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Dynamics of the interface between immiscible liquids of different densities with low Froude number

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ABSTRACT

Consideration is given to slow interfacial dynamics in a two-layer system of viscous fluids of comparable density. The fluid flow is governed by the two-dimensional Navier–Stokes equations where it is assumed that the inertial terms can be disregarded. A numerical methodology is presented which allows the study of the dynamics in the nonlinear regime. The numerical method is based on rewriting the interface conditions in terms of appropriate Green's functions which are independent on the governing equations.

1. Introduction

Consider a system of two viscous fluids with different but comparable density, separated by a sharp interface. One example of such a configuration is the case where a denser fluid lies above the lighter fluid, and the interface is initially plane, and oriented such that gravity is acting normal to the interface. This configuration was studied by Rayleigh [1] and Taylor [2], and the resulting instability is known as the Rayleigh–Taylor instability. A related problem also studied here is the problem of a bubble of a dense liquid enclosed inside a lighter liquid. Such a bubble will have the tendency to sink to the bottom. While the mean velocity of the bubble can be approximately computed using relatively simple arguments, describing the concurrent deformation of the interface is no simple task.

The chief purpose of the present paper is the presentation of a straightforward numerical method for computation of interfacial dynamics in situations like the two mentioned above. The two main assumptions used here are that the fluid motion is essentially two-dimensional, and that the difference in density between the two fluids is so small that the inertial terms of the governing equations can be disregarded. Two-layer fluid systems with very small density gradient can be found in seismology problems when flows in the uppermost part of the mantle of the earth are studied [3–6]. Such problems also arise in other subsurface flows and flows in porous media.

The governing equations in the case at hand are the Navier–Stokes equations

$$\rho \frac{d\mathbf{u}}{dt} + \nabla p + \rho \mathbf{g} = \mu \Delta \mathbf{u}, \tag{1}$$

where $\nabla = (\partial_x, \partial_z)$, $\mathbf{u} = (u, v)$, u is the horizontal, and v is the vertical velocity, and $\frac{d}{dt} = \partial_t + \langle \mathbf{u}, \nabla \rangle$ is the material derivative. The constant μ denotes the dynamic viscosity (the same constant for both fluids) and $\mathbf{g} = (0, g)$ is the gravitational acceleration.

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We shall denote by ρ_l and ρ_u densities of the lower and upper fluids, respectively, and by $\rho_0 = (\rho_u + \rho_l)/2$ the mean density. We rewrite the pressure in the form

$$p = \text{const.} - g\rho_0 z + P,$$

and use the dimensionless coordinates

$$x = L_0\bar{x}, \qquad z = L_0\bar{z}, \quad \text{and} \quad t = T\bar{t},$$

and the dimensionless velocity

$$u = \frac{L_0}{T}\bar{u}, \qquad v = \frac{L_0}{T}\bar{v}.$$

The scaling factors L_0 and T will have to be chosen such that they represent a relevant length and time scale in any given physical setup. We define the relative density difference Δ_0 and the Froude number $\mathcal{F}r$ by

$$\Delta_0 = \frac{\rho_u - \rho_l}{\rho_0} \quad \text{and} \quad \mathcal{F}r = \frac{u_0}{\sqrt{L_0 g}}.$$

In general, small density differences lead to slow fluid velocities, so that the Froude number $\mathcal{F}r$ and the density difference Δ_0 are of similar magnitude. Therefore, an appropriate definition of dimensionless density $\bar{\rho}$ and pressure $\bar{p}$ is given by

$$\rho = \rho_0(1 + \Delta_0\bar{\rho}), \quad \text{and} \quad P = g\rho_0 L_0 \mathcal{F}r\,\bar{p}.$$

As shown in [7], assuming $\mathcal{F}r = \Delta_0$ for simplicity, leads to the dimensionless Navier–Stokes equations

$$\mathcal{F}r(1 + \Delta_0\bar{\rho})\frac{d\bar{\mathbf{u}}}{d\bar{t}} + \nabla\bar{p} + \vec{e}_2\bar{\rho} = \frac{\mathcal{F}r}{\mathcal{R}e}\Delta_{(\bar{x},\bar{z})}\bar{u}, \tag{2}$$

where $\mathcal{R}e = \frac{\rho_0 L_0 U_0}{\mu}$ is the Reynolds number, and $\vec{e}_2 = (0, 1)$.

