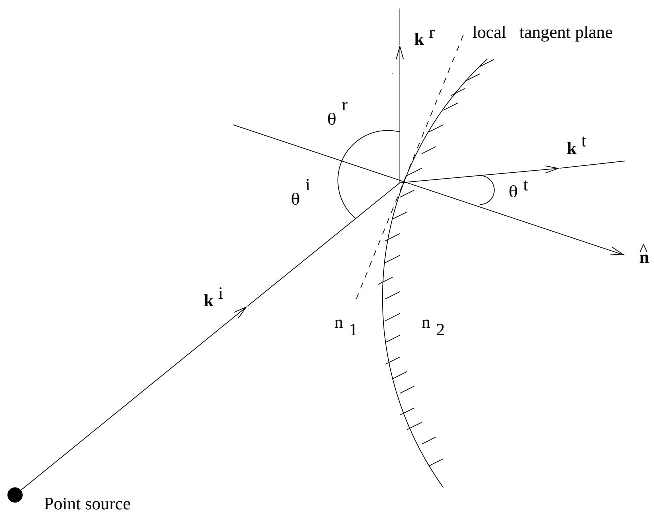

Figure 9.3: Generalisation of Snell's law and the reflection law to include non-planar waves that are incident upon a curved interface.

9.3 Reflection and refraction of plane electromagnetic waves

Note that the reflection law and the refraction law apply to all types of plane waves, i.e. to acoustic, electromagnetic, and elastic waves. In the derivation we have only used that $\mathbf{k}^q \cdot \mathbf{r} - \omega t$ $(q = i, r, t)$ shall be the same for $q = i$, $q = r$, and $q = t$. Now we take a closer look at the reflection and refraction of plane electromagnetic waves in order to determine how much of the energy in the incident wave that is reflected and transmitted.

We know that a plane electromagnetic wave is transverse, i.e. that both $\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$ are normal to the propagation direction $\mathbf{k} = k\hat{\mathbf{s}}$. In Fig. 9.1 we have chosen the z axis in the direction of the interface normal. If $\mathbf{E}$ is normal to the plane of incidence, we have s polarisation (from German, "Senkrecht") or TE polarisation ("transverse electric" relative to the plane of incidence or the z axis). And if $\mathbf{E}$ is parallel with the plane of incidence, we have p polarisation or TM polarisation, since in this case $\mathbf{B}$ is normal to the plane of incidence or the z axis; hence the use of the term TM or "transverse magnetic".

A general time-harmonic, plane electromagnetic wave consists of both a TE and a TM component. With the time dependence $e^{-i\omega t}$ suppressed, we have for the spatial part of the field

$$\mathbf{E} = \mathbf{E}^{TE} + \mathbf{E}^{TM} \quad ; \quad \mathbf{B} = \mathbf{B}^{TE} + \mathbf{B}^{TM}, \tag{9.3.1}$$

$$\mathbf{E}^{TE} = E^{TE}\frac{\mathbf{k}_t \times \hat{\mathbf{e}}_z}{k_t}e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{9.3.2}$$

$$\mathbf{E}^{TM} = E^{TM}\frac{\mathbf{k} \times (\mathbf{k}_t \times \hat{\mathbf{e}})_z}{kk_t}e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{9.3.3}$$

$$\mathbf{B}^{TE} = \frac{1}{k_0}\mathbf{k} \times \mathbf{E}^{TE} = E^{TE}\frac{\mathbf{k} \times (\mathbf{k}_t \times \hat{\mathbf{e}}_z)}{k_0 k_t}e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{9.3.4}$$

$$\mathbf{B}^{TM} = \frac{1}{k_0}\mathbf{k} \times \mathbf{E}^{TM} = E^{TM}\frac{1}{k_0 k k_t}\mathbf{k} \times [\mathbf{k} \times (\mathbf{k}_t \times \hat{\mathbf{e}}_z)]e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{9.3.5}$$

But since $\mathbf{k} \times [\mathbf{k} \times (\mathbf{k}_t \times \hat{\mathbf{e}}_z)] = \mathbf{k}[\mathbf{k} \cdot (\mathbf{k}_t \times \hat{\mathbf{e}}_z)] - (\mathbf{k}_t \times \hat{\mathbf{e}}_z)\mathbf{k} \cdot \mathbf{k} = -k^2\mathbf{k}_t \times \hat{\mathbf{e}}_z$, we get

$$\mathbf{B}^{TM} = \frac{-k}{k_0}E^{TM}\frac{\mathbf{k}_t \times \hat{\mathbf{e}}_z}{k_t}e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{9.3.6}$$

