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Delta shock waves in shallow water flow

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ABSTRACT

The shallow-water equations for two-dimensional hydrostatic flow over a bottom bathymetry $b(x)$ are

$$h_t + (uh)_x = 0,$$
$$u_t + \left(gh + u^2/2 + gb\right)_x = 0.$$

It is shown that the combination of discontinuous free-surface solutions and bottom step transitions naturally lead to singular solutions featuring Dirac delta distributions. These singular solutions feature a Rankine–Hugoniot deficit, and can readily be understood as generalized weak solutions in the variational context, such as defined in [13,22]. Complex-valued approximations which become real-valued in the distributional limit are shown to extend the range of possible singular solutions. The method of complex-valued weak asymptotics [22,23] is used to provide a firm link between the Rankine–Hugoniot deficit and the singular parts of the weak solutions. The interaction of a surface bore (traveling hydraulic jump) with a bottom step is studied, and admissible solutions are found.

1. Introduction

The standard theory of hyperbolic conservation laws in one spatial dimension can be applied to systems which are strictly hyperbolic and genuinely nonlinear. If initial data are given which have small enough total variation, then it can be shown that there is a solution which exists for all times [16,18,33]. This solution will in general be discontinuous, featuring a number of jumps. However, if one of the above hypotheses is not satisfied, the initial-value problem cannot in general be resolved (see e.g. [4–6,8,11,27,19,29,34]) and further restrictions on the data need to be introduced, such as for example in [34]. In fact, in some cases, even the Riemann problem cannot be solved.

Starting with the work reported on in [26], existence of solutions was shown to be possible if the space of solutions was extended to include Radon measures. In particular, such non-standard solutions were shown to contain Dirac δ-distributions attached to the location of certain discontinuities. As was shown in [25], the incorporation of such δ-shocks is equivalent to relaxing one or more of the required Rankine–Hugoniot conditions for classical shocks, and it may be shown that the strength of the Dirac δ-distribution associated to a certain shock is a precise measure of the deficit in the Rankine–Hugoniot conditions which are required to obtain a solution.

In the present work, we consider the shallow-water system, and show how δ-shocks arise naturally if this theory is to describe the physics of the underlying problem adequately. Indeed, unlike the situation from [23] where the δ-distribution was adjoined to the surface excursion, here we shall see that δ-naturally appears as part of the velocity as a measure of the Rankine–Hugoniot deficit. An alternative approach for physical explanation of the appearance of delta functions and Rankine–Hugoniot deficits in this context was given in [14], where a localized jet is considered. Singular solutions may also occur in shallow-water systems for two-layer flow [7,21] and in mixing closures for two-layer systems [20].

In the context of surface waves, the shallow-water system describes the flow of an inviscid fluid in a long channel of small uniform width, is used as a standard model in hydraulics, and is fundamental in the study of bores and storm surges in rivers and channels [18,35]. If the bottom is flat (such as in a laboratory situation), the system is usually written in the form

$$\partial_t h + \partial_x (uh) = 0, \text{ (mass conservation)}, \tag{1.1}$$

$$\partial_t (uh) + \partial_x \left(u^2 h + g\frac{h^2}{2}\right) = 0, \text{ (momentum balance)}, \tag{1.2}$$

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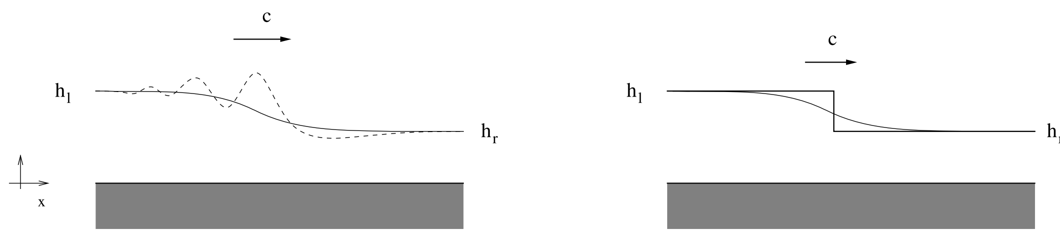

Fig. 1. Left panel: Surface profile of a traveling hydraulic jump (undular bore). Right panel: shallow-water approximation.

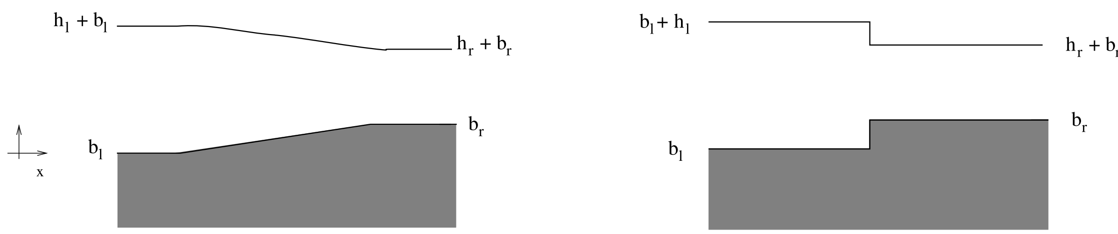

Fig. 2. Left panel: Surface profile over a bottom transition. Right panel: shallow-water approximation.

