
Extremal Widths on Homogeneous Groups

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Communicated by D.-C Chang

(Received 19 June 2003)

We define the extremal length and extremal width of horizontal vector measures on homogeneous groups and study capacities and modules associated with sub-elliptic equations. Coincidence between various modules of horizontal vector measure systems and some specific definitions of capacity is proved. As an application we deduce a reciprocal relation between the $p$-capacity and the $q$-module, $1/p + 1/q = 1$.

Keywords: Homogeneous groups; Carnot–Carathéodory metric; Extremal length; Extremal width; Vector measures; $p$-module of a family of curves

2000 Mathematics Subject Classifications: 31B15; 31C15; 22E30; 28A12

1. INTRODUCTION

The concept of extremal length and the module of a family of curves goes back to Grötzsch, Beurling, Ahlfors [1, 20]. In 1957 Fuglede [17] introduced the $p$-module of a measure system. These notions play an important role and have a lot of applications in analysis and potential theory. An interest to non-linear elliptic equations has inspired a more general notion of the module of a family of curves and the capacity associated with these types of equations [24, 25, 27, 31]. A question about the coincidence of the $p$-module and the $p$-capacity was considered in numerous papers (see, for instance, [5, 26, 39, 40, 46]). Aikawa and Ohtsuka [2] have made an effort to connect the definition of the $p$-capacity, associated with a linear equation of general type, to the definition of $p$-module of relevant vector measure systems.

Recently, the analysis of homogeneous groups (or in another terminology – Carnot groups) has been developed intensively. The fundamental role of such groups in analysis was pointed out by Stein [41, 42]. Briefly, a homogeneous group is a simply connected nilpotent Lie group, whose Lie algebra admits a grading. There is a natural family of dilations on the group under which the metric behaves like the Euclidean metric under the Euclidean dilation [9, 16]. The analysis on homogeneous groups is a test ground for the study of general sub-elliptic problems arising from vector fields.

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ISSN 0278-1077 print; ISSN 1563-5066 online © 2003 Taylor & Francis Ltd
DOI: 10.1080/02781070310001617592
satisfying the Hörmander hypoellipticity condition \cite{15,28}. An important motivation for the study of quasilinear sub-elliptic equations of the second order comes from the theory of quasiconformal and quasiregular mappings on stratified nilpotent groups \cite{10,19,23,37,45}.

In the present work, based on ideas of \cite{2}, we define a horizontal vector measure on homogeneous groups associated with linear sub-elliptic equations. The non-Riemannian geometry of the group and the structure of sub-elliptic equations introduces natural modifications. We prove some monotonicity properties of the Riemannian geometry of the group and the structure of sub-elliptic equations introduced by homogeneous groups. The non-nilpotent groups \cite{10,19,23,37,45}.

A Lie group is stratified and nilpotent if the corresponding Lie algebra is so. A Lie algebra \(G\) is called \emph{nilpotent of step} \(m\), if \(G_{m+1} = \{0\}\), but \(G_m \neq \{0\}\).

We call a Lie algebra to be \emph{graduated}, if it splits into the direct sum of vector spaces \(G = V_1 \oplus V_2 \oplus \ldots \oplus V_k \oplus \ldots\). Here \([V_i, V_j] \subset V_{i+j}\). A Lie algebra \(G\) is called \emph{stratified} if \(G\) is graduated and the subspace \(V_1 \subset G\) generates \(G\) as an algebra according to (2.1). For the nilpotent Lie algebra \(G\) of step \(m\), we have

\[
G = V_1 \oplus \ldots \oplus V_m; \quad [V_1, V_j] = V_{j+1}, \quad j = 1, \ldots, m-1; \quad [V_1, V_m] = \{0\}.
\]

A Lie group is stratified and nilpotent if the corresponding Lie algebra is so.

A homogeneous group \(G\) is a stratified simply connected nilpotent Lie group with the Lie algebra \(G\). Let \(X_{1i}, \ldots, X_{in_i}\) be a basis of the vector space \(V_1 \subset G\), \(n_1 = \dim V_1\). From now on we call \(V_1\) the \emph{horizontal space}. Since the vector fields \(X_{1i}, \ldots, X_{in_i}\) generate the Lie algebra \(G\), one can choose a basis \(X_{ij}, 1 \leq j \leq n_i = \dim V_i, 1 < i \leq m\) of space \(V_i\), such that \(X_{ij} \subset V_i\) are commutators of the vector fields \(X_{ij} \subset V_1, j = 1, \ldots, n_i\). The collection \(X_{11}, X_{12}, \ldots, X_{in_i}\) is an example of vector fields satisfying the Hörmander hypoellipticity condition \cite{28}.

It is known \cite{16}, that if \(G\) is a simply connected nilpotent Lie group with the Lie algebra \(G\), then the exponential map \(\exp : G \to G\) is a global diffeomorphism. Thus, \(dx \circ \exp^{-1}\) is a biinvariant Haar measure on \(G\), where \(dx\) is the Lebesgue measure on \(G\). We can identify the elements \(x \in G\) of the group with the elements \(x \in G\) of the algebra, and thus, with \(x \in \mathbb{R}^N, N = \sum_{i=1}^m \dim V_i\), by the exponential map \(x = \exp(\sum x_{ij} X_{ij})\). The numbers \(x = (x_{ij}), 1 \leq i \leq m, 1 \leq j \leq \dim V_i = n_i\) are called the coordinates of the point \(x\). There is a natural group of dilations, which is defined by

\section{Definitions and Statement of Main Results}

Let \(G\) be a Lie algebra and \(G\) be a corresponding simply connected Lie group. If \(U\) and \(V\) are some sets from \(G\), then we denote by \([U, V]\) the subspace of the algebra \(G\) generated by the elements \([X, Y] = XY - YX, X \in U, Y \in V\). By induction we define the following series

\[
G_1 = G, \quad G_j = [G, G_{j-1}]; \quad G_1 = G, \quad G_j = [G, G_{j-1}]. \tag{2.1}
\]

A Lie algebra \(G\) is called \emph{nilpotent of step} \(m\), if \(G_{m+1} = \{0\}\), but \(G_m \neq \{0\}\).

For the nilpotent Lie algebra \(G\) of step \(m\), we have

\[
G = V_1 \oplus \ldots \oplus V_m; \quad [V_1, V_j] = V_{j+1}, \quad j = 1, \ldots, m-1; \quad [V_1, V_m] = \{0\}.
\]
the rule \( \delta_i x = (r^j x_i)_j \), \( 1 \leq i \leq m \), \( 1 \leq j \leq n_i \). It is easy to see that \( d(\delta_i x) = r^Q dx \). The quantity \( Q = \sum_{i=1}^{m} i \cdot n_i \) is called the homogeneous dimension of the group \( G \).

