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Approximate conservation laws in the KdV equation

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ABSTRACT

The Korteweg–de Vries equation is known to yield a valid description of surface waves for waves of small amplitude and large wavelength. The equation features a number of conserved integrals, but there is no consensus among scientists as to the physical meaning of these integrals. In particular, it is not clear whether these integrals are related to the conservation of momentum or energy, and some researchers have questioned the conservation of energy in the dynamics governed by the equation. In this letter it is shown that while exact energy conservation may not hold, if momentum and energy densities and fluxes are defined in an appropriate way, then solutions of the Korteweg–de Vries equation give rise to *approximate* differential balance laws for momentum and energy.

1. Introduction

The Korteweg–de Vries (KdV) equation

$$\eta_t + \eta_x + \varepsilon\frac{3}{2}\eta\eta_x + \varepsilon\frac{1}{6}\eta_{xxx} = 0 \tag{1.1}$$

is one of the most widely studied equations in mathematical physics today, and it stands as a paradigm in the field of completely integrable partial differential equations [1,22]. The KdV equation admits a large number of closed-form solutions such as the solitary wave, the cnoidal periodic solutions, multisolitons and rational solutions [1,2,10]. It also features an infinite number of formally conserved integrals which is one of the hallmarks of a completely integrable system. Indeed the conservation can be made mathematically rigorous using the techniques developed in [9].

While our understanding of this model equation is generally rather complete, there appears to be one aspect which has not received much attention. Indeed it seems that the link between the invariant integrals of the equation and physical conservation laws has not been well understood. In the present note we explore the ramifications of imposing mechanical balance laws such as momentum and energy conservation in the context of the KdV equation.

To explain this point further, recall that if the equation is given in the form (1.1) then the first three conserved integrals are

$$\int_{-\infty}^{\infty} \eta\, dx, \qquad \int_{-\infty}^{\infty} \eta^2\, dx, \quad \text{and} \quad \int_{-\infty}^{\infty} \left(\tfrac{1}{3}\eta_x^2 - \eta^3\right) dx. \tag{1.2}$$

The first integral is found to be invariant with respect to time t as soon as it is recognized that the KdV equation can be written in the form

$$\frac{\partial}{\partial t}(\eta) + \frac{\partial}{\partial x}\left(\eta + \varepsilon\frac{3}{4}\eta^2 + \varepsilon\frac{1}{6}\eta_{xx}\right) = 0. \tag{1.3}$$

Recognizing that the unknown $\eta(x,t)$ represents an approximation of the deflection of the free surface from the rest position, it was shown in [6] that this relation can be interpreted as a statement of approximate mass conservation.

Invariance of the second and third integrals is obtained from the identities

$$\frac{\partial}{\partial t}\left(\eta^2\right) + \frac{\partial}{\partial x}\left(\eta^2 + \varepsilon\eta^3 + \frac{\varepsilon}{3}\eta\eta_{xx} - \frac{\varepsilon}{6}\eta_x^2\right) = 0, \tag{1.4}$$

$$\frac{\partial}{\partial t}\left(\eta^3 - \frac{1}{3}\eta_x^2\right) + \frac{\partial}{\partial x}\left(\eta^3 + \varepsilon\frac{9}{8}\eta^4 + \frac{2}{3}\eta_x\eta_t + \frac{1}{3}\eta_x^2 + \varepsilon\frac{1}{18}\eta_{xx}^2 + \varepsilon\frac{1}{2}\eta^2\eta_{xx}\right) = 0. \tag{1.5}$$

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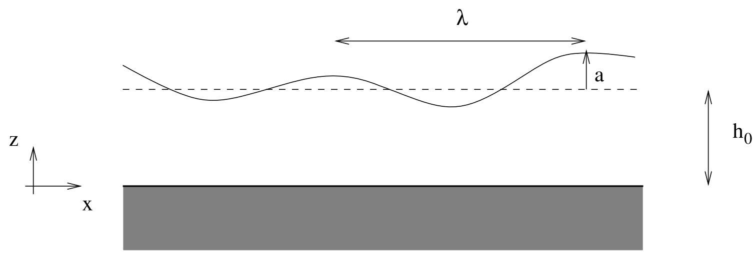

Fig. 1. The schematic elucidates the geometric setup of the problem. The free surface is described by a function $\eta(x,t)$, and the x-axis is aligned with the flat bed.

