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Models for Interfacial Capillary-Gravity Waves in the Long-Wave Limit

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Abstract: A matched asymptotic expansion is used to give a formal derivation of a system of partial differential equations modeling the evolution of the free interface in a two-fluid system. Under the assumption of one-way propagation, the system is reduced to a single equation.

INTRODUCTION

The object of this note is the systematic derivation of a number of model equations which are of use in the description of the evolution of long-crested internal waves in two-fluid systems. The equations put forward here are a system of two coupled evolution equations, similar to the Boussinesq equations describing waves at the surface of a fluid. The equations have the form

$$\begin{aligned} w_t + \eta_x + \sigma\, w w_x - \epsilon \frac{\rho_2}{\rho_1} \mathcal{T}_1 \eta_{xx} - \mu\, \eta_{xxx} &= 0, \\ \eta_t + w_x + \sigma\, (w\eta)_x &= 0. \end{aligned}$$

The quantities appearing in the equations will be defined momentarily. For now, note that μ is a parameter related to the interfacial tension and $\mathcal{T}_1$ is an integral operator. As it will turn out, this model is approximately valid for long interfacial waves, if one of the layers is thin, the other is deep, and the interface is subject to capillarity. Similar model equations have been obtained by Choi and Camassa (1996, 1999), Ostrovsky and Grue (2003), and Craig, Guyenne and Kalisch (2004). However, in these treatments capillarity was neglected. Since many of these last-mentioned model equations are ill-posed, i.e. unstable with respect to small perturbations, capillarity can be seen as a way to stabilize these equations. In fact, physical considerations indicate that capillarity should be included in the model if a sharp interface between two fluids is present, and this is the point of view taken here. The scientific interest in two-layer flows with a sharp interface originates from wave tank experiments as described in Koop and Butler (1981) and industrial coating processes (see Singh and Joseph).

There have been previous studies of interfacial capillary-gravity waves, mostly in the context of the full Euler equations. For example, the papers by Sha and Vanden-Broeck (1997) and Laget and Dias (1997) contain studies leaning on numerical approximation of steady waves of the Euler equations and bifurcation analysis. The work of Christodoulides and

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Dias (1995), contains a normal form analysis and a study of stability of steady waves via a Davey-Stewartson-type model. In fact, steady waves are a topic of special interest even in the case when capillarity is neglected. In the present situation, one of the best known equations admitting steady solutions is the so-called intermediate long-wave equation, as derived by Kubota, Ko and Dobbs (1979). This equation may be used if the principal direction of propagation of the waves to be studied is in one direction only. If one-way propagation is assumed in the situation under study in this article, the model equation

$$\eta_t + \eta_x + \sigma \frac{3}{2}\eta\eta_x - \epsilon \frac{1}{2}\frac{\rho_2}{\rho_1}\mathcal{T}_1\eta_{xx} - \frac{\mu}{2}\eta_{xxx} = 0. \qquad (1)$$

arises. This equation is similar to the so-called Benjamin equation which has been found as a model for interfacial capillary-gravity waves if one of the layers is extremely deep (Benjamin, 1992). If interfacial tension is negligible, the parameter μ is zero, and the equation (1) in fact reduces to the intermediate long-wave equation. In its full generality, the intermediate long-wave equation is valid for two-layer flows with arbitrary depths. Here we restrict to the situation when one of the two layers is thin when compared to the wavelength of a typical wave.

The next section is devoted to the description of the basic equations governing the fluid motion and the discussion of the linear dispersion relation in the situation at hand. Thereafter, the derivation of the evolution equations is given, and in the last section the evolution of essentially one-directional waves is treated.