We use the Carnot–Carathéodory metric based on the length of horizontal curves. A piecewise curve \( \gamma : [0, b] \to G \) is said to be horizontal if its tangent vector \( \dot{\gamma}(s) \) belongs to the space \( V_1 \), i.e., there exist functions \( a_j(s) \), \( s \in [0, b] \), such that

\[
\sum_{j=1}^{n_i} a_j^2 \leq 1 \quad \text{and} \quad \dot{\gamma}(s) = \sum_{j=1}^{n_i} a_j(s) X_j(\gamma(s)).
\]

The result of [8] implies that one can connect two arbitrary points \( x, y \in G \) by a horizontal curve. We fix on \( V_1 \) a non-degenerate quadratic form \( \langle \cdot, \cdot \rangle \), such that the vector fields \( X_{1j}(x), \ldots, X_{in_i}(x) \) are orthonormal with respect to this form at every \( x \in G \). Then the length \( l(\gamma) \) of a curve \( \gamma \) is defined by the formula

\[
l(\gamma) = \int_0^b \langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle^{1/2} ds = \int_0^b \left( \sum_{j=1}^{n_i} |a_j(s)|^2 \right)^{1/2} ds.
\]

The Carnot–Carathéodory distance \( d_c(x, y) \) is the infimum of the length over all horizontal curves connecting \( x \) and \( y \in G \). Since the quadratic form is left-invariant, the Carnot–Carathéodory metric is left-invariant as well. The group \( G \) is connected, therefore the metric \( d_c(x, y) \) is finite (see [43]). For a vector \( \xi \in V_1 \) we shall use the notation \( |\xi| = \langle \xi, \xi \rangle^{1/2} \). The Hausdorff dimension of the metric space \( (G, d_c) \) coincides with its homogeneous dimension \( Q \). By \( \text{mes}(E) \) we denote the measure of the set \( E \):

\[
\text{mes}(E) = \int_E dx.
\]

Our normalizing condition is such that the balls of radius one have the measure one: \( \text{mes}(B(0, 1)) = \int_{B(0, 1)} d\nu = 1 \). Since the Jacobian determinant of the dilation \( \delta_r \) is \( r^Q \), we have that \( \text{mes}(B(\cdot, r)) = r^Q \).

Example 1  The Euclidean space \( \mathbb{R}^n \) with the standard structure is an example of the Abelian Carnot group: the exponential map is the identity and the vector fields \( X_{1j} = \partial/\partial x_j \), \( j = 1, \ldots, n \), have only trivial commutators and form the basis of the corresponding Lie algebra.

Example 2  The simplest example of a non-abelian homogeneous group is the Heisenberg group \( \mathbb{H}^n \). The non-commutative multiplication is defined as

\[
pq = (x, y, t)(x', y', t') = (x + x', y + y', t + t' - 2xy' + 2yx'),
\]

where \( \in \mathbb{R}^n \), \( tx, x', y, y' \in \mathbb{R} \), and the left translation \( L_p(q) = pq \) is defined. The left-invariant vector fields

\[
X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad i = 1, \ldots, n, \quad T = \frac{\partial}{\partial t},
\]

form the basis of the Lie algebra of the Heisenberg group. All non-trivial relations are of the form \( [X_i, Y_i] = -4T, i = 1, \ldots, n \), and all other commutators vanish. Thus, the Heisenberg algebra has the dimension \( 2n + 1 \) and splits into the direct sum
\( G = V_1 \oplus V_2 \). The vector space \( V_1 \) is generated by the vector fields \( X_i, Y_i, i = 1, \ldots, n \), and the space \( V_2 \) is the one-dimensional center which is spanned by the vector field \( T \).

A curve \( \gamma : I = [0, l] \rightarrow G \) is called rectifiable if \( \sup \{ \sum_{k=1}^{p} d_\ast(\gamma(s_k), \gamma(s_{k-1})) \} \) is finite, where the supremum ranges over all partitions \( 0 = s_0 \leq s_1 \leq \ldots \leq s_p = l \) of the segment \( I \). We remark that the definition of a rectifiable curve is based on the Carnot–Carathéodory metric. That is why a curve is not rectifiable if it is not horizontal (see [29]). Thus, from now on we work only with horizontal curves.

Now we define an absolutely continuous function on curves of the horizontal fibration. For this we consider a family of horizontal curves \( X \) that form a smooth fibration of an open set \( U \subset G \). Usually, one can think of a curve \( \gamma \in X \) as an orbit of a smooth horizontal vector field \( X \in V_1 \). If we denote by \( \varphi_t \) the flow associated with this vector field, then the fiber is of the form \( \gamma(s) = \varphi_s(p) \). Here the point \( p \) belongs to the surface \( S \) which is transversal to the vector field \( X \). The parameter \( s \) ranges over an open interval \( J \subset \mathbb{R} \). One can assume that there is a measure \( d\gamma \) on the fibration \( X \) of the set \( U \subset G \). The measure \( d\gamma \) on \( X \) is equal to the inner product of the vector field \( X \in V_1 \) and a biinvariant volume form \( dx \) (for more information see, for instance [30, 44]). The measure \( d\gamma \) satisfies the inequality

\[
[s, t] \omega(B(x, R))^{(Q-1)/Q} \leq \int_{\gamma \in \text{measurable\,X, } \gamma \cap B(x, R) \neq \emptyset} d\gamma \leq k_1 \text{mes}(B(x, R))^{(Q-1)/Q}
\]

for sufficiently small balls \( B(x, R) \subset U \) with constants \( k_0, k_1 \) which do not depend on a ball \( B(x, R) \).

We use the symbol \( \Omega \) for a domain (open connected set) on the homogeneous group.

**Definition 2.1** A function \( u : \Omega \rightarrow \mathbb{R}, \Omega \subset G \), is said to be absolutely continuous on lines \( (u \in ACL(\Omega)) \) if for any domain \( U, \overline{U} \subset \Omega \), and any fibration \( X \) defined by a left-invariant vector field \( X_{1j}, j = 1, \ldots, n_1 \), the function \( u \) is absolutely continuous on \( \gamma \cap U \) with respect to the \( H^1 \)-Hausdorff measure for \( d\gamma \)-almost all curves \( \gamma \in X \).

The derivatives \( X_{1j}u, j = 1, \ldots, n_1 \), exist almost everywhere in \( \Omega \) for such function \( u \) [30]. If they belong to \( L_p(\Omega), p \geq 1 \), for all \( X_{1j} \in V_1 \), then \( u \) is said to be from \( ACL_p(\Omega) \).

