Mechanical balance laws in long wave models

Henrik Kalisch*
Department of Mathematics, University of Bergen
Postbox 7800, 5020 Bergen, Norway

Consider wave motion at the surface of an inviscid incompressible fluid of unit density in the absence of capillarity. Suppose the depth of the fluid in the undisturbed state is given by h_0, and gravity is denoted by g. For waves which respect an approximate relationship $\alpha \sim \beta$ between the nondimensional amplitude $\alpha = a/h_0$ and the long-wave parameter $\beta = h_0^2/\lambda^2$, there are a variety of Boussinesq-type equations which may be used to describe the wave motion for waves which have sufficiently long wavelength λ when compared to the undisturbed depth h_0.

The derivation of such systems is well understood [14], and there exist a large number of systems with various requisite properties. For instance, the systems may be optimized with respect to the description of shorter waves, or with respect to smoothing properties, or amenability to numerical study. An overview is given in [11]. Here we focus on a class of models derived and studied in [4, 5]. Denote the limiting long-wave speed by $c_0 = \sqrt{gh_0}$, and define non-dimensional variables by

$$\tilde{x} = \frac{x}{\lambda}, \quad \tilde{z} = \frac{z + h_0}{h_0}, \quad \tilde{\eta} = \frac{\eta}{a}, \quad \tilde{t} = \frac{c_0 t}{\lambda}, \quad \tilde{\phi} = \frac{c_0 \phi}{ga\lambda}.$$

Assuming irrotaional fluid motion, expanding the velocity potential ϕ in an asymptotic series, and substituting into the governing Euler equations and free-surface boundary conditions yields

$$\tilde{\eta}_{\tilde{t}} + \tilde{w}_{\tilde{x}} + \alpha(\tilde{\eta}\tilde{w})_{\tilde{x}} - \tfrac{1}{2}\big(\theta^2 - \tfrac{1}{3}\big)\beta\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{t}} = \mathcal{O}(\alpha\beta, \beta^2), \tag{1}$$

$$\tilde{w}_{\tilde{t}} + \tilde{\eta}_{\tilde{x}} + \alpha\tilde{w}\tilde{w}_{\tilde{x}} - \tfrac{1}{2}\big(1 - \theta^2\big)\beta\tilde{w}_{\tilde{x}\tilde{x}\tilde{t}} = \mathcal{O}(\alpha\beta, \beta^2). \tag{2}$$

From these relations it appears that if α and β are sufficiently small, terms of order $\mathcal{O}(\alpha\beta, \beta^2)$ can be disregarded, and one may use the following system as approximate equations of motion:

$$\begin{aligned} \eta_t + h_0 w_x + (\eta w)_x - b\eta_{xxt} &= 0, \\ w_t + g\eta_x + w w_x - d w_{xxt} &= 0. \end{aligned} \tag{3}$$

Here $\eta(x,t)$ represents the excursion of the free surface at a spatial point x and at time t, while $w(x,t)$ represents the horizontal velocity at a given height $h_0\theta$ in the fluid column. The parameters b and d are given in terms of $0 \le \theta \le 1$ by

$$b = \tfrac{1}{2}\big(\theta^2 - \tfrac{1}{3}\big)h_0^2, \qquad d = \tfrac{1}{2}\big(1 - \theta^2\big)h_0^2.$$

This point of view can also be made rigorous by proving that solutions of the free-surface problem based on the Euler equations converge to solutions of (3) in an appropriate sense on a certain time scale [6, 11].

Since the system (3) was obtained by a procedure which is not based on the conservation of mass and momentum (such as the derivation of the shallow-water system for example), one may

*henrik.kalisch@math.uib.no

ask whether the system (3) allows the conservation of mass, momentum, or indeed conservation of energy. As it happens, if $\theta^2 = \frac{2}{3}$, the system takes the form

$$\begin{aligned} \eta_t + h_0 w_x + (\eta w)_x - \tfrac{h_0^2}{6}\eta_{xxt} &= 0, \\ w_t + g\eta_x + w w_x - \tfrac{h_0^2}{6} w_{xxt} &= 0, \end{aligned} \tag{4}$$

and in this case, the system is Hamiltonian with Hamiltonian function

$$H = \int_{-\infty}^{\infty} \left\{ \tfrac{g}{2}\eta^2 + \tfrac{h_0}{2} w^2 + \tfrac{1}{2}\eta w^2 \right\} dx.$$

However, since the derivation of (4) was not based on preserving the Hamiltonian structure, it remains to be shown that this functional represents the total mechanical energy due to the wave motion. Moreover, the question also arises how to express the energy of the wave motion in the more general system (3).