When contemplating these formulas, the question naturally arises if the invariant integrals in (1.2) and the related quantities appearing in (1.4) and (1.5) have a definite physical meaning. As a matter if fact, it was noted by the authors of [3] that the densities and fluxes appearing in (1.4) and (1.5) do not represent any concrete physical quantities. More recent work also clearly casts doubt on the exact conservation of energy in the KdV equation [15,16,20]. In light of these findings, one may then ask what the appropriate densities and fluxes are if momentum and energy conservation are to be understood in the context of the KdV equation.

The main result of the present letter is that one may define densities and fluxes that represent momentum and energy conservation and lead to *approximate*, but not exact conservation. Indeed, we will show that if these quantities are chosen correctly, then momentum and energy conservation hold to the same order as the KdV equation is valid. To be more precise, if we call the non-dimensional momentum density by $I(\eta)$, and the non-dimensional flow force (momentum flux plus pressure force) by $q_I(\eta)$, we obtain the approximate local balance law

$$\frac{\partial}{\partial t}I(\eta)+\frac{\partial}{\partial x}q_I(\eta)=\mathcal{O}(\varepsilon^2). \tag{1.6}$$

Similarly, the approximate energy balance

$$\frac{\partial}{\partial t}E(\eta)+\frac{\partial}{\partial x}q_E(\eta)=\mathcal{O}(\varepsilon^2) \tag{1.7}$$

follows if the expressions for the energy density $E(\eta)$ and the energy flux plus work rate due to pressure force $q_E(\eta)$ are chosen appropriately.

The plan of the paper is as follows. In Section 2, we explain the physical background against which the KdV equation is used as an approximate water-wave model. The developments in Section 2 are based on firm mathematical theory which has been developed in the last two decades, and is summarized handily in [19]. Then using this background material, the statements of momentum and energy conservation introduced above will be made mathematically precise in sections 3 and 4.

2. The KdV equation in the context of surface water waves

We study the KdV equation as a model equation for waves at the free surface of an incompressible, inviscid fluid running in a narrow open channel where transverse effects can be neglected. Let h_0 be the depth of the undisturbed fluid. Denoting by λ a typical wavelength and by a a typical amplitude of a wavefield to be described, the number $\varepsilon=a/h_0$ represents the relative amplitude, and $\mu=h_0^2/\lambda^2$ measures the inverse relative wavenumber. The geometric setup of the problem is indicated in Fig. 1. In suitably non-dimensionalized variables, the motion of the interface and underlying fluid is described by the system

$$\begin{cases}\mu\partial_x^2\varphi+\partial_z\varphi^2=0 & \text{in} \quad \Omega_t,\\ \partial_z\varphi=0, & \text{at} \quad z=0,\\ \partial_t\zeta-\frac{1}{\mu}(-\mu\partial_x\zeta\,\partial_x\varphi+\partial_z\phi)=0 & \text{at} \quad z=1+\varepsilon\zeta,\\ \partial_t\varphi+\zeta+\frac{\varepsilon}{2}(\partial_x\varphi)^2+\frac{\varepsilon}{2\mu}(\partial_z\varphi)^2=0 & \text{at} \quad z=1+\varepsilon\zeta,\end{cases} \tag{2.1}$$

where $\Omega_t=\{(x,z)\,|\,0<z<1+\varepsilon\zeta(x,t)\}$ is the fluid domain bounded by the free surface $\{z=1+\zeta(x,t)\}$, and the bottom $\{z=0\}$, and $\varphi(x,z,t)$ is the velocity potential associated with the flow (i.e. the velocity field is given by $\mathbf{v}=(\partial_x\varphi,\partial_z\varphi)^T$).

