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Stability of traveling wave solutions to the Whitham equation

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ABSTRACT

The Whitham equation was proposed as an alternate model equation for the simplified description of unidirectional wave motion at the surface of an inviscid fluid. An advantage of the Whitham equation over the KdV equation is that it provides a more faithful description of short waves of small amplitude. Recently, Ehrnström and Kalisch [19] established that the Whitham equation admits periodic traveling-wave solutions. The focus of this work is the stability of these solutions. The numerical results presented here suggest that all large-amplitude solutions are unstable, while small-amplitude solutions with large enough wavelength L are stable. Additionally, periodic solutions with wavelength smaller than a certain cut-off period always exhibit modulational instability. The cut-off wavelength is characterized by $kh_0 = 1.145$, where $k = 2\pi/L$ is the wave number and h_0 is the mean fluid depth.

1. Introduction

The water-wave problem concerns the flow of an incompressible inviscid fluid on a horizontal impenetrable bed. The flow is described by the Euler equations and the dynamics of the free surface are of particular interest [26]. There are a number of models which allow the approximate description of the evolution of the free surface without having to provide a complete solution of the fluid flow below the surface. One of the best known of such models is the Korteweg–de Vries (KdV) equation. If the undisturbed depth of the fluid h_0 is taken as the unit of length and the ratio $\sqrt{h_0/g}$ is taken as the unit of time, then the KdV equation is given in non-dimensional form by

$$\eta_t + \eta_x + \frac{3}{2}\eta\eta_x + \frac{1}{6}\eta_{xxx} = 0. \tag{1}$$

It is well known that solutions of this equation provide a fair approximation to the free surface in the long-wave/shallow-water asymptotic limit [25,31]. For waves of amplitude a and wavelength L, this asymptotic limit is characterized by balancing the two small parameters h_0^2/L^2 and a/h_0. Unlike the full water-wave problem, the KdV equation can be solved exactly for a wide range of initial conditions using the inverse scattering transform [1]. One conspicuous difference between the water-wave problem for the full Euler equations and the KdV equation is the velocity of small disturbances of the form $\cos k(x-ct)$ in their respective linearizations about the zero solution. These linear phase speeds are given by

$$c_K = 1 - \frac{1}{6}k^2, \tag{2}$$

for the KdV equation, and

$$(c_E)^2 = \frac{\tanh(k)}{k} \tag{3}$$

for the dimensionless Euler equations. The parameter k is the wavenumber and the wavelength is given by $L = 2\pi/k$.

The linear phase speed in the KdV equation can be obtained from (3) by taking the first two terms in the Taylor expansion around $k = 0$. A comparison of these phase speeds is provided in Fig. 1. It appears immediately that the linear phase speed for the KdV equation approximates the Euler phase speed well for waves of small wavenumber (i.e. long waves), but does a poor job for waves of larger wavenumber (i.e. short waves).

Recognizing this shortcoming of the KdV equation as a water wave model, Whitham [30] proposed an alternative evolution equation featuring the same nonlinearity as the KdV equation, but one branch of the linear phase speed of the Euler equations in the linear part. The equation has the form

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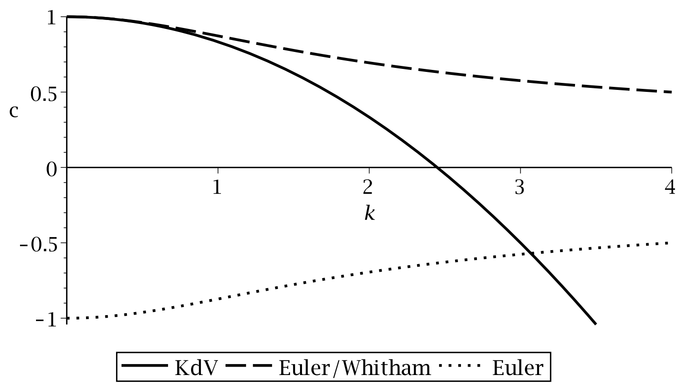

Fig. 1. Phase speed, c, plotted versus wave number, k, for the KdV, Euler, and Whitham equations. The curve for the Whitham phase speed is the same as the curve for the positive part of the Euler phase speed.

$$\eta_t + \frac{3}{2}\eta\eta_x + \frac{1}{2\pi}\int_{-\infty}^{\infty} ik\sqrt{\frac{\tanh(k)}{k}}\,\hat{\eta}(k,t)e^{ikx}\,dk = 0. \tag{4}$$

The linear phase speed of the Whitham equation is given by

$$c_W = \sqrt{\frac{\tanh(k)}{k}}. \tag{5}$$

Thus, except for the restriction to one-way propagation, the phase speed of the Whitham equation matches the phase speed of the Euler equations. The linear part of the equation is defined with the help of the Fourier transform,

$$\hat{\eta}(k,t) = \mathcal{F}\{\eta(x,t)\} = \int_{-\infty}^{\infty} \eta(y,t)e^{-iky}\,dy, \tag{6a}$$

and the inverse Fourier transform,

$$\eta(x,t) = \mathcal{F}^{-1}\{\hat{\eta}(k,t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\eta}(k,t)e^{ikx}\,dk. \tag{6b}$$

In the case where $\eta(\cdot,t)$ is not absolutely integrable, such as if $\eta(\cdot,t)$ is a spatially periodic function, the Fourier transform and the convolution integral in (4) have to be interpreted in the context of tempered distributions [19,28]. It is convenient to define the integral kernel K by

$$\hat{K}(k) = \sqrt{\frac{\tanh(k)}{k}}, \tag{7}$$

so that the Whitham equation can be written in the tidy form

$$\eta_t + \frac{3}{2}\eta\eta_x + K * \eta_x = 0. \tag{8}$$

Even though Eq. (8) has been known for a few decades, it has not been studied as much as the KdV equation. This is partially due to a lack of evidence (beyond the reasoning that led to its derivation) that the Whitham equation actually is a reasonable model for surface water waves. However, there are some recent studies that suggest that the Whitham equation models the evolution of certain waves more accurately than does the KdV equation. In particular, Carter and George [12] study the properties of the Whitham equation as an evolutionary equation and compare its solutions with data from physical experiments, while Borluk et al. [6] investigate the modeling properties in the context of steady waves.

