrm{KdV}} w_x^{\mathrm{KdV}} = \varepsilon^2 r_2 , \end{cases}$$

with $|r_1|_{H^s} + |r_2|_{H^s} \leqslant C\big(\varepsilon_{\max}, |\eta^{\mathrm{KdV}}|_{H^{s+7}}\big)$ on $[0, T/\varepsilon]$. Now by theorem 2, with $V = (\underline{\xi}, \underline{v}) \in C([0,T/\varepsilon]; X^{s+N-11})$ and $\widetilde{V} = (\eta^{\mathrm{KdV}}, w^{\mathrm{KdV}}) \in C([0,T/\varepsilon]; X^{s+N-10})$ and $\widetilde{F} = \varepsilon^2(r_1, r_2)$, the stability property (4.12) implies that

$$
\begin{aligned}
|V-\widetilde{V}|_{L^\infty([0,t],X^{s+N-11}(\mathbb{R}))} &= \left|\underline{\xi}-\eta^{\mathrm{KdV}}\right|_{L^\infty([0,t],H^{s+N-8}(\mathbb{R}))}+\left|\underline{v}-w^{\mathrm{KdV}}\right|_{L^\infty([0,t],H^{s+N-7}(\mathbb{R}))}\\
&\leqslant \varepsilon^2(1+t)\,C\left(h_{\min}^{-1},\left|\left(\eta^{\mathrm{KdV}},w^{\mathrm{KdV}}\right)\right|_{X^{s+N-10}},\left|\left(\underline{\xi},\underline{v}\right)\right|_{X^{s+N-11}},\left|(r_1,r_2)\right|_{H^{s+N-6}}\right)\\
&\leqslant \varepsilon^2(1+t)\,C\left(h_{\min}^{-1},\varepsilon_{\max},|\eta_0|_{H^{s+N+1}}\right)\ ,
\end{aligned}
$$

where for the latter inequality we used the size estimate (4.11). Finally, from the assumption made on N (mainly greater than 8) and the Sobolev embedding $H^{s-2}(\mathbb{R})\subset L^\infty(\mathbb{R})$, the desired estimates (4.19) and (4.20) holds. □

So far, proposition 3 states that the fully symmetric Boussinesq system (4.6) provide good approximation to the water wave equations (1.6), and proposition 8 states that the KdV equation (4.17) provide good approximation the fully symmetric Boussinesq system (4.6). Consequently, for small initial data we can deduce that the KdV equation (4.17) is also a good approximation for water wave equations (1.6).

Corollary 1. *Suppose that the assumption of theorem 3 is satisfied. Moreover, assume that* $\max(|\eta_0|_{H^{s+N-2}},|\psi_{0,x}|_{H^{s+N-2}})\leqslant\varepsilon$*, then for all* $0\leqslant t\leqslant T/\varepsilon$*, it holds that:*

$$|\eta^{\mathrm{Euler}}-\eta^{\mathrm{KdV}}|_{W^{2,\infty}}\leqslant|\eta^{\mathrm{Euler}}-\eta^{\mathrm{KdV}}|_{H^{s+N-8}}\leqslant\varepsilon^2(1+t)\,C\left(h_{\min}^{-1},\varepsilon_{\max},|\eta_0|_{H^{s+N+1}},|\psi_{0,x}|_{H^{s+N}}\right)\ , \tag{4.21}$$

$$|w^{\mathrm{Euler}}-w^{\mathrm{KdV}}|_{W^{2,\infty}}\leqslant\varepsilon^2(1+t)\,C\left(h_{\min}^{-1},\varepsilon_{\max},|\eta_0|_{H^{s+N+1}},|\psi_{0,x}|_{H^{s+N}}\right)\ , \tag{4.22}$$

and in addition we have

$$|\psi_x-w^{\mathrm{KdV}}|_{L^\infty}\leqslant\varepsilon^2(1+t)\,C\left(h_{\min}^{-1},\varepsilon_{\max},|\eta_0|_{H^{s+N+1}},|\psi_{0,x}|_{H^{s+N}}\right)\ . \tag{4.23}$$

Proof. The proof requires similar estimates between the solutions of the Boussinesq systems mentioned in propositions 3 and 8. By theorem 2, with $\widetilde{V}=(\underline{\xi},\underline{v})\in C([0,T/\varepsilon];X^{s+N-5})\subset C([0,T/\varepsilon];X^{s+N-6})$ and $V=(\xi,v)\in C([0,T/\varepsilon];X^{s+N-6})\subset C([0,T/\varepsilon];X^{s+N-7})$ and $\widetilde{F}=(0,0)$, the stability property (4.12) implies that

$$|\xi-\underline{\xi}|_{H^{s+N-4}}+|v-\underline{v}|_{H^{s+N+3}}\leqslant|\eta_0-\psi_{0,x}|_{H^{s+N-2}}+\varepsilon C\left(|\eta_0|_{H^{s+N+1}},|\psi_{0,x}|_{H^{s+N}}\right)\ .$$

The first two estimates follows from the Sobolev embedding $H^{s-2}(\mathbb{R})\subset L^\infty(\mathbb{R})$ combined with a triangular inequality. The last estimate is a direct consequence of the second one combined with (4.16) and the fact that $w=\psi_x+\frac{1}{2}\varepsilon\psi_{xxx}+\mathcal{O}(\varepsilon^2)$. □

Remark 2. The precision of the KdV approximation is $\mathcal{O}(\sqrt{\varepsilon})$ if no additional assumption was made on the initial data (see corollary 7.2, p 180, [26]). Otherwise, in order to improve the precision into $\mathcal{O}(\varepsilon)$, a smallness assumption is required on the initial data (as assumed here) or under an additional decay assumption on the initial data (see corollary 7.12, p 188, [26]) or one may assume that $\eta_0=\psi_{0,x}$.

5. Velocity field and pressure in the KdV equation

So far, in the context of the KdV equation, we have developed a formula for the horizontal velocity component at the bed. In the KdV equation, this variable is given in terms of the principal unknown η^{KdV} as $w^{\mathrm{KdV}}=\eta^{\mathrm{KdV}}-\frac{1}{4}\varepsilon(\eta^{\mathrm{KdV}})^2+\frac{1}{3}\varepsilon\eta^{\mathrm{KdV}}_{xx}$. To understand mass, momentum, and energy balances in the context of the KdV equation, approximate expressions for the velocity field and pressure in the entire fluid column must be available. These variables will thus be expressed not only in terms of x and t, but also in terms of z. The goal of this section is to create formulas for these variables and demonstrate that they converge to the appropriate quantities defined in the context of the full Euler equations as the small parameter ε approaches zero.

Here and throughout the rest of this paper we denote by C any constant depending on $h_{\min}^{-1}$, $\varepsilon_{\max}, |\eta_0|_{H^{s+N+1}}, |\psi_{0,x}|_{H^{s+N}}$ with $N \geqslant 8$.

5.1. Velocity field

We denote that approximate velocity field to be reconstructed from knowledge of the solution $\eta(x,t)$ of the KdV equation by (4.17) by $(\varphi_x^{\mathrm{KdV}}, \varphi_z^{\mathrm{KdV}})$. It will be shown presently that at any non-dimensional height z in the fluid column, the approximated velocities can be defined by

$$\varphi_x^{\mathrm{KdV}}(x,z,t) := \eta^{\mathrm{KdV}} - \frac{1}{4}\varepsilon\left(\eta^{\mathrm{KdV}}\right)^2 + \varepsilon\left(\frac{1}{3} - \frac{z^2}{2}\right)\eta_{xx}^{\mathrm{KdV}}\,, \tag{5.1}$$

$$\varphi_z^{\mathrm{KdV}}(x,z,t) := -\varepsilon z \eta_x^{\mathrm{KdV}}\,. \tag{5.2}$$

Consequently, we will prove the following.

