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On the stability of internal waves

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Abstract

The extended KdV equation $u_t + uu_x + \alpha u^2 u_x + u_{xxx} = 0$ is widely used as a model describing internal waves in ideal fluids. The equation admits a family of negative and positive solitary waves Φ_c. These solitary waves exhibit the typical broadening effect seen in internal waves. It is shown here that all solitary-wave solutions of the extended KdV equation are orbitally stable. The proof of stability is based on the general theory of Grillakis *et al* (1987 *J. Funct. Anal.* **74** 160) for equations of the form $u_t = JE'(u)$ which have two conserved integrals $E(u)$ and $V(u)$. A spectral analysis of the linear operator $\mathcal{L}_c = E''(\Phi_c) + cV''(\Phi_c)$ reduces the question of orbital stability to the question of whether the scalar function $d(c) = E(\Phi_c) + cV(\Phi_c)$ is convex. To prove the stability, an explicit calculation showing the convexity is performed.

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1. Introduction

Internal waves, sometimes with large amplitude, arise in various parts of the world such as oceans, large lakes and river estuaries. The existence of such waves has been established through numerous field studies, and due to their importance for the geophysical sciences, internal waves have been the subject of a great number of scientific works. Breaking internal waves and the ensuing large-scale mixing have been implicated as a possible significant factor in maintaining the stable background ocean stratification [18, 40]. Internal waves are also an important factor in the dynamics of benthic boundary layers [14], and are thus important for sediment resuspension and for the transport and distribution of nutrition to marine organisms.

Internal waves have been studied thoroughly in wave tank experiments and through numerical computations. For a small sample of results, the reader may refer to [10, 14, 18, 23, 32, 39]. The numerical study of internal waves using the full Euler or Navier–Stokes equations is very complex even in the two-dimensional case, and for some purposes, it can be advantageous to use simpler model equations which retain the basic competing effects of nonlinearity, dispersion and dissipation. In particular, in the inviscid case a modal analysis of horizontally propagating internal waves, when restricted to small amplitude and long wavelength leads to a number of model equations, such as the Korteweg–de Vries (KdV), Benjamin–Ono (BO) and intermediate long-wave equations [3, 6, 4, 12, 26, 31, 35, 36]. As the importance of these models has been recognized, there have been a number of studies focusing on the comparison of experimental data with these weakly nonlinear model equations [13, 23, 30].

One equation which has seen particular attention is the so-called extended Korteweg–de Vries (eKdV) equation, which is also known as the Gardner equation. In the non-dimensional form, the equation is given by

$$u_t + uu_x + \alpha u^2 u_x + u_{xxx} = 0. \tag{1.1}$$

Early derivations of this equation were given in [7, 33], and more recent work can be found in [22, 24] and the references contained therein. Equation (1.1) features a combination of the quadratic nonlinearity such as found in the KdV equation, and a cubic nonlinearity as associated with the modified KdV equation. Both these equations are known to be expressible as completely integrable Hamiltonian systems. Interestingly, (1.1) can also be expressed as a completely integrable Hamiltonian system. While this property of the equation does not figure into the analysis presented in this paper, it should be mentioned that it was in connection with the integrability that the equation was first found [34].

From a modeling point of view, one of the main advantages of equation (1.1) is that it appears to describe well the typical broadening of internal solitary waves with increasing amplitude which is observed in field measurements [16, 38]. Solitary-wave solutions for the eKdV equation (2.1) are known in the closed form. Using the ansatz $u(x, t) = \Phi_c(x - ct)$, it appears that a solution is given by the expression

$$\Phi_c(\xi) = \frac{A}{b + (1 - b)\cosh^2 K\xi}. \tag{1.2}$$

Here, $\xi = x - ct$, and the wave amplitude A, the wavelength K and the parameter b depend on c and α. The exact dependence of the solitary wave on these parameters is well known (see [22]), and will be reviewed in section 3, where it will also be shown that the equation does not admit solitary waves with $c < 0$.

In this work, the dynamic stability of internal and interfacial solitary waves is under review. It will be shown that internal solitary waves which can be described by equation (1.1) are nonlinearly stable with respect to finite-amplitude perturbations. The notion of stability used here is generally known as orbital stability. A precise definition of this concept is given in section 4. As explained above, the ingredients necessary for the validity of equation (1.1) limit its applicability to waves of great length and small excursion. Such waves are expected to be stable because of the physical effects of molecular viscosity and diffusion. However, the stability of internal waves described by the eKdV equation is of a more basic nature, requiring only the balance of nonlinearity and dispersion.