The Whitham equation admits the following conserved quantities

$$\mathcal{Q}_1 = \int_{-\infty}^{\infty} \eta\,dx, \tag{9a}$$

$$\mathcal{Q}_2 = \int_{-\infty}^{\infty} \eta^2\,dx, \tag{9b}$$

$$\mathcal{Q}_3 = \int_{-\infty}^{\infty} \left(\eta K * \eta - \eta^3\right)dx. \tag{9c}$$

These conservation laws are useful in the study of the equation from mathematical point of view [27] and can also be used to test numerical algorithms for the time-dependent problem. Moreover, if the Whitham equation is posed on the real line, then the existence of solitary waves has been recently proven [18] and this result depends strongly on the third conserved quantity (9) which is taken as a mathematical energy in the proof of existence. If periodic solutions are studied, then the domain of integration in the above integrals needs to be replaced by the fundamental periodic domain, such as $[0, L]$ if the solutions are periodic with spatial period L.

In the current work, the focus is on the stability of periodic waves in the Whitham equation and it will be shown numerically that waves of large enough amplitude are always unstable to sideband perturbations. To put this study into context, recall that periodic wavetrains in the full surface water-wave problem may be unstable with respect to modulation by waves of similar but not equal wavelength. In the case of instability, the amplitudes of the so-called sideband modes continue to grow, and a periodic wavetrain literally disintegrates into what seems to be a haphazard combination of waves of various wavelengths. This instability is today known as *modulational instability*, and appears not only in water waves, but also in a range of other dispersive systems. For instance, the instability was first found in nonlinear electromagnetic waves propagating through a liquid [4]. For a historical account of the modulational instability, the reader may consult the recent review by Zakharov and Ostrovsky [32]. In the context of waves on the surface of a body of fluid, Benjamin and Feir [3] established that small but finite-amplitude periodic wavetrains are unstable with respect to a modulational instability. Benjamin [2] found the cut-off separating stable and unstable, small-amplitude wavetrains occurs precisely when the ratio $2\pi h_0/L$ exceeds the value 1.363. For large and intermediate depths, the nonlinear Schrödinger equation and similar models can be used for describing the evolution of wavetrains in the so-called narrow-banded spectrum approximation, and it turns out that the nonlinear Schrödinger equation and most related models feature modulational instability of periodic wavetrains [13,29].

For shallow water waves in the long-wave/shallow-water asymptotic limit mentioned earlier, the generic model equation is the KdV equation, and this equation does not exhibit modulational instability of periodic wavetrains [7]. However, other similar model equations may feature modulational instability, and there have been a number of recent investigations into the modulational stability of periodic solutions of model equations which use mathematical analysis to give definite proofs of stability or instability. See for instance the analyses given in [5,9,23,24]. For a review of asymptotic results on modulational stability, such as the Whitham perturbation method the reader may consult [14].

As explained above, the Whitham equation belongs to a class of equations in which the dispersion relation has been improved (such as the models studied in [21]), but is similar to the KdV equation in the sense that it contains a quadratic nonlinearity. However, since the equation was specifically designed to better approximate short waves, the question arises whether steady

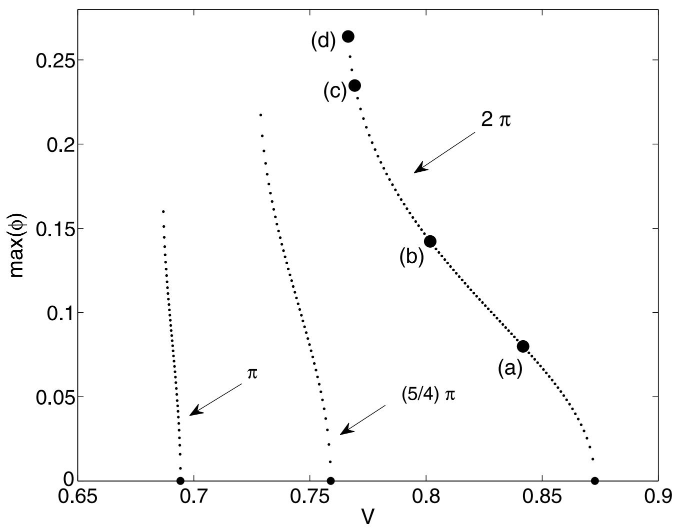

Fig. 2. Bifurcation diagrams for three wavelengths. Wave amplitude is plotted against wave speed. The leftmost curve represents waves with wavelength π, and the bifurcation point is $V^\dagger = 0.6943$. The center curve represents waves with wavelength $5/4\pi$, and the bifurcation point is $V^\dagger = 0.7590$. The rightmost curve represents waves with wavelength 2π, and the bifurcation point is $V^\dagger = 0.8727$. The labels (a), (b), (c) and (d) indicate the points on the bifurcation curve corresponding to the wave profiles shown in Fig. 3.

solutions of the Whitham equation experience modulational instability. The present article contains a numerical study of the stability of steady solutions of the Whitham equation and the results indicate that periodic wave solutions are stable or unstable depending on the wavelength and amplitude. In particular, the dependence of stability on the wavelength is similar to the criterion found by Benjamin for the stability of periodic solutions of the linear water-wave problem. In addition, the numerical results indicate that waves of large amplitude are always unstable.