While in the study of system of this type, the prevailing point of view is to consider conservation of functionals usually interpreted as total excess mass, momentum and energy[1] a different way to proceed is to focus on approximate local conservation. As explained in [2], this approach entails substituting the expansion for the velocity potential into the conservation equations based on the Euler description of the flow, and requiring the approximate balance law

$$\frac{\partial}{\partial \tilde{t}} \tilde{E}(\tilde{\eta}, \tilde{w}) + \frac{\partial}{\partial \tilde{x}} \tilde{q}_E(\tilde{\eta}, \tilde{w}) = \mathcal{O}(\alpha^2, \alpha\beta, \beta^2),$$

which defines the energy density E and energy flux q_E.

In the case of (3), the dimensional versions of the energy density and energy flux are obtained in the form

$$E_\theta = \tfrac{g}{2}\eta^2 + \tfrac{h_0}{2} w^2 + \tfrac{1}{2} w^2 \eta + \tfrac{h_0^3}{2}\big(\theta^2 - \tfrac{1}{3}\big) w w_{xx} + \tfrac{h_0^3}{6} w_x^2 \tag{5}$$

and

$$q_{E_\theta} = \tfrac{h_0}{2} w^3 + c_0^2 \eta w + \tfrac{c_0^2 h_0^2}{2}\big(\theta^2 - \tfrac{1}{3}\big)\eta w_{xx} - \tfrac{h_0^3}{3} w w_{xt} + \tfrac{c_0^2}{h_0} w \eta^2. \tag{6}$$

In particular, $q_{E_\theta}(x,t)$ gives the energy flux and work done by the pressure force due to the wave motion at a point x and a time t. Integrating $E_\theta(x,t)$ over an interval $[x_1, x_2]$ yields the energy due to the wave motion in the control interval shown in Figure 1 at a time t, and to the same order of approximation as the system (3) is valid.

If the surface disturbance is localized, so that η and w decay to zero at infinity, and the integration of E is taken over the entire real line, then the Hamiltonian of (4) is recovered in the case when $\theta^2 = 2/3$: $H = \int_{-\infty}^{\infty} E_\theta \, dx$.

Similar approximate balance laws can be sought for the mass density and flux, and for the momentum density and flux. Since it was already decided that the system (3) is the governing system in the current description, these balance laws will generally not hold exactly, but only up to some order in β and α.

One application of the analysis detailed above has been used to understand the energy budget in an undular bore as approximated by different model equations [1, 9].

[1]The general system (3) features conservation of total excess mass through the conserved integral $\int_{-\infty}^{\infty} \eta \, dx$. Moreover, for the system (4) the integral $\int_{-\infty}^{\infty} \eta w + b\eta_x w_x \, dx$ is also formally conserved. However, it is not clear if this last integral has any physical significance.

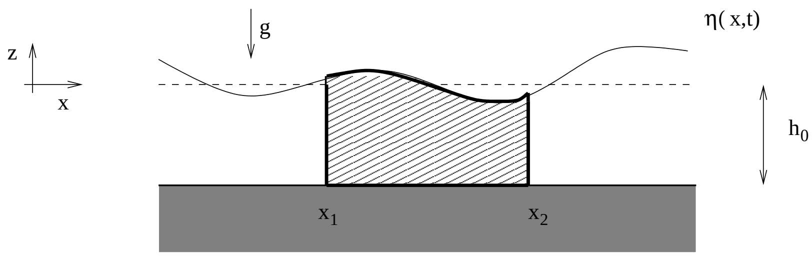

Figure 1: Geometric setup of the problem. The undisturbed water depth is h_0, and the x-axis is aligned with the free surface at rest. The free surface is described by a function $\eta(x,t)$. The figure shows a control interval delimited by x_1 and x_2 on the abscissa.