Lemma 3. *Suppose that the assumption of corollary 1 is satisfied. Then for all* $0 \leqslant t \leqslant T/\varepsilon$, *we have:*

$$||\varphi_x^{\mathrm{KdV}} - \varphi_x^{\mathrm{app}}||_{L^\infty H^s} \leqslant \varepsilon^2(1+t)\,C\,, \tag{5.3}$$

and

$$||\varphi_z^{\mathrm{KdV}} - \varphi_z^{\mathrm{app}}||_{L^\infty H^s} \leqslant \varepsilon^3(1+t)\,C\,. \tag{5.4}$$

Proof. From section 2, we have

$$\varphi^{\mathrm{app}} = \psi - \frac{1}{2}\varepsilon\left(z^2 - 1\right)\psi_{xx} + \varepsilon^2 \underline{r}\,,$$

where the residual term $\underline{r}$ depend on the error term R that appears in (2.8) given in terms of z and derivatives of (η,ψ) (see the proof of proposition 1). Therefore, as for (3.15), we have that $|\underline{r}|_{H^s} \leqslant C(|\eta|_{H^{s+2}})|\psi_x|_{H^{s+5}}$. Continuing, using (4.1) twice, we may re-write the expression to read

$$\varphi^{\mathrm{app}} = f - \varepsilon\frac{1}{2}z^2 f_{xx} + \varepsilon^2 \underline{r}\,,$$

Recalling that $w = f_x$ is the approximate horizontal velocity in the context of the Euler equations at the bottom, we write

$$\varphi_x^{\mathrm{app}} = w - \varepsilon\frac{1}{2}z^2 w_{xx} + \varepsilon^2 \underline{r}\,. \tag{5.5}$$

Inserting in (5.5) the w^{KdV} from the KdV theory (4.18), it holds that

$$\begin{aligned}\varphi_x^{\mathrm{app}} = &\left(w - w^{\mathrm{Euler}}\right) + \left(w^{\mathrm{Euler}} - w^{\mathrm{KdV}}\right) - \varepsilon\frac{1}{2}z^2\left[\left(w_{xx} - w_{xx}^{\mathrm{Euler}}\right) + \left(w_{xx}^{\mathrm{Euler}} - w_{xx}^{\mathrm{KdV}}\right)\right] \\ &+ w^{\mathrm{KdV}} - \varepsilon\frac{1}{2}z^2 w_{xx}^{\mathrm{KdV}} + \varepsilon^2 \underline{r}\,.\end{aligned}$$

Now, it is not hard to check that using (4.18) we have $w^{\mathrm{KdV}} - \varepsilon\frac{1}{2}z^2 w_{xx}^{\mathrm{KdV}} = \varphi_x^{\mathrm{KdV}} + \varepsilon^2 \underline{r}'$ with $|\underline{r}'|_{H^s} \leqslant C(|\eta^{\mathrm{KdV}}|_{H^{s+4}})$. Consequently, the latter equations combined with (4.16) and (4.22) yields the first estimate (5.3). Similarly, for the second estimate we have

$$\varphi_z^{\mathrm{app}} - \varphi_z^{\mathrm{KdV}} = \varepsilon z\left[\left(w^{\mathrm{KdV}} - w^{\mathrm{Euler}}\right)_x + \left(w^{\mathrm{Euler}} - w\right)_x\right] + \varepsilon^2 \underline{r} \leqslant \varepsilon^3(1+t)\,C\,.$$

□

We are interested in mechanical quantities due to the flow through a slice of the fluid domain and they are therefore integrated over the vertical coordinate. With this in mind we find it convenient to make a definition.

Definition 5. Let φ^{Euler} be a solution of the Euler equations, then we define its integral representation as

$$\overline{\varphi}^{\text{Euler}} = \int_0^{1+\varepsilon\eta^{\text{Euler}}} \varphi^{\text{Euler}}(x,z,t)\, dz.$$

and similarly for any other quantity $\overline{\Phi}^{(\,\cdot\,)}$.

Also, let us introduce the following inequality that we shall use in our analysis. By Fubini's theorem we have

$$\left| \int_{-1}^{0} f\, d\hat{z} \right|_{H^s} \leqslant \|f\|_{L^\infty H^s}\,. \tag{5.6}$$

We now turn to the main results of the section, needed to estimate energies associated to the KdV approximation.

Corollary 2. *Suppose that the assumption of corollary 1 is satisfied. Then for all* $0 \leqslant t \leqslant T/\varepsilon$, *we have:*

$$|\overline{\varphi_x}^{\text{Euler}} - \overline{\varphi_x}^{\text{KdV}}|_{H^s} \leqslant \varepsilon^2 (1+t)\, C\,, \tag{5.7}$$

and

$$|\overline{\varphi_z}^{\text{Euler}} - \overline{\varphi_z}^{\text{KdV}}|_{H^s} \leqslant \varepsilon^3 (1+t)\, C\,. \tag{5.8}$$

Proof. We start by proving the first estimate. We have to control the following integral

$$\begin{aligned}\overline{\varphi_x}^{\text{Euler}} - \overline{\varphi_x}^{\text{KdV}} &= \int_0^{1+\varepsilon\eta^{\text{Euler}}} \varphi_x^{\text{Euler}} - \varphi_x^{\text{app}}\, dz - \int_0^{1+\varepsilon\eta^{\text{Euler}}} \varphi_x^{\text{KdV}} - \varphi_x^{\text{app}}\, dz - \int_{1+\varepsilon\eta^{\text{Euler}}}^{1+\varepsilon\eta^{\text{KdV}}} \varphi_x^{\text{KdV}}\, dz \\ &= \mathbf{I} + \mathbf{II} + \mathbf{III}\,.\end{aligned}$$

We refer to (3.23) for estimating $\mathbf{I}$. For estimating $\mathbf{III}$, using the expression (5.1) we have

$$\begin{aligned}\mathbf{III} = \Big[&\eta^{\text{KdV}} - \frac{1}{4}\left(\eta^{\text{KdV}}\right)^2 + \frac{1}{3}\varepsilon\eta_{xx}^{\text{KdV}} - \frac{1}{6}\varepsilon\eta_{xx}^{\text{KdV}}\left(3 + 3\varepsilon\eta^{\text{Euler}} + 3\varepsilon\eta^{\text{KdV}}\right. \\ &\left.+\varepsilon^2\left(\eta^{\text{Euler}}\right)^2 + \varepsilon^2\left(\eta^{\text{KdV}}\right)^2 + \varepsilon^2\eta^{\text{Euler}}\eta^{\text{KdV}}\right)\Big]\,\varepsilon\left(\eta^{\text{Euler}} - \eta^{\text{KdV}}\right)\,.\end{aligned}$$

Therefore, using the assumption of the corollary and (4.21), we have $|\mathbf{III}|_{H^s} \leqslant \varepsilon^3(1+t)C$. For estimating $\mathbf{II}$, we change variables in the integral using the transformation Σ defined in (2.12). Consequently, denote by $h^{\text{Euler}} = 1 + \varepsilon\eta^{\text{Euler}}$, using (5.6) combined with (5.3) and (5.4), it holds that

$$|\mathbf{II}|_{H^s} = \left| \int_{-1}^{0} \left[h^{\text{Euler}}\left(\varphi_x^{\text{KdV}} - \varphi_x^{\text{app}}\right) + (\hat{z}+1)\,\varepsilon\eta_x^{\text{Euler}}\left(\varphi_{\hat{z}}^{\text{KdV}} - \varphi_{\hat{z}}^{\text{app}}\right)\right] \circ \Sigma\, d\hat{z} \right|_{H^s} \leqslant \varepsilon^2 (1+t)\, C\,.$$

For the second estimate we have to control the following integral

$$\overline{\varphi_z}^{\text{Euler}} - \overline{\varphi_z}^{\text{KdV}} = \int_0^{1+\varepsilon\eta^{\text{Euler}}} \varphi_z^{\text{Euler}} - \varphi_z^{\text{app}}\, dz - \int_0^{1+\varepsilon\eta^{\text{Euler}}} \varphi_z^{\text{KdV}} - \varphi_z^{\text{app}}\, dz - \int_{1+\varepsilon\eta^{\text{Euler}}}^{1+\varepsilon\eta^{\text{KdV}}} \varphi_z^{\text{KdV}}\, dz$$
$$= I + II + III\,.$$

We refer to (3.24) for estimating I. For estimating III, using the expression (5.2) combined with (4.21), we get

$$\big|III\big|_{H^s} = \frac{1}{2}\varepsilon^2 \Big|\eta_x^{\text{KdV}}\left(\eta^{\text{KdV}} - \eta^{\text{Euler}}\right)\left(2 + \varepsilon\eta^{\text{KdV}} + \varepsilon\eta^{\text{Euler}}\right)\Big|_{H^s} \leqslant \varepsilon^4 (1+t)\, C\,.$$

For estimating II, by changing variables in the integral using the transformation Σ defined in (2.12) and using (5.6) with (5.4), we get

$$|II|_{H^s} = \Big|\int_{-1}^{0} \varphi_{\hat z}^{\text{KdV}} - \varphi_{\hat z}^{\text{app}}\, d\hat z\Big|_{H^s} \leqslant \|\varphi_{\hat z}^{\text{KdV}} - \varphi_{\hat z}^{\text{app}}\|_{L^\infty H^s} \leqslant \varepsilon^3 (1+t)\, C\,.$$

□

Similarly we have another important result.