Note that the vectors

$$\hat{\mathbf{e}}^{TE} = \frac{\mathbf{k}_t \times \hat{\mathbf{e}}_z}{k_t} \quad ; \quad \hat{\mathbf{e}}^{TM} = \frac{\mathbf{k} \times (\mathbf{k}_t \times \hat{\mathbf{e}}_z)}{k k_t}, \tag{9.3.7}$$

are unit vectors in the directions of $\mathbf{E}^{TE}$ and $\mathbf{E}^{TM}$, respectively.

We represent each of the incident, reflected, and transmitted fields in the manner given above, so that $(q = i, r, t)$

$$\mathbf{E}^q = \mathbf{E}^{TEq} + \mathbf{E}^{TMq} \quad ; \quad \mathbf{B}^q = \mathbf{B}^{TEq} + \mathbf{B}^{TMq}, \tag{9.3.8}$$

$$\mathbf{E}^{TEq} = E^{TEq} \frac{\mathbf{k}_t \times \hat{\mathbf{e}}_z}{k_t} e^{i\mathbf{k}^q \cdot \mathbf{r}}, \tag{9.3.9}$$

$$\mathbf{E}^{TMq} = E^{TMq} \frac{\mathbf{k}^q \times (\mathbf{k}_t \times \hat{\mathbf{e}}_z)}{k^q k_t} e^{i\mathbf{k}^q \cdot \mathbf{r}}, \tag{9.3.10}$$

$$\mathbf{B}^{TEq} = \frac{k^q}{k_0} E^{TEq} \frac{\mathbf{k}^q \times (\mathbf{k}_t \times \hat{\mathbf{e}}_z)}{k^q k_t} e^{i\mathbf{k}^q \cdot \mathbf{r}}, \tag{9.3.11}$$

$$\mathbf{B}^{TMq} = \frac{-k^q}{k_0} E^{TMq} \frac{\mathbf{k}_t \times \hat{\mathbf{e}}_z}{k_t} e^{i\mathbf{k}^q \cdot \mathbf{r}}, \tag{9.3.12}$$

where

$$\mathbf{k}^i = \mathbf{k}_t + k_{z1} \hat{\mathbf{e}}_z \quad ; \quad \mathbf{k}_t = k_x \hat{\mathbf{e}}_x + k_y \hat{\mathbf{e}}_y, \tag{9.3.13}$$

$$\mathbf{k}^r = \mathbf{k}_t - k_{z1} \hat{\mathbf{e}}_z \quad ; \quad \mathbf{k}^t = \mathbf{k}_t + k_{z2} \hat{\mathbf{e}}_z, \tag{9.3.14}$$

$$k^q = \begin{cases} k_1 = n_1 k_0 & \text{for } q = i, r \\ k_2 = n_2 k_0 & \text{for } q = t. \end{cases} \tag{9.3.15}$$

The continuity conditions that must be satisfied at the interface $z = 0$ are that the tangential components of $\mathbf{E}$ and $\mathbf{H} = \frac{1}{\mu}\mathbf{B}$ be continuous, i.e.

$$\hat{\mathbf{e}}_z \times \left\{ \mathbf{E}^{TEi} + \mathbf{E}^{TEr} - \mathbf{E}^{TEt} + \mathbf{E}^{TMi} + \mathbf{E}^{TMr} - \mathbf{E}^{TMt} \right\} = 0, \tag{9.3.16}$$

$$\hat{\mathbf{e}}_z \times \left\{ \frac{1}{\mu_1} \left(\mathbf{B}^{TEi} + \mathbf{B}^{TEr} \right) - \frac{1}{\mu_2} \mathbf{B}^{TEt} + \frac{1}{\mu_1} \left(\mathbf{B}^{TMi} + \mathbf{B}^{TMr} \right) - \frac{1}{\mu_2} \mathbf{B}^{TMt} \right\} = 0. \tag{9.3.17}$$