A first step in establishing the stability of the solitary waves is the development of a robust well-posedness theory. With regard to this task, note that recent advances in the study of local well-posedness of nonlinear dispersive evolution equations in low-regularity function spaces can easily be extended to equation (1.1). In particular, the results obtained in [28, 29] will suffice to ensure the existence of a solution which for a small enough time t is in the space $H^1 = H^1(\mathbb{R})$ of square-integrable functions on the real line which have square-integrable weak derivatives. Note that equation (1.1) has the following two invariant integrals:

$$V(u) = \int_{-\infty}^{\infty} \frac{1}{2} u^2 \, \mathrm{d}x \tag{1.3}$$

and

$$E(u) = \int_{-\infty}^{\infty} \left\{ -\frac{1}{6}u^3 - \frac{\alpha}{12}u^4 + \frac{1}{2}u_x^2 \right\} \mathrm{d}x. \tag{1.4}$$

In fact, the equation has an infinite number of conserved integrals, but only $V(u)$ and $E(u)$ are needed for our analysis. The conservation of these two integrals in connection with local well-posedness yields global well-posedness in the space $C(0, T, H^1)$ of continuous functions of time t with values in $H^1(\mathbb{R})$, for any fixed T. The global existence of solutions in $C(0, \infty, H^1)$ also follows from standard arguments. The main contribution of this paper is the following theorem.

Theorem 1. *All solitary-wave solutions Φ_c of (1.1) with wave speed $c > 0$ are stable.*

The plan of the paper is as follows. In section 2, some modeling issues regarding (1.1) will be recalled. Then in section 3, it will be shown that all solitary-wave solutions of (1.1) are in fact given by the explicit formula (1.2). The notion of orbital stability will be reviewed in section 4, and a short account of the general theory of Grillakis *et al* in [19] for proving orbital stability will be given. Finally, the application of this theory to the situation at hand will be presented in section 5. Section 6 contains a brief conclusion.

2. Modeling considerations

Equation (1.1) can be derived as a model for interfacial waves in a two-fluid system, as well as in the case of horizontally propagating internal waves in a continuous density stratification. The derivation of the equation depends crucially on the assumptions of small amplitude, long wavelength and one-way propagation. In dimensional variables, the equation takes the form

$$\eta_\tau + c_0\eta_X + \alpha_1\eta\eta_X + \alpha_2\eta^2\eta_X + \beta_1\eta_{XXX} = 0. \tag{2.1}$$

In the two-layer case, the unknown function $\eta(X, \tau)$ is the deflection of the interface from its rest position, while in the case of continuous stratification, $\eta(X, \tau)$ is the isopycnal vertical displacement, i.e. the displacement of a line of constant density. The independent spatial variable X increases in the direction of wave propagation and τ denotes time. For a given stratification, it is important to use the correct values for the parameters c_0, α_1, α_2 and β_1. In the two-layer case, these parameters are well known [2, 11, 15, 24, 30] and are given by

$$c_0^2 = gh_1h_2\frac{\rho_1 - \rho_2}{\rho_2h_1 + \rho_1h_2}, \tag{2.2}$$

$$\alpha_1 = \frac{3}{2}c_0\frac{1}{h_1h_2}\frac{\rho_1h_2^2 - \rho_2h_1^2}{h_1\rho_2 + h_2\rho_1}, \tag{2.3}$$

$$\alpha_2 = \frac{3c_0}{(h_1h_2)^2}\left[\frac{7}{8}\left(\frac{\rho_1h_2^2 - \rho_2h_1^2}{\rho_1h_2 + \rho_2h_1}\right)^2 - \frac{\rho_1h_2^3 + \rho_2h_1^3}{\rho_1h_2 + \rho_2h_1}\right], \tag{2.4}$$

$$\beta_1 = \frac{c_0}{6}h_1h_2\frac{\rho_1h_1 + \rho_2h_2}{h_1\rho_2 + h_2\rho_1}. \tag{2.5}$$

Here, g is the gravitational acceleration, ρ_1 and ρ_2 are the densities of the lower and upper layers, respectively, and the depths of the lower and upper layers in the rest position are given by h_1 and h_2, respectively. Computation of the coefficients in the case of a continuous stratification is somewhat more complicated. An account of such a calculation may be found in [2, 20, 21, 33, 37].

When α_2 is set to zero, equation (2.1) reduces to the KdV equation. Both equations are valid model equations for internal waves propagating to the right (in the direction of increasing the values of X) in the case when the depths h_1 and h_2 are comparable. Suppose both h_1 and h_2 are similar to a mean height H, a is a measure of a typical wave amplitude and λ is a measure of a typical wavelength. Then the KdV equation arises from the assumption that the quantities a/H and $(H/\lambda)^2$ are small and of comparable magnitude, i.e. $(H/\lambda)^2 = \mathcal{O}(a/H) \ll 1$. Given this assumption, it can be seen immediately that the terms $\alpha_1\eta\eta_X$ and $\beta_1\eta_{XXX}$ are the small corrections to the first-order relation