The aim of the present paper is the development of a numerical method to study the evolution of a sharp interface between two fluids governed by (2). We focus on the case where the Froude number is much smaller than unity, so that the inertial terms in (2) can be disregarded. However, in general the ratio $\mathcal{F}r/\mathcal{R}e$ is not small, so viscous effects are kept in the description.

The method to be proposed is based on tracking the interface between the fluids by describing its dynamics by a conservation law which can be solved by computing the characteristics via Green functions connected to the biharmonic equation. This formulation enables us to avoid implicit numerical methods, and to describe the dynamics with rather simple numerical schemes.

The paper is organized as follows. In Section 2, the dynamics of the interface in terms of the Green functions are developed. In Section 3, we introduce the numerical procedures and apply the method to the situation of a bubble of heavy liquid sinking inside a lighter liquid.

2. Formulation of the problem in terms of Green functions

The basis for the procedure used here lies in the analytical work of Danilov and Omel'yanov [7]. While their analysis focused on vertical movements of the interface, we shall adapt their procedure in such a way which makes it possible to track horizontal movements, as well. The underlying idea of reformulating the flow problem in terms of the Greens functions has also been used in previous papers, notably in [8,9].

In order to present the numerical method, we first remove the factor $\mathcal{F}r/\mathcal{R}e$ by a further scaling (cf. [7]). Moreover, recall that the inertial terms in (2) are disregarded. Omitting overbars on the new variables, and keeping the notation ρ_u and ρ_l for the normalized densities of the upper and lower layer, respectively, the system (2) reduces to

$$\frac{\partial p}{\partial x} = \Delta u, \tag{3}$$

$$\rho + \frac{\partial p}{\partial z} = \Delta v. \tag{4}$$

These equations are supplemented with the conservation of mass and the component conservation equation in the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0, \tag{5}$$

$$\partial_t\rho + \partial_x(u\rho) + \partial_z(v\rho) = 0. \tag{6}$$

In order to specify the quantities and equations given in the introduction, we consider the process in the strip $\Pi = \{(x, z) \in \mathbb{R}^2 : x \in [-\pi, \pi], z \in (-\infty, \infty)\}$, and assume that the interface is x-periodic with period 2π, and even around $x = 0$. We assume that $z = \varphi(x, t)$ describes the location of the interface $\Gamma = \{(x, z) : z = \varphi(x, t)\}$. For the presentation of the method, this simpler case which assumes that the interface can be described by the graph of a function is assumed. However, the method can easily be extended to the case where the interface is given by a level set $\{\phi(t, x, z) = 0\}$ for a function ϕ, and indeed this case will be treated in Section 3.

Under the assumptions given above, the density may be written in the form

$$\rho = \rho_l + (\rho_u - \rho_l)H(z - \varphi) = \rho_l + \rho_0 \Delta_0 H(z - \varphi), \tag{7}$$

where H is the Heaviside function. Taking into account the form of the density ρ given by (7), we obtain from (6):

$$\frac{\partial \varphi}{\partial t}\delta(z - \varphi) + u\frac{\partial \varphi}{\partial x}\delta(z - \varphi) - v\delta(z - \varphi) = 0. \tag{8}$$

Evaluating the velocity on the interface can be written as

$$\frac{\partial \varphi}{\partial t} + u|_{z=\varphi}\frac{\partial \varphi}{\partial x} - v|_{z=\varphi} = 0. \tag{9}$$

Note that (9) is a first-order hyperbolic equation which could be solved by the method of characteristics. Indeed, the system of characteristics corresponds to the conservation law (9) is given by

$$\begin{aligned} &\dot{x} = u(x, z, t), \qquad \dot{\varphi} = v(x, z, t), \\ &x(0) = x_0, \qquad \varphi(0) = z_0, \end{aligned} \tag{10}$$

where x is a function of time t, and the initial position of a point in the interface is denoted by (x_0, z_0). It is well known that the characteristics, at least locally, define the classical solution to (9). However, the coefficients u and v depend on the unknown φ in a complicated and a priori unknown way. Therefore, the numerical algorithm used to solve the equation will be based on a different approach.