If attention is focused on waves that are predominantly propagating in the direction of increasing values of x, then the surface wave profile $\zeta(x,t)$ can be shown to satisfy the relation

$$\zeta_t+\zeta_x+\varepsilon\frac{3}{2}\zeta\zeta_x+\mu\frac{1}{6}\zeta_{xxx}=\mathcal{O}(\varepsilon^2,\varepsilon\mu,\mu^2).$$

If the wave motion is such that both ε and μ are small and of similar size, then we can take equation (1.1) to obtain an approximate description of the dynamics of the free surface. The approximation can be made rigorous using the techniques in [8,11,13,19,21] and others. Sometimes the Stokes number $\mathcal{S}=\varepsilon/\mu$ is introduced in order to quantify the applicability of the equation to a particular regime of surface waves. Let us assume for the time being that the Stokes number is equal to unity, so that we can work with a single small parameter ε. In this scaling, we may also assume that initial data η_0 are given, such that for any $k>0$, we have $\|\partial_x^k\eta_0\|_{L^2}\le\mathcal{O}(1)$.

Using the aforementioned techniques, it can be shown that the velocity field (φ_x,φ_z) can be approximated to second order accuracy in ε using an approximate potential ϕ. The approximate velocity field is then expressed solely in terms of a solution $\eta(x,t)$ of the KdV equation (1.1) by

$$\phi_x(x,z,t)=\eta-\varepsilon\frac{1}{4}\eta^2+\varepsilon\Big(\frac{1}{3}-\frac{z^2}{2}\Big)\eta_{xx}, \tag{2.2}$$

$$\phi_z(x,z,t)=-\varepsilon z\eta_x. \tag{2.3}$$

Similarly, the pressure can be approximated in terms of a solution η of the KdV equation as follows. First define the dynamic pressure by subtracting the hydrostatic contribution at rest:

$$p-p_{atm}=\varepsilon p'-(z-1).$$

Since the atmospheric pressure is of a magnitude much smaller than the significant terms in the equation, it will be assumed to be zero. As shown in [5], the dynamic pressure p' can the be approximated to second order in ε by

$$p'=\eta-\frac{1}{2}\varepsilon(z^2-1)\eta_{xx}. \tag{2.4}$$

3. Approximate momentum balance

The horizontal momentum balance for a control interval such as depicted in Fig. 2 can be written in terms of the non-dimensional variables of the full Euler equations (2.1) in the form

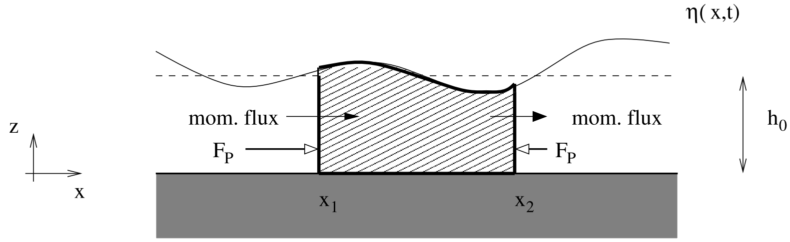

Fig. 2. The flow force q_I represents the sum of momentum flux and pressure force on a fluid element of unit width, reaching from the free surface to the bed.

$$\frac{\partial}{\partial t}\int_0^{1+\varepsilon\zeta} \varepsilon\varphi_x\,dz + \frac{\partial}{\partial x}\int_0^{1+\varepsilon\zeta}\left\{\varepsilon^2\varphi_x^2 + \varepsilon p' - (z-1)\right\}dz = 0.$$