where h denotes the total flow depth, u represents an average horizontal velocity, and g is the gravitational constant. For smooth solutions, an equivalent system is

$$\partial_t h + \partial_x (uh) = 0, \tag{1.3}$$

$$\partial_t u + \partial_x \Big(\frac{u^2}{2} + gh \Big) = 0, \tag{1.4}$$

and it is immediately clear that mass and momentum conservation in discontinuous solutions lead to a Rankine–Hugoniot deficit in (1.4). One might conclude that it would therefore be best to avoid the system (1.3)–(1.4) in favor of the system (1.1)–(1.2). The theory for this system is well developed, and both the initial-value problem and the Riemann problem can be solved [18]. It is well known that the conservation of energy is formulated as

$$\partial_t \left(h\frac{u^2}{2} + g\frac{h^2}{2} \right) + \partial_x \left(guh^2 + h\frac{u^3}{2} \right) = 0 \tag{1.5}$$

and this then serves as a mathematical entropy [2,3,35].

On the other hand, in many practical situations, the assumption of a flat bottom is too restrictive. If an uneven bed is introduced, the equations take the form

$$\partial_t h + \partial_x (uh) = 0 \quad \text{(mass conservation)} \tag{1.6}$$

$$\partial_t u + \partial_x \left(gh + \frac{u^2}{2} \right) = -gb_x \tag{1.7}$$

$$\partial_t (uh) + \partial_x \left(u^2 h + g\frac{h^2}{2} \right) = -ghb_x \quad \text{(momentum balance)} \tag{1.8}$$

$$\partial_t \left(h\frac{u^2}{2} + g\frac{h^2}{2} + bh \right) + \partial_x \left(guh(h+b) + h\frac{u^3}{2} \right) = 0 \quad \text{(energy balance)} \tag{1.9}$$

In this system, the function $b(x)$ measures the rise of the bed above a certain reference level at $z = 0$. The function $h(x,t)$ measures the flow depth of the fluid, so that $b(x) + h(x,t)$ measures the position of the free surface relative to the reference point $z = 0$ (see Fig. 1 and Fig. 2).

Again, for discontinuous solutions, mass and momentum conservation are to be satisfied, so that (1.7) and the energy equation (1.9) will feature a Rankine–Hugoniot deficit. In the case of a shock over a bottom step, momentum is not conserved because of the lateral pressure force appearing in (1.8), and in this case energy conservation needs to be specified. Therefore, in this case a Rankine–Hugoniot deficit will be introduced in (1.8).

In this paper we will address the relatively simple situation of a flow of a shock wave over a bottom step. The shock wave is governed by the Rankine–Hugoniot conditions originating from mass and momentum conservation, i.e. by (1.6) and (1.8). On the other hand, as explained above, a discontinuity over a bottom step is governed by the Rankine–Hugoniot conditions originating from mass and energy conservation, i.e. by (1.6) and (1.9). Thus it is plain that it is not possible to resolve the underlying physical problem with the use of only two governing equations. If the goal is to maintain the classical modeling approach of describing a situation with a certain fixed set of equations so that the number of equations and unknowns is the same, it is necessary to allow for Rankine–Hugoniot deficits and the corresponding incorporation of singular delta shocks.

Thus in order to salvage the classical modeling approach, we propose the following procedure. Use the system (1.6)–(1.7) as the system to be solved, and use the corresponding Rankine–Hugoniot conditions for momentum or energy conservation in the appropriate places. Since these can be made explicit via delta-shock waves, the system is self-sufficient. For further study, the system (1.6)–(1.7) can be cast in conservative form by writing

$$\left. \begin{aligned} \partial_t h + \partial_x (uh) &= 0, \\ \partial_t u + \partial_x \left(gh + \frac{u^2}{2} + gb \right) &= 0. \end{aligned} \right\} \tag{1.10}$$

The plan of the present paper is as follows. In Section 2, surface discontinuities over a flat bottom are studied, and it is shown that if these discontinuous solutions satisfy mass and momentum conservation, and the required energy loss, then the total head $\frac{1}{2g}u^2 + h$ cannot be conserved. Thus a Rankine–Hugoniot deficit is needed in the second equation in (1.10). The solution is verified both in the weak asymptotic context, and in the weak variational context. In Section 3, bottom step transitions are studied. In Section 4, the interaction of a discontinuous moving surface profile with a bottom step is investigated.

2. Surface discontinuities

In this section, we briefly review the theory surrounding discontinuous solutions of the shallow-water system, and we show that an admissible weak solution conserving mass and momentum, and dissipating mechanical energy must give rise to a Rankine–Hugoniot deficit for the conservation equation for the total head. Then, it is described how such a singular solution can be understood as a delta shock wave in the framework of the weak asymptotic method, and in the generalized variational framework.