THE DISPERSION RELATION

Attention is given to long-crested internal waves in a two-layer system of stably stratified fluids. The fluid in the lower layer has density ρ_1, while the one in the upper layer has density ρ_2, where it is assumed that $\rho_2 < \rho_1$. At rest, the depth of the lower layer is h_1, measured from the flat bottom located at $z = 0$. The depth of the upper layer is given by h_2, and the fluid is bounded above by a rigid lid at $z = h_2$. The case of a coupled surface-interface model is not part of our analysis here. We consider the case of a thin lower layer, and a deep upper layer. In other words, h_1 is assumed to be much smaller than h_2. The geometry of the system under study has been sketched in Figure 1. The case of a deep lower layer and thin upper layer can be treated by the change of variables

$$g \to -g, \qquad \eta \to -\eta, \qquad \rho_1 \leftrightarrow \rho_2.$$

Continuing the development, it is further assumed that the interface between the fluids is subject to interfacial tension proportional to its curvature with a proportionality constant T. Moreover, it is assumed that the interface between the two layers can be described by a function $\eta(x,t)$, measuring the deflection of the interface from its rest position. When viscosity is neglected, the basic continuum model is given by a pair of Euler equations coupled through boundary conditions at the common free interface. Irrotationality is also assumed, so that the Euler equations can be written in terms of velocity potentials ψ in the upper layer and ϕ in the lower layer. The equations take the form

$$\begin{aligned} \triangle\psi = 0 \quad &\text{in} \quad \eta < z < h_2, \\ \psi_z = 0 \quad &\text{at} \quad z = h_2, \end{aligned} \qquad (2)$$

$$\begin{aligned} \triangle\phi &= 0 \quad \text{in} \quad -h_1 < z < \eta, \\ \phi_z &= 0 \quad \text{at} \quad z = -h_1. \end{aligned} \tag{3}$$

At the interface, the following boundary conditions are in force:

$$\left.\begin{aligned} \eta_t + \psi_x \eta_x &= \psi_z \quad (4) \\ \eta_t + \phi_x \eta_x &= \phi_z \quad (5) \\ p_1 - p_2 &= -T\,\eta_{xx} \quad (6) \end{aligned}\right\} \text{ at } z = \eta.$$

Note that it is assumed that the waves are long-crested. This means that there is no significant variation in the direction transverse to the propagation of the waves, so that y-derivatives may be safely ignored. In particular, the Laplace operator in equations (2) and (3) is two-dimensional. The boundary conditions (4) and (5) are known as kinematic boundary conditions. The relation (6), involving the pressures p_2 and p_1 in the upper and lower layers, respectively, is called the dynamic boundary condition. This condition is posed under the assumption that the slope of the free interface varies slowly, as the second derivative is then a good approximation to the curvature of the interface. Since the focus in this work is on long waves, this is certainly a reasonable assumption. Using Bernoulli's law

$$\frac{p}{\rho} = -\phi_t - \frac{1}{2}(\phi_x^2 + \phi_z^2) - gz$$

in each layer, the boundary condition (6) may be rewritten as

$$\rho_1 \left(g\eta + \phi_t + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_z^2 \right) - \rho_2 \left(g\eta + \psi_t + \frac{1}{2}\psi_x^2 + \frac{1}{2}\psi_z^2 \right) = T\eta_{xx}. \tag{7}$$

If the equations (2) - (6) are linearized, the boundary conditions are applied on $z = 0$, and oscillatory solutions of the form $e^{ikx - i\omega t}$ are sought, then the dispersion relation

$$\omega^2(k) = k \frac{(\rho_1 - \rho_2)g + Tk^2}{\rho_1 \coth(kh_1) + \rho_2 \coth(kh_2)} \tag{8}$$

with phase speed

$$c^2(k) = \frac{\omega^2}{k^2} = \frac{(\rho_1 - \rho_2)g + Tk^2}{\rho_1\, k \coth(kh_1) + \rho_2\, k \coth(kh_2)}$$

may be found. Since the lower layer is assumed thin, h_1 is small, and at least for long waves, kh_1 is sufficiently small so that $kh_1 \coth(kh_1)$ can be developed in a Taylor series. After an asymptotic expansion in k, the square of the phase speed can be expressed as

$$c^2(k) = gh_1 \left(1 - \frac{\rho_2}{\rho_1}\right) \left[1 - \frac{\rho_2}{\rho_1} h_1 k \coth(kh_2) + \left\{ \frac{T}{g(\rho_1 - \rho_2)} + \left(\frac{{\rho_2}^2}{{\rho_1}^2} \coth^2(kh_2) - \frac{1}{3} \right) {h_1}^2 \right\} k^2 \right],$$