Using ideas from [4,5], we can define an approximate momentum density I by substituting the approximate free-surface profile $\eta(x,t)$ and the approximate horizontal velocity $\phi_x(x,t)$ given by (2.2) into the first integral in the above equality. This substitution leads to the expansion

$$\int_0^{1+\varepsilon\eta} \varepsilon\phi_x\,dz = \varepsilon\eta + \varepsilon^2\tfrac{3}{4}\eta^2 + \varepsilon^2\tfrac{1}{6}\eta_{xx} + \mathcal{O}(\varepsilon^3),$$

and using the asymptotic analysis delineated in [6], the horizontal momentum density in the KdV context is found to be

$$I = \varepsilon\eta + \varepsilon^2\tfrac{3}{4}\eta^2 + \varepsilon^2\tfrac{1}{6}\eta_{xx}. \tag{3.1}$$

The approximate momentum flux is found to be

$$\int_0^{1+\varepsilon\eta} \varepsilon^2\phi_x^2\,dz = \varepsilon^2\eta^2 + \mathcal{O}(\varepsilon^3), \tag{3.2}$$

and the approximate pressure force F_p can be constructed using the integral

$$\int_0^{1+\varepsilon\eta} p\,dz = \tfrac{1}{2} + \varepsilon\eta + \varepsilon^2\tfrac{1}{2}\eta^2 + \varepsilon^2\tfrac{1}{3}\eta_{xx} + \mathcal{O}(\varepsilon^3). \tag{3.3}$$

Thus the flow force (horizontal momentum flux plus pressure force, as defined by [7]) is

$$q_I = \tfrac{1}{2} + \varepsilon\eta + \varepsilon^2\tfrac{3}{2}\eta^2 + \varepsilon^2\tfrac{1}{3}\eta_{xx}. \tag{3.4}$$

The approximate local momentum balance can be formulated as follows.

Theorem 1. (Momentum balance) *Suppose η is a solution of (1.1) with initial data η_0 satisfying $\|\eta_0\|_{H^k} = \mathcal{O}(1)$ for some integer $k \geq 5$. Then there is a constant C, so that the estimate*

$$\left\| \frac{\partial}{\partial t}\left\{\varepsilon\eta + \varepsilon^2\tfrac{3}{4}\eta^2 + \varepsilon^2\tfrac{1}{6}\eta_{xx}\right\} + \frac{\partial}{\partial x}\left\{\tfrac{1}{2} + \varepsilon\eta + \varepsilon^2\tfrac{3}{2}\eta^2 + \varepsilon^2\tfrac{1}{3}\eta_{xx}\right\}\right\|_{L^2} \leq C\varepsilon^3$$

holds for all $t \in [0,\infty)$.

Proof. It was shown in Prop. 6 in [9] that given initial data $\eta_0 \in H^k$, there exists a solution $\eta(x,t)$ which is bounded in the space $C(0,\infty,H^k)$. Thus all ensuing computations hold rigorously since $\eta(\cdot,t) \in H^k$. Using the assumption that η satisfies the KdV equation and factoring out ε, we can write the integrand in the statement of the theorem in the following way.

$$\begin{aligned}
&\left\{\varepsilon\eta + \varepsilon^2\tfrac{3}{4}\eta^2 + \varepsilon^2\tfrac{1}{6}\eta_{xx}\right\}_t + \left\{\tfrac{1}{2} + \varepsilon\eta + \varepsilon^2\tfrac{3}{2}\eta^2 + \varepsilon^2\tfrac{1}{3}\eta_{xx}\right\}_x \\
&= \varepsilon\Big(\eta_t + \eta_x + \varepsilon\tfrac{3}{2}\eta\eta_x + \varepsilon\tfrac{1}{6}\eta_{xxx}\Big) \\
&\quad + \varepsilon^2\Big(\tfrac{3}{4}\eta^2 + \tfrac{1}{6}\eta_{xx}\Big)_t + \varepsilon^2\Big(\tfrac{3}{4}\eta^2 + \tfrac{1}{6}\eta_{xx}\Big)_x \\
&= 0 + \varepsilon^2\tfrac{3}{4}\Big(2\eta\eta_t + 2\eta\eta_x\Big) + \varepsilon^2\tfrac{1}{6}\Big(\eta_{xxt} + \eta_{xxx}\Big) \\
&= \varepsilon^2\tfrac{3}{2}\eta\Big(-\varepsilon\tfrac{3}{2}\eta\eta_x - \varepsilon\tfrac{1}{6}\eta_{xxx}\Big) + \varepsilon^2\tfrac{1}{6}\partial_x^2\Big(-\varepsilon\tfrac{3}{2}\eta\eta_x - \varepsilon\tfrac{1}{6}\eta_{xxx}\Big).
\end{aligned}$$