To obtain a convenient formulation of the problem, first recall that (5) guarantees the existence of the stream function F which at any time t satisfies

$$\frac{\partial F}{\partial z} = -u, \qquad \frac{\partial F}{\partial x} = v. \tag{11}$$

Taking *curl* of (3) and (4) shows that

$$\Delta^2 F = \frac{\partial \rho}{\partial x}. \tag{12}$$

Next, we shall analyze the boundary condition associated to (12) more closely. Recall that we assume that the solution of the problem is given in terms of periodic and even φ. As shown in [7], it can be concluded that ρ, v and p are periodic and even, while u is periodic and odd. With regard to F, observe that (5) shows that $\int_{-\pi}^{\pi} v\, dx$ is independent of z, and since the velocity is zero at great distances from the interface, the streamfunction F is periodic. Furthermore, assuming ρ to be even, $\frac{\partial \rho}{\partial x}$ is odd, and it follows that

$$\Delta^2(F(x, z) + F(-x, z)) = 0.$$

Since the function $(F(x, z) + F(-x, z))$ is bounded and it is a solution to the biharmonic equation, it must be equal to a constant. Thus we conclude that F is an odd function in x. Moreover, note that since u is periodic and odd, $u(\pm\pi) = 0$, so that F is constant on the boundaries $x = \pm\pi$ of the fundamental domain. Since the streamfunction is only defined up to a constant, we may assume that the streamfunction is zero on $x = \pm\pi$. For the numerical integration, we may also assume that F satisfies homogeneous Neumann conditions on the boundary of Π. Next, for a fixed $x_i \in (-\pi, \pi)$, we introduce an auxiliary function G_i satisfying

$$\Delta^2 G_i(x, z) = \tfrac{1}{2}\delta'(z)(\delta(x - x_i) - \delta(x + x_i)), \tag{13}$$

and we assume that G_i satisfies the same boundary conditions as F. The distribution $\delta'(z)(\delta(x - x_i) - \delta(x + x_i))$ denotes the tensor product of a distribution in the independent variable in z and a distribution in x (see [10, pg. 225], for example). In a similar fashion, we introduce Green's function G satisfying

$$\Delta^2 G = \delta(z)\delta'(x). \tag{14}$$

It is well known that solutions of the problems (13) and (14) exist (see [11,12]). However, since it is not completely obvious how to obtain expressions for the functions G_i and G in a concrete situation, we demonstrate how the function G may be found in the case of homogeneous Dirichlet conditions. In general, one may have to use numerical methods to find these Green's functions. First, notice that the fundamental solution of the equation $\Delta N = \delta(x)\delta(z)$ given by $N(x, z) = \frac{1}{2\pi}\log\left\{|x^2 + z^2|^{1/2}\right\}$. Therefore, the function $\partial_x N(x, z) = \frac{1}{2\pi}\frac{x}{x^2+z^2}$ solves the equation $\Delta \partial_x N = \delta'(x)\delta(z)$. From here and (14), it follows that if we find a function G such that $\Delta G = \frac{1}{2\pi}\frac{x}{x^2+z^2}$ which takes the required boundary conditions, we have obtained Green's function. This problem can be solved in many ways, but we choose an elegant approach from complex analysis.[1] Denote

$$\eta(w)=\tan(w/4),\qquad \eta^{-1}(w)=4\,\mathrm{atan}(w),\qquad (\eta^{-1})'(w)=\frac{4}{1+w^2},\quad w\in\mathbb{Z}.$$

It is well known that $\tan(w)$ maps the strip $-\pi/4\leq x\leq\pi/4, -\infty<z<\infty$ into the unit disc $D=D(0,1)$, and thus $\tan(4w)$ maps the strip $-\pi\leq x\leq\pi, -\infty<z<\infty$ into the unit disc.

