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# Instability of Solitary Waves for a Nonlinearly Dispersive Equation

Henrik Kalisch and Nguyet Thanh Nguyen

Abstract. Solitary-wave solutions of a nonlinearly dispersive evolution equation are considered. It is shown that these waves are unstable in a certain parameter range.

## 1. Introduction

Consideration is given to the dynamic stability of solitary-wave solutions of the nonlinearly dispersive model equation

$$u_t + \omega u_x + 3uu_x - u_{xxt} = \gamma(2u_x u_{xx} + uu_{xxx}), \tag{1.1}$$

where $\gamma \in \mathbb{R}$ and $\omega \geq 0$. Equation (1.1) is a fully nonlinear dispersive evolution equation similar to the so-called Camassa-Holm equation which emerges if the parameter $\gamma$ is set equal to 1. If also $\omega = 0$, the Camassa-Holm equation has an integrable bi-Hamiltonian structure, which fact has lead to intense activity regarding the equation. Results related to the integrable structure may be found in [**5, 6, 7, 11, 19, 20, 29**].

A formal derivation of the Camassa-Holm equation as a long-wave model for water waves in a long uniform channel was provided in [**24**]. In addition, there are now mathematical proofs available which show that solutions of the Camassa-Holm equation approximate solutions of the full water-wave problem in a certain sense [**13, 28**]. As shown in [**17**], if $\omega = 0$, and the range of the parameter $\gamma$ is roughly from $-29.5$ to 3.4, equation (1.1) may be used to study the evolution of wave packets of mechanical vibrations in compressible elastic rods.

Equations of Camassa-Holm type have been actively studied recently with regard to well-posedness, singularity formation and numerical approximation schemes. A small selection of results may be found in [**4, 8, 9, 10, 12, 23, 33**]. Recent results on stability for equation (1.1) with $\gamma = 0$ can be found in [**26, 27**]. For equation (1.1), Yin [**34**] has proved local well-posedness in $H^s$, when $s > \frac{3}{2}$. He also showed that global well-posedness is prohibited by the existence of smooth solutions that develop an infinite slope in finite time.

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The focus in the present article is on stability of solitary-wave solutions of (1.1). In particular, the instability of solitary waves in a certain parameter range will be proved. Let us first discuss some properties of solitary-wave solutions of (1.1). Solitary waves are solutions of (1.1) which have the special form $u(x,t) = \Phi_c(x-ct)$, where $\Phi_c(\xi)$, for $\xi = x-ct$, is a function which decays at infinity, and has a positive maximum.

As was already observed by one of the authors in [**25**], when $\gamma < 1$, equation (1.1) admits only smooth solitary waves with wave speed $c > \omega$. These waves were shown to be stable in [**25**] by a similar method as was used to show stability of the Camassa-Holm solitary waves in [**16**]. The notion of stability used in these works is orbital stability, as defined in [**2**], and the proof is based on the general theory of Grillakis, Shatah, and Strauss [**21**].

When $\gamma > 1$, equation (1.1) admits both peaked and smooth solitary waves, depending on the wave speed $c$. Solitary waves are smooth for $c$ in the range $\omega < c < \frac{\omega\gamma}{\gamma-1}$, while for $c = \frac{\omega\gamma}{\gamma-1}$, the solitary waves are peaked waves, similar to the peakons appearing in the Camassa-Holm equation with $\omega = 0$. It was proved in [**25**] that smooth solitary waves with $c > \omega$, are stable if $c$ is close enough to $\omega$. On the other hand, it was indicated that smooth solitary waves are unstable if $c < \frac{\omega\gamma}{\gamma-1}$, but $c$ is close to $\frac{\omega\gamma}{\gamma-1}$. It will be our purpose in the present paper to provide a full proof of the latter fact. Thus, the main result to be proved here is the following theorem.

THEOREM 1.1. *Suppose $\gamma > 1$, and let $\omega < c \leq \frac{\omega\gamma}{\gamma-1}$. For $c$ close enough but not equal to $\frac{\omega\gamma}{\gamma-1}$, solitary-wave solutions of* (1.1) *are unstable with respect to small perturbations.*

The proof proceeds along the lines of the general theory of instability outlined in [**21, 31**], and developed in [**3, 32**]. However, due to the fully nonlinear character of the equation (1.1), the proofs given in these works do not carry over to the situation at hand here, and a number of nontrivial modifications have to be made in the argument.

One important ingredient in the proof of Theorem 1.1, is the fact that (1.1) has three invariant integrals, namely

$$\begin{aligned} I(u) &= \int_{-\infty}^{\infty} u\, dx, \\ V(u) &= \tfrac{1}{2}\int_{-\infty}^{\infty} (u^2 + u_x^2) dx, \\ E(u) &= -\tfrac{1}{2}\int_{-\infty}^{\infty} (u^3 + \gamma u u_x^2 + \omega u^2) dx. \end{aligned}$$

Note that equation (1.1) can be written in the form

$$u_t = \frac{\partial_x}{(1-\partial_x^2)} \left[ -\omega u - \tfrac{3}{2}u^2 + \gamma \partial_x(u u_x) - \tfrac{\gamma}{2} u_x^2 \right], \tag{1.2}$$

or simply

$$u_t = JE'(u), \tag{1.3}$$

if the operator $J$ is defined by $J = \frac{\partial_x}{(1-\partial_x^2)}$ and it is recognized that the term in brackets in (1.2) is the variational derivative of $E(u)$. Now for a given wave speed $c$, the stability of the corresponding solitary wave $\Phi_c$ is determined by the convexity of the scalar function

$$d(c) = E(\Phi_c) + cV(\Phi_c).$$

In particular if $d''(c) > 0$, then it can often be shown that the solitary wave is stable, while if $d''(c) < 0$, the solitary wave is expected to be unstable. The applicability of these considerations depend on a certain spectral problem which will be recalled in Section 4.

While the conservation of $I(u)$ is unnecessary for the proofs of stability given in [**16, 25**], it is essential for the proof of instability. Indeed, $I(u)$ plays a crucial role in proving the estimate

$$\sup_{-\infty<x<\infty} \left| \int_x^\infty u(y,t)\,dy \right| \leq C(1+t^\zeta), \tag{1.4}$$

for some positive constant $C$, and for $0 < \zeta < 1$ and $t > 0$. This estimate in turn is intimately related to growth of the Lyapunov functional to be used in the proof of instability. The estimate (1.4) will be proved in Section 2. In Section 3, we will recall some properties of smooth solitary waves, and finally Section 4 contains the proof of Theorem 1.1.