Since $\|\eta_0\|_{H^5}$ is on the order of 1, and $\|\eta(\cdot,t)\|_{H^k}$ is bounded for all time, we have

$$\sup_t\left\| \frac{\partial}{\partial t}\left\{\varepsilon\eta + \varepsilon^2\tfrac{3}{4}\eta^2 + \varepsilon^2\tfrac{1}{6}\eta_{xx}\right\} + \frac{\partial}{\partial x}\left\{\tfrac{1}{2} + \varepsilon\eta + \varepsilon^2\tfrac{3}{2}\eta^2 + \varepsilon 62\tfrac{1}{3}\eta_{xx}\right\}\right\|_{L^2} \leq C^2\varepsilon^3$$

for some constant C for $t \in [0,\infty)$. $\square$

Remark 1. It should be noted that the L^2-norm used in the proof can be replaced by any Sobolev norm H^s so long as the initial data are regular enough. Indeed it can be seen immediately from the proof that if the approximate momentum balance is to be proved in $H^s(\mathbb{R})$, then the initial data should be given in $H^{s+5}(\mathbb{R})$.

Another important point is that the estimate in Theorem 1 is independent of the time t. In other words, the momentum balance is global in the sense that it holds as long as the solution of the KdV equation exists. This is in stark contrast to the proofs providing the approximation property of the KdV equation for the water-wave problem which are generally such that the error is bounded by $C\varepsilon^2(1+t)$, so that as t gets larger, the approximation degenerates, and if $t \sim 1/\varepsilon$, the approximation is only on the order of ε. Thus in this sense the momentum balance is self-consistent i.e. it is independent of the approximate nature of the KdV equation with regards to the water-wave problem.

4. Approximate energy balance

The energy balance for a control interval such as depicted in Fig. 3 can be written in terms of the non-dimensional variables of the full Euler equations (2.1) in the form

$$\frac{\partial}{\partial t}\int_0^{1+\varepsilon\zeta}\left\{\tfrac{\varepsilon^2}{2}\varphi_x^2 + \tfrac{\varepsilon}{2}\varphi_z^2 + z\right\}dz + \varepsilon\frac{\partial}{\partial x}\int_0^{1+\varepsilon\zeta}\left\{\tfrac{\varepsilon^2}{2}\varphi_x^3 + \tfrac{\varepsilon}{2}\varphi_z^2\varphi_x + z\varphi_x\right\}dz + \varepsilon\frac{\partial}{\partial x}\int_0^{1+\varepsilon\zeta} p\varphi_x\,dz.$$

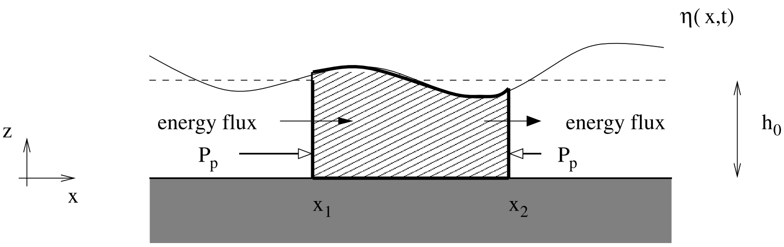

Fig. 3. Energy balance for a control volume. The rate of change of the mechanical energy in the control volume is balanced by the net flux of energy into the control volume due the fluid flow, and the work rate (power) due to the pressure forces on the control volume.