2.1. Traveling hydraulic jump

A traveling hydraulic jump traveling over an even bottom must respect the conservation of mass (1.1) and momentum (1.2). In the shallow-water theory, it is useful to consider the jump as having a discontinuity at the bore front. This modeling approach leads to the Rankine–Hugoniot conditions

$$\begin{aligned} c(h_r - h_l) &= u_r h_r - u_l h_l, \\ c(u_r h_r - u_l h_l) &= \left(u_r^2 h_r + \frac{1}{2} g h_r^2\right) - \left(u_l^2 h_l + \frac{1}{2} g h_l^2\right). \end{aligned} \tag{2.1}$$

Here the subscripts l and r indicate the left and right states of the shock, respectively. From these relations, the velocity can be expressed as

$$c = \frac{u_r h_r - u_l h_l}{h_r - h_l} = \frac{\left(u_r^2 h_r + \frac{1}{2} g h_r^2\right) - \left(u_l^2 h_l + \frac{1}{2} g h_l^2\right)}{u_r h_r - u_l h_l}. \tag{2.2}$$

Algebraic manipulation of equation (2.2) gives the expression

$$u_r - u_l = \pm (h_r - h_l) \sqrt{\frac{g}{2}\left(\frac{1}{h_r} + \frac{1}{h_l}\right)}. \tag{2.3}$$

In particular, we see from this relation that $u_r \neq u_l$ if and only if $h_r \neq h_l$. Inserting this relation into (2.2) gives

$$c = u_l \pm h_r \sqrt{\frac{g}{2}\left(\frac{1}{h_r} + \frac{1}{h_l}\right)} = u_r \pm h_l \sqrt{\frac{g}{2}\left(\frac{1}{h_r} + \frac{1}{h_l}\right)}. \tag{2.4}$$

It is clear that solving the square root gives rise to two possible solutions. In order to pick the one which is physically reasonable, use is made of the conservation of mechanical energy (1.5).

Mass conservation through the jump discontinuity is derived from the first equation in (2.1) and equation (2.4) as

$$m = h_r(u_r - c) = h_l(u_l - c) = \mp h_r h_l \sqrt{\frac{g}{2}\left(\frac{1}{h_r} + \frac{1}{h_l}\right)}. \tag{2.5}$$

Momentum conservation through the discontinuity is derived similarly from the second equation in (2.1) and equation (2.4) in the form

$$h_r(u_r - c)u_r + \frac{1}{2} g h_r^2 = h_l(u_l - c)u_l + \frac{1}{2} g h_l^2.$$

Using the expression in (2.5) simplifies this equation to

$$m u_r + \frac{1}{2} g h_r^2 = m u_l + \frac{1}{2} g h_l^2. \tag{2.6}$$

The mechanical energy dissipates in the jump but remains balanced in areas where the solution is smooth. Using the Rankine–Hugoniot condition for equation (1.9) which has the form

$$\begin{aligned} & c\left[\left(\frac{1}{2} u_r^2 h_r + \frac{1}{2} g h_r^2\right) - \left(\frac{1}{2} u_l^2 h_l + \frac{1}{2} g h_l^2\right)\right] \\ &\quad = \left(\frac{1}{2} u_r^3 h_r + g u_r h_r^2\right) - \left(\frac{1}{2} u_l^3 h_l + g u_l h_l^2\right), \end{aligned}$$

an expression for the mechanical energy loss per unit time is

$$\begin{aligned} \frac{1}{\rho Y} \Delta E = & \left(\frac{1}{2} u_r^2 + \frac{1}{2} g h_r\right) h_r(u_r - c) - \left(\frac{1}{2} u_l^2 + \frac{1}{2} g h_l\right) h_l(u_l - c) \\ & + \frac{g}{2}(u_r h_r^2 - u_l h_l^2). \end{aligned}$$

This can be simplified further by using equations (2.5) and (2.6) to obtain[1]

$$\frac{1}{\rho Y} \Delta E = -\frac{m g (h_r - h_l)^3}{4 h_r h_l}. \tag{2.7}$$

This shows that the mechanical energy, which should be chosen as the entropy condition for picking the valid solution, decreases as the solution passes through the discontinuity. As a consequence of (2.7), it is noted that

$$\Delta E < 0 \quad \text{if} \quad m(h_r - h_l) > 0. \tag{2.8}$$

Since energy gain is impossible, it is clear that these inequalities together with the expression in equation (2.5) lead to the relations

$$\begin{aligned} h_r > h_l &\Rightarrow u_r > c \text{ and } u_l > c, \\ h_r < h_l &\Rightarrow u_r < c \text{ and } u_l < c. \end{aligned} \tag{2.9}$$

What is not yet established is the relation between the left state, u_l, and the right state, u_r, variables. From equations (2.3) and (2.5) it is found that

$$u_r - u_l = -\frac{m(h_r - h_l)}{h_r h_l}, \tag{2.10}$$

and the inequalities in equation (2.8) require that this expression obeys the condition

$$u_r < u_l. \tag{2.11}$$

This condition plays an important role in analyzing the Rankine–Hugoniot jump condition for the equation (1.4). Indeed, as it turns out, energy per unit mass cannot be constant through the shock, and we have the following theorem.

Theorem 2.1. *An admissible shock-wave solution satisfying the Rankine–Hugoniot conditions arising from mass and momentum conservation, and featuring the required energy loss must feature a Rankine–Hugoniot deficit in equation (1.4).*

Equation (1.4) can be interpreted as a balance equation involving horizontal velocity and total hydraulic head $\mathcal{H} = \frac{u^2}{2g} + h$, and it is convenient to state the result in the following form:

$$g \Delta \mathcal{H} = (u_r - c)u_r - (u_l - c)u_l + \frac{1}{2}(u_l^2 - u_r^2) + g(h_r - h_l).$$

Using the expression for m in (2.5) gives[2]

$$g \Delta \mathcal{H} = m\left(\frac{u_r}{h_r} - \frac{u_l}{h_l}\right) + \frac{1}{2}\left(u_l^2 - u_r^2\right) + g(h_r - h_l). \tag{2.12}$$