Before we embark on the analysis, some notation is established. For $1 \leq p < \infty$, the space $L^p = L^p(\mathbb{R})$ is the set of measurable real-valued functions of a real variable whose $p^{th}$ powers are integrable over $\mathbb{R}$. For $f \in L^p$, the norm $|f|_p$ is defined by

$$|f|_p^p = \int_{-\infty}^\infty |f(x)|^p \; dx$$

as usual. For the case $p = \infty$, we say that $f \in L^\infty$ if there is a constant $A$, such that $|f(x)| \leq A$ almost everywhere. The norm in this case is defined by

$$|f|_\infty = \inf\big\{A : |f(x)| \leq A \; a.e\big\}.$$

For $s \geq 0$, the Sobolev space $H^s = H^s(\mathbb{R})$ is the subspace of $L^2(\mathbb{R})$ consisting of functions such that

$$\|f\|_{H^s}^2 = \int_{-\infty}^\infty (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi < +\infty,$$

where $\hat{f}$ denotes the Fourier transform of $f$. We will also use the space $\mathcal{C}([0,\infty); H^s)$ of continuous functions of $t$ with values in $H^s(\mathbb{R})$.

The duality pairing of a distribution with a test function is denoted by $\langle T, \phi\rangle$. For distributions in $L^2(\mathbb{R})$, this reduces to the $L^2$-inner product $\langle f, g\rangle = \int_{-\infty}^\infty f(x)g(x)dx$ since all functions considered here are real-valued. Finally, the convolution of two functions is defined as usual by $g * f(x) = \int_{-\infty}^\infty g(y)f(x-y)\,dy$.

## 2. Estimate on a Lyapunov functional

The aim of this section is to provide a proof of the estimate (1.4). Defining the operator $M = 1 - \partial_x^2$, it is elementary to check that $M$ is self-adjoint with respect to the $L^2$-inner product, and that the inverse $M^{-1}$ is given by convolution with the Green's function

$$G(x) = \tfrac{1}{2}e^{-|x|}. \tag{2.1}$$

For the proof of the estimate (1.4), a number of auxiliary results will be needed. The first is concerned with the following linear initial-value problem.

LEMMA 2.1. *Let $K$ be the propagator for the equation $\big[\partial_t + \omega M^{-1}\partial_x\big]w = 0$; that is, $K$ is the solution of*

$$\begin{cases} \big[\partial_t + \omega M^{-1}\partial_x\big]\, K = 0, & t > 0, \\ K(x,0) = \delta(x), \end{cases}$$

*where $\delta(x)$ is the Dirac delta function centered at 0. Then, there is an evolution operator $S(t)$ given in terms of $K(x,t)$, such that $w(x,t) = S(t)w_0(x) = K(\cdot,t) * w_0(x)$, where $w(x,0) = w_0$. Moreover, for all $t \geq 0$, there is a positive constant $k$ such that*

$$\big|K(\cdot,t) * w_0\big|_\infty \leq k(1+t)^{-1/4}\big(\|w_0\|_{H^1} + |w_0|_1\big).$$

This lemma can be proved exactly as in the analogous case of [**32**]. The proof is based on the van der Corput lemma, and is similar to the techniques used in [**1**]. In order to relate this linear initial-value problem to the equation under study, observe that (1.1) can be written in the form

$$\big[\partial_t + \omega M^{-1}\partial_x\big]u = -M^{-1}\partial_x g(u),$$

where, $g(u) = \frac{3}{2}u^2 + \frac{\gamma}{2}u_x^2 - \gamma\partial_x(uu_x)$. The next lemma provides an estimate for the $H^1$-norm and the $L^1$-norm of $M^{-1}g(u(\cdot,t))$.

LEMMA 2.2. *Let $M^{-1}$ be the inverse of the operator $M = 1 - \partial_x^2$ and let $g(u)$ be defined by*

$$g(u) = \tfrac{3}{2}u^2 + \tfrac{\gamma}{2}u_x^2 - \gamma\partial_x(uu_x). \tag{2.2}$$

*Suppose $u \in \mathcal{C}([0,\infty); H^1)$ is a solution of (1.1). Then there is a positive constant $k_1$ such that the estimate*

$$\|M^{-1}g(u(\cdot,t))\|_{H^1} + |M^{-1}g(u(\cdot,t))|_1 \leq k_1\|u_0\|_{H^1}^2$$

*holds for all $t > 0$.*

PROOF. First, the $H^1$-norm is estimated using the mapping properties of $M$. The dependence on $t$ is suppressed in the following computations.

$$\begin{aligned} \|M^{-1}g(u)\|_{H^1} &= \Big\|M^{-1}\big[\tfrac{3}{2}u^2 + \tfrac{\gamma}{2}u_x^2 - \gamma\partial_x(uu_x)\big]\Big\|_{H^1} \\ &\leq \tfrac{3}{2}\|M^{-1}u^2\|_{H^1} + \tfrac{|\gamma|}{2}\|M^{-1}u_x^2\|_{H^1} + |\gamma|\|M^{-1}\partial_x(uu_x)\|_{H^1} \\ &\leq \tfrac{3}{2}\|u^2\|_{H^{-1}} + \tfrac{|\gamma|}{2}\|u_x^2\|_{H^{-1}} + |\gamma|\big|(1+|\xi|^2)^{-1/2}\ |i\xi|\ \widehat{uu_x}\big|_2. \end{aligned}$$

Using the simple bilinear estimate $\|v^2\|_{H^{-1}} \leq k_2|v|_2^2$, and examining the growth of the weights in the $L^2$ norm in the last term, it is plain that we get

$$\|M^{-1}g(u)\|_{H^1} \leq \tfrac{3}{2}k_2|u|_2^2 + \tfrac{|\gamma|}{2}k_2|u_x|_2^2 + |\gamma|\big|\widehat{uu_x}\big|_2.$$

Finally using the standard Sobolev estimate

$$\sup_{x\in\mathbb{R}}|u(x)| \leq k_3\|u\|_{H^1},$$

and the time-invariance of $V(u) = \frac{1}{2}\|u\|_{H^1}^2$, the estimate

$$\begin{aligned} \|M^{-1}g(u)\|_{H^1} &\leq \tfrac{3}{2}k_2\|u\|_{H^1}^2 + \tfrac{|\gamma|}{2}k_2\|u\|_{H^1}^2 + |\gamma|k_3\|u\|_{H^1}^2 \\ &\leq \big(\tfrac{3}{2}k_2 + \tfrac{|\gamma|}{2}k_2 + |\gamma|k_3\big)\|u_0\|_{H^1}^2 \end{aligned} \tag{2.3}$$

appears. Next, the $L^1$-norm will be estimated using the triangle inequality as follows.