Corollary 3. *Suppose that the assumption of corollary 1 is satisfied. Then for all* $0 \leqslant t \leqslant T/\varepsilon$, *we have:*

$$|\overline{\varphi_x^2}^{\text{Euler}} - \overline{\varphi_x^2}^{\text{KdV}}|_{H^s} \leqslant \varepsilon^2 (1+t)\, C\,, \tag{5.9}$$

and

$$|\overline{\varphi_z^2}^{\text{Euler}} - \overline{\varphi_z^2}^{\text{KdV}}|_{H^s} \leqslant \varepsilon^3 (1+t)\, C\,. \tag{5.10}$$

Proof. We start by proving the first estimate. We have to control in H^s the following integral

$$\overline{\varphi_x^2}^{\text{Euler}} - \overline{\varphi_x^2}^{\text{KdV}} = \int_0^{1+\varepsilon\eta^{\text{Euler}}} \left(\varphi_x^{\text{Euler}}\right)^2 - \left(\varphi_x^{\text{app}}\right)^2\, dz + \int_0^{1+\varepsilon\eta^{\text{Euler}}} \left(\varphi_x^{\text{app}}\right)^2$$
$$- \left(\varphi_x^{\text{KdV}}\right)^2\, dz - \int_{1+\varepsilon\eta^{\text{Euler}}}^{1+\varepsilon\eta^{\text{KdV}}} \left(\varphi_x^{\text{KdV}}\right)^2\, dz =: \mathbb{I} + \mathbb{II} + \mathbb{III}\,.$$

We refer to (3.17) for estimating $\mathbb{I}$. For estimating $\mathbb{III}$, using the expression (5.1), it is not hard to check that as for **III** in the proof of corollary 2, we have $|\mathbb{III}|_{H^s} \leqslant \varepsilon^3(1+t)C$. For estimating $\mathbb{II}$, we change variables in the integral using the transformation Σ defined in (2.12). Consequently, denote by $h^{\text{Euler}} = 1 + \varepsilon\eta^{\text{Euler}}$ and using the identity (3.19), one may write

$$\mathbb{II} = \int_{-1}^{0} h^{\text{Euler}}\left(\varphi_x^{\text{app}} + \varphi_x^{\text{KdV}}\right)\left(\varphi_x^{\text{app}} - \varphi_x^{\text{KdV}}\right)\circ\Sigma\, d\hat z + 2\int_{-1}^{0} (\hat z + 1)\,\varepsilon\eta_x^{\text{Euler}}\left[\varphi_x^{\text{app}}\varphi_{\hat z}^{\text{app}} - \varphi_x^{\text{KdV}}\varphi_{\hat z}^{\text{KdV}}\right]\circ\Sigma\, d\hat z$$
$$+ \int_{-1}^{0} \frac{1}{h^{\text{Euler}}}(\hat z + 1)\,\varepsilon^2\left(\eta_x^{\text{Euler}}\right)^2\left(\varphi_{\hat z}^{\text{app}} + \varphi_{\hat z}^{\text{KdV}}\right)\left(\varphi_{\hat z}^{\text{app}} - \varphi_{\hat z}^{\text{KdV}}\right)\circ\Sigma\, d\hat z$$
$$=: \mathbb{II}_1 + \mathbb{II}_2 + \mathbb{II}_3\,.$$

To control $\mathbb{II}_1$, from the expression (5.2) we have that $\|\varphi_x^{\text{KdV}}\|_{L^\infty H^s} \leqslant C(|\eta^{KdV}|_{H^{s+2}})$, then combining (5.6) and (5.3), it holds that

$$|\mathbb{II}_1|_{H^s} \leqslant \left(|h^{\text{Euler}} - 1|_{H^s} + 1\right)\left(\|\varphi_x^{\text{KdV}} - \varphi_x^{\text{app}}\|_{L^\infty H^s} + 2\|\varphi_x^{\text{KdV}}\|_{L^\infty H^s}\right)\|\varphi_x^{\text{KdV}} - \varphi_x^{\text{app}}\|_{L^\infty H^s} \leqslant \varepsilon^2 (1+t)\, C\,.$$

To control $\mathbb{III}_2$, note that

$$\varphi_x^{\mathrm{app}}\varphi_{\hat{z}}^{\mathrm{app}} - \varphi_x^{\mathrm{KdV}}\varphi_{\hat{z}}^{\mathrm{KdV}} = \left(\varphi_x^{\mathrm{app}} - \varphi_x^{\mathrm{KdV}}\right)\left(\varphi_{\hat{z}}^{\mathrm{app}} - \varphi_{\hat{z}}^{\mathrm{KdV}}\right) + \left(\varphi_x^{\mathrm{app}} - \varphi_x^{\mathrm{KdV}}\right)\varphi_{\hat{z}}^{\mathrm{KdV}} + \left(\varphi_{\hat{z}}^{\mathrm{app}} - \varphi_{\hat{z}}^{\mathrm{KdV}}\right)\varphi_x^{\mathrm{KdV}} .$$

Now using the latter identity combined with (5.6), (5.3), (5.4) and expressions (5.1) and (5.2), it holds that $|\mathbb{III}_2|_{H^s} \leqslant \varepsilon^4(1+t)C$. Similarly, using in addition the non-cavitation condition (2.10), it holds that $|\mathbb{III}_3|_{H^s} \leqslant \varepsilon^5(1+t)C$. For the proof of the second estimate, we have to control in H^s the following integral

$$\overline{\varphi_{\hat{z}}^2}^{\mathrm{Euler}} - \overline{\varphi_{\hat{z}}^2}^{\mathrm{KdV}} = \int_0^{1+\varepsilon\eta^{\mathrm{Euler}}} \left(\varphi_{\hat{z}}^{\mathrm{Euler}}\right)^2 - \left(\varphi_{\hat{z}}^{\mathrm{app}}\right)^2 \, dz + \int_0^{1+\varepsilon\eta^{\mathrm{Euler}}} \left(\varphi_{\hat{z}}^{\mathrm{app}}\right)^2 - \left(\varphi_{\hat{z}}^{\mathrm{KdV}}\right)^2 \, dz - \int_{1+\varepsilon\eta^{\mathrm{Euler}}}^{1+\varepsilon\eta^{\mathrm{KdV}}} \left(\varphi_{\hat{z}}^{\mathrm{KdV}}\right)^2 \, dz .$$

Similarly, with (3.18) and (5.2) in hands, It is not hard to check that as above the desired estimate holds. □

Similarly we have another important result.

Corollary 4. *Suppose that the assumption of corollary 1 is satisfied. Then for all* $0 \leqslant t \leqslant T/\varepsilon$, *we have:*

$$|\overline{\varphi_x^3}^{\mathrm{Euler}} - \overline{\varphi_x^3}^{\mathrm{KdV}}|_{H^s} \leqslant \varepsilon^2 (1+t) C ,$$

and

$$|\overline{\varphi_z^3}^{\mathrm{Euler}} - \overline{\varphi_z^3}^{\mathrm{KdV}}|_{H^s} \leqslant \varepsilon^3 (1+t) C .$$

Proof. We start by proving the first estimate. We have to control in H^s the following integral

$$\overline{\varphi_x^3}^{\mathrm{Euler}} - \overline{\varphi_x^3}^{\mathrm{KdV}} = \int_0^{1+\varepsilon\eta^{\mathrm{Euler}}} \left(\varphi_x^{\mathrm{Euler}}\right)^3 - \left(\varphi_x^{\mathrm{app}}\right)^3 \, dz + \int_0^{1+\varepsilon\eta^{\mathrm{Euler}}} \left(\varphi_x^{\mathrm{app}}\right)^3 - \left(\varphi_x^{\mathrm{KdV}}\right)^3 \, dz - \int_{1+\varepsilon\eta^{\mathrm{Euler}}}^{1+\varepsilon\eta^{\mathrm{KdV}}} \left(\varphi_x^{\mathrm{KdV}}\right)^3 \, dz =: \mathbb{J}_1 + \mathbb{J}_2 + \mathbb{J}_3 .$$

We refer to (3.25) for estimating $\mathbb{J}_1$. For estimating $\mathbb{J}_3$, using the expression (5.1), it is not hard to check that as for **III** in the proof of corollary 2, we have $|\mathbb{J}_3|_{H^s} \leqslant \varepsilon^3(1+t)C$. For estimating $\mathbb{J}_2$, denote by $h = 1+\varepsilon\eta^{\mathrm{Euler}}$, we first write

$$\begin{aligned}\mathbb{J}_2 = &\int_0^{1+\varepsilon\eta} \left((\varphi_x^{\mathrm{app}})^2 - \left(\varphi_x^{\mathrm{KdV}}\right)^2\right)\left(\varphi_x^{\mathrm{app}} - \varphi_x^{\mathrm{KdV}}\right) dz + \int_0^{1+\varepsilon\eta} \varphi_x^{\mathrm{KdV}}\left((\varphi_x^{\mathrm{app}})^2 - \left(\varphi_x^{\mathrm{KdV}}\right)^2\right) dz \\ &+ \int_0^{1+\varepsilon\eta} \left(\varphi_x^{\mathrm{KdV}}\right)^2 \left(\varphi_x^{\mathrm{app}} - \varphi_x^{\mathrm{KdV}}\right) dz \\ = &\ \mathbb{J}_{21} + \mathbb{J}_{22} + \mathbb{J}_{23} .\end{aligned}$$