A function \( u : \Omega \rightarrow \mathbb{R} \) is said to belong to the Sobolev space \( L^1_p(\Omega) \) if its distributional derivatives \( X_{1j}u \) along the horizontal vector fields \( X_{1j}, j = 1, \ldots, n_1 \), exist, i.e., the equality \( \int_\Omega X_{1j}u \varphi dx = \int_\Omega u X_{1j} \varphi dx \) holds for all \( \varphi \in C_0^\infty(\Omega) \) and the next semi-norm \( \|u \|_p^1(\Omega) = \left( \int_\Omega |\nabla_0 u|^p(x) dx \right)^{1/p} \) is finite. Here \( \nabla_0 u = (X_{11}u, \ldots, X_{1n_1}u) \) is the horizontal gradient of \( u \) and \( |\nabla_0 u| = \left( \sum_{j=1}^{n_1} |X_{1j}u|^2 \right)^{1/2} \). If the function \( u \) belongs to \( L^1_p(\Omega) \), then there exists a function \( v \in ACL_p(\Omega) \), such that \( u = v \) almost everywhere.

Let \( A(x) = (a_{ij}(x)), x \in \Omega \), be a positive definite symmetric \((N \times N)\)-matrix, \( N = \sum_{i=1}^{n_1} \dim(V_i) \), with measurable components \( a_{ij}(x) \), such that

\[
A(x) \eta = \sum_{j=1}^{n_1} c(x) X_{1j}(x) \eta \quad \text{for any vector } \eta \in G, \quad (2.2)
\]

and

\[
\alpha^{-1}|\xi| \leq (A\xi, A\xi)^{1/2} = |A\xi| \leq \alpha|\xi|, \quad (2.3)
\]
for any \( \xi \in V_1 \subset G \) and some constant \( c \geq 1 \). Let \( \mathcal{B}(x) = (b_{ij}(x)) \) be the inverse matrix for \( A(x) \). The matrix \( B(x) \) also satisfies the inequality (2.3).

We recall the definition of the \( p \)-module of a system of measures [17]. Let \( f \) be a non-negative Borel measurable function and \( \mu \) be a non-negative Borel measure. If \( \int f \, d\mu \geq 1 \), then we say that the function \( f \) is admissible for the measure \( \mu \). Let \( \mathcal{E} \) be a system of non-negative Borel measures. If \( f \) is admissible for all \( \mu \in \mathcal{E} \), then we denote by \( \mathcal{F}M(\mathcal{E}) \) the set of admissible functions for the module of the system of measures \( \mathcal{E} \). The quantity

\[
M_p(\mathcal{E}) = \inf \left\{ \int f^p \, dx : f \geq 0, \, f \in \mathcal{F}M(\mathcal{E}) \right\}
\]

is called the \( p \)-module of \( \mathcal{E} \).

We define a system of vector measures on the homogeneous group, which is related to the stratified structure of the Lie algebra of \( G \). Let \( \mu = (\mu_1, \ldots, \mu_m) \) be a vector measure whose components \( \mu_i \) are signed measures defined for sets from \( G \). Our principal assumption is that the dimension of the vector measure is equal to \( n_1 \) and coincides with the dimension of \( V_1 \subset G \), so it is natural to call this measure the horizontal vector measure. The total variation \( |\mu| \) of \( \mu \) is defined by

\[
|\mu|(E) = \sup \sum_j |\mu(E_j)| = \sup \sum_j \left( \sum_i^{n_1} \mu_i(E_j) \right)^{1/2}
\]

for Borel sets \( E \), where the supremum is taken over all finite partitions of \( E \) into Borel sets \( E_j \). The total variation \( \mu \) is a non-negative measure. We define an exceptional set for a system of vector measures in terms of vanishing \( p \)-module of total variation of these measures.

**Definition 2.2** Let \( \mathcal{M} \) be a set of vector measures \( \mu \). We put \( |\mathcal{M}| = \{|\mu|: \mu \in \mathcal{M}\} \). If \( M_p(|\mathcal{M}|) = 0 \), then we say that \( \mathcal{M} \) is \( p \)-exceptional. If a statement with respect to vector measures is not satisfied only for a \( p \)-exceptional system \( \mathcal{M} \), then we say that it holds \( p \)-almost everywhere.

We put \( K_0 \) and \( K_1 \) to be closed non-empty disjoint sets, such that \( K_0 \cap \overline{\Omega} \neq \emptyset \) and \( K_1 \cap \overline{\Omega} \neq \emptyset \). The triple \((K_0, K_1; \Omega)\) we will call the condenser.

Let \([a, b]\) be an interval of one of the following types: \([a, b]\), \([a, b), (a, b]\), or \((a, b)\). From now on, we suppose that a horizontal curve \( \gamma: [a, b] \to G \) is parameterized by the length element. We set

\[
\Gamma = \Gamma(K_0, K_1; \Omega) = \left\{ \gamma: \gamma([a, b]) \cap K_i \neq \emptyset, \, i = 0, 1, \, \gamma(t) \in \Omega, \, t \in (a, b) \right\}
\]

and call by \( \Gamma(K_0, K_1; \Omega) \) the family of curves that connect the compacts \( K_0 \) and \( K_1 \) in the domain \( \Omega \).

Now we give two different definitions of \( A_p \)-capacity of a condenser.

**Definition 2.3** We denote by \( \mathcal{F}C(K_0, K_1; \Omega) \) a class of functions \( u \in ACL_p(\Omega) \), such that

\[
u(x) \to 0 \quad \text{as} \quad x \to K_0 \cap \overline{\Omega} \quad \text{along} \ p \text{-almost all curves from} \ \Gamma(K_0, K_1; \Omega),
\]

\[
u(x) \to 1 \quad \text{as} \quad x \to K_1 \cap \overline{\Omega} \quad \text{along} \ p \text{-almost all curves from} \ \Gamma(K_0, K_1; \Omega).
\]
The $A_p$-capacity of the condenser $(K_0, K_1; \Omega)$ is defined by

$$ \text{cap}_{A_p}(K_0, K_1; \Omega) = \inf \left\{ \int_{\Omega} |A\nabla u|^p \, dx \mid u \in FC(K_0, K_1; \Omega) \right\}. $$

**Definition 2.4** Let $\mathcal{F}C^*(K_0, K_1; \Omega)$ be a class of functions $u \in ACL_p(\Omega)$, such that

- $u(x) = 0$ on the intersection of $\Omega$ with a neighborhood of $K_0$,
- $u(x) = 1$ on the intersection of $\Omega$ with a neighborhood of $K_1$.

We define $A_p^*$-capacity by the next value

$$ \text{cap}_{A_p^*}(K_0, K_1; \Omega) = \inf \left\{ \int_{\Omega} |A\nabla u|^p \, dx \mid u \in \mathcal{F}C^*(K_0, K_1; \Omega) \right\}. $$

Let us observe that results from [32, 39] imply that $ACL_p$-function is absolutely continuous on $p$-almost all horizontal curves. So we can state that an admissible function is an absolutely continuous function on $p$-almost all horizontal curves and its horizontal gradient $|\nabla_0 u|$ belongs to $L_p(\Omega)$. Capacities associated with sub-elliptic equations were studied in [4, 6, 7, 11, 12, 21, 34–36].