$$\begin{aligned}\left|M^{-1}g(u)\right|_1 &= \left|M^{-1}\left\{\tfrac{3}{2}u^2+\tfrac{\gamma}{2}u_x^2-\gamma\partial_x(uu_x)\right\}\right|_1\\ &\le \tfrac{3}{2}\left|M^{-1}u^2\right|_1+\tfrac{|\gamma|}{2}\left|M^{-1}u_x^2\right|_1+|\gamma|\left|M^{-1}\partial_x(uu_x)\right|_1.\end{aligned}$$

Now from the definition (2.1) of the Green's function $G(x)$, it appears that $G(x)$ is in $L^1$. Therefore, it can be seen that

$$\left|M^{-1}u^2\right|_1=|G*u^2|_1\le|G|_1|u^2|_1\le k_4|u|_2^2=k_4\|u_0\|_{H^1}^2,$$

and that

$$\left|M^{-1}u_x^2\right|_1=|G*u_x^2|_1\le|G|_1|u_x^2|_1\le k_4|u_x|_2^2=k_4\|u_0\|_{H^1}^2.$$

Finally, to estimate the last term, note that also $G'(x)$ is in $L^1$. An integration by parts shows that

$$M^{-1}\partial_x(uu_x)=\int_{-\infty}^{\infty}G(x-y)\partial_y\left(uu_y\right)dy=\int_{-\infty}^{\infty}G'(x-y)\;uu_y\,dy.$$

Now one may estimate

$$\left|M^{-1}\partial_x(uu_x)\right|_1\le|G_x|_1\;|uu_x|_1\le k_5|u|_2|u_x|_2\le k_5\|u_0\|_{H^1}^2.$$

Putting together the last three inequalities and estimate (2.3), and collecting the constants finally proves the lemma. □

Lemma 2.1 and Lemma 2.2 are now put to use in the proof of the estimate (1.4). The precise statement is as follows.

THEOREM 2.3. *Assume that $u_0\in H^1(\mathbb{R})\cap L^1(\mathbb{R})$, and let $u(x,t)$ be the solution of (1.1) with initial data $u_0$. Then there exists a constant $C$ depending only on $u_0$, such that the estimate*

$$\sup_{-\infty<x<\infty}\left|\int_x^{\infty}u(y,t)\,dy\right|\le C(1+t^{3/4}),$$

*holds for $t\ge0$.*

PROOF. Recall that another form of the equation (1.1) is

$$\left[\partial_t+\omega M^{-1}\partial_x\right]u=-M^{-1}\partial_x g(u),$$

where $g(u)$ is defined in (2.2). Then, the solution $u$ of equation (1.1) may be expressed in the form

$$\begin{aligned}u(x,t) &= \int_{-\infty}^{\infty}K(x-y,t)u_0(y)\,dy\\ &\quad-\int_0^t\int_{-\infty}^{\infty}K(x-y,t-\tau)M^{-1}\partial_y g(u(y,\tau))\,dyd\tau\\ &= K(\cdot,t)*u_0(x)-\partial_x\int_0^t\left\{K(\cdot,t-\tau)*M^{-1}g(u(\cdot,\tau))(x)\right\}d\tau.\end{aligned}$$

Let

$$U(x,t) = \int_{-\infty}^{x} u(y,t)\,dy,$$
$$U_0(x) = \int_{-\infty}^{x} u_0(y)\,dy,$$

and

$$W(x,t) = \int_{-\infty}^{x} \{K(\cdot,t) * u_0(y)\}\,dy.$$

Then,

$$U(x,t) = W(x,t) - \int_0^t \Big\{K(\cdot,t-\tau) * M^{-1} g(u(\cdot,\tau))(x)\Big\}\,d\tau \tag{2.4}$$

Next, we will estimate the two terms on the right-hand side of equation (2.4) separately. First of all, observe that

$$\begin{aligned} K(\cdot,t) * u_0 &= K(\cdot,0) * u_0 + \int_0^t \partial_\tau \{K(\cdot,\tau) * u_0\}\,d\tau \\ &= u_0 + \int_0^t \{-\omega \partial_x M^{-1} K(\cdot,\tau) * u_0\}\,d\tau \\ &= u_0 - \omega\partial_x \int_0^t \{K(\cdot,\tau) * M^{-1} u_0\}\,d\tau. \end{aligned}$$

And thus,

$$W(x,t) = U_0(x) - \omega \int_0^t \{K(\cdot,\tau) * M^{-1} u_0(x)\}\,d\tau.$$

However, the first term of $W$ is estimated as

$$\big|U_0(x)\big| = \Big|\int_{-\infty}^{x} u_0(y)\,dy\Big| \le |u_0|_1$$

while using Lemma 2.1, the second term of $W$ is estimated as follows.

$$\begin{aligned} \omega\Big|\int_0^t \{K(\cdot,\tau) * M^{-1} u_0(x)\}\,d\tau\Big| &\le \omega \int_0^t \big|K(\cdot,\tau) * M^{-1} u_0(\cdot)\big|_\infty d\tau \\ &\le \omega k\Big(\|M^{-1}u_0\|_{H^1} + |M^{-1}u_0|_1\Big)\int_0^t (1+\tau)^{-1/4}\,d\tau \\ &\le \omega k\big(\|u_0\|_{H^1} + |u_0|_1\big)\int_0^t (1+\tau)^{-1/4}\,d\tau \\ &\le \tfrac{4}{3}\omega k\big(\|u_0\|_{H^1} + |u_0|_1\big)(1+t)^{3/4}, \end{aligned}$$

where the positive constant $k$ is defined in Lemma 2.1. Therefore, an estimate for $W$ is given by

$$\begin{aligned} \big|W(x,t)\big| &= \Big|U_0(x) - \omega\int_0^t \{K(\cdot,\tau) * u_0(x)\}\,d\tau\Big| \\ &\le |u_0|_1 + \tfrac{4}{3}\omega k\big(\|u_0\|_{H^1} + |u_0|_1\big)(1+t)^{3/4} \\ &\le \big(1 + \tfrac{4}{3}\omega k\big)\big(\|u_0\|_{H^1} + |u_0|_1\big)(1+t)^{3/4}. \end{aligned}$$