The KdV equation admits a four-parameter family of periodic traveling-wave solutions which are given in terms of cnoidal functions as

$$\eta = a_0 + \frac{4k^2\kappa^2}{3}\,\mathrm{cn}^2\big(\kappa(x - x_0 - Vt), k\big). \tag{10}$$

Here $\mathrm{cn}(\cdot, k)$ is a Jacobi elliptic function with elliptic modulus k [10] and a_0, m, κ, and x_0 are free parameters. The velocity of this solution is $V = \frac{1}{6}(6 + 9a_0 - 4\kappa^2 + 8k^2\kappa^2)$ and the spatial period is $L = 2\mathcal{K}(k)/\kappa$ where $\mathcal{K}(k)$ is the complete elliptic integral of the first kind.

For the study of steady solutions of the Whitham equation, it is convenient to make the transformation $\eta \mapsto \frac{3}{4}\eta$, so that the equation becomes

$$\eta_t + 2\eta\eta_x + K * \eta_x = 0. \tag{11}$$

Traveling-wave solutions of the Whitham equation of the form $\eta(x,t) = \phi(x - Vt)$ can be sought for certain V, and these are solutions of the equation

$$-V\phi + \phi^2 + K * \phi = 0. \tag{12}$$

In contrast to the KdV equation, there are no known exact solutions of the Whitham equation. However, in the case of periodic boundary conditions, Ehrnström and Kalisch proved the existence of a branch of 2π-periodic traveling-wave solutions to the Whitham equation [19,20]. They also computed numerical approximations of solutions along this branch using a branch-following method. A summary of this method is included in Appendix A. Fig. 2 contains a plot of the maximum of the steady solution versus wave speed for some branches of Whitham solutions. Note that for the purposes of this paper, we refer to the maximum value of the solution as the "amplitude". Fig. 3 contains plots of four representative solutions on the branch of 2π-periodic solutions. If the amplitude of a Whitham solution is small, then it is qualitatively similar to a KdV solution of the form given in Eq. (10). However, as the amplitude of a Whitham solution increases, it becomes much steeper than any KdV solution.

2. Stability analysis

Most mathematical results concerning stability of periodic waves fall into either of two classes. Either, nonlinear stability with respect to perturbations of the same wavelengths is proved, or linear stability or instability with respect to uniformly bounded perturbations is shown. In the context of the current article, works that deal with perturbations of arbitrary spatial period are most relevant. For instance, Bottman and Deconinck [7] proved that all KdV solutions of the form given in (10) are linearly stable regardless of the parameter values. Deconinck and Kapitula [15], establish that cnoidal waves are (nonlinearly) orbitally stable with respect to perturbations that are periodic with period any integer multiple of the cnoidal-wave period. Carter and Cienfuegos [11] numerically established that solutions of the Serre (Green–Naghdi) system with sufficiently small steepness are stable while those with large steepness are unstable. This result is somewhat similar to the results presented below for the Whitham equation. On the other hand, Bridges and Mielke [8] proved that periodic solutions of the full Euler equations are linearly unstable for certain parameter regimes. Very recently, Hur and Johnson [22] proved that small but finite amplitude periodic solutions of the Whitham equation are stable or unstable depending on the wavelength.

Of importance for the current spectral analysis is the recent work of Johnson [23], where modulational stability for a fractional evolution equation of KdV type is considered. In particular, Proposition 3.1 in [23] provides a tool which allows for the use of the Floquet theory in model equations featuring nonlocal operators of fractional type. While the analysis featured in [23] does not explicitly encompass operators such as K, the relevant results for the determination of spectral stability can be extended to apply also to such non-fractional operators. Thus relying on estimates similar to the results of Johnson [23] in the present work, the stability of the periodic Whitham solutions is investigated numerically. In particular, our numerical experiments suggest that the stability of small-amplitude solutions depends on the wavelength of the solutions, while large-amplitude solutions always exhibit modulational instability, regardless of the wavelength. In order to study the stability of solutions to the Whitham equation, we rely on spectral stability analysis and the Fourier–Floquet–Hill method [16]. This method has also been used to study instabilities in other equations, such as in the work of Deconinck and Oliveras [17] who established numerically that a family of stationary, periodic solutions of the one-dimensional Euler equations are unstable.

2.1. Spectral stability analysis

Let $\eta(x,t) = \phi(x - Vt)$ represent a traveling-wave solution of the Whitham equation with speed V. Apply the change of variables $y = x - Vt$ and $\tau = t$ in order to enter a frame moving with the speed of the solution. This change of variables causes the traveling-wave solution to become a stationary solution, $\eta(x,t) \to \eta(y)$, and the Whitham equation to appear as