where terms of order $|k|^3$ and higher have been neglected. For long waves, $|k|^2$ is sufficiently smaller than $|k|$ so that the even the terms of order $|k|^2$ can be omitted. However, if it is assumed that $\frac{Tk^2}{g(\rho_2 - \rho_1)} \sim h_1|k|$, then capillarity should be included, and the corresponding linear differential equation is

$$\eta_{tt} - gh_1 \left(1 - \frac{\rho_2}{\rho_1}\right) \left[\eta_{xx} - h_1 \frac{\rho_2}{\rho_1} \mathcal{T}_{h_2} \eta_{xxx} - \frac{T}{g(\rho_1 - \rho_2)} \eta_{xxxx} \right] = 0, \tag{9}$$

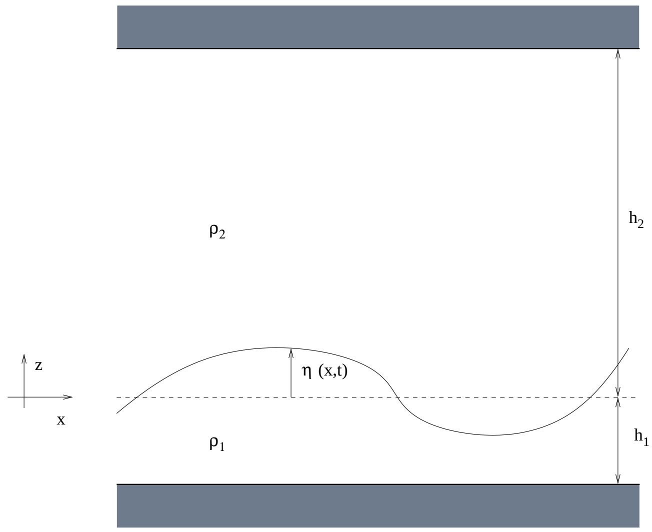

Fig. 1. A two-fluid system. Note that the height h_2 of the upper layer is approximately the same as the wavelength of a typical wave.

where $\mathcal{T}_{h_2}$ is the operator defined by

$$\mathcal{T}_{h_2} f(x) = \frac{1}{2h_2} \int_{-\infty}^{\infty} \coth\left(\frac{\pi}{2h_2} y\right) f(x-y)\, dy,$$

and the integral kernel is obtained by taking the inverse Fourier transform of the multiplier $-i\coth(kh_2)$.

A SYSTEM OF EVOLUTION EQUATIONS

Equation (9) is valid only for waves with infinitesimally small amplitude. To allow for the description of a wider range of waves, finite-amplitude effects must be considered. The goal is to obtain an equation which provides a good model for waves in a regime where nonlinear and dispersive effects are balanced. The usual point of view is to consider dispersive and nonlinear effects as small corrections to the basic linear hyperbolic system

$$\begin{aligned} w'_{t'} + C_0^2\, \eta'_{x'} &= 0, \\ \eta'_{t'} + w'_{x'} &= 0, \end{aligned} \tag{10}$$

with wavespeed C_0. Here $C_0 = c_0 v_0$, where $c_0 = \sqrt{gh_1}$ is the limiting long-wave speed for surface waves on homogeneous-layer flows, i.e. with $\rho_2 = 0$. The dimensionless quantity $v_0 = \sqrt{1 - \frac{\rho_2}{\rho_1}}$ is a correction in the case at hand, namely if $\rho_2 \neq 0$. The function w is related to the horizontal velocity of the fluid at the free interface. The system is written in terms of physical variables which are henceforth written with a prime.

As mentioned in the introduction, the significant quantities are the amplitude a and the wavelength λ of a typical wave. To quantify the meaning of 'small' and 'long', the depth h_1 of the lower layer is selected as a natural spatial scale. We then introduce the scaling factors $\sigma = \frac{a}{h_1}$ and $\epsilon = \frac{h_1}{h_2}$ which are assumed to be small and of the same order. Moreover, it is assumed that the wavelength is approximately on the order of the height of the upper layer, i.e. $\frac{\lambda}{h_2} \sim 1$. To quantify the interfacial tension, the parameter $\mu = \frac{T}{g(\rho_1-\rho_2)h_2^2}$ is introduced. As is evident, μ measures the strength of capillarity compared to the difference in density between the two layers and against the depth of the upper layer. In order to include capillary effects into the description, it will also be assumed that μ is of the same order as σ and ϵ. Note that the usual definition of the Bond number for homogeneous fluids would be $\mu = \frac{T}{g(\rho_1-\rho_2)h_1^2}$. This is an order-one quantity. However, for the two-fluid system, it is of limited applicability as gravity effects are present at a lower order in $|k|$ than the capillary effects.