To control $\mathbb{J}_{22} + \mathbb{J}_{23}$, first remark that by definition (5.1), it holds that

$$\Big| \sup_{z\in(0,1+\varepsilon\eta)} \varphi_x^{\mathrm{KdV}} \Big|_{H^s} \leqslant C\left(\varepsilon_{max}, |\eta|_{H^{s+2}}\right) . \tag{5.11}$$

Consequently, using the latter inequality combined with the estimates (3.17)–(3.23)–(5.7)–(5.9), we get that $|\mathbb{J}_{22}+\mathbb{J}_{23}|_{H^s} \leqslant \varepsilon^2(1+t)C$. To control $\mathbb{J}_{21}$, we start by using the transformation Σ. In view of (3.19) and (3.21), denote by $\Phi = \varphi^{\text{app}} - \varphi^{\text{KdV}}$, one may check that

$$\begin{aligned}
\mathbb{J}_{21} = &\int_{-1}^{0} h\left(\Phi_x + 2\varphi_x^{\text{KdV}}\right)\Phi_x^2\,d\hat{z} + 2\int_{-1}^{0}(\hat{z}+1)\,\varepsilon\eta_x\left(\Phi_x\Phi_{\hat{z}} - \Phi_x\varphi_{\hat{z}}^{\text{KdV}} + \Phi_{\hat{z}}\varphi_x^{\text{KdV}}\right)\Phi_x\,d\hat{z}\\
&+\int_{-1}^{0}\frac{1}{h}(\hat{z}+1)^2\varepsilon^2\eta_x^2\left(\Phi_{\hat{z}} + 2\varphi_{\hat{z}}^{\text{KdV}}\right)\Phi_{\hat{z}}\Phi_x\,d\hat{z} + \int_{-1}^{0}\varepsilon(\hat{z}+1)\,\eta_x\left(\Phi_x + 2\varphi_x^{\text{app}}\right)\Phi_x \mathsf{u}_{\hat{z}}\,d\hat{z}\\
&+2\int_{-1}^{0}\frac{1}{h}(\hat{z}+1)^2\varepsilon^2\eta_x^2\left(\Phi_x\Phi_{\hat{z}} - \Phi_x\varphi_{\hat{z}}^{\text{KdV}} + \Phi_{\hat{z}}\varphi_x^{\text{KdV}}\right)\Phi_{\hat{z}}\,d\hat{z}\\
&+\int_{-1}^{0}\frac{1}{h^2}(\hat{z}+1)^3\varepsilon^3\eta_x^3\left(\Phi_{\hat{z}} + 2\varphi_{\hat{z}}^{\text{KdV}}\right)\Phi_{\hat{z}}^2\,d\hat{z}\,.
\end{aligned}$$

The latter integrals can are controlled by $\varepsilon^3(1+t)C$. Indeed, we used (5.3), (5.4), (5.6), (2.10), and from expressions (5.1)–(5.2) the fact that $\|\varphi_x^{\text{KdV}}\|_{L^\infty H^s} \leqslant C(|\eta^{KdV}|_{H^{s+2}})$ and $\|\varphi_z^{\text{KdV}}\|_{L^\infty H^s} \leqslant \varepsilon C(|\eta^{KdV}|_{H^{s+1}})$. The proof of the second estimate follows similarly. □

Similarly we have another important result.

Corollary 5. *Suppose that the assumption of corollary 1 is satisfied. Then for all* $0 \leqslant t \leqslant T/\varepsilon$, *we have:*

$$\Big|\int_0^{1+\varepsilon\eta^{\text{Euler}}}\left(\varphi_z^{\text{Euler}}\right)^2\varphi_x^{\text{Euler}}\,dz - \int_0^{1+\varepsilon\eta^{\text{KdV}}}\left(\varphi_z^{\text{KdV}}\right)^2\varphi_x^{\text{KdV}}\,dz\Big|_{H^s} \leqslant \varepsilon^2(1+t)\,C\,.$$

Proof. We have to control in H^s the following integral

$$\begin{aligned}
\int_0^{1+\varepsilon\eta^{\text{Euler}}}&\left(\varphi_z^{\text{Euler}}\right)^2\varphi_x^{\text{Euler}}\,dz - \int_0^{1+\varepsilon\eta^{\text{KdV}}}\left(\varphi_z^{\text{KdV}}\right)^2\varphi_x^{\text{KdV}}\,dz = \int_0^{1+\varepsilon\eta^{\text{Euler}}}\left[\left(\varphi_z^{\text{Euler}}\right)^2\varphi_x^{\text{Euler}} - \left(\varphi_z^{\text{app}}\right)^2\varphi_x^{\text{app}}\right]dz\\
&+\int_0^{1+\varepsilon\eta^{\text{Euler}}}\left[\left(\varphi_z^{\text{app}}\right)^2\varphi_x^{\text{app}} - \left(\varphi_z^{\text{KdV}}\right)^2\varphi_x^{\text{KdV}}\right]dz\\
&-\int_{1+\varepsilon\eta^{\text{Euler}}}^{1+\varepsilon\eta^{\text{KdV}}}\left(\varphi_z^{\text{KdV}}\right)^2\varphi_x^{\text{KdV}}\,dz\\
&=\ell_1+\ell_2+\ell_3\,.
\end{aligned}$$

We refer to (3.28) for estimating ℓ_1. For estimating ℓ_3, using the expressions (5.1) and (5.2), it is not hard to check that as for **III** in the proof of corollary 2, we have $|\ell_3|_{H^s} \leqslant \varepsilon^3(1+t)C$. For estimating ℓ_2, remark that

$$\begin{aligned}
\ell_2 = &\int_0^{1+\varepsilon\eta^{\text{Euler}}}\left[\left(\varphi_z^{\text{app}}\right)^2 - \left(\varphi_z^{\text{KdV}}\right)^2\right]\left(\varphi_x^{\text{app}} - \varphi_x^{\text{KdV}}\right)dz + \int_0^{1+\varepsilon\eta^{\text{Euler}}}\left(\varphi_z^{\text{KdV}}\right)^2\left(\varphi_x^{\text{app}} - \varphi_x^{\text{KdV}}\right)dz\\
&+\int_0^{1+\varepsilon\eta^{\text{Euler}}}\varphi_x^{\text{KdV}}\left[\left(\varphi_z^{\text{app}}\right)^2 - \left(\varphi_z^{\text{KdV}}\right)^2\right]dz = \ell_{21}+\ell_{22}+\ell_{23}\,.
\end{aligned}$$

To control $\ell_{22}+\ell_{23}$, first remark that by definition (5.2), it holds that

$$\Big|\sup_{z\in(0,1+\varepsilon\eta)}\varphi_z^{\text{KdV}}\Big|_{H^s} \leqslant C\left(\varepsilon_{max}, |\eta|_{H^{s+1}}\right)\,.$$

Consequently, using the latter inequality combined with (5.11) and the estimates (3.18)–(3.23)–(5.7)–(5.10), we get that $|\ell_{22}+\ell_{23}|_{H^s} \leqslant \varepsilon^2(1+t)C$. To control ℓ_{21}, we start by using the transformation Σ. In view of (3.19), denote by $\Phi = \varphi^{\mathrm{app}} - \varphi^{\mathrm{KdV}}$, one may check that

$$\ell_{21} = \int_{-1}^{0} \left(\Phi_{\hat z} + 2\varphi_z^{\mathrm{KdV}}\right) \Phi_{\hat z}\Phi_x \, d\hat z + \varepsilon \int_{-1}^{0} \frac{1}{h^{\mathrm{Euler}}}\eta^{\mathrm{Euler}} \left(\Phi_{\hat z} + 2\varphi_z^{\mathrm{KdV}}\right) \Phi_{\hat z}^2 \, d\hat z \, .$$

The latter integrals can are controlled by $\varepsilon^3(1+t)C$. Indeed, we used (5.6), (5.3), (5.4), (2.10), and from expressions (5.1) and (5.2) the fact that $\|\varphi_x^{\mathrm{KdV}}\|_{L^\infty H^s} \leqslant C(|\eta^{KdV}|_{H^{s+2}})$ and $\|\varphi_z^{\mathrm{KdV}}\|_{L^\infty H^s} \leqslant \varepsilon C(|\eta^{KdV}|_{H^{s+1}})$. □