$$\eta_\tau + c_0\eta_X = 0, \tag{2.6}$$

if the coefficients α_1 and β_1 are of order 1 or smaller. On the other hand, in the eKdV equation (2.1), the quantities a/H and H/λ have comparable small magnitude, i.e. $H/\lambda = \mathcal{O}(a/H) \ll 1$. Indeed, writing the cubic term in (2.1) as

$$\eta^2\eta_X \sim (a/H)^3\frac{1}{\lambda/H},$$

and the dispersive term as

$$\eta_{XXX} \sim (a/H)\frac{1}{(\lambda/H)^3},$$

it becomes plain that these terms are balanced if $a/H \sim H/\lambda$. Thus, if the scaling $a/H \sim H/\lambda$ is assumed, then the terms $\alpha_2\eta^2\eta_X$ and $\beta_1\eta_{XXX}$ are the small corrections to the first-order relation (2.6). However, now a further assumption is necessary to ensure that the quadratic term $\alpha_1\eta\eta_X$ in (2.1) is a correction of the same order. Indeed, it becomes plain immediately that we need $\alpha_1 \ll 1$. Examining the coefficient α_1 given in (2.3), it appears that α_1 will be small if $\rho_1 h_1^2 - \rho_2 h_2^2$ is small. This means that the eKdV equation is a valid approximation for internal waves so long as $\rho_1 h_1^2 \sim \rho_2 h_2^2$. As variations in density are typically very small in stratified fluids, only a few percent at most, we see that the eKdV equation can be used as a model for small-amplitude long internal waves in a two-layer stratified fluid in which the two layers have approximately the same depth.

Converting the equation into the non-dimensional form, the cubic coefficient $\alpha = \alpha_2/\alpha_1^2$ appears. This coefficient can be either positive or negative depending on the sign of α_2. However, since we need $\rho_1 h_1^2 \sim \rho_2 h_2^2$ for the validity of the equation, in most situations where the equation is an appropriate model, a quick look at (2.4) shows that α_2 will typically be negative. In the KdV approximation, the non-dimensionalization shows that the polarity of the solitary waves is determined by the coefficient α_1. Moreover, the wavelength decreases with increasing amplitude, and does not capture the broadening of waves that is often observed in the field [24]. In contrast, eKdV solitary waves capture this shape when $\alpha_2 < 0$. Initially, the wavelength also narrows with increasing amplitude. However, as the upper limit is approached, the waves begin to broaden again until the wave of maximum amplitude $A_{\max} = -1/\alpha$ is reached. To illustrate this discussion, a number of solitary waves with different amplitudes are displayed in figure 1 for the cases of KdV ($\alpha_2 = 0$) and eKdV ($\alpha_2 = -1$). Finally, note that in the eKdV equation, the polarity of the solitary wave is dependent on the signs of both α_1 and α_2, thus affording a finer analysis.

3. Solitary-wave solutions

In order to prove the stability of solitary waves, it is convenient to study the eKdV equation in the non-dimensional form (1.1) given in the introduction. The purpose of this section is

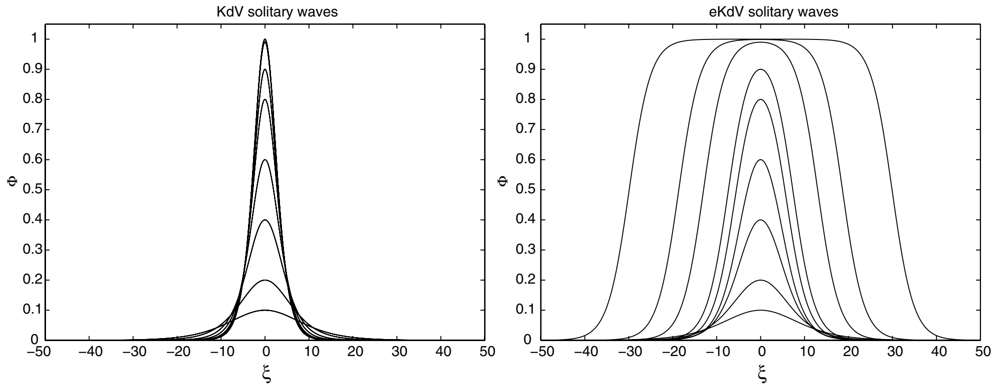

Figure 1. A comparison of KdV and eKdV solitary waves with non-dimensional amplitudes ranging from 0.1 to 0.999 99. Left panel: KdV solitary waves. Right panel: eKdV solitary waves.

to analyze the functions given by (1.2), and to show that all solitary waves are given by this formula.

By a solitary wave, we mean a progressive wave which propagates to the right without changing its profile over time. In mathematical terms, a solitary wave is a solution $u(x,t) = \Phi_c(x - ct)$, where $\xi = x - ct$, and $\Phi_c(\xi)$ is a bounded continuous function with a single maximum or a single minimum, and which decays to zero as $\xi \to \pm\infty$. The transformation to non-dimensional variables used in section 2 shows that wave speeds $c > 0$ correspond to supercritical wave velocities, i.e. to solitary waves that propagate faster than any linear wave. As shown in proposition 3.1, equation (1.1) does not admit subcritical solitary waves.