Considering different cases for h_r and h_l and using the above inequalities, it is not hard to check that $\Delta \mathcal{H}$ cannot be zero. From the above equation we obtain the expression

$$g \Delta \mathcal{H} = \frac{1}{2}\left((u_r - c)^2 - (u_l - c)^2 + 2g(h_r - h_l)\right). \tag{2.13}$$

To simplify this expression further we obtain from (2.5) the following relations

$$\begin{aligned} (u_r - c)^2 &= \frac{g h_l^2}{2}\left(\frac{1}{h_r} + \frac{1}{h_l}\right), \\ (u_l - c)^2 &= \frac{g h_r^2}{2}\left(\frac{1}{h_r} + \frac{1}{h_l}\right). \end{aligned}$$

[1] Note that the quantity on the left has been divided by the density ρ and the width of the channel Y in order to get the units of energy per unit time. It will be convenient in the following to assume that both ρ and Y are unity.

[2] It appears most convenient here to present this quantity as head loss $\Delta \mathcal{H}$ per unit time. The quantity has been multiplied by g in order to get the right units.

Insert these expressions into (2.13) to obtain

$$g\Delta\mathcal{H} = \frac{1}{2}\left(\frac{g}{2}(h_l^2 - h_r^2)\left(\frac{1}{h_r} + \frac{1}{h_l}\right) + 2g(h_r - h_l)\right),$$

which is simplified by algebraic manipulations to

$$g\Delta\mathcal{H} = \frac{g(h_l - h_r)^3}{4h_l h_r}. \tag{2.14}$$

From this expression it is obvious that $g\Delta\mathcal{H}$ is nonzero so long as $h_l \neq h_r$.

2.2. Weak asymptotics

One available tool for the description of singular shock waves is the method of weak asymptotics [9,10,12,15,28,30,32]. This method was recently extended to the case where complex-valued approximations are allowed which significantly extended its range of applicability [22–24].

Define $\mathcal{D}(\mathbb{R})$ to be the standard space of test functions, and let $\mathcal{D}'(\mathbb{R})$ be the dual space of distributions (see e.g. [31]). In order to define complex-valued weak asymptotic solutions of (1.10), we first recall the definition of a vanishing family of distributions.

Definition 2.1. Let $f_\varepsilon(x) \in \mathcal{D}'(\mathbb{R})$ be a family of distributions depending on $\varepsilon \in (0,1)$. We say that $f_\varepsilon = o_{\mathcal{D}'}(1)$ if for any test function $\phi(x) \in \mathcal{D}(\mathbb{R})$, we have the estimate

$$\langle f_\varepsilon, \phi\rangle = o(1), \quad \text{as } \varepsilon \to 0. \tag{2.15}$$

Thus a family of distributions vanishes in the sense defined above if for a given test function ϕ, the pairing $\langle f_\varepsilon, \phi\rangle$ converges to zero with ε. For families of distributions depending on t, we say $f_\varepsilon = o_{\mathcal{D}'}(1) \subset \mathcal{D}'(\mathbb{R})$ if (2.15) holds uniformly in t. In other words, f_ε vanishes if

$$\langle f_\varepsilon(\cdot,t), \varphi\rangle \leq C_T g(\varepsilon) \text{ for } t \in [0,T],$$

for a function g depending on $\varphi(x,t)$ and tending to zero with $\varepsilon \to 0$, and where the constant C_T should only depend on T. Next we define solutions of (1.10) in the weak asymptotic sense.

Definition 2.2. The collection of smooth complex-valued distributions (u_ε) and (h_ε) represent a weak asymptotic solution to (1.10) if there exist real-valued distributions $u, v \in C(\mathbb{R}_+; \mathcal{D}'(\mathbb{R}))$, such that for every fixed $t \in \mathbb{R}_+$

$$u_\varepsilon \rightharpoonup u, \quad h_\varepsilon \rightharpoonup h \text{ as } \varepsilon \to 0,$$

in the sense of distributions in $\mathcal{D}'(\mathbb{R})$, and

$$\left.\begin{aligned} \partial_t h_\varepsilon + \partial_x(u_\varepsilon h_\varepsilon) &= o_{\mathcal{D}'}(1), \\ \partial_t u_\varepsilon + \partial_x\left(gh_\varepsilon + \tfrac{u_\varepsilon^2}{2} + gb\right) &= o_{\mathcal{D}'}(1). \end{aligned}\right\} \tag{2.16}$$

In addition, if initial data are given, we require

$$u_\varepsilon(x,0) \rightharpoonup u(x,0) \text{ and } h_\varepsilon(x,0) \rightharpoonup h(x,0), \tag{2.17}$$

where the weak convergence designates convergence in the sense of distributions as ε tends to 0.