The derivation of the model equations is based upon the observation that to first order in ϵ, σ and μ the problem is linear in the upper layer. The first step is to nondimensionalize the variables. Remember that for the sake of notational convenience, the original variables appear with a prime. To make the difference in the z-scales in the two layers explicit, two different normalizations for the z-variable are used. In particular, we let

$$z' = h_2 Z \quad \text{in} \quad \eta' < z' < h_2,$$

and

$$z' = h_1 z \quad \text{in} \quad -h_1 < z' < \eta'.$$

The other variables are nondimensionalized by

$$x' = h_2 x, \qquad t' = \frac{h_2}{c_0 v_0} t,$$

$$\eta' = a\eta, \qquad \phi' = \frac{g h_2 a}{c_0} v_0\, \phi, \qquad \psi' = \frac{g h_2 a}{c_0} v_0\, \psi,$$

where $c_0 = \sqrt{gh_1}$ and $v_0 = \sqrt{1 - \frac{\rho_2}{\rho_1}}$ as before. The full equations in the new variables are

$$\begin{aligned} \psi_{xx} + \psi_{ZZ} &= 0 \quad \text{in} \quad \epsilon\,\sigma\,\eta < Z < 1, \\ \psi_Z &= 0 \quad \text{at} \quad Z = 1, \end{aligned}$$

$$\begin{aligned} \epsilon^2 \phi_{xx} + \phi_{zz} &= 0 \quad \text{in} \quad -1 < z < \sigma\,\eta, \\ \phi_z &= 0 \quad \text{at} \quad z = -1. \end{aligned}$$

Subtracting the kinematic boundary conditions (4) and (5) at the interface, and using the original primed variables yields

$$\phi'_{z'} = \psi'_{z'} + \eta'_{x'}(\phi'_{x'} - \psi'_{x'}).$$

In the new variables, this becomes

$$\frac{1}{h_1}\,\phi_z = \frac{1}{h_2}\psi_Z + \frac{a}{h_2^2}\eta_x(\phi_x - \psi_x),$$

$$\phi_z = \epsilon\,\psi_Z + \epsilon^2 \sigma\,\eta_x(\phi_x - \psi_x).$$

Thus the matching condition at the interface is

$$\psi_Z = \frac{1}{\epsilon}\phi_z + O(\epsilon\,\sigma\,) \quad \text{at} \quad Z = \epsilon\,\sigma\,\eta. \tag{11}$$

To derive the model equations, we first focus on the velocity potential ϕ in the lower layer. Since the lower layer is relatively thin, it is assumed that ϕ can be written as an expansion in the small parameter ϵ, namely

$$\phi = \sum_{k=0}^{\infty} \epsilon^{\,k}(z+1)^k f_k(x,t).$$

Laplace's equation and the boundary condition at the bottom of the lower layer give

$$\begin{aligned}\phi &= \sum_{k=0}^{\infty}(-1)^k \epsilon^{\,2k}\frac{(z+1)^{2k}}{(2k)!}\frac{\partial^k f_0}{\partial x^k}(x,t)\\ &= f - \epsilon^{\,2}\frac{(z+1)^2}{2}f_{xx} + O(\epsilon^{\,4}),\end{aligned}$$

where f_0 is now called f. This expression for ϕ gives

$$\phi_z = -\epsilon^{\,2}(z+1)f_{xx} + O(\epsilon^{\,4}). \tag{12}$$

Next, a representation for ψ in terms of ϕ will have to be found. The potential ψ satisfies Laplace's equation in the upper layer with a boundary condition given by matching with the lower layer at the interface. Combining (11) and (12) yields the problem

$$\begin{aligned}\triangle\psi &= 0 && \text{in} \quad \epsilon\,\sigma\,\eta < Z < 1,\\ \psi_Z &= -\epsilon\,(\sigma\,\eta + 1)f_{xx} + O(\epsilon^{\,2}, \epsilon\,\sigma\,) && \text{at} \quad Z = \epsilon\,\sigma\,\eta.\end{aligned}$$