Using ideas from [5,6], we can define an approximate energy density E by substituting the horizontal velocity ϕ_x given by (2.2) into the integral

$$\int_0^{1+\varepsilon\zeta} \left\{ \tfrac{\varepsilon^2}{2}\phi_x^2 + \tfrac{\varepsilon}{2}\phi_z^2 + z \right\} dz,$$

where the last term in the integrand represents the potential energy. In the same vein, the energy flux and work rate due to pressure force are constructed using the integrals

$$\varepsilon \int_{-0}^{1+\varepsilon\eta} \left\{ \tfrac{\varepsilon^2}{2}\phi_x^2 + \tfrac{\varepsilon}{2}\phi_z^2 + z \right\} \phi_x \, dz$$

and

$$\varepsilon \int_{-0}^{1+\varepsilon\eta} p\phi_x \, dz = \varepsilon^2 \frac{\partial}{\partial x} \int_0^{1+\varepsilon\zeta} p'\varphi_x \, dz + \varepsilon \frac{\partial}{\partial x} \int_0^{1+\varepsilon\zeta} (1-z)\varphi_x \, dz.$$

In this way, using again the asymptotic analysis explained in [6], the horizontal energy density in the KdV context is found to be

$$E = \frac{1}{2} + \varepsilon\eta + \varepsilon^2\eta^2, \tag{4.1}$$

the energy flux is found to be

$$\frac{\varepsilon}{2}\eta + \varepsilon^2 \frac{7}{8}\eta^2 + \varepsilon^2 \frac{1}{24}\eta_{xx},$$

and the work rate on the fluid due to pressure forces is

$$\varepsilon^2\eta^2 + \varepsilon\eta + \varepsilon^2 \frac{3}{4}\eta^2 + \varepsilon^2 \frac{1}{6}\eta_{xx} - \frac{\varepsilon}{2}\eta - \varepsilon^2 \frac{7}{8}\eta^2 - \varepsilon^2 \frac{1}{24}\eta_{xx}.$$

It is convenient to combine the last two expressions to find the horizontal energy flux plus work rate due to pressure force to be

$$q_E = \varepsilon\eta + \varepsilon^2 \frac{7}{4}\eta^2 + \varepsilon^2 \frac{1}{6}\eta_{xx}. \tag{4.2}$$

For the energy balance, we have the following theorem.

Theorem 2. (Energy balance) *Suppose η is a solution of (1.1) with initial data η_0 satisfying $\|\eta_0\|_{H^k} = \mathcal{O}(1)$ for some integer $k \geq 4$. Then there is a constant C, so that the estimate*

$$\left\| \frac{\partial}{\partial t} \left\{ \tfrac{1}{2} + \varepsilon\eta + \varepsilon^2\eta^2 \right\} + \frac{\partial}{\partial x} \left\{ \varepsilon\eta + \varepsilon^2 \tfrac{7}{4}\eta^2 + \varepsilon^2 \tfrac{1}{6}\eta_{xx} \right\} \right\|_{L^2} \leq \mathcal{O}(\varepsilon^3)$$

holds for all $t \in [0, \infty)$.

Proof. The proof is similar, but simpler. Observe that

$$\begin{aligned} &\frac{\partial}{\partial t} \left\{ \tfrac{1}{2} + \varepsilon\eta + \varepsilon^2\eta^2 \right\} + \frac{\partial}{\partial x} \left\{ \varepsilon\eta + \varepsilon^2 \tfrac{7}{4}\eta^2 + \varepsilon^2 \tfrac{1}{6}\eta_{xx} \right\} \\ &\quad = \varepsilon\eta_t + \varepsilon\eta_x + \varepsilon^2 \frac{3}{2}\eta\eta_x + \varepsilon^2 \frac{1}{6}\eta_{xxx} + 2\varepsilon^2\eta\eta_t + 2\varepsilon^2\eta\eta_x \\ &\quad = -3\varepsilon^3\eta^2\eta_x - \varepsilon^3 \frac{1}{3}\eta\eta_{xxx}. \end{aligned} \tag{4.3}$$