On the other hand, using both Lemma 2.1 and Lemma 2.2, the estimate for the second term on the right-hand side of equation (2.4) is given as follows

$$
\begin{aligned}
&\Big|\int_0^t \Big\{K(\cdot,t-\tau)*M^{-1}g(u(\cdot,\tau))(x)\Big\}\,d\tau\Big| \\
&\qquad\qquad \leq \int_0^t \big|K(\cdot,t-\tau)*M^{-1}g(u(\cdot,\tau))\big|_\infty d\tau \\
&\qquad \leq k\int_0^t \Big(\|M^{-1}g(u(\cdot,\tau))\|_{H^1}+|M^{-1}g(u(\cdot,\tau))|_1\Big)(1+t-\tau)^{-1/4}\,d\tau \\
&\qquad\qquad \leq kk_1\|u_0\|_{H^1}^2\int_0^t (1+t-\tau)^{-1/4}\,d\tau \leq \tfrac{4}{3}kk_1\|u_0\|_{H^1}^2(1+t)^{3/4},
\end{aligned}
$$

where $k$ and $k_1$ are defined in Lemma 2.1 and Lemma 2.2, respectively. Consequently, an upper bound for $U$ is

$$
\begin{aligned}
\big|U(x,t)\big| &= \Big|W(x,t)-\int_0^t \Big\{K(\cdot,t-\tau)*M^{-1}g(u(\cdot,\tau))(x)\Big\}\,d\tau\Big| \\
&\leq \Big[\big(1+\tfrac{4}{3}\omega k\big)\big(\|u_0\|_{H^1}+|u_0|_1\big)+\tfrac{4}{3}kk_1\|u_0\|_{H^1}^2\Big](1+t)^{3/4} \\
&\leq \big(1+\tfrac{4}{3}\omega k+\tfrac{4}{3}kk_1\big)\big(\|u_0\|_{H^1}^2+\|u_0\|_{H^1}+|u_0|_1\big)(1+t)^{3/4}.
\end{aligned}
$$

Now, using the last estimate and the fact that $I(u)=\int_{-\infty}^{\infty}u(x,t)dx$ is time-invariant, the final estimate is revealed as follows.

$$
\begin{aligned}
\Big|\int_x^\infty u(y,t)\,dy\Big| &= \big|I(u)-U(x,t)\big| \leq \big|I(u_0)\big|+\big|U(x,t)\big| \\
&\leq |u_0|_1+\big(1+\tfrac{4}{3}\omega k+\tfrac{4}{3}kk_1\big)\big(\|u_0\|_{H^1}^2+\|u_0\|_{H^1}+|u_0|_1\big)(1+t)^{3/4} \\
&\leq C(1+t^{3/4}),
\end{aligned}
$$

where $C$ is a positive constant which only depends on the initial data $u_0$. □

## 3. Solitary-wave solutions

Solitary-wave solutions of (1.1) will be reviewed in this section. Following the usual method of obtaining an equation for solitary waves, suppose there are solutions of the form

$$u(x,t)=\Phi_c(x-ct),$$

where $\Phi_c(\xi)$, for $\xi=x-ct$, is a function which decays at infinity, and has a positive maximum. Inserting this form into the equation (1.1), there appears the ordinary differential equation

$$(\omega-c)\Phi_c'+3\Phi_c\Phi_c'+c\Phi_c'''=\gamma(2\Phi_c'\Phi_c''+\Phi_c\Phi_c'''), \tag{3.1}$$

where $\Phi_c'=\frac{d\Phi_c}{d\xi}$. Since $\Phi_c(\xi)$ is assumed to approach zero as $\xi\to\pm\infty$, this equation can be integrated, and there appears

$$(\omega-c)\Phi_c+\frac{3}{2}\Phi_c^2+c\Phi_c''=\frac{\gamma}{2}\Phi_c'^2+\gamma\Phi_c\Phi_c''. \tag{3.2}$$

Multiplying by $\Phi_c'$, and integrating once more yields

$$(\omega-c)\Phi_c^2+\Phi_c^3+c\Phi_c'^2=\gamma\Phi_c\Phi_c'^2. \tag{3.3}$$

If $c \neq \omega$, rearranging the equation yields

$$(c-\omega)\Big[\frac{c}{c-\omega}\Phi_c'^2 - \Phi_c^2\Big] = \Phi_c(\gamma\Phi_c'^2 - \Phi_c^2). \tag{3.4}$$

It is apparent that when $\gamma = \frac{c}{c-\omega}$, the solutions of (3.4) are peaked solitary waves given by the formula

$$\Phi_c(\xi) = (c-\omega)e^{-\sqrt{\frac{c-\omega}{c}}|\xi|},$$

where the wave speed is

$$c = \frac{\gamma\omega}{\gamma-1}, \ \text{for } \gamma \neq 1 \text{ and } \omega \neq 0. \tag{3.5}$$

If $c$ does not have the special form (3.5), an explicit formula for the solution has not been found. However, as observed by one of the authors in [**25**], (3.3) may be rearranged in the form

$$(c-\omega-\Phi_c)\Phi_c^2 = (c-\gamma\Phi_c)\Phi_c'^2, \tag{3.6}$$

and a phase plane analysis of this equation shows that $\Phi_c$ is a positive smooth function of maximal height $c-\omega$, symmetric around and monotonically decreasing from its crest. Moreover, $\Phi_c, \Phi_c'$, and $\Phi_c''$ are all exponentially decaying at infinity. Moreover, when $\gamma > 1$, the relation (3.6) implies $\omega < c < \frac{\omega\gamma}{\gamma-1}$. The peaked solitary waves arise as the limiting case of this relation in the case $c = \frac{\omega\gamma}{\gamma-1}$. Figure 1 summarizes the stability properties of the solitary waves in this range.

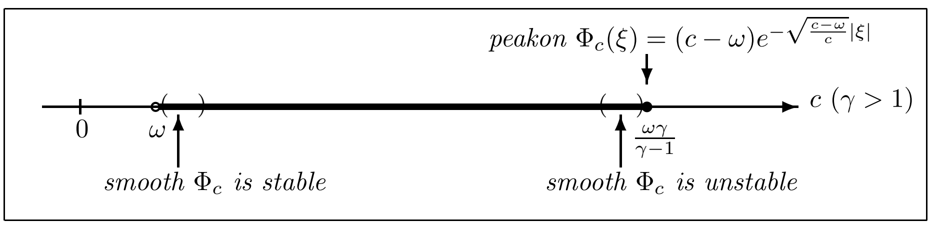


FIGURE 1. If $\gamma > 1$, solitary waves exist only in the range $\omega < c \leq \frac{\omega\gamma}{\gamma-1}$. The peaked solitary wave occurs at the maximum value $c = \frac{\omega\gamma}{\gamma-1}$. For $c$ close to the lower limit $\omega$, solitary waves $\Phi_c$ are stable. On the other hand, for $c$ close to the upper limit $\frac{\omega\gamma}{\gamma-1}$, solitary waves $\Phi_c$ are unstable.