Further, we have

$$\hat{\mathbf{e}}_z \times [\mathbf{k}^q \times (\mathbf{k}_t \times \hat{\mathbf{e}}_z)] = (\mathbf{k}^q \cdot \hat{\mathbf{e}}_z) \, \hat{\mathbf{e}}_z \times \mathbf{k}_t, \tag{9.3.18}$$

$$\hat{\mathbf{e}}_z \times (\mathbf{k}_t \times \hat{\mathbf{e}}_z) = \mathbf{k}_t. \tag{9.3.19}$$

By substituting from (9.3.9)-(9.3.12) into the boundary conditions (9.3.16)-(9.3.17) and using (9.1.2) and (9.3.18)-(9.3.19), we get

$$\mathbf{k}_t \left\{ E^{TEi} + E^{TEr} - E^{TEt} \right\} + \hat{\mathbf{e}}_z \times \mathbf{k}_t \left\{ \frac{k_{z1}}{k_1} E^{TMi} - \frac{k_{z1}}{k_1} E^{TMr} - \frac{k_{z2}}{k_2} E^{TMt} \right\} = 0, \tag{9.3.20}$$

$$\begin{aligned} &\hat{\mathbf{e}}_z \times \mathbf{k}_t \left\{ \frac{1}{\mu_1} \left(\frac{k_{z1}}{k_0} E^{TEi} - \frac{k_{z1}}{k_0} E^{TEr} \right) - \frac{1}{\mu_2} \frac{k_{z2}}{k_0} E^{TEt} \right\} \\ &+ \mathbf{k}_t \left\{ \frac{1}{\mu_1} \left(\frac{-k_1}{k_0} E^{TMi} - \frac{k_1}{k_0} E^{TMr} \right) - \frac{-k_2}{k_0} E^{TMt} \right\} = 0. \end{aligned} \tag{9.3.21}$$

Since $\mathbf{k}_t$ and $\hat{\mathbf{e}}_z \times \mathbf{k}_t$ are orthogonal vectors, the expression inside each of the {} parentheses in (9.3.20) and (9.3.21) must vanish, i.e.

$$E^{TEi} + E^{TEr} = E^{TEt}, \tag{9.3.22}$$

$$k_{z1}\mu_2 \left(E^{TEi} - E^{TEr}\right) = k_{z2}\mu_1 E^{TEt}, \tag{9.3.23}$$

$$k_{z1}k_2 \left(E^{TMi} - E^{TMr}\right) = k_{z2}k_1 E^{TMt}, \tag{9.3.24}$$

$$k_1\mu_2 \left(E^{TMi} - E^{TMr}\right) = k_2\mu_1 E^{TMt}. \tag{9.3.25}$$

Now we define reflection and transmission coefficients as

$$R^{TE} = \frac{E^{TEr}}{E^{TEi}} \quad ; \quad T^{TE} = \frac{E^{TEt}}{E^{TEi}}, \tag{9.3.26}$$

$$R^{TM} = \frac{E^{TMr}}{E^{TMi}} \quad ; \quad T^{TM} = \frac{E^{TMt}}{E^{TMi}}, \tag{9.3.27}$$

so that (9.3.22)-(9.3.25) give

$$1 + R^{TE} = T^{TE} \quad ; \quad 1 - R^{TE} = \frac{k_{z2}\mu_1}{k_{z1}\mu_2} T^{TE}, \tag{9.3.28}$$

$$1 - R^{TM} = \frac{k_{z2}k_1}{k_{z1}k_2} T^{TM} \quad ; \quad 1 + R^{TM} = \frac{k_2\mu_1}{k_1\mu_2} T^{TM}. \tag{9.3.29}$$

The two equations in (9.3.28) have the following solution

$$R^{TE} = \frac{\mu_2 k_{z1} - \mu_1 k_{z2}}{\mu_2 k_{z1} + \mu_1 k_{z2}} \quad ; \quad T^{TE} = \frac{2\mu_2 k_{z1}}{\mu_2 k_{z1} + \mu_1 k_{z2}}, \tag{9.3.30}$$

whereas the two equations in (9.3.29) give

$$R^{TM} = \frac{k_2^2 \mu_1 k_{z1} - k_1^2 \mu_2 k_{z2}}{k_2^2 \mu_1 k_{z1} + k_1^2 \mu_2 k_{z2}} \quad ; \quad T^{TM} = \frac{2k_1 k_2 \mu_2 k_{z1}}{k_2^2 \mu_1 k_{z1} + k_1^2 \mu_2 k_{z2}}. \tag{9.3.31}$$

The interpretation of the reflection and transmission coefficients follow from (9.3.26)-(9.3.27). Thus, the reflection coefficient represents the amplitude ratio between the reflected and the incident **E** field, whereas the transmission coefficient represents the amplitude ratio between the transmitted and the incident **E** field.