When the ansatz $u(x,t) = \Phi_c(x - ct)$ is substituted into (1.1), there appears the ordinary differential equation

$$-c\Phi_c' + \Phi_c\Phi_c' + \alpha\Phi_c^2\Phi_c' + \Phi_c''' = 0, \tag{3.1}$$

where $\Phi_c' = \frac{d\Phi_c}{d\xi}$, for $\xi = x - ct$. Integrating once yields

$$-c\Phi_c + \frac{1}{2}\Phi_c^2 + \frac{\alpha}{3}\Phi_c^3 + \Phi_c'' = 0. \tag{3.2}$$

Note that the constant of integration has been set to zero because decay to zero at infinity is assumed. A solution of (3.2) can be written in the form (1.2)

$$\Phi_c(\xi) = \frac{A}{b + (1-b)\cosh^2 K\xi},$$

with the amplitude given by $A = A^+$ or $A = A^-$, where

$$A^+ = \frac{-1 + \sqrt{1 + 6c\alpha}}{\alpha}, \tag{3.3}$$

$$A^- = \frac{-1 - \sqrt{1 + 6c\alpha}}{\alpha}. \tag{3.4}$$

The parameters K and b are given by

$$K = \frac{\sqrt{c}}{2} \quad \text{and} \quad b = -\frac{\alpha A^2}{6c}. \tag{3.5}$$

As can be gleaned from the expressions above, this formula is valid only if $c > 0$ and $1 + 6c\alpha > 0$. In this case, it can be checked by differentiation that the functions defined by (1.2) and (3.3)–(3.5) are the solutions of (3.2). What is more, the following results show that all solitary-wave solutions of (1.1) are given by this formula. First, however, let us show that there are no solitary waves with wave speeds $c < 0$.

Lemma 1. *For any $\alpha \in \mathbb{R}$, there is no solitary wave with wave speed $c < 0$.*

Proof. Suppose that there is such a solitary-wave solution. By assumption, this function has a single maximum or minimum, and decays to zero as $x \to \pm\infty$. Multiply (3.2) by Φ_c' and integrate to obtain the equation

$$\left(\Phi_c'\right)^2 = c\,\Phi_c^2 - \frac{1}{3}\Phi_c^3 - \frac{\alpha}{6}\Phi_c^4. \tag{3.6}$$

By the supposed decay at infinity, choosing x large enough will make the cubic and quartic terms much smaller than the quadratic term. Let us say that x is chosen so large that $|\frac{1}{3}\Phi_c^3(x)| + |\frac{\alpha}{6}\Phi_c^4(x)| \leqslant \frac{1}{2}|c\Phi_c^2(x)|$. Then the left-hand side of (3.6) is positive while the right-hand side is negative. Since this is impossible, we conclude that no solitary wave exists if $c < 0$. □

Since negative wave speeds are now ruled out, attention is focused on the case $c > 0$.

Lemma 2. *If $\alpha < 0$, then all solitary-wave solutions of (1.1) must be non-negative.*

Proof. Suppose Φ_c is a solution of (3.2) which decays to zero as $x \to \pm\infty$. Note that equation (3.2) shows that Φ_c is smooth. If Φ_c is negative somewhere, then it must have a minimum $\phi_m = \Phi_c(x_m) < 0$. Since Φ_c is smooth, the first derivative of Φ_c is zero at the minimum, so that we obtain the equation

$$0 = \phi_m^2 \left\{ c - \frac{1}{3}\phi_m - \frac{\alpha}{6}\phi_m^2 \right\}$$

from (3.6). Now it appears that either $\phi_m = 0$, in which case $\Phi_c \geqslant 0$, or $c = \frac{1}{3}\phi_m + \frac{\alpha}{6}\phi_m^2$. But we assumed that $c > 0$, while the right-hand side of the equation is negative since ϕ_m is negative, and α is negative as well. Since this is impossible, Φ_c must be positive or zero. □

The proof of the lemma 2 shows in particular that if $\alpha < 0$, then there exist no solitary-wave solutions of (1.1) which are strictly negative. The next lemma further restricts the range of possible solitary waves in the case when $\alpha < 0$.