## 4. Proof of instability

After a short review of the concept of orbital stability, the proof of the instability is given. As is plain from examining the time evolution of two solitary waves of similar but unequal height and speed, a solitary wave cannot be Lyapunov stable in the usual sense. In the situation just alluded to, the two waves will drift apart over time because their speeds are not equal. Recognizing this behavior, Benjamin introduced the notion of orbital stability in [**2**]. In the situation just described, it is evident that two solitary waves with slightly differing heights will stay similar in shape during the time evolution, even though their peaks will be located at different positions. We say the solitary wave is orbitally stable, if a solution $u$ of the equation (1.1) that is initially sufficiently close to a solitary-wave will always stay close to a translation of the solitary-wave during the time evolution. A formal definition can be given using an $\varepsilon$-neighborhood of the collection of all translates of $\Phi_c$. To be precise, for any $\varepsilon > 0$, consider

$$U_\varepsilon = \{u \in H^1 : \inf_s \|u - \tau_s \Phi_c\|_{H^1} < \varepsilon\},$$

where $\tau_s \Phi_c(x) = \Phi_c(x - s)$ is a translation of $\Phi_c$.

Definition 4.1. The solitary wave is stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $u_0 = u(\cdot, 0) \in U_\delta$, then $u(\cdot, t) \in U_\varepsilon$ for all $t \geq 0$. The solitary wave $\Phi_c$ is unstable if $\Phi_c$ is not stable.

Determining the optimal translation $\tau_\alpha$ for a given solitary wave and a perturbation can be achieved by choosing $\alpha \in \mathbb{R}$, such that

$$\int_{-\infty}^{\infty} \Big\{u(\xi + \alpha(u)) - \Phi_c(\xi)\Big\}^2 d\xi = \inf_{a \in \mathbb{R}} \int_{-\infty}^{\infty} \Big\{u(\xi + a) - \Phi_c(\xi)\Big\}^2 d\xi$$

if this infimum exists. If the integral on the right is a differentiable function of $a$, and $\|u\|_{L^2} = \|\Phi_c\|_{L^2}$, then $\alpha(u)$ can be determined by solving the equation

$$\big\langle u(\cdot + \alpha(u)), \Phi_c' \big\rangle = 0. \tag{4.1}$$

This idea is summarized in the following proposition.

Proposition 4.2. There is $\varepsilon > 0$, such that there exists a $C^1$-mapping $\alpha : U_\varepsilon \longrightarrow \mathbb{R}$, with the property that $\big\langle u(\cdot + \alpha(u)), \Phi_c' \big\rangle = 0$ for every $u \in U_\varepsilon$.

The proof of this fact is well known, and can be found for instance in [**3**]. Next we establish a few facts which are important for the proof of instability. First, observe that the differential equation (3.2) defining the solitary waves can be written in terms of the functionals $E$ and $V$ in variational form as

$$E'(\Phi_c) + cV'(\Phi_c) = 0, \tag{4.2}$$

where $E'(\Phi_c) = -\frac{3}{2}\Phi_c^2 + \frac{\gamma}{2}\Phi_c'^2 + \gamma\Phi_c\Phi_c'' - \omega\Phi_c$ and $V'(\Phi_c) = \Phi_c - \Phi_c''$ are the Fréchet derivatives at $\Phi_c$ of $E$ and $V$, respectively. The functional derivative of $E'(\Phi_c) + cV'(\Phi_c)$ is given by the linear operator

$$\mathcal{L}_c \equiv E''(\Phi_c) + cV''(\Phi_c) = (\gamma\Phi_c - c)\partial_x^2 + \gamma\Phi_c'\partial_x - 3\Phi_c + \gamma\Phi_c'' + (c - \omega).$$

Since $\Phi_c$, $\Phi_c'$ and $\Phi_c''$ are exponentially decaying, the spectral equation $\mathcal{L}_c v = \lambda v$ can be transformed by the Liouville transformation

$$z = \int_0^x \frac{1}{\sqrt{2c - 2\gamma\Phi_c(y)}}\, dy,$$

and

$$\psi(z) = \big(2c - 2\gamma\Phi_c(x)\big)^{\frac{1}{4}} v(x),$$

into

$$\begin{aligned} \mathcal{H}_c\psi(z) &= \big(-\partial z^2 + q(z) + 2(c - \omega)\big)\psi(z) \\ &= \lambda\psi(z), \end{aligned}$$

where

$$q_c(z) = -6\Phi_c(x) + \frac{3\gamma}{2}\Phi_c''(x) - \frac{\gamma^2\big[\Phi_c'(x)\big]^2}{4\big(c - \gamma\Phi_c(x)\big)}.$$

Since $q_c$ has exponential decay, it can be shown that the operator $\mathcal{H}_c$ has continuous spectrum $[2(c - \omega), \infty)$, and there are finitely many eigenvalues below $2(c - \omega)$.

Moreover, the $n$-th eigenvalue in increasing order from the left has an associated eigenfunction with exactly $(n-1)$ zeroes (cf. Dunford and Schwartz [**18**]). These considerations carry over to the operator $\mathcal{L}_c$. Note that (3.1) shows that $\mathcal{L}_c(\Phi_c')=0$, and we know that $\Phi_c'$ has exactly one zero. Therefore 0 is the second eigenvalue from the left, and it appears that there is exactly one negative eigenvalue for the operator $\mathcal{L}_c$, with a corresponding eigenfunction $\chi_c$ which can be taken to be strictly positive, and normalized so that $\chi_c(0)=1$. Finally, note the following relation involving $\mathcal{L}_c$ and the derivative of $\Phi_c$ with respect to $c$.

LEMMA 4.3. *In the notation established above, the following relation holds.*

$$\mathcal{L}_c\left(d\Phi_c/dc\right)=-V'(\Phi_c). \tag{4.3}$$

PROOF. The relation (4.3) follows from (4.2) after the following computation.

$$\begin{aligned} 0 &= \partial_c\big[E'(\Phi_c)+cV'(\Phi_c)\big]=\big[E''(\Phi_c)+cV''(\Phi_c)\big]d\Phi_c/dc+V'(\Phi_c)\\ &= \mathcal{L}_c(d\Phi_c/dc)+V'(\Phi_c). \end{aligned}$$

□

The instability of the solitary wave $\Phi_c$ will follow from the fact that the functional $E$ has a constrained maximum at the critical point $\Phi_c$,. This fact will be established in the following lemma.