We give the definition of the $A_p$-module of a system of horizontal vector measures associated with Definitions 2.3 and 2.4. Let $\zeta(x) = (\zeta_1(x), \ldots, \zeta_n(x))$ be a vector-valued function at each $x \in \Omega$, $\Omega \subset \mathbb{G}$. If $\int |\zeta_i| \, d\mu < \infty$ for all $i = 1, \ldots, n$, then we define $\int \zeta \cdot d\mu = \sum_{i=1}^n \int \zeta_i \, d\mu_i$. If $\zeta$ is such that $\int \zeta \cdot d\mu \geq 1$ for all $\mu \in \mathcal{M}$, then we call $\zeta$ the admissible (vector-valued) function for the system $\mathcal{M}$ and write $\zeta \in \mathcal{F}M(\mathcal{M})$.

**Definition 2.5** Let $\xi$ be an admissible vector-valued function and let $\mu \in \mathcal{M}$ be a complete horizontal vector measure on $\Omega \subset \mathbb{G}$. We define the $A_p$-module as

$$ M_{A_p}(\mathcal{M}) = \inf \left\{ \int_{\Omega} |A\xi|^p \, dx \mid \xi \in \mathcal{F}M(\mathcal{M}), \ p\text{-almost everywhere} \right\}. $$

We put the condition $p$-almost everywhere to avoid nonsense. For example, let us choose some horizontal vector field $X_i$, its orbit $\beta_i$, and the one-dimensional Hausdorff measure $d\beta_i$ on $\beta_i$. We fix an arc $C \subset \beta_i$ of finite length. Let us consider the horizontal vector measure system $\mathcal{M} = \{(0, \ldots, d\beta_i|_{c}, \ldots, 0), (0, \ldots, -d\beta_i|_{c}, \ldots, 0)\}$. There is no admissible vector-valued function $\xi$ for $\mathcal{M}$. However, since $M_p(|\mathcal{M}|) = 0$, the $p$-exceptional set coincides with $\mathcal{M}$, and therefore, $M_{A_p}(\mathcal{M}) = 0$.

**Example 3** If $\Gamma$ is a family of horizontal curves, then we have, naturally, horizontal vector measures $d\gamma$, $\gamma \in \Gamma$, and measures $|d\gamma| = (d\gamma, d\gamma)^{1/2}$. We write $d\Gamma = \{d\gamma \mid \gamma \in \Gamma\}$, and $|d\Gamma| = \{|d\gamma| \mid \gamma \in \Gamma\}$.

**Example 4** The horizontal gradient of an $ACL$-function is another example of a horizontal vector measure. We will work with $\nabla_0 C^* = \{\nabla_0 u \mid u \in \mathcal{F}C^*(K_0, K_1; \Omega)\}$. More generally, for a positive definite $(N \times N)$-matrix $Q(x) = (q_{ij}(x))$, we write...
\[ |Q d\Gamma| = \{ |Q d\gamma| = |Q d\gamma, Q d\gamma|^{1/2} : \gamma \in \Gamma \} \text{ and } |Q \nabla u| = |Q \nabla u, Q \nabla u|^{1/2} ; u \in F C^* (K_0, K_1; \Omega). \]

In [33] we have obtained the next relationships between capacities and modules

\[ \text{cap}_{A_p}^* (K_0, K_1; \Omega) = \text{cap}_{A_p} (K_0, K_1; \Omega) = M_{A_p} (d\Gamma) = M_p (|B d\Gamma|) < \infty. \tag{2.5} \]

If the capacity \( \text{cap}_{A_p}^* (K_0, K_1; \Omega) \) is strictly positive, then

\[ \left( \text{cap}_{A_p}^* (K_0, K_1; \Omega) \right)^{1/p} (M_{B_p} (\nabla C^*))^{1/q} = 1, \quad 1/p + 1/q = 1. \tag{2.6} \]

If \( \text{cap}_{A_p}^* (K_0, K_1; \Omega) = 0 \), then \( M_{B_p} (\nabla C^*) = \infty. \)

In the present article we study families of measures, that in some sense “separate” compacts \( K_0, K_1 \in \Omega \). We use the definition for a function of bounded variation on homogeneous groups from [3]. In the case \( G = \mathbb{R}^n \ X_{ij} = \partial / \partial x_j, j = 1, \ldots n \), this definition reduces to a classical definition by De Giorgi [13]. Let

\[ F(\Omega) = \left\{ \xi(x) = (\xi_1(x), \ldots, \xi_n(x)) : \xi_j(x) \in C^1_0 (\Omega), \ \sup_{x \in \Omega} |\xi(x)| \leq 1 \right\}. \]

For a given \( u \in L_{1, \text{loc}} (\Omega) \) the variation of \( u \) in \( \Omega \) is defined as

\[ \text{Var}(u) = \sup \left\{ \int_{\Omega} u(x) \sum_{j=1}^n X_{ij} \xi_j(x) dx : \xi \in F(\Omega) \right\}. \]

A function \( u \) is said to have bounded variation in \( \Omega \) if \( \text{Var}(u) < \infty \). In this case we shall write \( u \in BV (\Omega) \). The divergence theorem easily gives the equality \( \text{Var}(u) = \int_{\Omega} |\nabla u(x)| dx \) for functions \( u \in W_1^1 (\Omega) \) (see [18]). We shall continue to use the notation \( \nabla u(x) \) for the horizontal distributive gradient of a function of bounded variation. As in [14] one can show that \( \nabla u(x) \) is a vector measure on \( G \) and \( \text{Var}(u) \) is the total variation of the distributive gradient \( \nabla u(x) \). If \( E \subset G \) is measurable, then the perimeter of \( E \) relative to \( \Omega \) is defined by \( P(E; \Omega) = \text{Var} (\chi_E) \), where \( \chi_E \) denotes the characteristic function of \( E \).