Denote $G(w)=Q(\eta(w)),\ w\in\mathbb{Z}$. Since tan is an analytic function, we have

$$\partial_x N(w)=\Delta G(w)=|\eta'(w)|^2\Delta Q(\eta(w))$$

implying

$$\Delta Q(w)=\partial_x N(\eta^{-1}(w))|(\eta^{-1})'(w)|^2,\quad w\in D(0,1).$$

The solution of this equation is

$$Q(w)=\int_D \partial_x N(\eta^{-1}(p+iq))|(\eta^{-1})'(p+iq)|^2\ln\left|\frac{w-(p+iq)}{1-(p-iq)w}\right|dpdq,\quad w\in\mathbb{Z}.$$

From here, it follows that

$$G(w)=\int_D \partial_x N(\eta^{-1}(p+iq))|(\eta^{-1})'(p+iq)|^2\ln\left|\frac{\tan(w)-(p+iq)}{1-(p-iq)\tan(w)}\right|dpdq.$$

On the boundaries of the strip, we have $G=0$ since $|\frac{\tan(w)-(x+iy)}{1-(x-iy)\tan(w)}|=1$ if $|\tan(w)|=1$ and $\lim_{\mathrm{Im}(w)\to\pm\infty}|\frac{\tan(w)-(x+iy)}{1-(x-iy)\tan(w)}|=1$ ($|\tan(w)|\to 1$ as $\mathrm{Im}(w)\to\infty$).

In exactly the same manner we obtain the function G_i. More precisely, denoting

$$\partial_z\hat{G}(x,z)=\frac{1}{4\pi}\left(\frac{z}{(x-x_i)^2+z^2}-\frac{z}{(x+x_i)^2+z^2}\right),$$

G_i is given by

$$G_i(w)=\int_U \partial_z\hat{G}(\eta^{-1}(p+iq))|(\eta^{-1})'(p+iq)|^2\ln\left|\frac{\tan(w)-(p+iq)}{1-(p-iq)\tan(w)}\right|dpdq.$$

Now, we can derive the main equations describing the motion of the interface. Below, we assume that the interface is well defined by the equation $z=\varphi(x,t)$, i.e. that the function $x\mapsto\varphi(x,t)$ is not multi-valued for any $t\in\mathbb{R}^+$. This is not always the case, but it does not change the situation significantly. In fact, computations with multi-valued φ will be presented in Section 3.

Consider again the Eqs. (10) for the characteristics. Let $(x_i,z_i)\in\Pi$ be the position of a particle in the interface. Then using the definition of the streamfunction and the fact that it is an odd function in x, we find

$$\frac{dx_i}{dt}=u(x_i,z_i,t)=\left\langle\frac{1}{2}\left(\delta(x-x_i)-\delta(x+x_i)\right)\delta'(z-z_i),F(x,z,t)\right\rangle.\tag{15}$$

Using (13), identity (15) can be continued as follows.

$$\begin{aligned}\frac{dx_i}{dt}&=\left\langle\Delta^2 G_i(x,z-z_i),F(x,z,t)\right\rangle\\&=\left\langle G_i(x,z-z_i),\Delta^2 F(x,z,t)\right\rangle\\&=\left\langle G_i(x,z-z_i),\partial_x\rho(x,z,t)\right\rangle.\end{aligned}$$

To evaluate this expression, note that

$$-\left\langle\rho(x,z,t),\partial_x G_i(x,z-z_i)\right\rangle=(\rho_u-\rho_l)\int_{-\pi}^{\pi}\int_{\varphi(x,t)}^{\infty}\partial_x G_i(x,z-z_i)\,dzdx.$$

We consider a subinterval (x_0,x_1) of $(-\pi,\pi)$ where the function $\varphi(x,t)$ is monotone in x, so that it can be inverted. Denoting the inverse at a given t by $x_\varphi(z,t)$, the integral can be rewritten as

$$\int_{x_0}^{x_1}\int_{\varphi(x,t)}^{\infty}\partial_x G_i(x,z-z_i)\,dzdx=\int_{z_0}^{z_1}G_i(x_\varphi(x,t),z-z_i)\,dz.$$

[1] We would like to thank to D. Kalaj for drawing our attention to this approach.