LEMMA 4.4. *Let $c$ close to but less than $\frac{\omega\gamma}{\gamma-1}$ be fixed. If $d''(c)<0$, then there exists a curve $\nu\mapsto\Psi_\nu$ in a neighborhood of $c$, such that $\Psi_c=\Phi_c$, $V(\Psi_\nu)=V(\Phi_c)$ for all $\nu$, and $E(\Psi_\nu)<E(\Phi_c)$ for $\nu\neq c$.*

PROOF. Consider a mapping $\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ given by $(\nu,s)\mapsto V(\Phi_\nu+s\chi_c)$, where $\chi_c$ is the eigenfunction corresponding to the negative eigenvalue of the operator $\mathcal{L}_c$. Note that $(c,0)$ maps to $V(\Phi_c)$. To obtain the curve $\nu\mapsto\Psi_\nu$, we first apply the implicit function theorem to find a mapping $\nu\to s(\nu)$, such that $V(\Phi_\nu+s(\nu)\chi_c)$ is constant. To this end, it has to be shown that

$$\frac{\partial}{\partial s}\big\{V(\Phi_\nu+s\chi_c)\big\}\Big|_{\nu=c,\,s=0}=\big\langle V'(\Phi_c),\chi_c\big\rangle$$

is nonzero. This expression can be evaluated explicitly when $\Phi_c(\xi)=(c-\omega)e^{-\sqrt{\frac{c-\omega}{c}}\,|\xi|}$ (recall that then $c=\frac{\omega\gamma}{\gamma-1}$, $\gamma>1$ and $\omega\neq0$). First record the derivatives of $\Phi_c$ as

$$\Phi_c'(\xi)=-\sqrt{\tfrac{c-\omega}{c}}\;(c-\omega)\;\mathrm{sgn}(\xi)\;e^{-\sqrt{\frac{c-\omega}{c}}\,|\xi|},$$

and

$$\Phi_c''(\xi)=-\sqrt{\tfrac{c-\omega}{c}}\;(c-\omega)\Big\{2\delta(\xi)e^{-\sqrt{\frac{c-\omega}{c}}\,|\xi|}-\sqrt{\tfrac{c-\omega}{c}}e^{-\sqrt{\frac{c-\omega}{c}}\,|\xi|}\Big\}.$$

Therefore, it can be seen that

$$\begin{aligned} \big\langle V'(\Phi_c),\chi_c\big\rangle &= \big\langle\Phi_c(\xi),\chi_c(\xi)\big\rangle-\big\langle\Phi_c''(\xi),\chi_c(\xi)\big\rangle\\ &= (c-\omega)2\sqrt{\tfrac{c-\omega}{c}}\,\Big\langle\delta(\xi)\;e^{-\sqrt{\frac{c-\omega}{c}}\,|\xi|},\chi_c\Big\rangle\\ &\qquad+(c-\omega)\Big(1-\tfrac{c-\omega}{c}\Big)\int_{-\infty}^{\infty}e^{-\sqrt{\frac{c-\omega}{c}}\,|\xi|}\chi_c(\xi)\;d\xi\\ &= (c-\omega)\left\{2\sqrt{\tfrac{c-\omega}{c}}\;\chi_c(0)+\Big(1-\tfrac{c-\omega}{c}\Big)\int_{-\infty}^{\infty}e^{-\sqrt{\frac{c-\omega}{c}}\,|\xi|}\;\chi_c\;d\xi\right\}. \end{aligned}$$

Observe that $1 - \frac{c-\omega}{c} > 0$ since $c > \omega > 0$. Now, since $\chi_c$ is normalized so that $\chi_c(0) = 1$, we see that last expression in the above string of equalities is bounded away from zero for values of $c$ close to $\frac{\omega\gamma}{\gamma-1}$. Consequently, $\big\langle V'(\Phi_c), \chi_c \big\rangle$ is positive for $c$ close enough to but less than $\frac{\omega\gamma}{\gamma-1}$. Now the implicit function theorem may be used to find the mapping $\nu \to s(\nu)$, and $\Psi_\nu$ is defined by $\Psi_\nu = \Phi_\nu + s(\nu)\chi_c$.

Next, we show that $c$ is a critical point of $\nu \to E(\Psi_\nu)$. Since $V(\Psi_\nu)$ is constant near $c$, we have

$$\frac{d}{d\nu} E(\Psi_\nu) = \frac{d}{d\nu}\big\{E(\Psi_\nu) + cV(\Psi_\nu)\big\}, \tag{4.4}$$

and in light of (4.2), the above expression is zero when evaluated at $\nu = c$. Furthermore, as will be shown next, at this critical point, the curve $\nu \to E(\psi_\nu)$ is strictly concave, i.e, $\frac{d^2}{d\nu^2} E(\Psi_\nu)\Big|_{\nu=c} < 0$, and hence has a local maximum. Differentiating equation (4.4) and using (4.2) gives

$$\frac{d^2}{d\nu^2} E(\Psi_\nu)\Big|_{\nu=c} = \Big\langle \big[E''(\Phi_c) + cV''(\Phi_c)\big] \frac{d\Psi_\nu}{d\nu}\Big|_{\nu=c}, \frac{d\Psi_\nu}{d\nu}\Big|_{\nu=c} \Big\rangle.$$

Recall now that $\mathcal{L}_c = E''(\Phi_c) + cV''(\Phi_c)$, and $\chi_c$ is an eigenfunction corresponding to the negative eigenvalue $-\lambda^2$. Therefore, if we define

$$y = \frac{d\Psi_\nu}{d\nu}\Big|_{\nu=c} = \frac{d\Phi_c}{dc} + s'(c)\chi_c, \tag{4.5}$$

then

$$\frac{d^2}{d\nu^2} E(\Psi_\nu)\Big|_{\nu=c} = \big\langle \mathcal{L}_c y, y \big\rangle.$$

Thus, the proof of Lemma 4.4 will be completed if it can be shown that $\big\langle \mathcal{L}_c y, y \big\rangle < 0$. First observe that

$$\langle V'(\Phi_c), y \rangle = 0. \tag{4.6}$$

This can be seen from differentiating $\nu \to V(\Psi_\nu)$ as follows.

$$0 = \frac{d}{d\nu} V(\Psi_\nu)\Big|_{\nu=c} = \Big\langle V'(\Phi_c), \frac{d\Psi_\nu}{d\nu}\Big|_{\nu=c} \Big\rangle = \big\langle V'(\Phi_c), y \big\rangle.$$

Combining (4.6) and Lemma 4.3, we obtain

$$\begin{aligned} \big\langle \mathcal{L}_c y, y \big\rangle &= \Big\langle \mathcal{L}_c\big(d\Phi_c/dc + s'(c)\chi_c\big), y \Big\rangle \\ &= \Big\langle -V'(\Phi_c) + s'(c)\mathcal{L}_c\chi_c, y \Big\rangle \\ &= s'(c)\big\langle \mathcal{L}_c\chi_c, y \big\rangle. \end{aligned}$$

Since $\mathcal{L}_c$ is self-adjoint, we obtain further

$$\begin{aligned} \big\langle \mathcal{L}_c y, y \big\rangle &= s'(c)\big\langle \chi_c, \mathcal{L}_c y \big\rangle \\ &= s'(c)\Big\langle \chi_c, \mathcal{L}_c\big(d\Phi_c/dc + s'(c)\chi_c\big) \Big\rangle \\ &= s'(c)\big\langle \chi_c, -V'(\Phi_c) + s'(c)\mathcal{L}_c\chi_c \big\rangle \\ &= -s'(c)\big\langle \chi_c, V'(\Phi_c) \big\rangle + [s'(c)]^2\big\langle \chi_c, \mathcal{L}_c\chi_c \big\rangle. \end{aligned}$$