5.2. *Pressure*

The forces that cause the momentum to change are the fluid's forces on itself, which is the pressure force. At any point, cut the liquid. The liquid on each side of your cut exerts a force equal to the pressure on the opposite side of the cut. The force is always directed normal to the cut. In this sense, the weight is distributed equally in all directions. Because the fluid is pushed far from where the pressure is high, the gradient of pressure per unit volume is the opposite of these forces. The dynamic pressure of the fluid is defined as

$$(P')^{\mathrm{Euler}} = P - P_{atm} + gz.$$

Since the atmospheric pressure is of a magnitude much smaller than the significant terms in the equation, it will be assumed to be zero. Therefore $(P')^{\mathrm{Euler}}$ can be written by using Bernoulli's dimensionless form of (1.5) of the water wave problem as:

$$(P')^{\mathrm{Euler}} := -\varphi_t^{\mathrm{Euler}} - \frac{1}{2}\left(\varepsilon\left(\varphi_x^{\mathrm{Euler}}\right)^2 + \left(\varphi_z^{\mathrm{Euler}}\right)^2\right). \tag{5.12}$$

We show in the next proposition that $(P')^{\mathrm{Euler}}$ can be approximated by the pressure defined in the context of KdV equation as follows:

$$(P')^{\mathrm{KdV}} := \eta^{\mathrm{KdV}} - \frac{1}{2}\varepsilon\left(z^2-1\right)\eta_{xx}^{\mathrm{KdV}}, \tag{5.13}$$

in the meaning of convergence. Indeed, we obtain:

Proposition 9. *Let η^{KdV} be a solution of the KdV equation* (4.17)*, and $(\eta^{\mathrm{Euler}}, \psi_x)$ be a solution of the water-wave problem* (1.6)*. Define the non-dimensional pressure by* (5.13) *in the KdV approximation. Then for all time $t \in [0, T/\varepsilon]$ we have the estimate*

$$|\overline{P'}^{\mathrm{Euler}} - \overline{P'}^{\mathrm{KdV}}|_{H^s} \leqslant \varepsilon^2 (1+t)\, C\, .$$

Proof. Consider the pressure formulated in the full Euler equations is given by (5.12). We may rewrite this equation by adding and subtracting convenient terms,

$$\overline{P'}^{\mathrm{Euler}} = -\left[\int_0^{1+\varepsilon\eta^{\mathrm{Euler}}} \left(\varphi_t^{\mathrm{Euler}} - \varphi_t^{\mathrm{app}}\right) dz + \frac{1}{2}\varepsilon\left(\overline{\varphi_x^2}^{\mathrm{Euler}} - \overline{\varphi_x^2}^{\mathrm{KdV}}\right) + \frac{1}{2}\left(\overline{\varphi_z^2}^{\mathrm{Euler}} - \overline{\varphi_z^2}^{\mathrm{KdV}}\right)\right]$$
$$- \int_0^{1+\varepsilon\eta^{\mathrm{KdV}}} \varphi_t^{\mathrm{app}}\, dz - \frac{1}{2}\varepsilon \int_0^{1+\varepsilon\eta^{\mathrm{KdV}}} \left(\varphi_x^{\mathrm{KdV}}\right)^2 dz - \int_{1+\varepsilon\eta^{\mathrm{KdV}}}^{1+\varepsilon\eta^{\mathrm{Euler}}} \varphi_t^{\mathrm{app}}\, dz - \frac{1}{2}\int_0^{1+\varepsilon\eta^{\mathrm{KdV}}} \left(\varphi_z^{\mathrm{KdV}}\right)^2 dz\, .$$

We then find that the H^s-norm of each term within the brackets are controlled by $\varepsilon^2(1+t)C$ as a result of proposition 1 and corollary 3. While the H^s-norm of the last term, defined by (5.2) is also of the same order after a simple integration.

To complete the proof we need to prove that the remaining terms given by

$$-\varphi_t^{\text{app}} - \frac{1}{2}\varepsilon\left(\varphi_x^{\text{KdV}}\right)^2 , \tag{5.14}$$

corresponds to the definition of $(P')^{\text{KdV}}$. From its definition (2.1) and using (4.1) twice, we obtain

$$\begin{aligned}\varphi_t^{\text{app}} &= \psi_t - \frac{1}{2}\varepsilon\left(z^2-1\right)\psi_{xxt} + \varepsilon^2\underline{R}\\ &= f_t - \frac{1}{2}\varepsilon z^2 f_{xxt} + \varepsilon^2\underline{R} ,\end{aligned}$$

where the residual term $\underline{R}$ depend on the error term $\partial_t R$ that appears in (2.8) given in terms of z and derivatives of (η,ψ) (see the proof of proposition 1). Therefore, as for (3.15) and (3.16), we have that $|\underline{R}|_{H^s} \leqslant C(|\eta|_{H^{s+7}})|\psi_x|_{H^{s+6}}$.
Recognizing the formulation of η^{Euler} by (4.3) and using the usual trick $\eta_x = -(f_x)_t + \mathcal{O}(\varepsilon)$, gives

$$\begin{aligned}\varphi_t^{\text{app}} &= -\eta^{\text{Euler}} + \frac{\varepsilon}{2}\left(z^2-1\right)f_{xxt} - \frac{1}{2}\varepsilon f_x^2 + \varepsilon^2\left(\underline{R}+\underline{R}'\right) ,\\ &= -\eta^{\text{Euler}} + \frac{\varepsilon}{2}\left(z^2-1\right)\eta_{xx}^{\text{Euler}} - \frac{1}{2}\varepsilon f_x^2 + \varepsilon^2\left(\underline{R}+\underline{R}'\right) .\end{aligned}$$

where $\underline{R}'$ depending on the same remainders R_1 and R_2 of definition 3 stemming from the proof of proposition 6. Consequently, we have $|\underline{R}'|_{H^s} \leqslant C(|\eta|_{H^{s+7}})|\psi_x|_{H^{s+6}}$. Again, adding and subtracting convenient terms and noting that $f_x = w$, it holds that

$$\begin{aligned}\varphi_t^{\text{app}} = &-\left(\eta^{\text{Euler}} - \eta^{\text{KdV}}\right) - \eta^{\text{KdV}} + \frac{\varepsilon}{2}\left(z^2-1\right)\left(\eta_{xx}^{\text{Euler}} - \eta_{xx}^{\text{KdV}}\right) + \frac{\varepsilon}{2}\left(z^2-1\right)\eta_{xx}^{\text{KdV}}\\ &-\frac{1}{2}\varepsilon\left(\left(w^{\text{Euler}}\right)^2 - \left(w^{\text{KdV}}\right)^2\right) - \frac{1}{2}\varepsilon\left(w^{\text{KdV}}\right)^2 + \varepsilon^2\left(\underline{R}+\underline{R}'\right) .\end{aligned}$$

As a result of corollary 1, most terms can be neglected in the sense of converging in the L^∞-norm, we therefore consider relation (5.14) in the following way

$$-\varphi_t^{\text{app}} - \frac{1}{2}\varepsilon\left(\varphi_x^{\text{KdV}}\right)^2 = \eta^{\text{KdV}} - \frac{1}{2}\varepsilon\left(z^2-1\right)\eta_{xx}^{\text{KdV}} + \varepsilon^2\mathbf{R} , \tag{5.15}$$

where we also used the relations (5.1) and (4.18) to deal with remaining terms so that $|\mathbf{R}|_{H^s} \leqslant (1+t)C$. Thus, by definition of the dynamic pressure in the KdV (5.13) the proof is complete.
With this in hands, it remains to control the following integral as follows

$$-\int_{1+\varepsilon\eta^{\text{KdV}}}^{1+\varepsilon\eta^{\text{Euler}}} \varphi_t^{\text{app}}\, dz = \int_{1+\varepsilon\eta^{\text{KdV}}}^{1+\varepsilon\eta^{\text{Euler}}} \left[\frac{1}{2}\varepsilon\left(\varphi_x^{\text{KdV}}\right)^2 + \eta^{\text{KdV}} - \frac{1}{2}\varepsilon\left(z^2-1\right)\eta_{xx}^{\text{KdV}} + \varepsilon^2\mathbf{R}\right] dz = I_1 + .. + I_4 .$$

As for **III** in the proof of corollary 2, we have $|I_1+I_2+I_3|_{H^s} \leqslant \varepsilon^3(1+t)C$ by a simple integration combined with (4.21). Finally, it is not hard to check that $I_4 \leqslant \varepsilon^2(1+t)C$. □