Note that (9.3.22)-(9.3.23) and (9.3.28) contain only TE quantities, whereas equations (9.3.24)-(9.3.25) and (9.3.29) contain only TM quantities. This implies that these two wave types are independent or de-coupled upon reflection and refraction. Thus, an incident TE plane wave produces a reflected TE plane wave and a transmitted TE plane wave, whereas an incident TM plane wave produces a reflected TM plane wave and a transmitted TM plane wave. Upon reflection and refraction there is no coupling between TE and TM waves.

From Fig. 9.1 it follows that

$$k_{z1} = \mathbf{k}^i \cdot \hat{\mathbf{e}}_z = k_1 \cos\theta^i \quad ; \quad k_{z2} = \mathbf{k}^t \cdot \hat{\mathbf{e}}_z = k_2 \cos\theta^t, \tag{9.3.32}$$

so that if $\mu_1 = \mu_2 = 1$ the reflection and transmission coefficients become

$$T^{TM} = \frac{2n_1 \cos\theta^i}{n_2 \cos\theta^i + n_1 \cos\theta^t} \quad ; \quad R^{TM} = \frac{n_2 \cos\theta^i - n_1 \cos\theta^t}{n_2 \cos\theta^i + n_1 \cos\theta^t}, \tag{9.3.33}$$

$$T^{TE} = \frac{2n_1 \cos\theta^i}{n_1 \cos\theta^i + n_2 \cos\theta^t} \quad ; \quad R^{TE} = \frac{n_1 \cos\theta^i - n_2 \cos\theta^t}{n_1 \cos\theta^i + n_2 \cos\theta^t}, \tag{9.3.34}$$

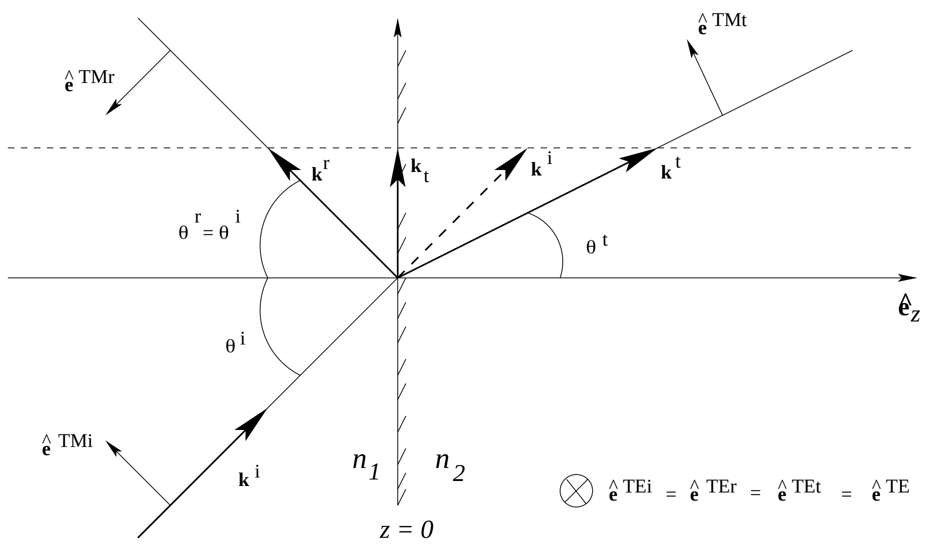

Figure 9.4: Reflection and refraction of a plane electromagnetic wave at a plane interface between two different media. Illustration of TE and TM components of the electric field.