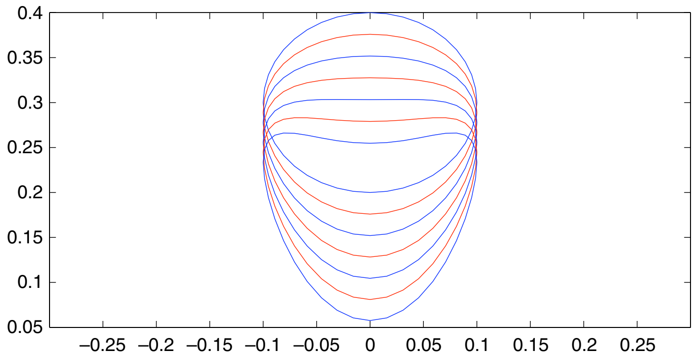

Fig. 1. Dynamics of a bubble of heavier liquid in the space filled with lighter liquid. The bubble is moving downwards.

As shown in [7], one may obtain a global representation by using the Leray measure on the interface Γ, i.e. a 1-form ω such that $\omega \wedge d(z-\varphi) = dx \wedge dz$ (cf. [13]). Since $d(z-\varphi) = dz - \partial_x \varphi dx$ and the manifold is given by $z = \varphi(x)$, we can take $\omega = dz/\partial_x\varphi$ (where $\partial_x\varphi \neq 0$). Thus one may formulate the global relation in the form

$$\frac{dx_i}{dt} = u(x_i, z_i, t) = (\rho_u - \rho_l)\int_\Gamma G_i(x, z - z_i)\partial_x\varphi\,\omega. \tag{16}$$

The vertical velocity may be represented in a similar way using Green's function G. The global formulation is

$$\frac{dz_i}{dt} = v(x_i, z_i, t) = -(\rho_u - \rho_l)\int_\Gamma G(x - x_i, z - z_i)\partial_x\varphi\,\omega. \tag{17}$$

3. Discretization procedure

In this section, we shall show how to use (16) and (17) to simulate dynamics of the interface between the two fluids. For each fixed $t \in \mathbb{R}^+$ we split the codomain of the function φ on subintervals (α_k, α_{k+1}), $k = 1, \ldots, l$, such that the function $\varphi(\cdot, t)$ is monotone on each subinterval. From (16) and (17), we conclude that the coordinates of the point $(x_i, z_i) \in \Gamma$ are governed by the system

$$\frac{dx_i}{dt} = -(\rho_u - \rho_l)\sum_{p=1}^{l}\int_{\alpha_p}^{\alpha_{p+1}} G_i(x_\varphi(z,t), z - z_i)dz, \tag{18}$$

and

$$\frac{dz_i}{dt} = (\rho_u - \rho_l)\sum_{p=1}^{l}\int_{\alpha_p}^{\alpha_{p+1}} G(x_\varphi(z,t) - x_i, z - z_i)dz, \tag{19}$$

where $x_\varphi(z, t)$ is the inverse function of $\varphi(x, t)$ on the appropriate interval.

Is is apparent that the problem has now been reduced to tracking a level set, and this can be achieved by computing the time evolution of a sufficient number of points in the interface. Accordingly, we split the interface at the initial time moment t_0 into N grid points (x_k, z_k), $k = 1, \ldots, N$, and approximate (19) by

$$\frac{dz_i}{dt} = -(\rho_u - \rho_l)\sum_{k=1}^{N}(z_{k+1} - z_k)G(x_\varphi(z_k, t) - x_i, z_k - z_i)dz.$$

Note that the integral has been approximated by evaluating the integrands of the left endpoint of a given interval. A similar approximation is used for (18). To solve the resulting system of first-order ODE, the forward Euler scheme is used. In particular, to forward from t to $t + \Delta t$, we use the prescription

$$z_i(t + \Delta t) = z_i(t) - \Delta t\,(\rho_u - \rho_l)\sum_{k=1}^{N}(z_{k+1} - z_k)G(x_\varphi(z_k, t) - x_i, z_k - z_i)dz$$

for the z component and a similar expression for the x component of the point (x_i, z_i). Dynamics of wide range of initial positions can be simulated using this scheme. As an example, we simulate dynamics of a bubble of heavier liquid placed in a lighter liquid (Fig. 1). The density difference is normalized so that $\rho_u - \rho_l = 1$, and the figure shows position of the bubble at $t = 0, 0.3, 0.6, \ldots$. The bubble moves downwards, and its shape changes from a perfect circle at the beginning to a figure in the shape of a drop. For more in-depth studies of bubbles sinking in lighter liquids, we refer the reader to [14,15].

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