Now we introduce some new families of horizontal vector measures and then formulate relations between them. Let \( \Omega \subset G \) be a bounded domain. We let \( S = S(K_0, K_1; \Omega) \) be a family of functions \( u \in BV (\Omega) \), such that

\[ u(x) = 0 \text{ on the intersection of } \Omega \text{ with a neighborhood of } K_0, \]

\[ u(x) = 1 \text{ on the intersection of } \Omega \text{ with a neighborhood of } K_1. \]

Let \( U \subset G \) be an open set, such that \( K_1 \subset U, K_0 \cap \overline{U} = \emptyset \), and \( E = U \setminus \Omega \). We denote by \( \Sigma = \Sigma (K_0, K_1; \Omega) \) the family of characteristic functions \( \chi_E \) of the set \( E \subset \Omega \), \( P(E; \Omega) \ll \infty \). We put

\[ \nabla_0 S = \left\{ \nabla_0 u : u \in S(K_0, K_1; \Omega) \right\}, \quad |\nabla_0 S| = \left\{ |\nabla_0 u| : u \in S(K_0, K_1; \Omega) \right\}, \]

\[ \nabla_0 \Sigma = \left\{ \nabla_0 \chi_E : \chi_E \in \Sigma (K_0, K_1; \Omega) \right\}, \quad |\nabla_0 \Sigma| = \left\{ |\nabla_0 \chi_E| : \chi_E \in \Sigma (K_0, K_1; \Omega) \right\}. \]
More generally, we set
\[
|Q\nabla_0 S| = \left\{ |Q\nabla_0 u| : u \in S(K_0, K_1; \Omega) \right\}, \quad |Q\nabla_0 \Sigma| = \left\{ |Q\nabla_0 \chi_E| : \chi_E \in \Sigma(K_0, K_1; \Omega) \right\},
\]
for a positive definite symmetric \((N \times N)\)-matrix \(Q\).

**Theorem 2.1** Let \(\Omega \subset \mathbb{G}\) be a bounded domain, then
\[
M_{B_q}(\nabla_0 S) = M_{B_q}(\nabla_0 \Sigma) = M_q(|A\nabla_0 S|) = M_q(|A\nabla_0 \Sigma|) = M_{B_q}(\nabla_0 C^*) > 0.
\]

As a result of Theorem 2.1 and the relations (2.5) and (2.6) we obtain the following reciprocal relations between the extremal length and the extremal width.

**Corollary 2.1** Suppose that \(B\) is uniformly continuous in a bounded domain \(\Omega \subset \mathbb{G}\), \(1/p + 1/q = 1\), and \(\Gamma\) is the family of curves (2.4). If \(M_{A_q}(d\Gamma) = 0\), then \(M_{B_q}(\nabla_0 \Sigma) = \infty\). If \(M_{A_q}(d\Gamma) > 0\), then
\[
\left( M_{A_q}(d\Gamma) \right)^{1/p} \left( M_{B_q}(\nabla_0 \Sigma) \right)^{1/q} = 1.
\]

If \(A = B\) are the identity matrix, then we get

**Corollary 2.2** Let \(\Omega\) be a bounded domain and \(1/p + 1/q = 1\). If \(M_p(|d\Gamma|) = 0\), then \(M_q(|\nabla_0 \Sigma|) = \infty\). If \(M_p(|d\Gamma|) > 0\), then \(\left( M_p(|d\Gamma|) \right)^{1/p} \left( M_q(|\nabla_0 \Sigma|) \right)^{1/q} = 1\).

3. PRELIMINARY RESULTS

By definition, the \(M_{A_q}(\mathcal{M})\) is monotone. For the completeness we give the proof of the next property (see also [2, 46]).

**Lemma 3.1** Let \(\mathcal{M}_i\) be an increasing sequence of horizontal vector measures, such that \(\mathcal{M} = \bigcup_{i=1}^{\infty} \mathcal{M}_i\). Then \(\lim_{i \to \infty} M_{A_q}(\mathcal{M}_i) = M_{A_q}(\mathcal{M})\).

**Proof** Let us denote by \(c\) the limit of \(M_{A_q}(\mathcal{M}_i)\) as \(i \to \infty\). It is sufficient to show that \(M_{A_q}(\mathcal{M}) \leq c\). Fix \(\varepsilon > 0\). There exists \(\xi_i \in \mathcal{F}M(\mathcal{M}_i)\), such that
\[
M_{A_q}(\mathcal{M}_i) \leq \int_\Omega |A\xi_i|^p \, dx \leq M_{A_q}(\mathcal{M}_i) + \varepsilon \tag{3.1}
\]
for \(p\)-almost all \(\mu \in \mathcal{M}\) and for each \(i \in \mathbb{N}\). Since \((\xi_i + \xi_k)/2 \in \mathcal{F}M(\mathcal{M}_i)\) \(p\)-almost everywhere for \(i \leq k\), we also have \(M_{A_q}(\mathcal{M}_i) \leq \int_\Omega |A((\xi_i + \xi_k)/2)|^p \, dx\). The sequence \(\{\xi_i\}\) is a Cauchy sequence in \(L_p(\Omega)\), because of
\[
\int_\Omega \left| \frac{\xi_i - \xi_k}{2} \right|^p \, dx \leq \alpha \int_\Omega \left| A\frac{\xi_i - \xi_k}{2} \right|^p \, dx \leq \alpha \left( \frac{1}{2} \int_\Omega |A\xi_i|^p \, dx + \frac{1}{2} \int_\Omega |A\xi_k|^p \, dx - \int_\Omega \left| A\frac{\xi_i + \xi_k}{2} \right|^p \, dx \right) \to 0.
\]
We used (2.3) and Clarkson’s inequalities. Therefore, the sequence \( \{\xi_i\} \) tends to some \( \xi \) in \( L_p(\Omega) \) with \( \int_\Omega |A\xi|^p \, dx = c \) by (3.1). By a property of measure system we can find a subsequence \( \{\xi_{i_m}\} \), such that \( \int |\xi - \xi_{i_m}| \, d\mu \rightarrow 0 \) for \( p \)-almost all \( \mu \in \mathcal{M} \) [17 Theorem 3 (f)]. This implies that \( \int (\xi - \xi_{i_m}) \cdot d\mu \rightarrow 0 \) \( p \)-almost everywhere and

\[
\int \xi \cdot d\mu = \int \xi_{i_m} \cdot d\mu + \int (\xi - \xi_{i_m}) \cdot d\mu \leq \liminf \int \xi_i \cdot d\mu \geq 1 \quad \text{for } p - \text{a.a. } \mu \in \mathcal{M}.
\]

Since \( \xi \in \mathcal{F}\mathcal{M}(\mathcal{M}) \), finally, we obtain \( M_{A_p}(\mathcal{M}) \leq \int_\Omega |A\xi|^p \, dx = c \).

For a moment, let us denote by \( F \) one of the sets \( C^* \), \( S \), \( \Sigma \), and by \( \nabla_0 F \) one of the sets \( \nabla_0 C^*, \nabla_0 S, \nabla_0 \Sigma \), respectively.

**Corollary 3.1** Let \( K_0, K_1 \subset \Omega \) be disjoint compacts and let \( K^j_0, K^j_1 \in \Omega \) be sequences of compact sets, such that \( K^0_0 \cap K^0_1 = \emptyset, K^{j+1}_1 \subset \text{int } K^j_1, K^{j+1}_1 \subset \text{int } K^j_1, K_0 = \bigcap_{i=0}^\infty K^0_i, K_1 = \bigcap_{i=0}^\infty K^1_i \). Then, \( M_{A_p}(\nabla_0 F(\Omega_0, K_1; \Omega)) = \lim_{i \to \infty} M_{A_p}(\nabla_0 F(\Omega_i, K_1; \Omega)) \).