Corollary 6. *Suppose that the assumption of corollary 1 is satisfied. Then for all* $0 \leqslant t \leqslant T/\varepsilon$, *we have:*

$$\left|\int_0^{1+\varepsilon\eta^{\text{Euler}}} (P')^{\text{Euler}}\varphi_x^{\text{Euler}}\, dz - \int_0^{1+\varepsilon\eta^{\text{KdV}}} (P')^{\text{KdV}}\varphi_x^{\text{KdV}}\, dz\right|_{H^s} \leqslant \varepsilon^2(1+t)C .$$

Proof. Recall the pressure formulated in the full Euler equations given by (5.12). We may rewrite this equation by adding and subtracting convenient terms

$$\begin{aligned}\int_0^{1+\varepsilon\eta^{\text{Euler}}} (P')^{\text{Euler}}\varphi_x^{\text{Euler}}\,dz = &-\Big[\int_0^{1+\varepsilon\eta^{\text{Euler}}} (\varphi_t^{\text{Euler}}\varphi_x^{\text{Euler}} - \varphi_t^{\text{app}}\varphi_x^{\text{app}})\,dz + \frac{1}{2}\varepsilon\left(\overline{\varphi_x^3}^{\text{Euler}} - \overline{\varphi_x^3}^{\text{KdV}}\right)\\
&+\frac{1}{2}\int_0^{1+\varepsilon\eta^{\text{Euler}}} (\varphi_z^{\text{Euler}})^2\varphi_x^{\text{Euler}}\,dz - \frac{1}{2}\int_0^{1+\varepsilon\eta^{\text{KdV}}} (\varphi_z^{\text{KdV}})^2\varphi_x^{\text{KdV}}\,dz\Big]\\
&-\int_0^{1+\varepsilon\eta^{\text{KdV}}} \varphi_t^{\text{app}}\varphi_x^{\text{app}}\,dz - \frac{1}{2}\varepsilon\int_0^{1+\varepsilon\eta^{\text{KdV}}} (\varphi_x^{\text{KdV}})^3\,dz\\
&-\int_{1+\varepsilon\eta^{\text{KdV}}}^{1+\varepsilon\eta^{\text{Euler}}} \varphi_t^{\text{app}}\varphi_x^{\text{app}}\,dz - \frac{1}{2}\int_0^{1+\varepsilon\eta^{\text{KdV}}} (\varphi_z^{\text{KdV}})^2\varphi_x^{\text{KdV}}\,dz\\
= &\,\Upsilon_1 + .. + \Upsilon_8\,.\end{aligned}$$

As a result of proposition 5 and corollary 5 we have $|\Upsilon_1 + .. + \Upsilon_4|_{H^s} \leqslant \varepsilon^2(1+t)C$. While the H^s-norm of the last term Υ_8, defined by (5.1) and (5.2) is also of the same order after a simple integration. To complete the proof we need first to write φ_x^{app} in terms of φ_x^{KdV}. From definition (2.1) and using (4.1) twice with $f_x = w^{\text{Euler}}$ combined with (4.18) and (4.22), we obtain

$$\begin{aligned}\varphi_x^{\text{app}} &= w^{\text{KdV}} - \frac{1}{2}\varepsilon z^2 w_{xx}^{\text{KdV}} + \left(w^{\text{Euler}} - w^{\text{KdV}}\right) - \frac{1}{2}\varepsilon z^2\left(w_{xx}^{\text{Euler}} - w_{xx}^{\text{KdV}}\right) + \varepsilon^2 r\\
&= \varphi_x^{\text{KdV}} + \varepsilon^2\mathbf{r}\,,\end{aligned}\tag{5.16}$$

such that $|\mathbf{r}|_{H^s} \leqslant C$. Now, combining the two approximate equations (5.15) and (5.16) yields the desired estimations to complete the proof. □

6. Estimates for densities and fluxes

In the present section, we will prove that for some mechanical quantities of time, the approximation between the equation of KdV and Euler system may be rendered mathematically rigorous. The results in the following theorems shows therefore that the mechanical laws in the Euler equations converges to the mechanical laws defined in terms of the function of solution of the KdV equation for a perfect fluid when the physical parameter ε goes to zero. We recall that C is any constant depending on $h_{\min}^{-1}$, $\varepsilon_{\max}$, $|\eta_0|_{H^{s+N+1}}$, $|\psi_{0,x}|_{H^{s+N}}$ with $N \geqslant 8$.

6.1. Mass balance

We now look at the convergence of the mass density and flux for the KdV equation. Recall from [4] that the (depth-integrated) mass density for the KdV equation in non-dimensional variables is given by $\mathcal{M}^{\text{KdV}} = 1 + \varepsilon\eta^{\text{KdV}}$. On the other hand, the mass density in the full Euler approximation is given by $\mathcal{M}^{\text{Euler}} = \int_0^{1+\varepsilon\eta^{\text{Euler}}} dz$. Taking the difference of these two quantities yields the estimate

$$|\mathcal{M}^{\text{Euler}} - \mathcal{M}^{\text{KdV}}|_{L^\infty} \leqslant \varepsilon|\eta^{\text{Euler}} - \eta^{KdV}|_{L^\infty}\,.\tag{6.1}$$

Thus we have the following theorem.

Theorem 4. *Suppose that the assumption of corollary 1 is satisfied. Let* $(\eta^{\text{Euler}}, \psi_x)$ *be a solution of the full water-wave problem* (1.6)*, with initial data regular enough given by* $(\eta_0^{\text{Euler}}, \psi_{0,x})$*. Let* η^{KdV} *be a solution of the KdV equation* (4.17) *with corresponding initial data. Then there exists a constant C, so that we have the estimate*

$$\left|\mathcal{M}^{\text{Euler}} - \mathcal{M}^{\text{KdV}}\right|_{L^\infty} \leqslant \varepsilon^3 (1+t) C\,. \tag{6.2}$$

Proof. Using the above inequality (6.1) in connection with the estimate (4.21) yields the result. □

Next we consider the mass flux through a section of the fluid which is defined in the context of the Euler equations as ${\mathcal{Q}_\mathcal{M}}^{\text{Euler}} = \int_0^{1+\varepsilon\eta^{\text{Euler}}} \varphi_x^{\text{Euler}}\, dz$. It was shown in [4] that the mass flux in the KdV approximation is given by ${\mathcal{Q}_\mathcal{M}}^{\text{KdV}} = \eta^{\text{KdV}} + \varepsilon\frac{3}{4}(\eta^{\text{KdV}})^2 + \varepsilon\frac{1}{6}\eta_{xx}^{\text{KdV}}$. As noted in [4], this expression is identical with the approximate momentum density $\mathcal{I}$ defined in (1.4).

Theorem 5. *Suppose that the assumption of corollary 1 is satisfied. Let* $(\eta^{\text{Euler}}, \varphi^{\text{Euler}})$ *be a solution of the water-wave problem defined below, with initial data given by* $(\eta_0^{\text{Euler}}, \varphi_0^{\text{Euler}})$ *which is regular enough. Let* η^{KdV} *be a solution of the KdV equation* (4.17) *with initial data* $\eta_0^{\text{KdV}} = \eta_0^{\text{Euler}}$*. Then there exists a constant , so that we have the estimate*

$$\left|{\mathcal{Q}_\mathcal{M}}^{\text{Euler}} - {\mathcal{Q}_\mathcal{M}}^{\text{KdV}}\right|_{L^\infty} \leqslant \varepsilon^2 (1+t) C\,. \tag{6.3}$$

Proof. Firstly, let us denote by

$$\mathcal{Q}_\mathcal{M}^* := \int_0^{1+\varepsilon\eta^{\text{KdV}}} \varphi_x^{\text{KdV}}\, dz\,.$$

Then one can use corollary 2 to ensure the existence of a constant C independent of ε. Such that

$$|{\mathcal{Q}_\mathcal{M}}^{\text{Euler}} - {\mathcal{Q}_\mathcal{M}}^*|_{L^\infty} \leqslant \varepsilon^2 (1+t) C\,.$$

To complete the proof, we approximate ${\mathcal{Q}_\mathcal{M}}^*$ by ${\mathcal{Q}_\mathcal{M}}^{\text{KdV}}$. Recall equation (5.1), it is clear from direct computation that

$$\begin{aligned}{\mathcal{Q}_\mathcal{M}}^* - {\mathcal{Q}_\mathcal{M}}^{\text{KdV}} &= \varepsilon\eta^{\text{KdV}}\left(\eta^{\text{Euler}} - \eta^{\text{KdV}}\right) - \frac{1}{4}\varepsilon^2\eta^{\text{Euler}}\left(\eta^{\text{KdV}}\right)^2 - \frac{1}{6}\varepsilon^2\eta^{\text{Euler}}\eta_{xx}^{\text{KdV}} \\ &\quad - \frac{1}{2}\varepsilon^3\left(\eta^{\text{Euler}}\right)^2\eta_{xx}^{\text{KdV}} - \frac{1}{6}\varepsilon^4\left(\eta^{\text{Euler}}\right)^3\eta_{xx}^{\text{KdV}}.\end{aligned}$$