Lemma 3. *Let $\alpha < 0$. There are no solitary-wave solutions of (1.1) if $c \geqslant -1/6\alpha$.*

Proof. Suppose Φ_c is a solution of (3.2), which decays to zero as $x \to \pm\infty$. We know from the previous lemma that Φ_c must be non-negative. Multiply (3.2) by Φ_c' and integrate to obtain equation (3.6) as above. Now if Φ_c is not identically zero, then it must have a maximum $\phi_M = \Phi_c(x_M)$. Since Φ_c is smooth, the first derivative of Φ_c is zero at the maximum, so that we obtain the equation

$$0 = \phi_M^2 \left\{ c - \frac{1}{3}\phi_M - \frac{\alpha}{6}\phi_M^2 \right\}.$$

If $\phi_M = 0$, then Φ_c is the trivial solution. Hence we must have $c - \frac{1}{3}\phi_M - \frac{\alpha}{6}\phi_M^2 = 0$. Now it appears that this quadratic equation for ϕ_M has solutions

$$-\frac{1}{\alpha}\left(1 - \sqrt{1 + 6c\alpha}\right) \quad \text{and} \quad -\frac{1}{\alpha}\left(1 + \sqrt{1 + 6c\alpha}\right).$$

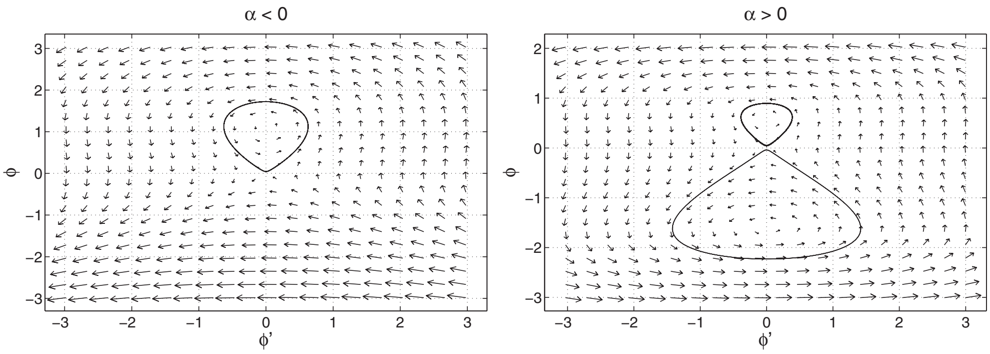

Figure 2. Phase plane diagram for solutions of (3.2). In the left panel, $\alpha < 0$, and $c = \frac{1}{2}$. There is only one homoclinic orbit. In the right panel, $\alpha > 0$, and $c = \frac{1}{2}$. There are two homoclinic orbits.

These solutions are real-valued only if $1 + 6c\alpha \geqslant 0$. Thus, Φ_c can be bounded only if $c \leqslant -1/6\alpha$. Thus, we conclude that we must have $1 + 6c\alpha > 0$ for a non-trivial solitary wave. □

The properties of solitary waves in the cases $\alpha < 0$ and $\alpha > 0$ are summarized in the following two propositions.

Proposition 1. *Let $\alpha < 0$. Non-trivial solitary-wave solutions of (1.1) exist for c in the range $0 < c < -1/6\alpha$. The solitary waves are given by formula (1.2) with amplitude $A^+ > 0$ given by (3.3), and K and b given by (3.5).*

Proof. Solitary waves with $c < 0$ are ruled out by lemma 1. On the other hand, lemma 3 states that there are solitary waves only for those positive velocities c satisfying $c \leqslant -1/6\alpha$. By lemma 2 these solitary waves must be non-negative functions. Now for given α and c, a phase plane analysis of the pair (Φ_c, Φ_c') (the phase plane for a particular case is shown on the left in figure 2) shows that there is a saddle point at the origin. The existence and uniqueness theorem for ordinary differential equations [25] guarantees the existence of a unique maximal solution connecting the unstable and stable directions of the saddle point. Since the function defined by (1.2) with $A = A^+$ satisfies equation (3.2), it is this unique solution. In the case $c = 0$, the solitary waves is identically zero, while in the case $c = -1/6\alpha$, the solution is given by a non-zero constant, which does not constitute a solitary wave according to our definition, since it does not decay to zero as $x \to \pm\infty$. □

Remark 1. With regard to the parameter b, note that from the restriction on c just expounded, an analysis of formula (3.5) shows that b lies between $0 < b < 1$ if $\alpha < 0$.

Proposition 2. *Let $\alpha > 0$. Non-trivial solitary-wave solutions of (1.1) exist for c in the range $0 \leqslant c < \infty$. For a given $c \geqslant 0$, there is a positive and a negative solitary wave. These solitary-wave solutions are given by formulas (1.2) and (3.5) with amplitude $A^+ > 0$ given by (3.3) and amplitude $A^- < 0$ given by (3.4).*

Proof. Again, solitary waves with $c < 0$ are ruled out by lemma 1. As in the proof of proposition 1, a phase plane analysis of the pair (Φ_c, Φ_c') shows that there is a saddle point at the origin. Since solutions Φ_c may now be either positive or negative, the existence and uniqueness theorem for ordinary differential equations [25] guarantees the existence of a