The estimate now follows in the same way as for the momentum balance above. □

In some cases, it is convenient to normalize the potential energy differently, so that the undisturbed state has zero potential energy. The approximate energy density E^* and energy flux q_E^* are then defined by substituting the horizontal velocity ϕ_x given by (2.2) into the energy formula

$$\begin{aligned} &\frac{\partial}{\partial t} \int_0^{1+\varepsilon\zeta} \left\{ \tfrac{\varepsilon^2}{2}\varphi_x^2 + \tfrac{\varepsilon}{2}\varphi_z^2 + z - 1 \right\} dz \\ &\quad + \varepsilon \frac{\partial}{\partial x} \int_0^{1+\varepsilon\zeta} \left\{ \tfrac{\varepsilon^2}{2}\varphi_x^3 + \tfrac{\varepsilon}{2}\varphi_z^2\varphi_x + (z-1)\varphi_x \right\} dz + \varepsilon \frac{\partial}{\partial x} \int_0^{1+\varepsilon\zeta} p\varphi_x \, dz. \end{aligned}$$

As shown in [6], in this case, the energy density and energy flux (plus work rate due to pressure forces) can then be found to have the respective form

$$\begin{aligned} E^* &= \varepsilon^2\eta^2 + \frac{1}{4}\varepsilon^3\eta^3 + \frac{1}{6}\varepsilon^3\eta\eta_{xx} + \frac{1}{6}\varepsilon^3\eta_x^2, \\ q_E^* &= \varepsilon^2\eta^2 + \frac{5}{4}\varepsilon^3\eta^3 + \frac{1}{2}\varepsilon^3\eta\eta_{xx}. \end{aligned}$$

Theorem 3. (Energy balance) *Suppose η is a solution of (1.1) with initial data η_0 satisfying $\|\eta_0\|_{H^k} = \mathcal{O}(1)$ for some integer $k \geq 6$. Then there is a constant C such that the estimate*

$$\begin{aligned} &\bigg\| \frac{\partial}{\partial t} \left\{ \eta^2 + \frac{1}{4}\varepsilon\eta^3 + \frac{1}{6}\varepsilon\eta\eta_{xx} + \frac{1}{6}\varepsilon\eta_x^2 \right\} \\ &\quad + \frac{\partial}{\partial x} \left\{ \eta^2 + \frac{5}{4}\varepsilon\eta^3 + \frac{1}{2}\varepsilon\eta\eta_{xx} \right\} \bigg\|_{L^2} \leq C\varepsilon^2 \end{aligned}$$

holds for all $t \in [0, \infty)$.

Proof. The key computation is as follows.

$$\begin{aligned} &\frac{\partial}{\partial t} \left\{ \eta^2 + \varepsilon\frac{1}{4}\eta^3 + \varepsilon\frac{1}{6}\eta\eta_{xx} + \frac{1}{6}\varepsilon\eta_x^2 \right\} + \frac{\partial}{\partial x} \left\{ \eta^2 + \varepsilon\frac{5}{4}\eta^3 + \varepsilon\frac{1}{2}\eta\eta_{xx} \right\} \\ &\quad = 2\eta(\eta_t + \eta_x) + \varepsilon\frac{3}{4}\eta^2\eta_t + \varepsilon\frac{1}{6}\eta_t\eta_{xx} + \varepsilon\frac{1}{6}\eta\eta_{xxt} + \varepsilon\frac{1}{3}\eta_x\eta_{xt} \\ &\qquad + \varepsilon\frac{15}{4}\eta^2\eta_x + \varepsilon\frac{1}{2}\eta_x\eta_{xx} + \varepsilon\frac{1}{2}\eta\eta_{xxx} \end{aligned}$$