$$\eta_\tau - V\eta_y + 2\eta\eta_y + K * \eta_y = 0. \tag{13}$$

We consider small perturbations of the stationary solution by assuming

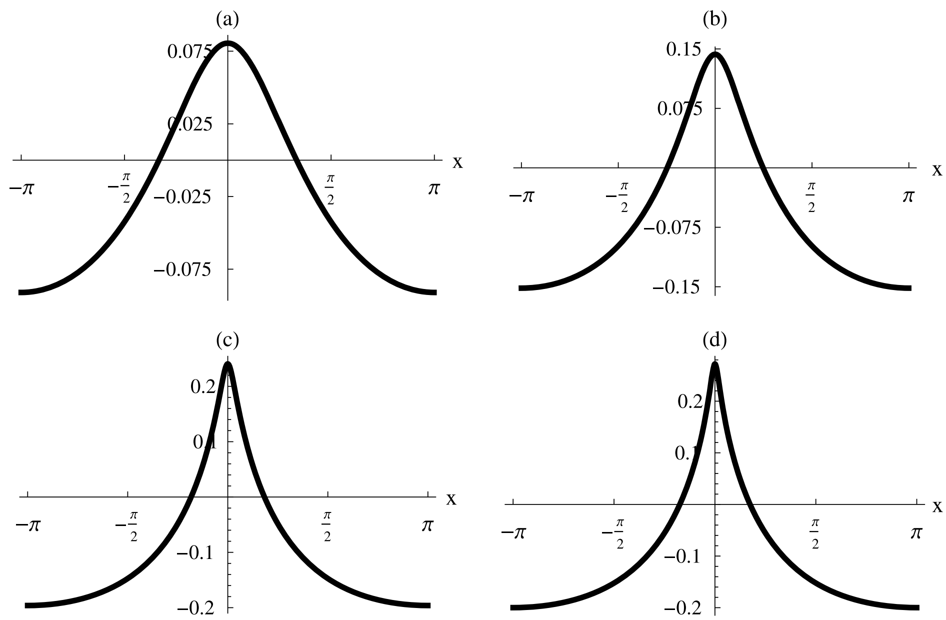

Fig. 3. Four representative 2π-periodic solutions of the Whitham equation. The wave speeds of these solutions are (a) $V = 0.8415$, (b) $V = 0.8002$, (c) $V = 0.7689$, and (d) $V = 0.7665$.

$$\eta(y,\tau) = \eta(y) + \varepsilon w(y,\tau) + \mathcal{O}(\varepsilon^2). \tag{14}$$

Here $w(y,\tau)$ is a real-valued function which will be simply called the *perturbation* and ε is a small positive constant. Substituting (14) into (13), linearizing, and simplifying gives

$$w_\tau + (2\eta - V)w_y + 2w\eta' + K * w_y = 0 \tag{15}$$

This equation is linear and autonomous in τ, so that no generality is lost by assuming that w can be written in the form

$$w(y,\tau) = W(y)\,\mathrm{e}^{\lambda\tau} + \text{c.c.}, \tag{16}$$

where W is a complex-valued function, λ is a complex constant, and c.c. denotes complex conjugate. Substituting (16) into Eq. (15) gives

$$-2\eta' W + (V - 2\eta)W' - K * W' = \lambda W. \tag{17}$$

For both physical and mathematical reasons, we only consider bounded W. This boundedness condition combined with Eq. (17) forms an integro-differential spectral problem for W and λ. If a certain eigenvalue λ solving (17) has positive real part, then there exists at least one perturbation that grows exponentially in time. In turn, this means that the Whitham solution η is spectrally unstable. If all eigenvalues are purely imaginary, then all perturbations oscillate in time and the corresponding solution is spectrally stable.

Since all of the coefficient functions in Eq. (17) are periodic, we use the Fourier–Floquet–Hill method (also known as Bloch theory or Hill's method) [16] in order to numerically solve this problem. First, expand the coefficient functions, $-2\eta'$ and $(V - 2\eta)$, in Fourier series

$$-2\eta' = f_0(y) = \sum_{m=-\infty}^{\infty} \hat{f}_{0,m}\,\mathrm{e}^{imy}, \tag{18a}$$

$$(V - 2\eta) = f_1(y) = \sum_{m=-\infty}^{\infty} \hat{f}_{1,m}\,\mathrm{e}^{imy}. \tag{18b}$$

Second, use Floquet's theorem, which states that the eigenfunctions can be written as

$$W(y) = \mathrm{e}^{i\mu y} \sum_{l=-\infty}^{\infty} \hat{W}_l\,\mathrm{e}^{ily}, \tag{19}$$

where $\mu \in (-\frac{1}{2}, \frac{1}{2}]$ is known as the Floquet parameter and the $\hat{W}_l$ are the to-be-determined Fourier coefficients of the eigenfunction. Note that the standard form of Floquet's theorem applies only to linear ordinary differential equations. By using Proposition 3.2 of [25], one can prove that all bounded eigenfunctions of the system given in Eq. (17) are of the form given in Eq. (19). Substituting (18) and (19) into (17) and separating by Fourier modes leads to

$$\hat{\mathcal{L}}\hat{W} = \lambda\hat{W} \tag{20}$$

where λ is the eigenvalue, $\hat{W}$ is the eigenvector $\hat{W} = (\ldots, \hat{W}_{-2}, \hat{W}_{-1}, \hat{W}_0, \hat{W}_1, \hat{W}_2, \ldots)^T$ and $\hat{\mathcal{L}}$ is the bi-infinite matrix defined by

$$\hat{\mathcal{L}}_{nm} = \begin{cases} \hat{f}_{0,n-m} + i(\mu+m)\hat{f}_{1,n-m} & \text{if } m \neq n, \\ \hat{f}_{0,0} + i(\mu+n)\hat{f}_{1,0} + i(\mu+n)\sqrt{\frac{\tanh(\mu+n)}{\mu+n}} & \text{if } m = n. \end{cases} \tag{21}$$

2.2. Numerical results

In this section we present results from numerical computations of solutions to Eq. (20) for a range of parameter values. As will come to light, a periodic solution of the Whitham equation is generally spectrally stable if it is located on the lower part of the bifurcation branch, that is if it has small enough amplitude. On the other hand, solutions which are higher up on the branch will generally experience modulational instabilities. While the numerical experiments described here are mostly for steady solutions with wavelength 2π, extensive numerical experiments have also been performed for solutions with different wavelengths. It appears that these are similar in the sense that the instability always appears for large enough amplitudes.