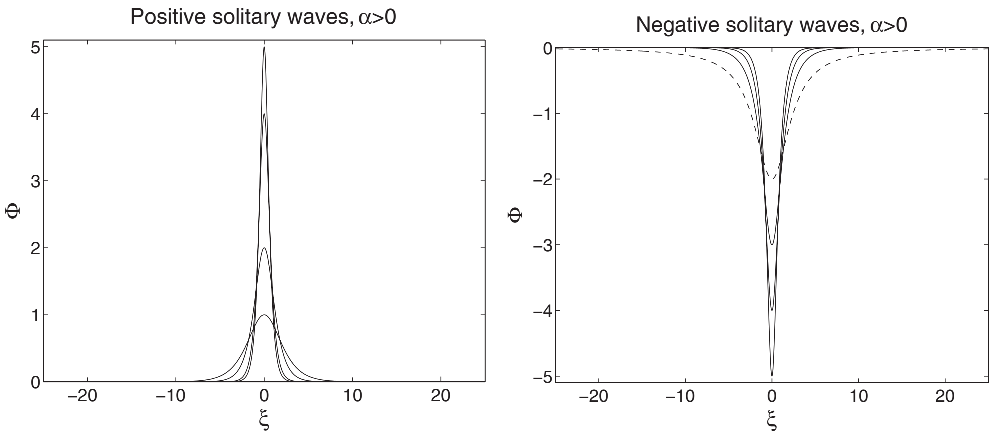

Figure 3. Comparison of positive and negative solitary waves in the case of positive α. The left panel shows a family of solitary waves with $A = A^+$, while the right panel shows a family of solitary waves with $A = A^-$.

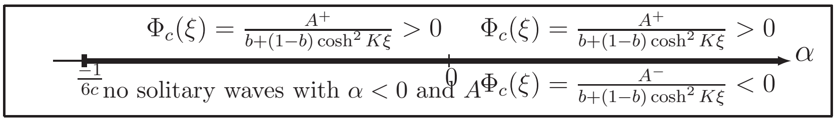

Figure 4. eKdV solitary waves. Here, $\xi = x - ct$ and $K = \frac{1}{2}\sqrt{c}$ for a wave speed $c > 0$. The wave amplitude A is equal to either A^+ or A^-.

unique positive and a negative maximal solution connecting the unstable and stable directions of the saddle point. Since it can be checked that (1.2) provides a solution of (3.2) for both $A = A^+$ and $A = A^-$, we see that the solitary waves are given by (1.2). For $c = 0$, there is the trivial wave $\phi = 0$, and the algebraic solitary wave $\phi = -48/(24\alpha + \alpha^2 A^2 x^2)$ obtained by letting $c \to 0$ in the family of functions with $A = A^-$ □

Remark 2. As seen from (3.5), the parameter b is strictly negative in the case $\alpha > 0$.

Figure 3 shows a family of positive and negative waves in the case $\alpha > 0$. We note that the negative solitary waves are limited *below* in amplitude, and the dashed line in the right panel of figure 3 indicates the limiting algebraic soliton.

The dependence of negative and positive solitary waves on the parameters α and c are summarized in figures 4 and 5.

4. Review of stability theory

In this section, a short review of the concept of orbital stability is given, and the main points of the general theory of stability for solitary-wave solutions of partial differential equations are sketched. As already observed by Benjamin and others [1, 4, 5, 8], a solitary wave cannot be stable in the strictest sense of the word. To understand this, consider two solitary waves of different heights, centered initially at the same point. Since the two waves have different amplitudes, they also have different velocities according to the formulas (3.3) and (3.4). As

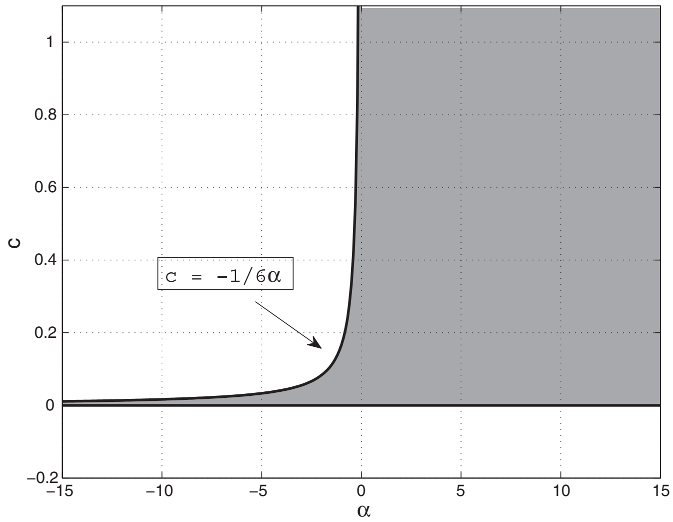

Figure 5. Relation between the positive speed c and the cubic nonlinear coefficient α. The shaded region indicates the parameter space for which solitary-wave solutions exist.