These expressions are called the *Fresnel formulas*. By using Snell's law (9.1.20), we can rewrite them as (Exercise 9)

$$T^{TM} = \frac{2\sin\theta^t\cos\theta^i}{\sin(\theta^i+\theta^t)\cos(\theta^i-\theta^t)} \quad ; \quad R^{TM} = \frac{\tan(\theta^i-\theta^t)}{\tan(\theta^i+\theta^t)}, \tag{9.3.35}$$

$$T^{TE} = \frac{2\sin\theta^t\cos\theta^i}{\sin(\theta^i+\theta^t)} \quad ; \quad R^{TE} = -\frac{\sin(\theta^i-\theta^t)}{\sin(\theta^i+\theta^t)}. \tag{9.3.36}$$

At normal incidence where $\theta^i = \theta^t = 0$, we get from (9.3.33) and (9.3.34)

$$T^{TE} = T^{TM} = \frac{2}{n+1} \quad ; \quad R^{TM} = -R^{TE} = \frac{n-1}{n+1} \quad ; \quad n = \frac{n_2}{n_1}. \tag{9.3.37}$$

The fact that $R^{TM} = -R^{TE}$ at normal incidence follows from the way in which $\mathbf{E}^{TE}$ and $\mathbf{E}^{TE}$ are defined. From Fig. 9.4 we see that these two vectors point in opposite directions at normal incidence.

9.3.1 Reflectance and transmittance

Fig. 9.4 shows the polarisation vectors $\hat{\mathbf{e}}^{TMq}$ $(q = i, r, t)$ and $\hat{\mathbf{e}}^{TE}$ for TM and TE polarisation. These unit vectors are parallel with the electric field and follow from (9.3.9)-(9.3.12)

$$\hat{\mathbf{e}}^{TEi} = \hat{\mathbf{e}}^{TEr} = \hat{\mathbf{e}}^{TEt} = \hat{\mathbf{e}}^{TE} = \frac{\mathbf{k}_t \times \hat{\mathbf{e}}_z}{k_t} \quad ; \quad |\hat{\mathbf{e}}^{TE}| = 1, \tag{9.3.38}$$

$$\hat{\mathbf{e}}^{TMq} = \frac{\mathbf{k}^q \times (\mathbf{k}_t \times \hat{\mathbf{e}}_z)}{k^q k_t} \quad ; \quad |\hat{\mathbf{e}}^{TMq}| = 1. \tag{9.3.39}$$

Let the angle between $\mathbf{E}^q$ and the plane of incidence spanned by $\mathbf{k}^q$ and $\hat{\mathbf{e}}^{TMq}$, be α^q [see Fig. 9.5], so that

$$\mathbf{E}^q = \hat{\mathbf{e}}^{TE} E^q \sin\alpha^q + \hat{\mathbf{e}}^{TMq} E^q \cos\alpha^q. \tag{9.3.40}$$

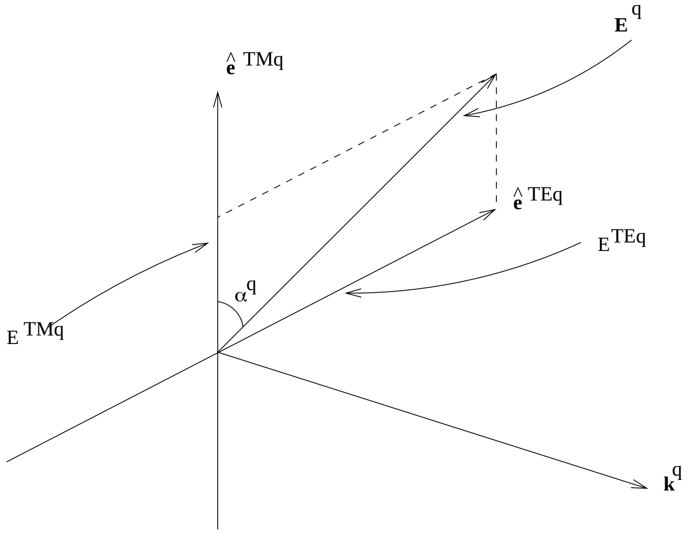

Figure 9.5: Illustration of the angle α^q between the electric vector $\mathbf{E}^q$ and the plane of incidence spanned by $\mathbf{k}^q$ and $\hat{\mathbf{e}}^{TMq}$.