2.2.1. Spectra

The spectra corresponding to the 2π-periodic traveling-wave solutions of the Whitham equation shown in Fig. 3 are depicted in Fig. 4. The spectrum in (a) is confined to the imaginary axis, indicating spectral stability of the corresponding solution. The

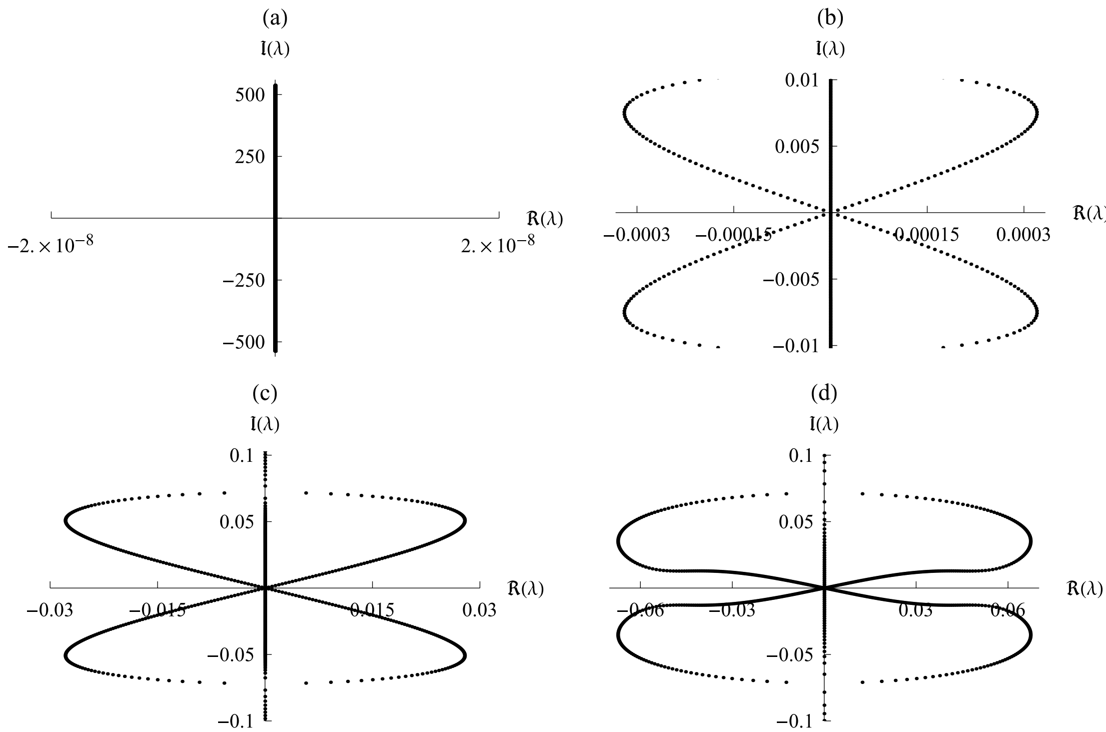

Fig. 4. Spectra corresponding to the solutions shown in Fig. 3.

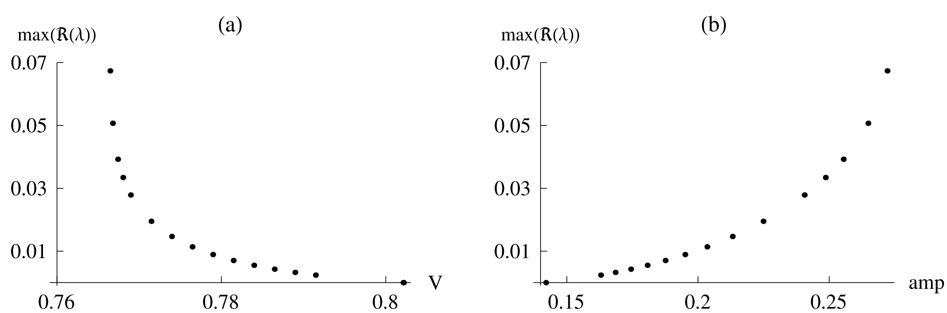

Fig. 5. (a) Maximum growth rate versus solution wave speed for solutions with period 2π. (b) Maximum growth rate versus amplitude for solutions with period 2π.

spectra shown in panels (b)–(d) include positive real parts which are indicative of instability. Together, these four spectra suggest that when the period of the Whitham solution is 2π, small-amplitude solutions are stable while large-amplitude solutions are unstable. Phrased in terms of the wave speed, this means that steady 2π-periodic solutions of the Whitham solution with a large enough speed are spectrally stable while solutions with smaller speeds are unstable. The eigenvalues with the largest real part in Fig. 4(b)–(d) are $0.0003200 \pm 0.007502i$, $0.02788 \pm 0.05078i$, and $0.06735 \pm 0.03530i$ respectively. These eigenvalues are achieved with $\mu = \pm 0.0406$, $\mu = \pm 0.410$, and $\mu = \pm 0.378$ respectively.