time passes the two waves will drift apart, no matter how small the initial difference was. However, in the situation just described, it is evident that two solitary waves with slightly differing heights will keep their original shape. Thus, if the variance in the translation in space is disregarded, the two waves will stay close. Factoring out the space translation will therefore give an acceptable notion of stability. This sense of orbital stability was introduced by Benjamin [4]. We say that the solitary wave is orbitally stable, if a solution u of equation (1.1) that is initially sufficiently close to a solitary wave will always stay close to a translation of the solitary wave during the time evolution. A mathematically precise definition is as follows. For a given solitary wave Φ_c and $\varepsilon > 0$, consider the set

$$U_\varepsilon = \{u \in H^1 : \inf_s \|u - \tau_s \Phi_c\|_{H^1} < \varepsilon\},$$

where $\tau_s \Phi_c(x) = \Phi_c(x - s)$ is a translation of Φ_c. Evidently, the set U_ε is an ε-neighborhood of the collection of all translates of Φ_c.

Definition 1. *The solitary wave Φ_c is 'stable' if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $u_0 = u(\cdot, 0) \in U_\delta$, then $u(\cdot, t) \in U_\varepsilon$, for all $t \in \mathbb{R}$.*

The stability theory of [19] applies to the situation of an evolutionary partial differential equation of the form

$$u_t = J E'(u), \tag{4.1}$$

where J is a skew-symmetric operator. Note that the case at hand, the operator $J = \partial_x$ and E is given by (1.4). The assumptions on J in [19] include that J be surjective, which is not the case for $J = \partial_x$. However, this property is only needed for the proof of instability [9], as is the assumption that the solitary waves are strictly positive [27]. Since the equation is given in the form (4.1), it can easily be shown that the functional $E(u)$ is time invariant. To see this, consider the following computation:

$$\frac{\mathrm{d}E(u)}{\mathrm{d}t} = \left\langle E'(u), \frac{\mathrm{d}u}{\mathrm{d}t} \right\rangle = \langle E'(u), \partial_x E'(u) \rangle = 0.$$

Now suppose there is another time-invariant functional $V(u)$, and that the solitary-wave solutions of (4.1) satisfy the equation

$$E'(\Phi_c) + cV'(\Phi_c) = 0.$$

Then the orbital stability of Φ_c can be determined if the operator

$$\mathcal{L}_c = E''(\Phi_c) + cV''(\Phi_c)$$

has only one simple negative eigenvalue, one simple zero eigenvalue and continuous positive spectrum bounded away from zero. As shown in [19], zero is always an eigenvalue with the eigenfunction Φ_c'. Now if the scalar function $d(c) = E(\Phi_c) + cV(\Phi_c)$ is convex at a certain value of c, then it follows that the corresponding solitary wave is orbitally stable. The proof entails finding a positive constant κ such that the conditional coercivity of the bilinear form $\langle \mathcal{L}_c y, y\rangle \geqslant \kappa \|y\|_{H^1}^2$ holds for all nonzero $y \in H^1(\mathbb{R})$ which are orthogonal in $L^2(\mathbb{R})$ to both $V'(\Phi_c)$ and Φ_c'. Then it can be shown that the functional $E(u)$ attains a local minimum at Φ_c subject to the constraint that $V(u)$ be constant. The latter condition can be removed by an additional scaling argument.

In view of the above, to prove the unconditional orbital stability of the solitary waves of (1.1), it is only necessary to obtain some spectral information about $\mathcal{L}_c = E''(\Phi_c) + cV''(\Phi_c)$ and to establish that the function $d(c) = E(\Phi_c) + cV(\Phi_c)$ is convex. This will be done in the next section.

5. Proof of stability

We will show that the problem falls within the framework of the general theory of Grillakis *et al* [19], which was reviewed in the previous section. First, observe that, in term of the functionals E and V introduced in the introduction, equation (3.2) can be written in the variational form as

$$E'(\Phi_c) + cV'(\Phi_c) = 0, \tag{5.1}$$

where $E'(\Phi_c) = -\frac{1}{2}\Phi_c^2 - \frac{\alpha}{3}\Phi_c^3 - \Phi_c''$ and $V'(\Phi_c) = \Phi_c$ are the Fréchet derivatives at Φ_c of E and V, respectively. The functional derivative of $E'(\Phi_c) + cV'(\Phi_c)$ is given by the self-adjoint unbounded linear operator

$$\mathcal{L}_c = E''(\Phi_c) + cV''(\Phi_c) = -\partial_x^2 + c - \Phi_c - \alpha\Phi_c^2.$$