**Proof** Since the measure system \( \nabla_0 F(K^j_0, K^j_1; \Omega) \) is increasing and \( \nabla_0 F(K_0, K_1; \Omega) = \bigcup_j \nabla_0 F(K^j_0, K^j_1; \Omega) \), Corollary 3.1 follows from Lemma 3.1.

**Lemma 3.2** Let \( \Omega_0 \) be an open subset of \( \Omega \), such that \( K_0, K_1 \subset \overline{\Omega}_0 \). Then,

\[
M_{A_p}(\nabla_0 F(K_0, K_1; \Omega_0)) \leq M_{A_p}(\nabla_0 F(K_0, K_1; \Omega_0)).
\] (3.2)

**Proof** If \( M_{A_p}(\nabla_0 F(K_0, K_1; \Omega_0)) = \infty \), then there is nothing to prove. Let us assume that the right-hand side of (3.2) is finite and \( \xi \in \mathcal{F}\mathcal{M}(\nabla_0 F(K_0, K_1; \Omega_0)) \). We note that if \( u \in F(\Omega_0, K_1; \Omega_0) \), then \( u|_{\Omega_0} \in F(\Omega_0, K_1; \Omega_0) \). The function

\[
\eta(x) = \begin{cases} 
\xi(x) & \text{if } x \in \Omega_0, \\
0 & \text{otherwise}
\end{cases}
\]

is admissible for \( M_{A_p}(\nabla_0 F(K_0, K_1; \Omega)) \) in view of \( \int_\Omega \eta(x) \cdot \nabla_0 u = \int_{\Omega_0} \xi \cdot \nabla_0(u|_{\Omega_0}) \geq 1 \). Hence,

\[
M_{A_p}(\nabla_0 F(K_0, K_1; \Omega)) \leq \int_\Omega |A\eta|^p \, dx = \int_{\Omega_0} |A\xi|^p \, dx.
\]

Taking the infimum with respect to \( \xi \in \mathcal{F}\mathcal{M}(\nabla_0 F(K_0, K_1; \Omega_0)) \), we obtain (3.2).

**Lemma 3.3** Let \( K_0, K_1 \subset \Omega_0 \subset \overline{\Omega}_0 \subset \Omega \). There is a sequence of open sets \( \Omega_0 \subset \Omega_1 \subset \ldots \subset \overline{\Omega}_1 \subset \ldots \subset \overline{\Omega} \subset \Omega \), such that

\[
\lim_{i \to \infty} M_{A_p}(\nabla_0 F(K_0, K_1; \Omega_i)) = M_{A_p}(\nabla_0 F(K_0, K_1; \overline{\Omega})).
\] (3.3)

**Proof** We choose a positive monotone function \( r < d_e(\Omega_0, \Omega) \) that tends to 0. The sequence of sets \( \Omega(r) = \{ x \in \Omega : d_e(x, \partial \Omega) > r \} \) exhausts \( \Omega \) as \( r \to 0 \). Since \( m(r) = M_{A_p}(\nabla_0 F(K_0, K_1; \Omega(r))) \) is a non-decreasing function, it is continuous from the right.
except for some countable set of \( r \). Therefore, there is a value \( r = \tilde{r} > 0 \), such that \( \lim_{r \to \tilde{r}} m(r) = m(\tilde{r}) \). We put \( \tilde{\Omega} = \Omega(\tilde{r}) \) and complete the proof. \( \blacksquare \)

From now on, we will say that \( \tilde{\Omega} \) can be approximated from inside with respect to the module \( M_{A_p}(\nabla_0 F(K_0, K_1; \tilde{\Omega})) \) if \( \tilde{\Omega} \) satisfies (3.3).

**Lemma 3.4** Let \( K_0, K_1 \subset \Omega \) and the domain \( \Omega \) can be approximated from inside with respect to the module \( M_{A_p}(\nabla_0 C^*(K_0, K_1; \Omega)) \). Then,

\[
M_{A_p}(\nabla_0 \Sigma(K_0, K_1; \Omega)) \leq M_{A_p}(\nabla_0 S(K_0, K_1; \Omega)) \leq M_{A_p}(\nabla_0 C^*(K_0, K_1; \Omega)).
\]

**Proof** The first inequality of (3.4) is a consequence of the inclusion \( \nabla_0 \Sigma \subset \nabla_0 S \).

To prove the second one we fix \( \varepsilon > 0 \). The condition of the theorem implies that there is \( \tilde{\Omega} \subset \Omega \), such that

\[
M_{A_p}(\nabla_0 C^*(K_0, K_1; \tilde{\Omega})) \leq M_{A_p}(\nabla_0 C^*(K_0, K_1; \Omega)) + \varepsilon.
\]

We can find compact sets \( K_0^0, K_1^0 \subset \Omega \) that possesses the inequality

\[
M_{A_p}(\nabla_0 S(K_0, K_1; \Omega)) \leq M_{A_p}(\nabla_0 S(K_0^0, K_1^0; \Omega)) + \varepsilon
\]

by Corollary 3.1. If we prove

\[
M_{A_p}(\nabla_0 S(K_0^0, K_1^0; \Omega)) \leq M_{A_p}(\nabla_0 C^*(K_0, K_1; \tilde{\Omega})),
\]

then the second inequality of 3.4 will follow from (3.5)–(3.7) by arbitrariness of \( \varepsilon \).

To show (3.7) we take the domain \( \tilde{\Omega} = \{ x \in \Omega : d_e(x, \partial \Omega) > (1/2)d_e(\partial \tilde{\Omega}, \partial \Omega) \} \) and a positive number \( r < \min\{(1/2)d_e(\partial \tilde{\Omega}, \partial \Omega), d_e(\partial K_0, \partial K_0^0), d_e(\partial K_1, \partial K_1^0)\} \). We choose a function \( \xi \in \mathcal{F}M(\nabla_0 C^*(K_0, K_1; \tilde{\Omega})) \) and define

\[
\eta(x) = \begin{cases} 
\xi(x) & \text{if } x \in \tilde{\Omega}, \\
0 & \text{otherwise.}
\end{cases}
\]

Let \( \eta_r(x) = \int_{B(0,1)} \eta(x, y) \psi(y) dy \), where \( \psi(y) \) is a non-negative \( C^\infty \)-function supported in \( B(0,1) \). The function \( \eta_r \) belongs to \( C^\infty(\tilde{\Omega}) \) by definition and we also claim

\[
\eta_r \in \mathcal{F}M(\nabla_0 C^*(K_0^0, K_1^0; \Omega)) \quad \text{and} \quad \eta_r \in \mathcal{F}M(\nabla_0 S(K_0^0, K_1^0; \Omega)).
\]