As the wave amplitude of the Whitham solution is increased (i.e. as the wave speed is decreased) the spectra undergo a bifurcation where the solutions go from stable to unstable. This bifurcation occurs at an amplitude, A^*, approximately equal to 0.142, which corresponds to a phase speed, V^*, approximately equal to 0.8002. The bifurcation point is characterized by the genesis of a *figure eight* of eigenvalues, and it is well known that such a figure eight is typically associated with the modulational instability [32]. Fig. 5(a) and (b) contain plots of the maximum instability growth rate, $\max(\Re(\lambda))$, versus wave speed and wave amplitude respectively.

Fig. 6 contains plots of $\Re(\lambda)$ versus μ for the solutions shown in Fig. 3. As the stability/instability threshold is crossed, a narrow range of μ values around 0, but not including 0, leads to instabilities. As the amplitude of the solution increases, the maximum growth rate increases and the range of μ values which leads to instabilities increases until all nonzero μ values lead to instability. The periodicity of μ can be seen in Figs. 6(c) and 6(d) where the figure eights can be seen to overlap. One interesting result of our analysis is that $\mu = 0$ never has an eigenvalue with nonzero real part. This means that there are no unstable perturbations having the same spatial period as the Whitham solution.

Fig. 7(a) contains a plot of a Whitham solution close to the limiting case (i.e. it is nearly a cusped wave). Fig. 7(b) contains a plot of the corresponding spectrum and Fig. 7(c) contains a plot of $\Re(\lambda)$ versus μ. The results are somewhat similar to those obtained for less steep solutions. The maximum growth rate is 0.115. Although the simple “figure-eight” structure becomes more complicated, there are no unstable perturbations with the same period as the solution. An additional difference is that there exists non-oscillatory instabilities, i.e. ones that correspond to real, positive eigenvalues, when the Whitham solution is steep enough.

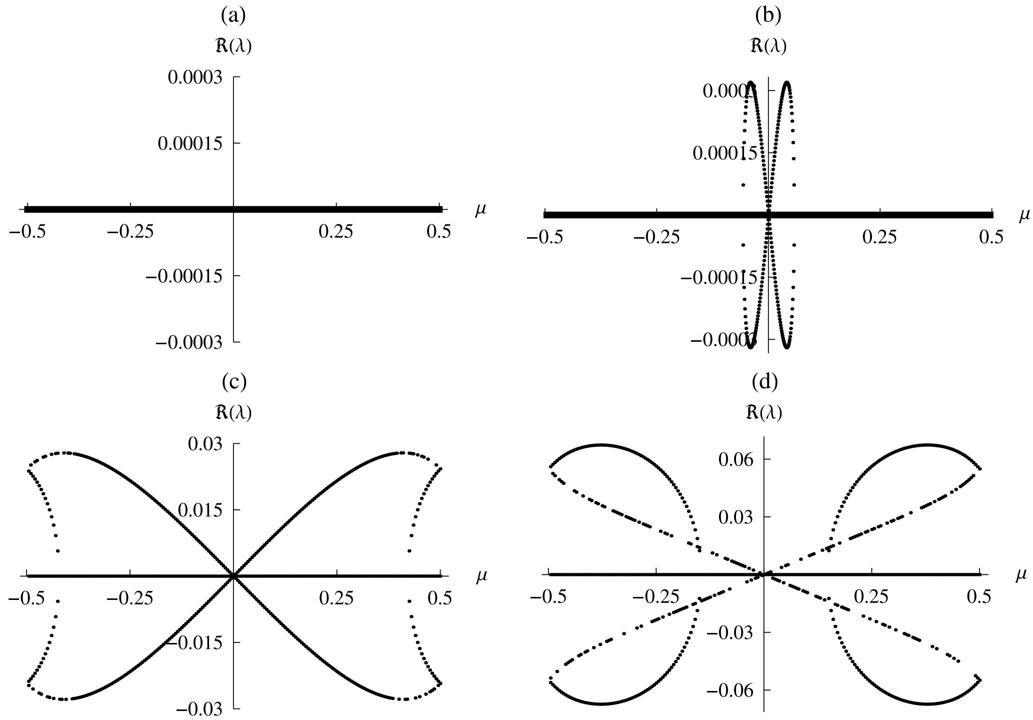

Fig. 6. Plots of $\Re(\lambda)$ versus μ corresponding to Whitham solutions shown in Fig. 3.

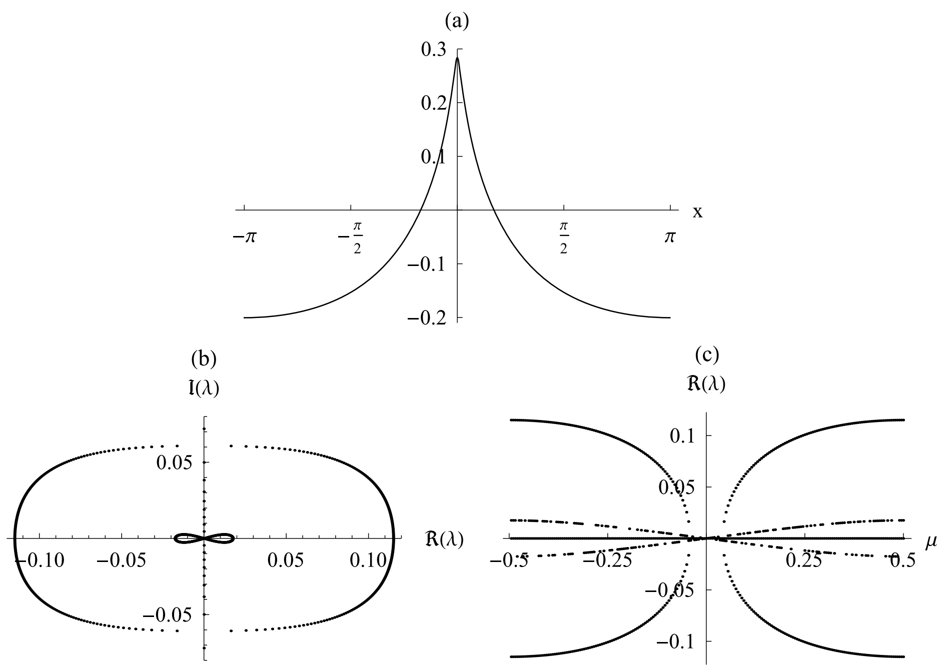

Fig. 7. A nearly cusped wave solution of the Whitham equation (a), the corresponding spectrum (b), and the corresponding plot of $\Re(\lambda)$ versus μ (c).