The boundary condition shows that ψ is of order ϵ. Moreover, ψ_t also satisfies Laplace's equation, so that when terms of $O(\epsilon^{\,2})$ and $O(\epsilon\,\sigma\,)$ are neglected, the following elliptic problem appears.

$$\begin{aligned}\triangle\psi_t &= 0 && \text{in} \quad 0 < Z < 1,\\ \psi_{tZ} &= -\,\epsilon\,f_{xxt} && \text{at} \quad Z = 0,\\ \psi_{tZ} &= 0 && \text{at} \quad Z = 1.\end{aligned}$$

As shown in the appendix, the boundary value of ψ_t at $z = 0$ is given by

$$\psi_t(Z=0) = -\epsilon\;\mathcal{T}_1 f_{xt} + O(\epsilon^{\,2}, \epsilon\,\sigma\,),$$

where $\mathcal{T}_1$ is defined analogously to $\mathcal{T}_{h_2}$. Next, the expressions for ϕ and ψ are inserted into the boundary conditions (7) and (5). In normalized variables the boundary condition (7) is written as

$$\rho_1\left(\eta + v_0^2\phi_t + \frac{1}{2}\sigma\,v_0^2\phi_x^2 + \frac{1}{2}\frac{\sigma}{\epsilon^{\,2}}v_0^2\phi_z^2\right) = \rho_2\left(\eta + v_0^2\psi_t + \frac{1}{2}\sigma\,v_0^2\psi_x^2 + \frac{1}{2}\sigma\,v_0^2\psi_Z^2\right) + \frac{T}{gh_2^2}\eta_{xx}.$$

Substituting the expansion for ϕ and the expression for ψ_t and keeping terms up to $O(\epsilon\,,\sigma\,)$ gives

$$\eta + f_t + \frac{1}{2}\sigma\,f_x^2 = -\epsilon\;\frac{\rho_2}{\rho_1}\,\mathcal{T}_1 f_{tx} + \frac{T}{g\rho_1 h_2^2}\frac{1}{v_0^2}\eta_{xx} + O(\epsilon^{\,2}, \epsilon\,\sigma\,).$$

Differentiating with respect to x yields

$$\eta_x + f_{tx} + \sigma\, f_x f_{xx} = -\epsilon\, \frac{\rho_2}{\rho_1} \mathcal{T}_1 f_{txx} + \frac{T}{g\rho_1 h_2^2} \frac{1}{v_0^2} \eta_{xxx} + O(\epsilon^2, \epsilon\,\sigma\,).$$

Setting $w = f_x$ and using $\mu = \frac{T}{g(\rho_1 - \rho_2)h_2^2}$, there obtains

$$\eta_x + w_t + \sigma\, w w_x + \epsilon\, \frac{\rho_2}{\rho_1} \mathcal{T}_1 w_{tx} - \mu\, \eta_{xxx} = O(\epsilon^2, \epsilon\,\sigma\,). \tag{13}$$

In particular, it follows from (13) that

$$\eta_x + w_t = O(\epsilon\,, \mu, \sigma\,). \tag{14}$$

If it is assumed that differentiation and application of the operator $\mathcal{T}_1$ do not alter the order of this relation, it may be used to find

$$w_t + \eta_x + \sigma\, w w_x - \epsilon\, \frac{\rho_2}{\rho_1} \mathcal{T}_1 \eta_{xx} - \mu\, \eta_{xxx} = O(\epsilon^2, \epsilon\,\sigma\,, \epsilon\,\mu).$$

This is the first equation of the system. The second equation is obtained in the usual way as described in Whitham (1975) from the kinematic boundary condition. In the new variables, (5) becomes

$$\eta_t + \sigma\, \phi_x \eta_x - \frac{1}{\epsilon^2} \phi_z = 0 \quad \text{at} \quad y = \sigma\, \eta.$$