In order to state an existence theorem in the context of the above definitions, we define the functions

$$H_0(x) = \begin{cases} h_l, & \text{if } x < 0, \\ h_r, & \text{if } x > 0, \end{cases} \tag{2.18}$$

and

$$U_0(x) = \begin{cases} u_l, & \text{if } x < 0, \\ u_r, & \text{if } x > 0. \end{cases} \tag{2.19}$$

Theorem 2.2. *If the constants h_l, h_r, u_l and u_r are chosen such that the functions $H_0(x-ct)$ and $U_0(x-ct)$ (with c given by (2.2)) represent an admissible (energy-dissipating) shock wave which conserves both mass and momentum, then there are weak asymptotic solutions h_ε and u_ε of the system (1.3)–(1.4), such that the families (h_ε) and (u_ε) have distributional limits*

$$h(x,t) = H_0(x-ct), \tag{2.20}$$

$$u(x,t) = U_0(x-ct) + \alpha(t)\delta(x-ct), \tag{2.21}$$

where

$$\alpha'(t) = g\Delta\mathcal{H}.$$

In order to prove this theorem, let $\rho \in C_c^\infty(\mathbb{R})$ be non-negative, smooth, compactly supported even function such that

$$\operatorname{supp}\rho \subset (-1,1), \quad \int_{\mathbb{R}} \rho(z)dz = 1, \quad \rho \geq 0.$$

Let $C_{\rho,2} = \int_{\mathbb{R}} \rho^2(z)dz$, and take

$$\delta_\varepsilon(x,t) = \frac{1}{2\varepsilon}\rho\left(\frac{x-ct-4\varepsilon}{\varepsilon}\right) + \frac{1}{2\varepsilon}\rho\left(\frac{x-ct+4\varepsilon}{\varepsilon}\right),$$

$$R_\varepsilon(x,t) = \frac{i}{2\varepsilon}\rho\left(\frac{x-ct-2\varepsilon}{\varepsilon}\right) - \frac{i}{2\varepsilon}\rho\left(\frac{x-ct+2\varepsilon}{\varepsilon}\right),$$

$$S_\varepsilon(x,t) = \frac{1}{\sqrt{\varepsilon}}\frac{1}{\sqrt{C_{\rho,2}}}\rho\left(\frac{x-ct}{\varepsilon}\right).$$

Now let the functions U_ε and H_ε be defined by

$$U_\varepsilon(x,t) = \begin{cases} u_l, & x < ct - 30\varepsilon, \\ 0, & ct - 20\varepsilon \leq x \leq ct + 20\varepsilon, \\ u_r, & x \geq ct + 30\varepsilon, \end{cases}$$

$$H_\varepsilon(x,t) = \begin{cases} h_l, & x < ct - 30\varepsilon, \\ 0, & ct - 20\varepsilon \leq x \leq ct + 20\varepsilon, \\ h_r, & x \geq ct + 30\varepsilon. \end{cases}$$

Notice in particular that we have

$$R_\varepsilon \rightharpoonup 0, \text{ and } S_\varepsilon \rightharpoonup 0.$$

Moreover, we also have the identities

$$U_\varepsilon\delta_\varepsilon = 0, \quad U_\varepsilon R_\varepsilon = 0, \quad U_\varepsilon S_\varepsilon = 0, \quad \delta_\varepsilon R_\varepsilon = 0, \quad \delta_\varepsilon S_\varepsilon = 0, \text{ and } R_\varepsilon S_\varepsilon = 0.$$

Furthermore, it is not hard to check that

$$H_\varepsilon\delta_\varepsilon = 0, \quad H_\varepsilon R_\varepsilon = 0, \text{ and } H_\varepsilon S_\varepsilon = 0.$$

In addition, the following limit is obtained:

$$S_\varepsilon^2 \rightharpoonup \delta. \tag{2.22}$$

Now make the ansatz

$$h_\varepsilon(x,t) = H_\varepsilon(x-ct),$$

$$u_\varepsilon(x,t) = U_\varepsilon(x-ct) + \alpha(t)(\delta_\varepsilon(x-ct) + R_\varepsilon(x-ct)) + \sqrt{2c\alpha(t)}S_\varepsilon(x-ct),$$

and substitute it into equations (1.10). Notice first of all that

$$u_\varepsilon^2(x,t) = U_\varepsilon^2 + \alpha^2(t)(R_\varepsilon^2 + \delta_\varepsilon^2) + c\alpha(t)S^2.$$

Focusing on the expression $R_\varepsilon^2 + \delta_\varepsilon^2$, we take $\varphi \in C_c^\infty(\mathbb{R})$ and consider the integral

$$\int_{\mathbb{R}} (R_\varepsilon^2 + \delta_\varepsilon^2)\varphi \, dx.$$

Noting the relation

$$\int_{\mathbb{R}} \tfrac{1}{\varepsilon^2} \Big(\rho^2((x-ct+\alpha\varepsilon)/\varepsilon) + \rho^2((x-ct-\beta\varepsilon)/\varepsilon) \Big) \varphi(x) \, dz$$
$$= \int_{\mathbb{R}} \tfrac{1}{\varepsilon} \rho^2(z) \left(\varphi(ct+\varepsilon(z-\alpha)) + \varphi(ct+\varepsilon(z+\beta)) \right) \, dz$$
$$= \int_{\mathbb{R}} \tfrac{1}{\varepsilon} \rho^2(z) \big(2\varphi(ct) + \varepsilon\varphi'(ct)(\beta-\alpha) \big) \, dz + \mathcal{O}(\varepsilon),$$

for $\alpha, \beta \in \mathbb{R}$,

which follows by making the changes of variables $(x-ct+\alpha\varepsilon)/\varepsilon = z$ and $(x-ct-\beta\varepsilon)/\varepsilon = z$, and observing that $\int z\rho^2(z)dz = 0$ since ρ is an even function, the above integral can be rewritten as

$$\frac{1}{4} \int_{\mathbb{R}} \tfrac{1}{\varepsilon^2} \Big(-\rho^2((x-ct+2\varepsilon)/\varepsilon) - \rho^2((x-ct-2\varepsilon)/\varepsilon)$$
$$+ \rho^2((x-ct+4\varepsilon)/\varepsilon) + \rho((x-ct-4\varepsilon)/\varepsilon) \Big) \varphi \, dx = \mathcal{O}(\varepsilon).$$