Observe that the first term on the right of this equation is exactly $d''(c)$. Indeed, since $d(c) = E(\Phi_c) + cV(\Phi_c)$, we have

$$d'(c) = \big\langle E'(\Phi_c) + cV'(\Phi_c), d\Phi_c/dc \big\rangle + V(\Phi_c) = V(\Phi_c),$$

and hence,

$$d''(c) = \big\langle V'(\Phi_c), d\Phi_c/dc \big\rangle = -s'(c)\big\langle V'(\Phi_c), \chi_c \big\rangle, \tag{4.7}$$

in light of (4.5) and equation (4.6). Therefore,

$$\big\langle \mathcal{L}_c y, y \big\rangle = d''(c) + [s'(c)]^2 \big\langle \chi_c, \mathcal{L}_c \chi_c \big\rangle = d''(c) - \lambda^2 [s'(c)]^2 \|\chi_c\|^2_{L^2} < 0,$$

since $d''(c)$ is assumed to be negative. Therefore we have shown that $\frac{d^2}{d\nu^2} E(\Psi_\nu)\Big|_{\nu=c} = \big\langle \mathcal{L}_c y, y \big\rangle < 0$, and thus $\nu \mapsto E(\Psi_\nu)$ has a local maximum at $\nu = c$. □

Next, an auxiliary operator $B$ is defined. For $u \in U_\varepsilon$, define $B(u)$ by the formula

$$B(u) = y(\cdot - \alpha(u)) - \big\langle Mu, y(\cdot - \alpha(u)) \big\rangle M^{-1} \partial_x \alpha'(u).$$

The next lemma provides a connection between $B$ and the fact that $E$ has a constrained maximum near $\Phi_c$. It can be proved exactly as in the analogous case of [**3**], and is therefore stated without proof.

Lemma 4.5. *Let $c$ close to but less than $\frac{\omega\gamma}{\gamma-1}$ be fixed. If $d''(c) < 0$, there is a $C^1$-functional $\Lambda : D_\varepsilon \to \mathbb{R}$, where $D_\varepsilon = \{v \in U_\varepsilon : V(v) = V(\Phi_c)\}$, such that $\Lambda(\Phi_c) = 0$, and if $v \in D_\varepsilon$ and $v$ is not a translate of $\Phi_c$, then*

$$E(\Phi_c) < E(v) + \Lambda(v)\big\langle E'(v), B(v) \big\rangle.$$

*Furthermore, $\big\langle E'(\Psi_\nu), B(\Psi_\nu) \big\rangle$ changes sign as $\nu$ passes through $c$, where $\nu \mapsto \Psi_\nu$ is the curve constructed in Lemma 4.4.*

With these auxiliary results in hand, we may attack the proof of the main theorem of this paper. Note that Lemma 4.4 and Lemma 4.5 depended on the condition that $d''(c) < 0$. Thus it will the necessary first to establish the concavity of $d(c)$. As shown in [**25**], the derivative of $d(c)$ is given by

$$d'(c) \;=\; 2\int_\omega^c (c-y) \frac{(y-\omega) + (1-\gamma)c + \gamma y}{\sqrt{y-\omega}\,\sqrt{(1-\gamma)c + \gamma y}}\, dy$$

for $\omega < c < \frac{\gamma\omega}{\gamma-1}$. Taking the second derivative and evaluating at the endpoint $c = \frac{\gamma\omega}{\gamma-1}$ yields

$$\begin{aligned} d''\Big(\frac{\omega\gamma}{\gamma-1}\Big) &= k_0 - \Big(\frac{-2\omega\gamma + (\gamma+1)\omega}{\gamma^{\frac{3}{2}}}\Big) \int_\omega^{\frac{\gamma\omega}{\gamma-1}} \frac{dy}{y-\omega}, \\ &= k_0 - \Big(\frac{\omega(1-\gamma)}{\gamma^{\frac{3}{2}}}\Big) \int_\omega^{\frac{\gamma\omega}{\gamma-1}} \frac{dy}{y-\omega}, \end{aligned}$$

where $k_0$ is a constant depending on $\omega$ and $\gamma$. Since $d''(\frac{\omega\gamma}{\gamma-1}) = -\infty$ for $\gamma > 1$, it appears that $d''(c)$ will be negative for values of $c$ close to $\frac{\omega\gamma}{\gamma-1}$.

We choose a solitary wave $\Phi_c$ with wavespeed $c$ in the range where $d''(c) < 0$, and let $\varepsilon > 0$ sufficiently small be given. By Lemma 4.4 and Lemma 4.5, we can choose $u_0 \in H^1 \cap L^1$ arbitrary close to $\Phi_c$, such that $u_0 \in U_\varepsilon$, $V(u_0) = V(\Phi_c)$, $E(u_0) < E(\Phi_c)$, and $\big|\big\langle E'(u_0), B(u_0) \big\rangle\big| > 0$. Note that the last condition guarantees that $u_0$ is not a translate of $\Phi_c$ [1].

[1]For example, let $u_0 = \Phi_\nu + s(\nu)\chi_c$, for an arbitrary $\nu$ close to $c$, but not exactly equal to $c$.

Now, if $u(x,t)$ is the solution of equation (1.1) with initial condition $u_0$, let $[0,t_1)$ denote the maximal time interval for which $u(\cdot,t) \in U_\varepsilon$. Instability of the solitary-wave will be demonstrated by showing that $t_1 < \infty$.

Let $\beta(t) = \alpha(u(\cdot,t))$, where $\alpha$ was defined in equation (4.1), and $Y(x) = \int_{-\infty}^{x}(1-\partial_z^2)y(z)\,dz$, where $y$ was defined in (4.5). Then define

$$L(t) = \int_{-\infty}^{\infty} Y(x-\beta(t))u(x,t)dx,$$

which will serve as a Lyapunov functional. First, it will be shown that $L(t)$ is finite, and grows no more rapidly than $t^{3/4}$ over time.