We note that if \( u(x) \in \mathcal{F}C^*(K_0^0, K_1^0; \Omega) \), then \( u(x(\delta_e y)^{-1}) |_{\Omega} \in \mathcal{F}C^*(K_0, K_1; \tilde{\Omega}) \) for \( |y| \leq 1 \). Hence,

\[
\int_{\Omega} \eta_r(x) \nabla_0 u(x) \, dx = \int_{B(0,1)} \psi(y) dy \int_{\Omega} \eta(x, y) \nabla_0 u(x) \, dx
\]

\[
= \int_{B(0,1)} \psi(y) dy \int_{\Omega} \xi(x) \nabla_0 u(x, \delta_e y)^{-1} \, dx \geq 1
\]

and the first assertion of (3.8) is obtained.
To show the second statement of (3.8), we choose \( u \in S(K_0^0, K_1^0; \Omega) \). In [18] it was proved that for \( u \in BV(\Omega) \) there exists a sequence \( \{u_k\}_{k \in \mathbb{N}} \) in \( C^{\infty}(\Omega) \), such that the following holds: \( \lim_{k \to \infty} \|u_k - u\|_{L^1(\Omega)} = 0 \), \( \lim_{k \to \infty} \text{Var}(u_k, \Omega) = \text{Var}(u, \Omega) \), and \( \int_{\Omega} \varphi \cdot \nabla u_k \, dx \to \int_{\Omega} \varphi \cdot \nabla u \) for any vector-valued function \( \varphi \in C_0^\infty(\Omega) \). Obviously, we can find \( u_k \) that vanishes in some neighborhood of \( K_0^0 \) and \( u_k = 1 \) in a neighborhood of \( K_1^0 \).

We should consider two options. If \( \nabla_0 u_k \in L_p(\Omega) \), then \( u_k \in FC^*(K_0^0, K_1^0; \Omega) \) and

\[
1 \leq \lim_{k \to \infty} \int_{\Omega} \eta_r \nabla_0 u_k \, dx = \int_{\Omega} \eta_r \cdot \nabla_0 u
\]

by the first assertion of (3.8). The horizontal vector measure \( \nabla_0 u \) does not need to be absolutely continuous with respect to \( dx \). This proves the second relation of (3.8). Now, we can conclude that \( M_{A_0}(\nabla_0 S(K_0^0, K_1^0; \Omega)) \leq \int_{\Omega} |A \eta_r|^p \, dx \). Letting \( r \to 0 \), we obtain the inequality \( M_{A_0}(\nabla_0 S(K_0^0, K_1^0; \Omega)) \leq \int_{\Omega} |A \eta|^p \, dx = \int_{\Omega} |A \xi|^p \, dx \). Taking the infimum with respect to \( \xi \in \mathcal{F}M(\nabla_0 C^*(K_0^0, K_1^0; \Omega)) \) we complete the lemma in this case.

If \( \nabla_0 u_k \notin L_p(\Omega) \), then we take a function \( v \in C_0^\infty(\Omega) \), \( 0 \leq v \leq 1 \) in \( \Omega \), and \( v = 1 \) in \( \tilde{\Omega} \). Since \( vu_k \in C_0^\infty(\Omega) \) we obtain \( vu_k \in FC^*(K_0^0, K_1^0; \Omega) \) and complete the lemma as above.

\[\square\]

4. PROOF OF THEOREM 2.1

To prove Theorem 2.1 we will show four inequalities:

\[
M_{B_q}(\nabla_0 S) \leq M_{B_q}(\nabla_0 C^*), \tag{4.1}
\]

\[
M_q(|A \nabla_0 \Sigma|) \geq M_{B_q}(\nabla_0 C^*), \tag{4.2}
\]

\[
M_{B_q}(\nabla_0 S) \geq M_q(|A \nabla_0 S|) \geq M_q(|A \nabla_0 \Sigma|), \tag{4.3}
\]

\[
M_{B_q}(\nabla_0 S) \geq M_{B_q}(\nabla_0 \Sigma) \geq M_q(|A \nabla_0 \Sigma|). \tag{4.4}
\]

We start with the first one. Let \( K_0^0, K_1^0 \) be sequences of compacts, such that \( K_0^0 \cap K_1^0 = \emptyset \), \( K_0^{i+1} \subset \text{int} K_0^i \), \( K_1^{i+1} \subset \text{int} K_1^i \), \( K_0 = \bigcap_{i=0}^{\infty} K_0^i \), \( K_1 = \bigcap_{i=0}^{\infty} K_1^i \), and let \( \Omega^i = \Omega \cup (\text{int} K_0^i) \cup (\text{int} K_1^i) \). Let us show \( \bigcup_{i=1}^{\infty} \nabla_0 S(K_0^i, K_1^i; \Omega^i) = \nabla_0 S(K_0, K_1; \Omega) \). Really, if \( u \in S(K_0^i, K_1^i; \Omega^i) \), then \( u \in S(K_0^{i+1}, K_1^{i+1}; \Omega^{i+1}) \) and \( u \in S(K_0, K_1; \Omega) \). Moreover, \( \text{supp}(\nabla_0 u)|_{\Omega^i} \subset \Omega \setminus (K_0^i \cup K_1^i) \subset \Omega^i \). Thus, \( \nabla_0 S(K_0^i, K_1^i; \Omega^i) \) is increasing and \( \bigcup_{i=1}^{\infty} \nabla_0 S(K_0^i, K_1^i; \Omega^i) \subset \nabla_0 S(K_0, K_1; \Omega) \). Now, we take \( u \in S(K_0, K_1; \Omega) \) and note that for sufficiently big \( i \in \mathbb{N} \) the function \( u \) belongs to \( S(K_0^i, K_1^i; \Omega^i) \). We put

\[
v = \begin{cases} 
0 & \text{if } x \in \text{int } K_0^i, \\
1 & \text{if } x \in \text{int } K_1^i, \\
u & \text{if } x \in \Omega, 
\end{cases}
\]
then $v \in S(K_0^i, K_1^i; \Omega^i)$, supp$(\nabla_v v)_{|_{\Omega^i}} = \text{supp} (\nabla_v u)_{|_{\Omega}}$. Hence, the reverse inclusion also holds and \(\lim_{i \to \infty} M_{\mathcal{B}_i}(\nabla_0 S(K_0^i, K_1^i; \Omega^i)) = M_{\mathcal{B}_i}(\nabla_0 S(K_0, K_1; \Omega))\) by Lemma 3.1.