Hence, using the estimate (4.21), we conclude

$$|{\mathcal{Q}_\mathcal{M}}^* - {\mathcal{Q}_\mathcal{M}}^{\text{KdV}}|_{L^\infty} \leqslant \varepsilon^2 (1+t) C\,.$$

□

6.2. Momentum balance

This section is devoted to finding a rigorous approximate expression for momentum density and flux. The momentum density associated to the full Euler equations $\mathcal{I}^{\text{Euler}} = \int_0^{1+\varepsilon\eta^{\text{Euler}}} \varphi_x^{\text{Euler}}\, dz$ corresponds with the mass flux $\mathcal{Q}_\mathcal{M}^{\text{Euler}}$. On the other hand, momentum density for the KdV equation in non-dimensional variables is given by $\mathcal{I}^{\text{KdV}} = \mathcal{Q}_\mathcal{M}^{\text{KdV}}$ and is therefore covered by theorem 5. Moreover, one may prove that the momentum density $\mathcal{I}^{\mathrm{KdV}} = \mathcal{Q}_{\mathcal{M}}^{\mathrm{KdV}}$ converges to the physical momentum density defined in terms of the governing Euler equation for a perfect fluid if ε tends to zero. The precise statement is as follows:

Theorem 6. *Suppose that the assumption of corollary 1 is satisfied. Let η^{KdV} be a solution of the KdV equation (4.17), and $(\eta^{\mathrm{Euler}}, \varphi^{\mathrm{Euler}})$ be a solution of the water-wave problem (1.5). Then for all time $t \in [0, T/\varepsilon]$ we have the estimate*

$$\left|\partial_x \varphi_{|z=0} - \eta^{\mathrm{KdV}} - \varepsilon \frac{3}{4}\left(\eta^{\mathrm{KdV}}\right)^2 - \varepsilon \frac{1}{6}\eta^{\mathrm{KdV}}_{xx}\right|_{L^\infty} \leqslant \varepsilon^2 (1+t) C, \tag{6.4}$$

and

$$\left|\psi_x - \eta^{\mathrm{KdV}} - \varepsilon \frac{3}{4}\left(\eta^{\mathrm{KdV}}\right)^2 - \varepsilon \frac{1}{6}\eta^{\mathrm{KdV}}_{xx}\right|_{L^\infty} \leqslant \varepsilon^2 (1+t) C, \tag{6.5}$$

One can deduce that

$$\left|\partial_x \varphi_{|z=0} - \psi_x\right|_{L^\infty} \leqslant \varepsilon^2 (1+t) C.$$

Proof. Remark that using (4.18) we have

$$\left|\partial_x \varphi_{|z=0} - \eta^{\mathrm{KdV}} - \varepsilon \frac{3}{4}\left(\eta^{\mathrm{KdV}}\right)^2 - \varepsilon \frac{1}{6}\eta^{\mathrm{KdV}}_{xx}\right|_{L^\infty} \leqslant |w^{\mathrm{Euler}} - w^{\mathrm{KdV}}|_{L^\infty} + \varepsilon \left| \left(\eta^{\mathrm{KdV}}\right)^2 + \frac{1}{6}\eta^{\mathrm{KdV}}_{xx}\right|_{L^\infty}.$$

Then using (4.22) the first estimate holds. The second estimate follows similarly using (4.23). The last estimate is a direct outcome of the latter two estimates. □

Regarding the momentum flux we have the following theorem.

Theorem 7. *Suppose that the assumption of corollary 1 is satisfied. Let η^{KdV} be a solution of the KdV equation (4.17), and $(\eta^{\mathrm{Euler}}, \varphi^{\mathrm{Euler}})$ be a solution of the water-wave problem (1.5). Let $(P')^{\mathrm{Euler}}$ be the corresponding dynamic pressure. Define the non-dimensional momentum flux*

$$\mathcal{Q}_{\mathcal{I}}{}^{\mathrm{KdV}} = \frac{1}{2} + \varepsilon \eta^{\mathrm{KdV}} + \frac{3\varepsilon^2}{2}\left(\eta^{\mathrm{KdV}}\right)^2 + \frac{\varepsilon^2}{3}\eta^{\mathrm{KdV}}_{xx}.$$

Then for all time $t \in [0, T/\varepsilon]$ we have the estimate

$$\left| \int_0^{1+\varepsilon\eta^{\mathrm{Euler}}} \left(\varepsilon^2 \left(\varphi_x^{\mathrm{Euler}}\right)^2 + \varepsilon (P')^{\mathrm{Euler}} - (z-1)\right) dz - \mathcal{Q}_{\mathcal{I}}{}^{\mathrm{KdV}} \right|_{L^\infty} \leqslant \varepsilon^3 (1+t) C. \tag{6.6}$$

Proof. Firstly, let us denote by

$$\mathcal{Q}_{\mathcal{I}}{}^{*} := \int_0^{1+\varepsilon\eta^{\mathrm{KdV}}} \left(\varepsilon^2 \left(\varphi_x^{\mathrm{KdV}}\right)^2 + \varepsilon (P')^{KdV} - (z-1)\right) dz.$$

Then one can use the proposition 9 and corollary 3 to ensure the existence of a constant C independent of ε. Such that

$$|\mathcal{Q}_{\mathcal{I}}{}^{\mathrm{Euler}} - \mathcal{Q}_{\mathcal{I}}{}^{*}|_{L^\infty} \leqslant \varepsilon^3 (1+t) C.$$

To complete the proof, we approximate $\mathcal{Q}_{\mathcal{I}}{}^{*}$ by $\mathcal{Q}_{\mathcal{I}}{}^{\mathrm{KdV}}$. Recall equation (5.1) and (5.13), it is clear from direct computation that

$$\mathcal{Q_I}^* = \int_0^{1+\varepsilon\eta^{\mathrm{KdV}}} \varepsilon^2 \left(\eta^{\mathrm{KdV}}\right)^2 + \varepsilon\eta^{\mathrm{KdV}} - \frac{1}{2}\varepsilon^2 \left(z^2-1\right)\eta^{\mathrm{KdV}}_{xx} - (z-1)\ dz + \varepsilon^3 R$$
$$= \mathcal{Q_I}^{\mathrm{KdV}} + \varepsilon^3 R\,.$$

Hence, as R depend on $\eta^{\mathrm{KdV}}, \eta^{\mathrm{KdV}}_x$ and η^{KdV}_{xx}, we conclude

$$|\mathcal{Q_I}^{\mathrm{Euler}} - \mathcal{Q_I}^{\mathrm{KdV}}|_{L^\infty} \leqslant C\varepsilon^3 (1+t)\,.$$

□

Attention is now turned to the energy balance in the fluid.

6.3. Energy balance

Suppose that the assumption of corollary 1 is satisfied. In this section we would like to prove results analogous to the work presented in [16] concerning the energy formulated in the KdV approximation.

Theorem 8. *Let η^{KdV} be a solution of the KdV equation (4.17), let $(\eta^{\mathrm{Euler}}, \varphi^{\mathrm{Euler}})$ be a solution of the water-wave problem (1.5). Define the energy density of the wave by*

$$\mathcal{E}^{\mathrm{KdV}} = \frac{1}{2} + \varepsilon\eta^{\mathrm{KdV}} + \varepsilon^2 \left(\eta^{\mathrm{KdV}}\right)^2. \tag{6.7}$$

Then for all time $t \in [0, T/\varepsilon]$, we have the estimate

$$\Big|\int_0^{1+\varepsilon\eta^{\mathrm{Euler}}} \left(\frac{\varepsilon^2}{2}\left(\varphi^{\mathrm{Euler}}_x\right)^2 + \frac{\varepsilon}{2}\left(\varphi^{\mathrm{Euler}}_z\right)^2 + z\right) dz - \mathcal{E}^{\mathrm{KdV}}\Big|_{L^\infty} \leqslant \varepsilon^3 (1+t)\, C\,. \tag{6.8}$$

Proof. The estimate is again a direct consequence of corollary 3, using the general formulation of the energy density in terms of φ^{KdV} by

$$\mathcal{E}^* := \int_0^{1+\varepsilon\eta^{\mathrm{KdV}}} \left(\frac{\varepsilon^2}{2}\left(\varphi^{\mathrm{KdV}}_x\right)^2 + \frac{\varepsilon}{2}\left(\varphi^{\mathrm{KdV}}_z\right)^2 + z\right) dz\,.$$