Finally, collecting terms, we have

$$\partial_t U_\varepsilon + \frac{1}{2}\partial_x U_\varepsilon^2 + \partial_x H_\varepsilon + \alpha'(t)\delta_\varepsilon - c\alpha(t)\delta' + c\alpha\partial_x S_\varepsilon^2 = o_{\mathcal{D}'}(1). \tag{2.23}$$

Note that the last two terms on the left cancel by (2.22). Therefore, taking into account Definition 2.2, we see that the Rankine–Hugoniot deficit is

$$\alpha'(t) = (u_r - u_l)c + \frac{1}{2}(u_l^2 - u_r^2) + g(h_r - h_l) = g\Delta\mathcal{H}.$$

From Theorem 2.1, we see that $\alpha'(t)$ must be nonzero. The first equation in (2.16) is verified in a similar fashion, but this is even simpler thanks to the choice of the constant c which was found from the Rankine–Hugoniot condition corresponding to the first equation.

2.3. Generalized weak solutions

We will show that the weak asymptotic solutions constructed above represent solutions to the shallow-water system also in the framework introduced in [13]. Following [13], we let $\Gamma = \{\gamma_i \mid i \in I\}$ be a graph in the closed upper half plane, consisting of Lipschitz curves γ_i, $i \in I$, with I a finite index set. I_0 is the subset of I containing the indices of all curves which touch the x-axis, and $\Gamma_0 = \{x_k^0 \mid k \in I_0\}$ is the set of initial points of the curves γ_k with $k \in I_0$. We denote the singular part of the solution by $\alpha(x,t)\delta(\Gamma) = \sum_{i\in I} \alpha_i(x,t)\delta(\gamma_i)$. The expression $\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}$ designates the tangential derivative of a function φ on the arc γ_i, and $\int_{\gamma_i}$ denotes the line integral over the set γ_i.

Definition 2.3. A graph Γ and a couple of distributions (h, u) where U is given by

$$u(x,t) = U(x,t) + \sum_{i\in I} \alpha_i(x,t)\delta(\gamma_i),$$

with $h, U \in L^\infty(\mathbb{R}\times\mathbb{R}_+)$, $\alpha_i \in C^1(\Gamma)$, $i \in I$, is called a generalized δ-shock wave solution of system (1.10) with initial data $h_0(x)$ and $U_0(x) + \sum_{I_0} \alpha_k(x_0^k, 0)\delta\big(x - x_k^0\big)$ if the integral identities

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} (h\partial_t\varphi + (Uh)\partial_x\varphi) \, dxdt + \int_{\mathbb{R}} h_0(x)\varphi(x,0) \, dx = 0, \tag{2.24}$$

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \Big(U\partial_t\varphi + \Big(\tfrac{U^2}{2} + g(h+b)\Big)\partial_x\varphi \Big) \, dxdt \tag{2.25}$$
$$+ \sum_{i\in I} \int_{\gamma_i} \alpha_i(x,t) \tfrac{\partial\varphi(x,t)}{\partial \mathbf{l}} + \int_{\mathbb{R}} U^0(x)\varphi(x,0) \, dx$$
$$+ \sum_{k\in I_0} \alpha_k(x_k^0, 0)\varphi(x_k^0, 0) = 0,$$

hold for all test functions $\varphi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}_+)$.

It is straightforward to check the solutions defined by (2.20) and (2.21) satisfy this weak definition. Indeed, the requirement (2.24) is exactly of the same form as the usual definition of a weak solution. Requirement (2.25) contains the more interesting singular part. As above, consider the case of a flat bed at $b = 0$. Using (2.20) and (2.21), standard computations lead to the identity

$$\int_{\mathbb{R}_+} \Big(c[U] - [U^2/2 + gh] \Big) \varphi(ct,t) \, dt - \int_{\mathbb{R}_+} \alpha'(t)\varphi(ct,t) \, dt = 0,$$

where $[U] = u_r - u_l$ and similarly $[U^2/2 + gh] = \big(u_r^2/2 + gh_r\big) - \big(u_l^2/2 + gh_l\big)$. Since $\alpha(0) = 0$, the conclusion follows from the form of $\alpha(t)$ defined in (2.14). Thus we have the following theorem:

Theorem 2.3. *If the constants h_l, h_r, u_l and u_r are chosen such that the functions $H_0(x-ct)$ and $U_0(x-ct)$ (with c given by (2.2)) represent an admissible (energy-dissipating) shock wave which conserves both mass and momentum, and $\alpha(t)$ is given by (2.14), then the functions defined in (2.20) and (2.21) represent a solution of the Riemann problem corresponding to the system (1.10) in the sense of Definition 2.3.*

3. Bottom step transitions

Consider a smooth bottom topography function defined by

$$b(x) = \begin{cases} b_l, & \text{if } x < 0, \\ b_r, & \text{if } x > 0. \end{cases} \tag{3.1}$$

For this bottom step, in the shallow-water approximation, the mass and energy of a flow have to be conserved. Since the shock-wave solution over a bottom step is stationary, the Rankine–Hugoniot conditions are written as

$$u_l h_l = u_r h_r \tag{3.2}$$

$$g u_r h_r (h_r + b_r) + h_r \frac{u_r^3}{2} = g u_l h_l (h_l + b_l) + h_l \frac{u_l^3}{2}. \tag{3.3}$$

As it turns out, the second condition can be replaced by the simpler condition

$$g(h_r + b_r) + \frac{u_r^2}{2} = g(h_l + b_l) + \frac{u_l^2}{2}, \tag{3.4}$$

and the conditions (3.2) and (3.4) are the standard relations in hydraulic theory [17]. Since the hydraulic fall over a step does not require a Rankine–Hugoniot deficit in the second equation of (1.10), it is clear that this is a weak solution in the classical sense, and satisfies Definition 2.3 without the singular part.