2.2.2. Eigenfunctions

Fig. 8 contains a plot of $w(y,0)$ corresponding to the solution in Fig. 3(c) with $\lambda = 0.01162 \pm 0.01552i$ and $\mu = 1/8$. This function is periodic on $y \in [-4L, 4L]$. Note that we are plotting $w(y,0)$, not $W(y)$, because $w(y,0)$ is a real-valued function and represents a physical perturbation. See Eq. (16) for the relationship between these two functions. Fig. 9 contains a plot of $w(y,0)$ corresponding to the solution in Fig. 3(c) with $\lambda = 0.02784 \pm 0.04953i$ and $\mu = 2/5$. This function is periodic on $y \in [-\frac{5}{2}L, \frac{5}{2}L]$. A plot of the most unstable perturbation is not included due to its large period.

As an independent test on these stability results, we used the function

$$\eta(x,0) = \eta(x) + 10^{-10} w(y,0), \tag{22}$$

where $\eta(x)$ was the solution given in Fig. 3(c) and $w(y,0)$ was the perturbation given in Fig. 9 as an initial condition in numerical simulations of Eq. (11), the full Whitham equation. The Whitham equation was solved numerically on the interval $x \in [-\frac{5}{2}\pi, \frac{5}{2}\pi]$ using operator splitting so that the linear portion of the PDE could be solved "exactly" in Fourier space. Fig. 10 contains a plot of the logarithm of the amplitude of the perturbation versus t. The slope

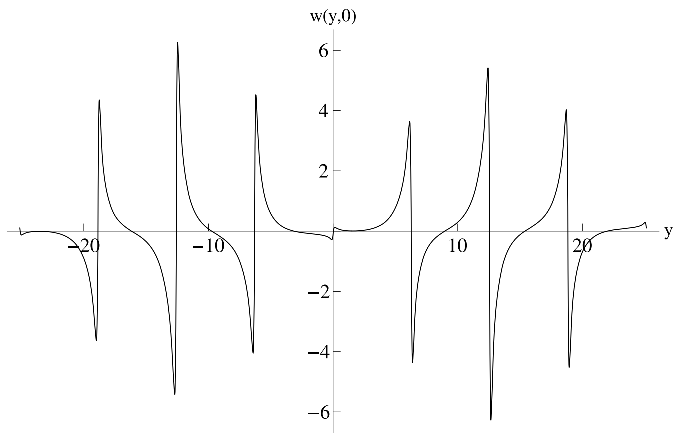

Fig. 8. $w(y,0)$ corresponding to the solution in Fig. 3(c) with $\lambda = 0.01162 \pm 0.01552i$ and $\mu = 1/8$.

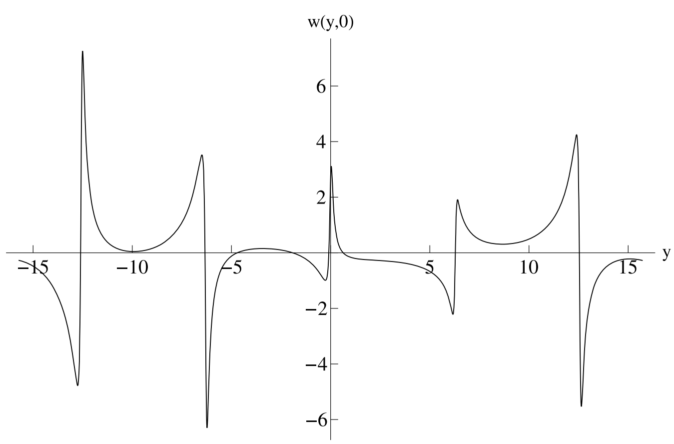

Fig. 9. $w(y,0)$ corresponding to the solution in Fig. 3(c) with $\lambda = 0.02784 \pm 0.04953i$ and $\mu = 2/5$.

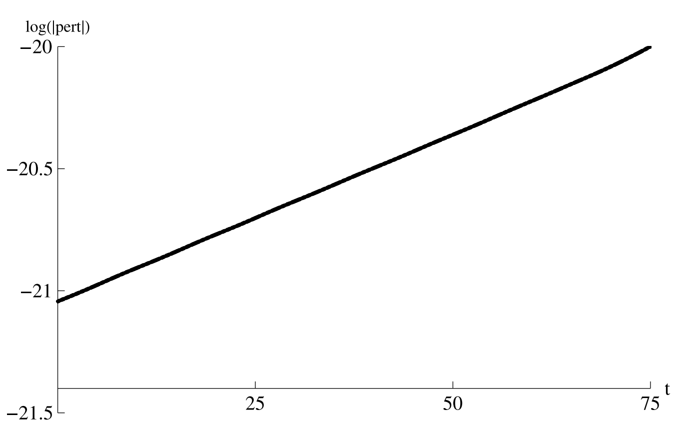

Fig. 10. A plot of the logarithm of the amplitude of the perturbation versus t from the full Whitham equation.

of the line, i.e. the growth rate of the perturbation, is approximately 0.0278 which is very close to the growth rate predicted by the stability analysis. As expected, the exponential growth of the perturbation does not continue indefinitely due to nonlinear effects. In this simulation, nonlinear effects begin to play a role around $t = 78$ (not shown).