Further, we let J^i, J^r, and J^t denote the energy flows of respectively the incident, reflected, and transmitted fields per unit area of the interface. Then we have

$$J^{pq} = S^{pq} \cos\theta^q \quad ; \quad p = TE, TM \quad ; \quad q = i, r, t, \tag{9.3.41}$$

where S^{pq} is the absolute value of the Poynting vector, given by

$$S^{pq} = \frac{c}{4\pi} |\mathbf{E}^{pq} \times \mathbf{H}^{pq}| = \frac{c}{4\pi} E^{pq} H^{pq} = \frac{c}{4\pi} \sqrt{\frac{\varepsilon^q}{\mu^q}} \left(E^{pq}\right)^2. \tag{9.3.42}$$

Here we have used the relation $\sqrt{\varepsilon^q} E^{pq} = \sqrt{\mu^q} H^{pq}$. The *reflectance* $\mathcal{R}^p$ ($p = TE, TM$ is the ratio between the reflected and incident energy flows. From (9.3.41)-(9.3.42) we have

$$\mathcal{R}^{TM} = \frac{J^{TMr}}{J^{TMi}} = \frac{|E^{TMr}|^2}{|E^{TMi}|^2} = (R^{TM})^2. \tag{9.3.43}$$

$$\mathcal{R}^{TE} = \frac{J^{TEr}}{J^{TEi}} = \frac{|E^{TEr}|^2}{|E^{TEi}|^2} = (R^{TE})^2, \tag{9.3.44}$$

Thus, the reflectance $\mathcal{R}^p$ is equal to the square of reflection coefficient R^p.

The *transmittance* $\mathcal{T}^p$ ($p = TE, TM$) is the ratio between the transmitted and incident energy flows, and (9.3.41)-(9.3.42) give

$$\mathcal{T}^{TM} = \frac{J^{TMt}}{J^{TMi}} = \frac{n_2}{n_1} \frac{\mu_1}{\mu_2} \frac{\cos\theta^t}{\cos\theta^i} (T^{TM})^2, \tag{9.3.45}$$

$$\mathcal{T}^{TE} = \frac{J^{TEt}}{J^{TEi}} = \frac{n_2}{n_1} \frac{\mu_1}{\mu_2} \frac{\cos\theta^t}{\cos\theta^i} (T^{TE})^2. \tag{9.3.46}$$

Thus, the transmittance $\mathcal{T}^p$ is proportional to the square of the transmission coefficient T^p ($p = TE, TM$). When $\mu_2 = \mu_1 = 1$, we find on substitution from (9.3.35)-(9.3.36) into (9.3.43)-(9.3.46) the following expressions for the reflectance and the transmittance

$$\mathcal{R}^{TM} = \frac{\tan^2(\theta^i - \theta^t)}{\tan^2(\theta^i + \theta^t)}, \tag{9.3.47}$$

$$\mathcal{R}^{TE} = \frac{\sin^2(\theta^i - \theta^t)}{\sin^2(\theta^i + \theta^t)}, \tag{9.3.48}$$

$$\mathcal{T}^{TM} = \frac{\sin 2\theta^i \sin 2\theta^t}{\sin^2(\theta^i + \theta^t)\cos^2(\theta^i - \theta^t)}, \tag{9.3.49}$$

$$\mathcal{T}^{TE} = \frac{\sin 2\theta^i \sin 2\theta^t}{\sin^2(\theta^i + \theta^t)}. \tag{9.3.50}$$

By use of these formulas one can show that

$$\mathcal{R}^{TM} + \mathcal{T}^{TM} = 1 \quad ; \quad \mathcal{R}^{TE} + \mathcal{T}^{TE} = 1, \tag{9.3.51}$$

so that for each of the two polarisations the sum of the reflected energy and the transmitted energy is equal to the incident energy.