It is elementary to check that $\mathcal{L}_c : H^2 \to L^2$ is self-adjoint with respect to the L^2-inner product. Furthermore, since the potential $c - \Phi_c - \alpha\Phi_c^2$ approaches the asymptotic value c exponentially as $x \to \pm\infty$, it is well known that $\mathcal{L}_c$ has essential spectrum $[c, \infty)$ and there are at most finitely many eigenvalues located to the left of c. Moreover, the nth eigenvalue (counted from the left) has a unique eigenfunction (up to a constant multiple) with precisely $(n-1)$ zeros [17]. However, since $\mathcal{L}_c\Phi_c' = 0$ as can be seen by equation (3.1) and Φ_c' has exactly one zero it is immediate that zero is an eigenvalue of $\mathcal{L}_c$, and there is exactly one negative simple eigenvalue. Under these circumstances, the general theory provided by [19] can be applied, and the stability of Φ_c is determined by the convexity of the scalar function

$$d(c) = E(\Phi_c) + cV(\Phi_c).$$

Differentiating $d(c)$ and taking equation (5.1) into account, we find

$$d'(c) = \frac{1}{2}\int_{-\infty}^{\infty} \Phi_c^2 \,\mathrm{d}\xi = \frac{1}{2}\int_{-\infty}^{\infty} \left\{\frac{A}{b + (1-b)\cosh^2 K\xi}\right\}^2 \mathrm{d}\xi.$$

Since the integrand is an even function, it can be seen that

$$d'(c) = \frac{A^2}{K}\int_0^{\infty} \left\{\frac{1}{b + (1-b)\cosh^2 z}\right\}^2 \mathrm{d}z.$$

Since $b < 1$ by the discussion in section 3, the change of variables $y = b + (1 - b)\cosh^2 z$ yields the simpler integral

$$d'(c) = \frac{A^2}{2K}\int_1^\infty \frac{1}{y^2\sqrt{(y-b)(y-1)}}\,\mathrm{d}y.$$

Since $b < 0$ for $\alpha > 0$ and $b > 0$ for $\alpha < 0$ (cf. (3.5)), rewrite this integral as

$$d'(c) = \begin{cases} \dfrac{A^2}{2K}\displaystyle\int_1^\infty \frac{1}{y^2\sqrt{(y-b)(y-1)}}\,\mathrm{d}y, & \text{if } \alpha < 0 \\ \dfrac{A^2}{2K}\displaystyle\int_1^\infty \frac{1}{y^2\sqrt{(y+|b|)(y-1)}}\,\mathrm{d}y, & \text{if } \alpha > 0. \end{cases}$$

Once these integrals are evaluated, it is revealed that

$$d'(c) = \begin{cases} \dfrac{A^2}{4K}\left\{(b^{-3/2} + b^{-1/2})\ln\dfrac{1+\sqrt{b}}{1-\sqrt{b}} - \dfrac{2}{b}\right\}, & \text{if } \alpha < 0, \\ \dfrac{A^2}{8K}\left\{(-|b|^{-3/2} + |b|^{-1/2})\left(\pi + 2\arctan\dfrac{-1+|b|}{2\sqrt{|b|}}\right) + \dfrac{4}{|b|}\right\}, & \text{if } \alpha > 0. \end{cases}$$

Taking the definition of A, K and b in equations (3.3) or (3.4) and (3.5), respectively, into account, and then differentiating this expression with respect to c yields

$$d''(c) = \frac{72\sqrt{c}\{2 + 12c\alpha + 9c^2\alpha^2 \mp (2 + 6c\alpha)\sqrt{1+6c\alpha}\}}{\sqrt{1+6c\alpha}(1 + 6c\alpha \mp \sqrt{1+6c\alpha})(\mp 1 + \sqrt{1+6c\alpha})^3} \quad \text{for } A = A^{\pm}.$$

Verification of the positivity of this second derivative falls into three cases. First of all, it is obvious that $d''(c)$ is positive for $A = A^-$ in the case when $\alpha > 0$. On the other hand, for $A = A^+$, and $\alpha > 0$, the positivity of both the numerator and the denominator can also be verified by elementary algebra. Finally, if $\alpha < 0$, it can again be shown that the expression is positive as long as $\alpha > -1/6c$ and $c > 0$.

6. Conclusion

As shown by the theorem proved in this paper, solitary-wave solutions of the eKdV equation are nonlinearly stable with respect to small but finite-amplitude perturbations. Thus, these waves are observable on the same level of description on which the equation is valid. Since model (1.1) does not feature viscous or diffusive effects, stability of its solitary-wave solution means that the model is inherently consistent, as neither viscosity nor diffusion are needed to guarantee the stability of its solitary-wave solutions.

In the absence of changes in bathymetry or variations in the background stratification, internal solitary waves appear to be remarkably stable in nature. Indeed, internal solitary waves travel a great distance at sea, and also appear to be exceedingly stable in wave tank experiments. As mentioned in the introduction, this stability does not depend on viscous or diffusive effects, but is a consequence of the balance of nonlinearity and dispersion, which is indeed captured by the model (1.1).

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