4. Flow of a bore over a bottom step

To be concrete, we study a bore (traveling hydraulic jump) approaching a bottom step from the left. In order to describe the interaction of the traveling jump with the bottom step, we need to solve the Riemann problem over a bottom step. In [1], it was found that there are 26 different solutions, but the authors did not investigate the admissibility of these solutions. Here, we find *admissible* solutions involving two shocks, one propagating to the left and the other propagating to the right of the step.

Definition 4.1. The shock defined by

$$u(x,t) = \begin{cases} u_l, & \text{if } x < ct, \\ u_r, & \text{if } x > ct, \end{cases} \tag{4.1}$$

connecting a left state (h_l, u_l) and a right state (h_r, u_r) is i-admissible if the shock speed c satisfies the Lax entropy conditions

$$\lambda_i(h_r, u_r) \le c \le \lambda_i(h_l, u_l), \tag{4.2}$$

for $i = 1, 2$.

Consider a bottom step function where $b_l = 0$ and $b_r = 1$ and the initial data

$$h\,|_{t=0} = \begin{cases} 4, & \text{if } x < -1, \\ 1, & \text{if } x > -1, \end{cases} \qquad u\,|_{t=0} = \begin{cases} 5.14, & \text{if } x < -1, \\ -2.29, & \text{if } x > -1. \end{cases} \tag{4.3}$$

For the given initial data the shock

$$h(t,x) = \begin{cases} 4, & \text{if } x < c_1 t - 1, \\ 1, & \text{if } x > c_1 t - 1, \end{cases}$$

$$u(t,x) = \begin{cases} 5.14, & \text{if } x < c_1 t - 1, \\ -2.29, & \text{if } x > c_1 t - 1, \end{cases} \tag{4.4}$$

where $c_1 = 7.61$ is a 2-shock and reaches the step at $t = 1/c_1$. At that moment we need to solve the system (1.10) at the step. It is important to note that out of the step the process is still governed by (1.1) and (1.2). The shock (4.4) will be split into a 1-shock corresponding to (1.1) and (1.2), a 2-shock corresponding to (1.1) and (1.2) and a stationary shock (SS) corresponding to (1.10). The goal is to obtain one system which describes the entire flow phenomenon. This can be done through the δ-shock concept where δ will actually describe deficiency of the model. To achieve this we consider the system (1.10) with initial data given in (4.3). The next task is to find two constant states (h_2, u_2) and (h_3, u_3) which are located to the left and right of the bottom step respectively. These states are obtained by simultaneously solving the equations

$$\left.\begin{aligned} u_2 - u_l &= -(h_2 - h_l)\sqrt{\frac{g}{2}\left(\frac{1}{h_2} + \frac{1}{h_l}\right)}, \\ u_r - u_3 &= (h_r - h_3)\sqrt{\frac{g}{2}\left(\frac{1}{h_r} + \frac{1}{h_3}\right)}, \\ g(h_2 + b_l) + \frac{u_2^2}{2} &= g(h_3 + b_r) + \frac{u_3^2}{2}, \\ u_2 h_2 &= u_3 h_3. \end{aligned}\right\} \tag{4.5}$$

The first two equations in (4.5) are obtained from the Rankine–Hugoniot conditions that describe the relations between the states on both sides of the left and right going shocks and the last two equations describe the bottom condition. To the left of the bottom step, the constant state (h_2, u_2) is connected to (h_l, u_l) by a left going 1-shock whereas to the right of the step the constant state (h_3, u_3) is connected to (h_r, u_r) by a right going 2-shock and a stationary shock is located at the step $x = 0$ as shown in Fig. 3. The state-wave diagram for this case is

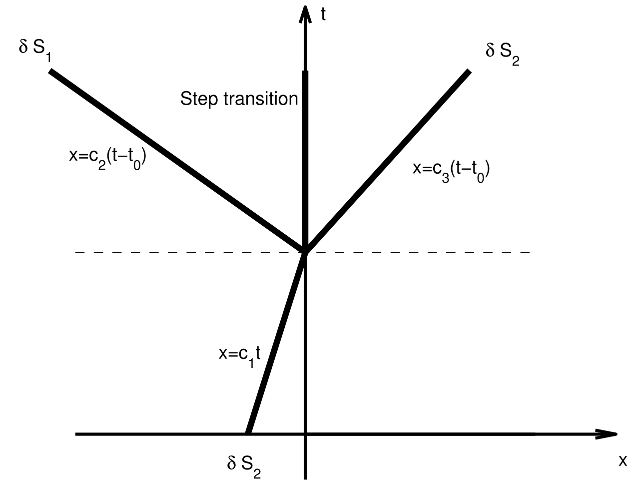

Fig. 3. Flow of a bore over a bottom step. The incoming shock δS_2 meets the bottom step at time $t_0 = 1/c_1$. For $t > t_0$, δS_1 is a left going delta shock, the bottom transition is at $x = 0$, and δS_2 is a right going delta shock.