From (9.3.41) and (9.3.42) we have

$$J^{pi} = \frac{c}{4\pi}\sqrt{\frac{\varepsilon_1}{\mu_1}}|E^{pi}|^2 \cos\theta^i, \tag{9.3.52}$$

which by the use of (9.3.40) gives

$$J^{TEi} = \cos\theta^i \frac{c}{4\pi}\sqrt{\frac{\varepsilon_1}{\mu_1}}|E^{TEi}|^2 = \cos\theta^i \frac{c}{4\pi}\sqrt{\frac{\varepsilon_1}{\mu_1}}E^2 \sin^2\alpha^i, \tag{9.3.53}$$

$$J^{TMi} = \cos\theta^i \frac{c}{4\pi}\sqrt{\frac{\varepsilon_1}{\mu_1}}|E^{TMi}|^2 = \cos\theta^i \frac{c}{4\pi}\sqrt{\frac{\varepsilon_1}{\mu_1}}E^2 \cos^2\alpha^i. \tag{9.3.54}$$

But since the total incident energy flow is given by

$$J^i = \cos\theta^i \frac{c}{4\pi}\sqrt{\frac{\varepsilon_1}{\mu_1}}E^2, \tag{9.3.55}$$

we find

$$J^{TEi} = J^i \sin^2\alpha^i \quad ; \quad J^{TMi} = J^i \cos^2\alpha^i. \tag{9.3.56}$$

Thus, we have

$$\mathcal{R} = \frac{J^r}{J^i} = \frac{J^{TMr} + J^{TEr}}{J^i} = \frac{J^{TMr}}{J^{TMi}}\cos^2\alpha^i + \frac{J^{TEr}}{J^{TEi}}\sin^2\alpha^i, \tag{9.3.57}$$

which gives

$$\mathcal{R} = \mathcal{R}^{TM}\cos^2\alpha^i + \mathcal{R}^{TE}\sin^2\alpha^i, \tag{9.3.58}$$

and similarly we find

$$\mathcal{T} = \mathcal{T}^{TM}\cos^2\alpha^i + \mathcal{T}^{TE}\sin^2\alpha^i. \tag{9.3.59}$$

At normal incidence, $\theta^i = \theta^t = 0$, and the distinction between TE and TM polarisation disappears. From (9.3.43)-(9.3.46) combined with (9.3.33)-(9.3.34), we find (when $\mu_1 = \mu_2 = 1$)

$$\mathcal{R} = \mathcal{R}^{TM} = \mathcal{R}^{TE} = (R^{TE})^2 = \left(R^{TM}\right)^2 = \left(\frac{n-1}{n+1}\right)^2 \quad ; \quad n = \frac{n_2}{n_1}, \tag{9.3.60}$$

$$\mathcal{T} = \mathcal{T}^{TM} = \mathcal{T}^{TE} = \left(T^{TE}\right)^2 = \left(T^{TM}\right)^2 = \frac{4n}{(n+1)^2} \quad ; \quad n = \frac{n_2}{n_1}. \tag{9.3.61}$$

When $n \to 1$, we see that $\mathcal{R} \to 0$ and $\mathcal{T} \to 1$, as expected. Similarly, we find from (9.3.47)-(9.3.50) that $\mathcal{R}_\parallel \to 0$, $\mathcal{R}_\perp \to 0$, $\mathcal{T}_\parallel \to 1$, $\mathcal{T}_\perp \to 1$ when $n \to 1$.

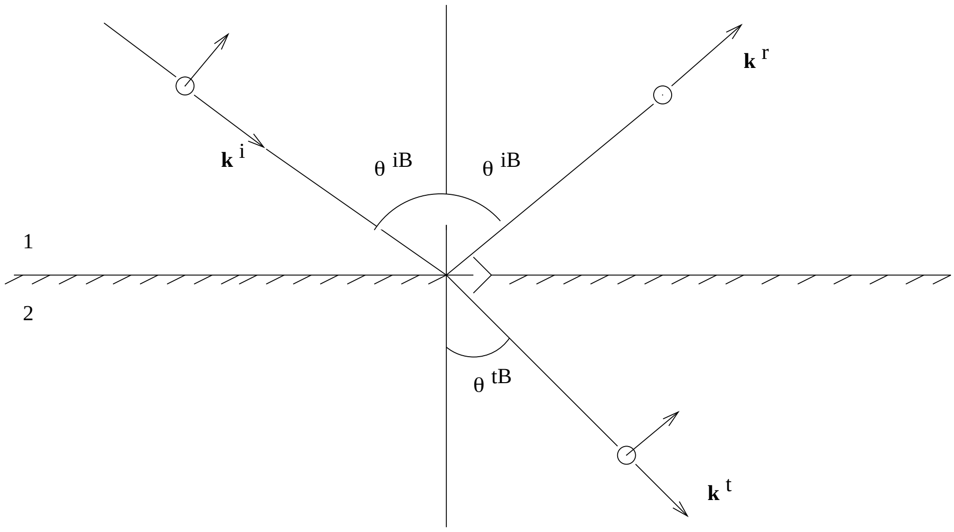

Figure 9.6: Illustration of Brewster's law.