LEMMA 4.6. *There is a positive constant $D$ such that $|L(t)| \leq D(1+t^{3/4})$ for $0 \leq t < t_1$.*

*Proof.* Let $\mathcal{H}$ be the Heaviside function, and define $\kappa = \int_{-\infty}^{\infty} y(x)\,dx$, and $F(x) = \int_{-\infty}^{x} y(\xi)\,d\xi$. Then the following equality appears after integration by parts.

$$L(t) = \int_{-\infty}^{\infty} \Big[F\big(x-\beta(t)\big) - \kappa\mathcal{H}\big(x-\beta(t)\big)\Big]u(x,t)\,dx + \int_{-\infty}^{\infty} y\big(x-\beta(t)\big)u_x(x,t)\,dx + \kappa\int_{\beta(t)}^{\infty} u(x,t)\,dx.$$

Using the Cauchy-Schwarz inequality on the first and second integrals, and applying Theorem 2.3 to the last integral, an upper bound for $|L(t)|$ is estimated as follows.

$$|L(t)| \leq \big(|F-\kappa\mathcal{H}|_2 + |y|_2\big)\|u(t)\|_{H^1} + |\kappa|C(1+t^{3/4}). \tag{4.8}$$

Next, $F - \kappa\mathcal{H}$ can be shown to belong to $L^2(\mathbb{R})$, as follows. First of all, note that

$$F(x) - \kappa\mathcal{H}(x) = \begin{cases} F(x), & \text{if } x < 0 \\ F(x) - \kappa, & \text{if } x \geq 0. \end{cases}$$

Thus in order to investigate $|F - \kappa\mathcal{H}|_2$, it is expedient to consider the cases $x < 0$ and $x > 0$ separately. When $x < 0$, Minkowski's inequality can be used to show that

$$|F-\kappa\mathcal{H}|_2 = |F(x)|_{L^2(-\infty,0)} = \sqrt{\int_{-\infty}^{0}\Big\{\int_{-\infty}^{x} y(\xi)d\xi\Big\}^2 dx} \leq \int_{-\infty}^{0} \sqrt{|\xi|}|y(\xi)|\,d\xi. \tag{4.9}$$

Recall that phase plane analysis of equation (3.3) shows that $\Phi_c$, decays exponentially at infinity. An analysis similar to the one given in [**30**] shows that $\frac{d\Phi_c}{dc}$ also decays exponentially at infinity. Finally, note that since $\chi_c$ is an eigenfunction of $\mathcal{L}_c$, it features exponential decay at infinity, as well (cf. Hislop and Sigal [**22**]). Now, since $y$ is defined in terms of $d\Phi_c/dc$ and $\chi_c$, it is immediate that the last term in the above string of inequalities (4.9) is finite. An analogous argument holds for $x > 0$. Therefore the inequality (4.8) can be written as

$$|L(t)| \leq D(1+t^{3/4}),$$

with the positive constant $D$ defined by $D = \big(|F-\kappa\mathcal{H}|_2 + |y|_2\big)\|u_0\|_{H^1} + |\kappa|C$, where $C$ was defined in the statement of Theorem 2.3. □

The previous lemma provides an upper bound on the growth of $L(t)$. Next, we will obtain a lower bound by giving and estimate of the derivative of $L$.

LEMMA 4.7. *There is a positive constant $m$ such that $|L'(t)| > m$, for all $t \in [0, t_1)$.*

PROOF. We have

$$L'(t) = -\beta'(t)\big\langle My(\cdot - \beta(t)), u(\cdot, t)\big\rangle + \big\langle Y(\cdot - \beta(t)), u_t(\cdot, t)\big\rangle.$$

Since $\beta'(t) = \big\langle \alpha'(u), u_t\big\rangle$, this derivative is equal to

$$\Big\langle -\big\langle My(\cdot - \beta(t)), u(\cdot, t)\big\rangle \alpha'(u), u_t\Big\rangle + \big\langle Y(\cdot - \beta(t)), u_t(\cdot, t)\big\rangle.$$

Since $M$ is self-adjoint, this derivative can be written in the form

$$\Big\langle -\big\langle y(\cdot - \beta(t)), Mu(\cdot, t)\big\rangle \alpha'(u) + Y(\cdot - \beta(t)), u_t\Big\rangle.$$

In view of equation (1.3), this derivative turns out to be

$$\Big\langle -\big\langle y(\cdot - \beta(t)), Mu(\cdot, t)\big\rangle \alpha'(u) + Y(\cdot - \beta(t)), \partial_x M^{-1}E'(u)\Big\rangle.$$

Using integration by parts together with the fact that $M^{-1}$ is self-adjoint and $\partial_x$ is skew-adjoint, this expression is equal to

$$\Big\langle \big\langle y(\cdot - \beta(t)), Mu(\cdot, t)\big\rangle \partial_x M^{-1}\alpha'(u) - y(\cdot - \beta(t)), E'(u)\Big\rangle.$$

In view of the definition of $B$, it is clear that $L'(t)$ has the compact expression

$$L'(t) = -\big\langle B(u), E'(u)\big\rangle. \tag{4.10}$$

Recall that for $t \in [0, t_1)$, the solution $u(\cdot, t) \in U_\varepsilon$ is not a translation of $\Phi_c$ since its initial solution is not. However, $V(u(t)) = V(\Phi_c)$ since both are equal to $V(u_0)$. On the other hand, Lemma 4.4 together with Lemma 4.5 imply that

$$0 < E(\Phi_c) - E(u_0) = E(\Phi_c) - E(u(t)) < \Lambda(u(t))\big\langle E'(u(t)), B(u(t))\big\rangle. \tag{4.11}$$

Using the continuity of $\Lambda$ and the fact that $\Lambda(\Phi_c) = 0$, which follows from the construction of the functional $\Lambda$ in Lemma 4.5, and recalling the assumption that $u(t) \in U_\varepsilon$, for $t \in [0, t_1)$, we may assume that $|\Lambda(u(t))| < 1$, possibly by choosing $\varepsilon$ smaller if necessary. Therefore, in view of equations (4.10) and (4.11), we have

$$\big|L'(t)\big| = \Big|\big\langle E'(u(t)), B(u(t))\big\rangle\Big| > \big[E(\Phi_c) - E(u(t))\big] = E(\Phi_c) - E(u_0) \;=\; m.$$

for all $t \in [0, t_1)$. □

Finally, we are in a position to complete the proof of Theorem 1.1. In view of Lemma 4.6 and Lemma 4.7, it turns out that

$$2D(1 + t^{3/4}) \geq |L(t)| + |L(0)| \geq \int_0^t |L'(s)| ds > \int_0^t m\, ds \;=\; m\, t,$$

for $t \in [0, t_1)$. However, since $3/4 < 1$, the rate of growth of the curve $f(t) = 2D(1 + t^{3/4})$ is less than the rate of growth of the line $l(t) = mt$. Therefore, $t_1$ must be the point where these two curves meet, and thus $t_1 < \infty$.

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Department of Mathematics, University of Bergen, 5008 Bergen, Norway
*E-mail address*: henrik.kalisch@math.uib.no

Department of Mathematics, University of Bergen, 5008 Bergen, Norway
*E-mail address*: nguyet.nguyen@math.uib.no