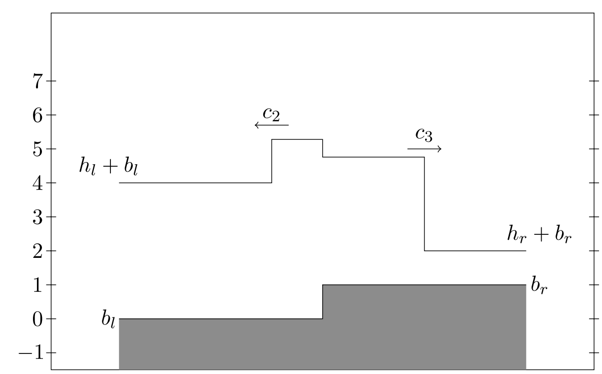

Fig. 4. A 1-shock moving left, a stationary step transition located at $x = 0$ and a 2-shock moving right.

$$(h_l, u_l) \xrightarrow{\delta S_1} (h_2, u_2) \xrightarrow{\text{SS}} (h_3, u_3) \xrightarrow{\delta S_2} (h_r, u_r). \tag{4.6}$$

The flow pattern corresponding to the above diagram is shown in Fig. 4. The undisturbed water surface is located at $\eta(x,t) = h_j + b(x)$ for $j \in \{l, r, 2, 3\}$. The left going shock is travelling at the speed $c_2 = -2.62$ and the right going shock has the approximate speed $c_3 = 7.07$. The physically relevant solution has the form

$$\begin{aligned} h(t,x) = {} & \chi_{[0,t_0]}(t) \begin{cases} 4, & \text{if } x < c_1 t - 1, \\ 1, & \text{if } x > c_1 t - 1, \end{cases} \\ & + \chi_{(t_0,+\infty)}(t) \begin{cases} 4, & \text{if } x < c_2(t - t_0), \\ 5.28, & \text{if } c_2(t - t_0) < x < 0, \end{cases} \\ & + \chi_{(t_0,+\infty)}(t) \begin{cases} 3.76, & \text{if } 0 < x < c_3(t - t_0), \\ 1, & \text{if } c_3(t - t_0) < x, \end{cases} \end{aligned} \tag{4.7}$$

and

$$u(t,x) = \chi_{[0,t_0]}(t)\alpha_1 \delta(x - c_1 t - 1)$$

$$
\begin{aligned}
&+ \chi_{[0,t_0]}(t) \begin{cases} 5.14, & \text{if } x < c_1 t - 1, \\ -2.29, & \text{if } x > c_1 t - 1, \end{cases} \\
&+ \chi_{(t_0,+\infty)}(t) \begin{cases} 5.14, & \text{if } x < c_2(t - t_0), \\ 3.25, & \text{if } c_2(t - t_0) < x < 0, \end{cases} \\
&+ \chi_{(t_0,+\infty)}(t) \begin{cases} 4.58, & \text{if } 0 < x < c_3(t - t_0), \\ -2.29, & \text{if } c_3(t - t_0) < x, \end{cases} \\
&+ \chi_{(t_0,+\infty)}(t)\alpha_2\delta(x - c_2(t - t_0)) \\
&+ \chi_{(t_0,+\infty)}(t)\alpha_3\delta(x - c_3(t - t_0)),
\end{aligned} \tag{4.8}
$$

where the Rankine–Hugoniot deficits are given by

$$
\begin{aligned}
\alpha_1 &= \Big(c_1[u] - [gh + u^2/2 + gb]\Big)\, t, \\
\alpha_2 &= \Big(c_2[u] - [gh + u^2/2]\Big)\,(t - t_0) \\
&\quad + \Big(c_2[u] - [gh + u^2/2 + gb]\Big)\, t_0, \\
\alpha_3 &= \Big(c_3[u] - [gh + u^2/2 + 1]\Big)\,(t - t_0).
\end{aligned}
$$

That is

$$
\begin{aligned}
\alpha_1 &= \Big(c_1(u_l - u_r) - (gh_l + u_l^2/2) + (gh_r + u_r^2/2 + 1)\Big)\, t, \\
\alpha_2 &= \Big(c_2(u_l - u_2) - (gh_l + u_l^2/2) + (gh_2 + u_2^2/2)\Big)\,(t - t_0) \\
&\quad + \Big(c_2(u_l - u_r) - (gh_l + u_l^2/2) + (gh_r + u_r^2/2 + 1)\Big)\, t_0, \\
\alpha_3 &= \Big(c_3(u_3 - u_r) - (gh_3 + u_3^2/2 + 1) + (gh_r + u_r^2/2 + 1)]\Big) \\
&\quad \times (t - t_0).
\end{aligned}
$$

It is now straightforward to show that these functions define a solution in the sense of Definition 2.3. Moreover, evaluating the eigenvalues of the derivative matrix for the states (h_l, u_l), (h_2, u_2), (h_3, u_3) and (h_r, u_r) reveals that all three delta shocks δS_1, δS_2 and δS_3, are admissible, and so is the bottom step transition.

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