Inserting the expansion for ϕ, there appears

$$\eta_t + \sigma\, \left(f_x + O(\epsilon^2)\right)\eta_x + \,(1 + \sigma\,\eta) f_{xx} = O(\epsilon^2, \epsilon\,\sigma\,).$$

Finally, substituting $w = f_x$ gives

$$\eta_t + w_x + \sigma\,(w\eta)_x = O(\epsilon^2, \epsilon\,\sigma\,).$$

Disregarding terms of $O(\epsilon^2)$, $O(\epsilon\,\sigma\,)$ and $O(\epsilon\,\mu)$, the system

$$\begin{aligned} w_t + \eta_x + \sigma\, w w_x - \epsilon\, \frac{\rho_2}{\rho_1} \mathcal{T}_1 \eta_{xx} + \mu\, \eta_{xxx} &= 0, \\ \eta_t + w_x + \sigma\,(w\eta)_x &= 0, \end{aligned} \tag{15}$$

appears as an approximate equation for waves which are governed by (2) - (6) in the case when the amplitude a and the wavelength λ satisfy the conditions set forth in the beginning of this section.

One may linearize the equations (15), and compare the result to the linear equation (9) derived from the dispersion relation in the previous section. The linearized system corresponding to (15) is

$$\begin{aligned} w_t + \eta_x - \epsilon\, \frac{\rho_2}{\rho_1} \mathcal{T}_1 \eta_{xx} - \mu \eta_{xxx} &= 0, \\ \eta_t + w_x &= 0. \end{aligned}$$

Differentiating the first equation with respect to x, the second with respect to t, and substituting, the single second-order linear equation

$$\eta_{tt} - \eta_{xx} - \epsilon\, \frac{\rho_2}{\rho_1}\, \mathcal{T}_1 \eta_{xxx} - \mu \eta_{xxxx} = 0$$

appears. After rescaling to the original variables and remembering that $\epsilon = \frac{h_1}{h_2}$ and $\mu = \frac{T}{g(\rho_1-\rho_2)h_2^2}$, this becomes

$$\eta'_{t't'} - C_0^2\left[\eta'_{x'x'} - h_1\,\frac{\rho_2}{\rho_1}\,\mathcal{T}_{h_2}\eta'_{x'x'x'} - \frac{T}{g(\rho_1-\rho_2)}\eta'_{x'x'x'x'}\right] = 0.$$

This equation is the same as (9), which was derived using different methods. The fact that they agree lends additional credibility to the foregoing development. If the geometry of the two-fluid system is reversed, i.e. if the lower layer with density ρ_1 is deep, and the upper layer with density ρ_2 is thin, then the following system appears.

$$\begin{aligned} w_t - \eta_x + \sigma\, ww_x - \epsilon\,\frac{\rho_1}{\rho_2}\mathcal{T}_1\eta_{xx} - \mu\,\eta_{xxx} &= 0,\\ \eta_t - w_x + \sigma\,(w\eta)_x &= 0. \end{aligned} \tag{16}$$

The derivation of this system follows the same pattern as the derivation of (15). We now return to the original geometry of a thin lower layer and a deep upper layer, and restrict our attention to waves that travel in one direction only.

A SINGLE MODEL EQUATION

In this section, the focus will be on waves which are mainly propagating in one direction. In particular, it will be assumed that waves propagate in the direction of increasing values of x. In this case, the wave motion can be approximately described by a single equation. First, observe that if terms of order ϵ, σ and μ are neglected, the system (15) reduces to the linear hyperbolic system

$$\begin{aligned} w_t + \eta_x &= 0,\\ \eta_t + w_x &= 0. \end{aligned}$$

Note that since the variables are non-dimensional, the wave speed C_0 appearing in (10) is no longer present. Solutions of this system propagate to the right, i.e. in the direction of increasing values of x, if and only if $w = \eta$. Therefore, to find solutions of (15) which propagate to the right to first order in ϵ, σ and μ, we propose solutions of the form $w = \eta + \epsilon\, A$, where A is a function of η and its spatial derivatives. Note that in order to keep the exposition as tidy as possible, it is assumed here that $\epsilon = \sigma = \mu$. The development in the general case differs only by the presence of additional constants. Substituting into (15), the system