While the formula (6.7) is derived from the formulas for φ^{KdV}_x and φ^{KdV}_z given by (5.1) and (5.2) respectively. Indeed,

$$\mathcal{E}^* = \int_0^{1+\varepsilon\eta^{\mathrm{KdV}}} \left(\frac{\varepsilon^2}{2}\left(\eta^{\mathrm{KdV}}\right)^2 + z\right) dz + \mathcal{O}\left(\varepsilon^3\right)$$
$$= \frac{1}{2} + \varepsilon\eta^{\mathrm{KdV}} + \varepsilon^2\left(\eta^{\mathrm{KdV}}\right)^2 + \mathcal{O}\left(\varepsilon^3\right).$$

Therefore one can find a constant C independent of ε such that for all time $t \in [0, T/\varepsilon]$

$$|\mathcal{E}^{\mathrm{Euler}} - \mathcal{E}^{\mathrm{KdV}}|_{L^\infty} \leqslant \varepsilon^3 (1+t)\, C\,.$$

□

Theorem 9. *Suppose that the assumption of corollary 1 is satisfied. Let η^{KdV} be a solution of the KdV equation (4.17), and $(\eta^{\mathrm{Euler}}, \varphi^{\mathrm{Euler}})$ be a solution of the water-wave problem (1.5). Let $(P')^{Euler}$ be the corresponding pressure. Define the non-dimensional energy flux*

$$\mathcal{Q_E}^{\mathrm{KdV}} = \varepsilon\eta^{\mathrm{KdV}} + \frac{7\varepsilon^2}{4}\left(\eta^{\mathrm{KdV}}\right)^2 + \frac{\varepsilon^2}{6}\eta^{\mathrm{KdV}}_{xx}.$$

Then for all time $t \in [0, T/\varepsilon]$ *we have the estimate*

$$\Big| \int_0^{1+\varepsilon\eta^{\text{Euler}}} \left(\left(\frac{\varepsilon^3}{2} \left(\varphi_x^{\text{Euler}}\right)^2 + \frac{\varepsilon^2}{2} \left(\varphi_z^{\text{Euler}}\right)^2 + \varepsilon^2 \left(P'\right)^{\text{Euler}} \right) \varphi_x^{\text{Euler}} + \varepsilon \varphi_x^{\text{Euler}} \right) dz - \mathcal{Q}_{\mathcal{E}}{}^{\text{KdV}} \Big|_{L^\infty} \leqslant \varepsilon^3 (1+t) C\,. \tag{6.9}$$

Proof. The proof follows by the same argument as in theorem 7 by turning to the general form of $\mathcal{Q}_{\mathcal{E}}{}^*$ and apply corollaries 2, 4 and 5, combined with corollary 6 to handle the cross terms. Moreover we have the following equality up to $\mathcal{O}(\varepsilon^3)$:

$$\mathcal{Q}_{\mathcal{E}}{}^* := \int_0^{1+\varepsilon\eta^{\text{KdV}}} \left(\frac{\varepsilon^3}{2} \left(\varphi_x^{\text{KdV}}\right)^3 + \frac{\varepsilon^2}{2} \left(\varphi_z^{\text{KdV}}\right)^2 \varphi_x^{\text{KdV}} + \varepsilon^2 \left(P'\right)^{\text{KdV}} \varphi_x^{\text{KdV}} + \varepsilon \varphi_x^{\text{KdV}} \right) dz$$
$$= \mathcal{Q}_{\mathcal{E}}{}^{\text{KdV}} + \mathcal{O}\left(\varepsilon^3\right).$$

□

Data availability statement

No new data were created or analysed in this study.

Acknowledgments

B K is funded by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Grant Agreement No. 101034309. H K was supported by the Research Council of Norway under Grant No. 239033/F20. S I would like to thank the department of mathematics at UiB, especially, H K for his kind hospitality when part of this work was undertaken.

ORCID iD

Bashar Khorbatly https://orcid.org/0000-0001-9912-1707

References

[1] Ablowitz M J and Segur H 1979 On the evolution of packets of water waves *J. Fluid Mech.* **92** 691–715

[2] Ali A and Kalisch H 2010 Energy balance for undular bores *C. R. Mécanique* **338** 67–70

[3] Ali A and Kalisch H 2012 Mechanical balance laws for Boussinesq models of surface water waves *J. Nonlinear Sci.* **22** 371–98

[4] Ali A and Kalisch H 2014 On the formulation of mass, momentum and energy conservation in the KdV equation *Acta Appl. Math.* **133** 113–31

[5] Amick C J 1984 Regularity and uniqueness of solutions to the Boussinesq system of equations *J. Differ. Equ.* **54** 231–47

[6] Bona J L, Chen M and Saut J-C 2004 Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II: the nonlinear theory *Nonlinearity* **17** 925–52

[7] Bona J L, Colin T and Lannes D 2005 Long wave approximations for water waves *Arch. Ration. Mech. Anal.* **178** 373–410

[8] Bona J L and Smith R 1975 The initial value problem for the Korteweg-de Vries equation *Proc. R. Soc.* A **278** 555–601

[9] Borluk H and Kalisch H 2012 Particle dynamics in the KdV approximation *Wave Motion* **49** 691–709
[10] Carter J D, Curtis C W and Kalisch H 2020 Particle trajectories in nonlinear Schrödinger models *Water Waves* **2** 31–57
[11] Constantin A and Lannes D 2009 The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations *Arch. Ration. Mech. Anal.* **192** 165–86
[12] Craig W 1985 An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits *Commun. PDE* **10** 787–1003
[13] Deny J and Lions J-L 1953–54 Les espaces de Beppo Levi *Ann. Inst. Fourier Grenoble* **5** 497–522
[14] Düll W-P 2012 Validity of the Korteweg-de Vries approximation for the two-dimensional water wave problem in the arc length formulation *Commun. Pure Appl. Math.* **65** 381–429
[15] Israwi S 2010 Variable depth KdV equations and generalizations to more nonlinear regimes *M2AN Math. Model. Numer. Anal.* **44** 347–70
[16] Israwi S and Kalisch H 2021 A mathematical justification of the momentum density function associated to the KdV equation *C. R. Mathématique* **359** 39–45
[17] Israwi S and Kalisch H 2019 Approximate conservation laws in the KdV equation *Phys. Lett.* A **383** 854–8
[18] Israwi S and Talhouk R 2013 Local well-posedness of a nonlinear KdV-type equation *C. R. Mathématiques* **351** 895–9
[19] Karczewska A, Rozmej P and Infeld E 2015 Energy invariant for shallow-water waves and the Korteweg-de Vries equation: doubts about the invariance of energy *Phys. Rev.* E **92** 053202
[20] Karczewska A, Rozmej P, Infeld E and Rowlands G 2017 Adiabatic invariants of the extended KdV equation *Phys. Lett.* A **381** 270–5
[21] Khorbatly B and Israwi S 2020 A conditional local existence result for the generalized nonlinear Kawahara equation *Math. Meth. Appl. Sci.* **43** 5522–31
[22] Khorbatly B and Israwi S 2023 Full justification for the extended Green-Naghdi system for an uneven bottom with/without surface tension *Publ. Res. Inst. Math. Sci.* **59** 587–631
[23] Khorbatly B and Kalisch H 2023 Rigorous estimates on mechanical balance laws in the Boussinesq-Peregrine equations *Stud. Appl. Math.* 1–21
[24] Khorbatly B, Zaiter I and Israwi S 2018 Derivation and Well-Posedness of the extended Green-Naghdi system for flat bottoms with surface tension *J. Math. Phys.* **59** 071501
[25] Lannes D 2005 Well-posedness of the water-waves equations *J. Am. Math. Soc.* **18** 605–54
[26] Lannes D 2013 *The Water Wave Problem* (*Mathematical Surveys and Monographs* vol 188) (American Mathematical Society)
[27] Saut J-C and Xu L 2012 The Cauchy problem on large time for surface waves Boussinesq systems *J. Math. Pures Appl.* **97** 635–62
[28] Schneider G and Wayne C E 2000 The long-wave limit for the water wave problem. I. The case of zero surface tension *Commun. Pure Appl. Math.* **53** 1475–535
[29] Schneider G and Wayne C E 2002 The rigorous approximation of long-wavelength capillary gravity waves *Arch. Ration. Mech. Anal.* **162** 247–85
[30] Schonbek M E 1981 Existence of solutions for the Boussinesq system of equations *J. Differ. Equ.* **42** 325–52
[31] Wu S 1997 Well-posedness in Sobolev spaces of the full water wave problem in 2-D *Invent. Math.* **130** 39–72
[32] Wu S 1999 Well-posedness in Sobolev spaces of the full water wave problem in 3-D *J. Am. Math. Soc.* **12** 445–95