2.2.3. Growth rates for varying wavelengths

As explained in the introduction, if the wavelength of periodic solutions of the linearized water-wave problem is reduced to the point where kh_0 exceeds the value 1.363, the waves become unstable. Recently, Hur and Johnson [22] established that a similar mathematical result holds for small-amplitude periodic solutions of the Whitham equation and the critical wavenumber for the Whitham equation is given by $kh_0 = 1.145$. While the main focus of the current work has been on stability and instability of steady wave solutions of finite amplitude, it is also interesting to check if the results of [22] can be corroborated numerically.

Several runs with $h_0 = 1$ fixed, and varying wavenumber k were performed. These experiments confirm the findings of Hur and Johnson that small-amplitude Whitham solutions with $kh_0 < 1.145$ are stable while solutions with $kh_0 > 1.145$ are unstable. However, if the amplitude of the solution is large enough, then the solution is unstable regardless of the value of k.

3. Conclusion

The stability of periodic traveling-wave solutions of the Whitham equation (4) has been studied. It has been found that the stability depends on the amplitude, velocity, and wavelength of the traveling wave. As a representative case, the stability of solutions of the Whitham equation with wavelength 2π and undisturbed depth normalized to $h_0 = 1$ has been illustrated in detail. In this case, solutions with velocity less than $V^* \approx 0.8002$ are unstable, while solutions with larger wave speeds are stable. The critical speed corresponds to an amplitude of $A^* \approx 0.142$, so that solutions with wavelength 2π and amplitude greater than A^* are unstable, while smaller amplitude 2π-periodic solutions are stable. The picture changes if the wavelength of the traveling wave is smaller than 1.75π. In this case, even small amplitude waves are unstable.

This finding is in accordance with a recent study of stability of periodic solutions of the Whitham equation [22] which found that small-amplitude solutions with $kh_0 > 1.145$ are always unstable, while those with $kh_0 < 1.145$ are stable. The difference between the corresponding stability criterion for periodic solutions of the water-wave problem which features a critical value of $kh_0 = 1.363$ may be accredited to features which remain unresolved in the Whitham approximation.

It should also be noted that the results are qualitatively different from the picture concerning the stability of periodic solutions of the KdV equation, where all traveling-wave solutions are stable. The qualitative difference in the behavior of the KdV and Whitham equations suggests that dispersion likely plays a key role in the instabilities of the full Euler equations.

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Appendix A. Numerical bifurcation method

Solutions of (12) are approximated using a spectral cosine collocation method. To define the cosine-collocation projection, first define the subspace

$$S_N = \mathrm{span}_{\mathbb{R}}\left\{\cos(lx) \,\middle|\, 0 \le l \le N-1\right\}$$

of $L^2(0,\pi)$, and the collocation points $x_n = \pi \frac{2n-1}{2N}$ for $n = 1, \ldots, N$. The discretization is defined by seeking ϕ_N in S_N satisfying the equation

$$-V\phi_N + \phi_N^2 + K^N\phi_N = 0, \tag{A.1}$$

where the operator K^N is the discrete form of K. The discrete cosine representation of ϕ_N given by

$$\phi_N(x) = \sum_{l=0}^{N-1} w(l)\Phi_N(l)\cos(lx),$$

where

$$w(l) = \begin{cases} \sqrt{1/N}, & l = 0, \\ \sqrt{2/N}, & l \geq 1, \end{cases}$$

is a normalization constant, and $\Phi_N(l)$ are the discrete cosine coefficients given by

$$\Phi_N(l) = w(l)\sum_{n=1}^{N}\phi_N(x_n)\cos(lx_n), \quad \text{for } l = 0, \ldots, N-1.$$

Now if Eq. (A.1) is enforced at the collocation points x_n, the term $K^N\phi_N$ may be practically evaluated with the help of the matrix $[K^N](m,n)$ by

$$\left[K^N\right]\phi_N(x_m) = \sum_{n=1}^{N}\left[K^N\right](m,n)\phi_N(x_n),$$

where the matrix $[K^N](m,n)$ is defined by

$$\left[K^N\right](m,n) = \sum_{l=0}^{N-1} w^2(l)\sqrt{\frac{1}{l}\tanh l}\cos(lx_n)\cos(lx_m).$$

Thus, Eq. (A.1) enforced at the collocation points x_n yields a system of N nonlinear equations, which can be efficiently solved using a Newton method. The cosine expansion effectively removes the singularities of the Jacobian matrix due to translational invariance and symmetry of the solutions. The non-dimensional speed V is used as the bifurcation parameter. The computation can be started for V close to but smaller than the critical speed V^*, and with an initial guess for the first computation given by a corresponding cos function. In order to compute solutions which are periodic with a wavelength L, we define a scaling $\phi(x) \to \phi(ax)$, where $a = \frac{L}{2\pi}$. Then the equation for ϕ transforms into

$$-V\phi + \phi^2 + \sqrt{a}K_{1/a} * \phi = 0,$$

where $K_{1/a}$ is defined by $\hat{K}_{1/a}(k) = \sqrt{\frac{\tanh(k/a)}{k}}$.

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