We have $M_{\mathcal{B}_i}(\nabla_0 S(K_0^i, K_1^i; \Omega^i)) \leq M_{\mathcal{B}_i}(\nabla_0 S(K_0, K_1; \Omega))$ in view of the inclusion $S(K_0^i, K_1^i; \Omega^i) \subset S(K_0, K_1; \Omega)$. Let us observe that since $K_0, K_1 \subset \Omega^i$, we may modify $\Omega^i$ such that $\Omega^i$ can be approximated from inside with respect to the module $M_{\mathcal{B}_i}(\nabla_0 C^*(K_0, K_1; \Omega^i))$. Lemma 3.4 implies

$$M_{\mathcal{B}_i}(\nabla_0 S(K_0, K_1; \Omega)) \leq M_{\mathcal{B}_i}(\nabla_0 C^*(K_0, K_1; \Omega^i)).$$

Finally, we conclude

$$M_{\mathcal{B}_i}(\nabla_0 S(K_0, K_1; \Omega)) = \lim_{i \to \infty} M_{\mathcal{B}_i}(\nabla_0 S(K_0^i, K_1^i; \Omega^i)) \leq \lim_{i \to \infty} M_{\mathcal{B}_i}(\nabla_0 S(K_0, K_1; \Omega^i)) \leq \lim_{i \to \infty} M_{\mathcal{B}_i}(\nabla_0 C^*(K_0, K_1; \Omega^i)) \leq M_{\mathcal{B}_i}(\nabla_0 C^*(K_0, K_1; \Omega)),$$

where the last inequality follows from $\Omega \subset \Omega^i$.

Now, we start to prove (4.2). In view of (2.6) it is sufficient to show that the capacity $\text{cap}_{\mathcal{A}_0}^q(K_0, K_1; \Omega)$ is strictly positive and $(\text{cap}^q_{\mathcal{A}_0}(K_0, K_1; \Omega))^{1/p} (M_q(|A\nabla_0 \Sigma|))^{1/q} \geq 1$. Let $u \in \nabla_0 C^*$ and $0 \leq u \leq 1$. In this case $u \in W_0^1(\Omega)$ and inclusion $W_0^1(\Omega) \subset BV(\Omega)$ (see [18]) implies $u \in BV(\Omega)$. We put $E_t = \{x \in \Omega : u(x) > t\}$. In [18] it was proved that $\chi_{E_t} \in BV(\Omega)$ for almost all $t$. We will use the co-area formula for homogeneous groups [22,38]

$$\int_{\Omega} f(x)|\nabla_0 u| \, dx = \int_0^1 \int_{E_t} u(y) |n_0| \, dS \, dt,$$

where $f(x)$ is a non-negative measurable function, $n_0$ is the horizontal component of the unit normal to $\{u = t\}$ and $dS$ is the Riemannian area element on $\{u = t\}$. In our case $n_0 dS = \nabla_0 \chi_{E_t}$ and $|n_0| = \nabla_0 \chi_{E_t} / \nabla_0 u$. Let us take $\rho \in \mathcal{F}\mathcal{M}(|A\nabla_0 \Sigma|)$. Then $\int_{E_t} \rho |A\nabla_0 \chi_{E_t}| \, dS \geq 1$ for almost all $t$. Hence, the co-area formula yields

$$1 \leq \int_0^1 \int_{E_t} \rho |A\nabla_0 \chi_{E_t}| \, dS \, dt = \int_0^1 \int_{E_t} \rho \frac{|A\nabla_0 u|}{|\nabla_0 u|} \frac{|\nabla_0 u|}{|\nabla_0 u|} \, dS \, dt = \int_{\Omega} \rho(x)|A\nabla_0 u| \, dx \leq \left( \int_{\Omega} |A\nabla_0 u|^p \, dx \right)^{1/p} \left( \int_{\Omega} \rho^q \, dx \right)^{1/q}.$$

Taking the infimum with respect to $u \in \nabla_0 C^*$ and $\rho \in \mathcal{F}\mathcal{M}(|A\nabla_0 \Sigma|)$ we obtain the required inequality.

We will prove (4.3). The second inequality follows from the inclusion $A\nabla_0 \Sigma \subset A\nabla_0 S$. To show the first one, we choose $\xi \in \mathcal{F}\mathcal{M}(\nabla_0 S)$ and will obtain that $B\xi \in \mathcal{F}\mathcal{M}(A\nabla_0 S)$. The inequality $\int \xi \cdot \nabla_0 u \geq 1$ holds for all $\nabla_0 u \in \nabla_0 S$ except for some family $U \subset \nabla_0 S$ with $M_q(|U|) = 0$. We state that $M_q(|U|) = 0$ implies $M_q(|A\nabla U|) = 0$. For given $\varepsilon > 0$ we can find non-negative $\rho \in \mathcal{F}\mathcal{M}(|U|)$, such that $\int \rho^q \, dx \leq \varepsilon$. Then, $1 \leq \int \rho |\nabla_0 u| \leq \alpha \int \rho |A\nabla_0 u|$ by (2.3) for $\nabla_0 u \in U$ and we conclude that $\alpha \rho \in \mathcal{F}\mathcal{M}(|A\nabla u|)$. Finally,
we have $M_q(\|Au\|) \leq \int (\alpha \rho)^q \, dx \leq (\alpha)^q \varepsilon$ that proves $M_q(\|Au\|) = 0$. Since we have

$$
1 \leq \int_\Omega \xi \cdot \nabla u = \int_\Omega B\xi : A\nabla_0 u \leq \int_\Omega |B\xi| \|A\nabla_0 u|,
$$

we conclude, that $|B\xi|$ belongs to $\mathcal{F}M(\|A\nabla_0 S\|) \, q$-almost everywhere. Therefore,

$$
M_q(\|A\nabla_0 S\|) \leq \int_\Omega |B\xi|^q \, dx.
$$

Taking the infimum with respect to $\xi \in \mathcal{F}M(\nabla_0 S)$ we obtain the first inequality of (4.3). The proof of the statement (4.4) is similar. Theorem 2.1 is complete.

5. PROOF OF COROLLARIES 2.1 AND 2.2

Theorem 2.1 gives $M_{B_q}(\nabla_0 C^\ast) = M_{B_q}(\nabla_0 \Sigma)$. We obtain $\text{cap}^\ast_{A_p}(K_0, K_1; \Omega) = M_{A_p}(d\Gamma)$ from (2.5). These equalities and relation (2.6) prove Corollary 2.1.

To show Corollary 2.2, we note that if $A, B$ are unity matrices, then (2.5) and Theorem 2.1 imply $M_p(d\Gamma) = M_p(\|d\Gamma\|)$ and $M_q(\nabla_0 \Sigma) = M_q(\|\nabla_0 \Sigma\|)$, respectively. From this and Corollary 2.1 we obtain Corollary 2.2.

Acknowledgements

This work was supported by Projects Fondecyt (Chile) \# 1020067 and \# 1030373.

References