$$
\begin{aligned}
&= \varepsilon\frac{3}{4}\eta^2\eta_t + \varepsilon\frac{1}{6}\eta_t\eta_{xx} + \varepsilon\frac{1}{6}\eta\eta_{xxt} + \varepsilon\frac{1}{3}\eta_x\eta_{xt} + \varepsilon\frac{3}{4}\eta^2\eta_x \\
&\quad + \varepsilon\frac{1}{2}\eta_x\eta_{xx} + \varepsilon\frac{1}{6}\eta\eta_{xxx} \\
&= \varepsilon\frac{3}{4}\eta^2(\eta_t+\eta_x) + \varepsilon\frac{1}{6}\eta\partial_x^2(\eta_t+\eta_x) + \varepsilon\frac{1}{6}\eta_t\eta_{xx} + \varepsilon\frac{1}{3}\eta_x\eta_{xt} \\
&\quad + \varepsilon\frac{1}{2}\eta_x\eta_{xx} \\
&= \mathcal{O}(\varepsilon^2) + \mathcal{O}(\varepsilon^2) + \varepsilon\frac{1}{2}(\eta_t+\eta_x)\eta_{xx} + \varepsilon\frac{1}{3}(-\eta_t\eta_{xx} + \eta_x\partial_x\eta_t) \\
&= \mathcal{O}(\varepsilon^2) + \mathcal{O}(\varepsilon^2) + \mathcal{O}(\varepsilon^2) + \varepsilon\frac{1}{3}\Big(\big(\eta_x + \mathcal{O}(\varepsilon)\big)\eta_{xx} \\
&\quad + \eta_x\partial_x\big(-\eta_x + \mathcal{O}(\varepsilon)\big)\Big).
\end{aligned}
$$

The proof now proceeds along the same lines as above. □

It should be noted that also in the case of the energy balance, the L^2 norm used in the proofs can be replaced by any Sobolev norm H^k as long as the initial data are regular enough.

5. Discussion

In the present letter, it was shown that momentum and energy conservation hold *approximately* in the context of the KdV equation. The approximate momentum and energy balances can be made rigorous solely by using well-posedness results for the KdV equation such as provided in [9] and in many other contributions.

One interesting aspect of these approximate balance laws is that they hold independently of the fidelity of the KdV solutions as an approximation of a solution of the water-wave problem based on the full Euler equations. To explain this further, note that it was shown in [14] that the momentum density $I(\eta)$ defined above in (3.1) approximates the corresponding quantity in the full Euler equations as long as the solution of the KdV equation is a close approximation of the full Euler equations. Indeed it was possible to show that the estimate

$$
\Big\| \int_0^{1+\varepsilon\zeta} \varphi_x\,dz - I(\eta) \Big\|_{H^s} \le C\varepsilon^2(1+t) \tag{5.1}
$$

holds. Similar estimates can probably be shown for the quantities q_I, E and q_E. Note however, that the estimate (5.1) degenerates as t gets larger and approaches $t \sim 1/\varepsilon$. The results in Theorems 1, 2 and 3 in the present note have no such restriction. Indeed, they hold *globally* for all $t \ge 0$.

We should point out that having the correct form of quantities such as I, q_I, E and q_E can be useful when applying the KdV equation in situations where knowledge of the free surface profile is insufficient. Indeed, there are situations where the internal dynamics of the flow are an important factor, and these quantities enter into the analysis. An example of such an application is the study of the energy balance in undular bores [4] which cannot be properly understood in the context of the KdV equation without knowledge of the approximate quantities E and q_E. The energy flux q_E was also used in a decisive way in a study of nonlinear shoaling in [18].

Finally, it should also be noted that approximate conservation laws can sometimes be utilized in the study of existence of solutions for differential equations. Examples for the use of approximate global conservation laws in two different cases can be found in [12,17].

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