Similar considerations can be applied to the KdV equation

$$\eta_t + c_0\eta_x + \tfrac{3}{2}\tfrac{c_0}{h_0}\eta\eta_x + \tfrac{c_0 h_0^2}{6}\eta_{xxx} = 0, \tag{7}$$

which is a unidirectional model for surface waves. In this case, it was found in [3] that the energy density and flux are given by

$$E = c_0^2\left(\tfrac{1}{h_0}\eta^2 + \tfrac{1}{4h_0^2}\eta^3 + \tfrac{h_0}{6}\eta\eta_{xx} + \tfrac{h_0}{6}\eta_x^2\right),$$

and

$$q_E = c_0^3\left(\tfrac{1}{h_0}\eta^2 + \tfrac{5}{4h_0^2}\eta^3 + \tfrac{h_0}{2}\eta\eta_{xx}\right). \tag{8}$$

One interesting application where these quantities can be put to use concerns the the shoaling of periodic wavetrains and solitary waves. Consider a wave which experiences a decrease in depth over a gentle slope with no variation in the transverse direction. From a practical point of view, the waveheight of the shoaling waves is of particular interest, and one may use the conservation of energy flux in an adiabatic setting to obtain a first approximation for the waveheight. The linear theory is well known [7], and there have also been many studies making use of cnoidal wave solutions of (7) for periodic shoaling.

However, there is a deep-water limit beyond which cnoidal solutions of the the KdV equation cannot be used to describe periodic wave trains. Because of this limitation, it is necessary in the shoaling problem to compute the initial transition from deep water to intermediate depths by linear wave theory [12].

However, one problem which the authors of [12] faced was that at the point where linear and cnoidal theory were to be matched, a discontinuity in waveheight appeared in the shoaling curve. This problem was overcome later in [13] by imposing continuity in waveheight directly, but at the cost of incurring a discontinuity in the energy flux. Using the nonlinearly defined energy flux q_E in the shoaling equation

$$\int_0^T q_{E_A}\,dt = \int_0^T q_E\,dt, \tag{9}$$

eliminates the problem of discontinuities in waveheight or energy flux at the matching point between linear and cnoidal theory [10].

A comparison between the shoaling computations based on (9) and the numerical results for the full water-wave problem [8] is shown in Figure 2 for a wave of initial wavelength L_0 and waveheight H_0. It can be seen that the waveheight increases initially more slowly than predicted by Green's law, but the shoaling curve then turns up, and reaches a slope similar to Boussinesq's law. The curve based on (9) matches the curve obtained in [8] rather well. One aspect in which

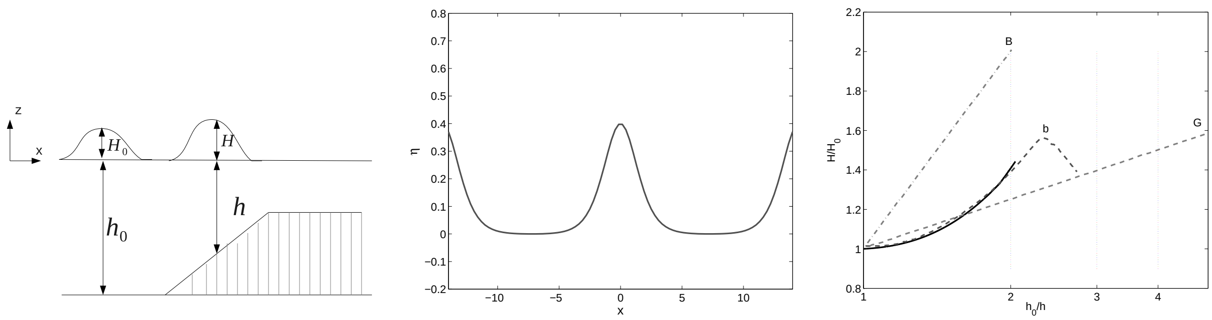

Figure 2: Left panel: definition sketch for h_0, h, H_0 and H. Center panel: incident wave profile with $\frac{L_0}{h_0} = 14.5$, $\frac{H_0}{h_0} = 0.4$. Right panel: b: shoaling curve after Grilli et al., black solid curve: shoaling curve based on (8) and (9), G: Green's law and B: Boussinesq's law.

the comparison is not favorable is the termination of the shoaling curve based on (9) before the breaking point. This issue has not been investigated further so far.

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