$$\begin{aligned} \eta_t + \eta_x + \epsilon\,(\eta\eta_x + A_t - \frac{\rho_2}{\rho_1}\mathcal{T}_1\eta_{xx} - \eta_{xxx}) &= O(\epsilon^2),\\ \eta_t + \eta_x + \epsilon\,(2\eta\eta_x + A_x) &= O(\epsilon^2), \end{aligned} \tag{17}$$

appears. Since A is a function of η and its derivatives, we may invoke (14) to replace A_t by $-A_x$ to order ϵ^2. Then, for the two equations to be consistent, we need

$$A = -\frac{1}{4}\eta^2 - \frac{1}{2}\frac{\rho_2}{\rho_1}\mathcal{T}_1\eta_x - \frac{1}{2}\eta_{xx}.$$

Hence

$$w = \eta - \epsilon\,\frac{1}{4}\eta^2 - \epsilon\,\frac{1}{2}\frac{\rho_2}{\rho_1}\mathcal{T}_1\eta_x - \frac{\epsilon}{2}\eta_{xx}.$$

Substituting this expression into the second equation of (17), and disregarding terms of higher order than $O(\epsilon)$ yields

$$\eta_t + \eta_x + \epsilon\,\frac{3}{2}\eta\eta_x - \epsilon\,\frac{1}{2}\frac{\rho_2}{\rho_1}\mathcal{T}_1\eta_{xx} - \frac{\epsilon}{2}\eta_{xxx} = 0.$$

In the case that interfacial tension can be neglected, the well known intermediate long-wave equation

$$\eta_t + \eta_x + \epsilon\,\frac{3}{2}\eta\eta_x - \epsilon\,\frac{1}{2}\frac{\rho_2}{\rho_1}\mathcal{T}_1\eta_{xx} = 0.$$

appears. This equation may be used for instance if the fluid system is stratified with a thin transition layer, so that capillarity does not play a role.

CONCLUSION

In this paper, a system of two coupled evolution equations has been found as a model for the time development of the interface between two fluids. The model is approximately valid for long-crested waves of small amplitude and long wavelength. Special attention has been paid to the effect of capillarity which can be seen as a stabilizing term in the equations. The system can be further simplified to a single evolution equation if the waves to be described are mainly propagating in only one direction. If capillarity is neglected, this evolution equation reduces to the intermediate long-wave equation.

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APPENDIX: THE NEUMANN PROBLEM IN A STRIP

Consider the elliptic problem

$$\begin{array}{rcll} u_{xx} + u_{zz} & = & 0 & \text{in} \quad 0 < z < 1, \infty < x < \infty \\ u_z & = & g(x) & \text{at} \quad z = 0, \\ u_z & = & 0 & \text{at} \quad z = 1. \end{array}$$

Let $\hat{u}(\xi, z)$ denote the Fourier transform of u in the x-variable, defined by

$$\hat{u}(\xi, z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x, z)e^{-ix\xi}\,dx.$$

Formally taking the Fourier transform in the x-variable, we obtain

$$-\xi^2\hat{u} + \hat{u}_{zz} = 0.$$

A solution of this ordinary differential equation is given by

$$\hat{u}(\xi, z) = -\frac{\hat{g}(\xi)}{\xi}\frac{\cosh[\xi(z-1)]}{\sinh\xi}.$$

At the lower boundary $z = 0$, we have

$$\hat{u}(\xi, 0) = -\frac{\hat{g}(\xi)}{\xi}\coth\xi.$$

This can be written as

$$\hat{u}(\xi, 0) = \frac{1}{i\xi} \left(-i \coth(\xi)\right) \hat{g}(\xi).$$

Hence $u(x, 0)$ is given by

$$u = \partial_x^{-1} \mathcal{T}_1 g,$$

where $\mathcal{T}_1$ is the integral operator

$$\mathcal{T}_1 g(x) = \text{p.v. } \frac{1}{2} \int_{-\infty}^{\infty} \coth\left(\frac{\pi}{2} y\right) g(x-y)\, dy.$$

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