9.3.2 Brewster's law

From (9.3.47) it follows that $\mathcal{R}^{TM} = 0$ when $\theta^i + \theta^t = \frac{\pi}{2}$, since then $\tan(\theta^i + \theta^t) = \infty$. We call this particular angle of incidence θ^{iB} and the corresponding refraction or transmission angle θ^{tB}. By using Snell's law (9.1.20), we find

$$n_2 \sin\theta^{tB} = n_2 \sin\left(\frac{\pi}{2} - \theta^{iB}\right) = n_2 \cos\theta^{iB} = n_1 \sin\theta^{iB}, \tag{9.3.62}$$

so that $\mathcal{R}^{TM} = 0$ when $\theta^i = \theta^{iB}$, where θ^{iB} is given by

$$\tan\theta^{iB} = \frac{n_2}{n_1} = n. \tag{9.3.63}$$

The angle θ^{iB} is called the *polarisation angle* or the *Brewster angle*. When the angle of incidence is equal to θ^{iB}, the **E** vector of the reflected light has no component in the plane of incidence (Fig. 9.6). This fact is exploited in sunglasses with polarisation filter. The filter is oriented such that only light that is polarised vertically (Fig. 9.6) is transmitted. Thus, one avoids to a certain degree annoying reflections from e.g. a water surface.

Note that $\mathbf{k}^r \cdot \mathbf{k}^t = 0$, i.e. $\mathbf{k}^r$ and $\mathbf{k}^t$ are normal to one another when $\theta^i = \theta^{iB}$, as shown in Fig. 9.6.

9.3.3 Unpolarised light (natural light)

For natural light, e.g. light from an incandescent lamp, the direction of the **E** vector varies very rapidly in an arbitrary or irregular manner, so that no particular direction is given preference. The average reflectance $\overline{\mathcal{R}}$ is obtained by averaging over all directions α. Since the average value of both $\sin^2\alpha$ and $\cos^2\alpha$ is $\frac{1}{2}$, we find from (9.3.56) that

$$\overline{J}^{TMi} = J^i\overline{\cos^2\alpha^i} = \overline{J}^{TEi} = J^i\overline{\sin^2\alpha^i} = \frac{1}{2}J^i. \tag{9.3.64}$$

For the reflected components we find

$$\overline{J}^{TMr} = \frac{\overline{J}^{TMr}}{\overline{J}^{TMi}} \cdot \overline{J}^{TMi} = \frac{\overline{J}^{TMr}}{\overline{J}^{TMi}} \cdot \frac{1}{2}J^i = \frac{1}{2}\mathcal{R}^{TM}J^i, \tag{9.3.65}$$

$$\overline{J}^{TEr} = \frac{\overline{J}^{TEr}}{\overline{J}^{TEi}} \cdot \frac{1}{2} J^i = \frac{1}{2} \mathcal{R}^{TE} J^i, \tag{9.3.66}$$

which shows that the *degree of polarisation* for the reflected light can be defined as

$$P^r = \left| \frac{\mathcal{R}^{TM} - \mathcal{R}^{TE}}{\mathcal{R}^{TM} + \mathcal{R}^{TE}} \right| = \frac{|J^{TMr} - J^{TEr}|}{J^{TMr} + J^{TEr}}. \tag{9.3.67}$$

The average reflectance is given by

$$\overline{\mathcal{R}} = \frac{\overline{J}^r}{\overline{J}^i} = \frac{\overline{J}^{TMr} + \overline{J}^{TEr}}{\overline{J}^i} = \frac{\overline{J}^{TMr}}{2\overline{J}^{TMi}} + \frac{\overline{J}^{TEr}}{2\overline{J}^{TEi}} = \frac{1}{2} \left(\mathcal{R}^{TM} + \mathcal{R}^{TE} \right), \tag{9.3.68}$$

so that the degree of polarisation becomes

$$P^r = \frac{1}{\overline{\mathcal{R}}} \frac{1}{2} |\mathcal{R}^{TM} - \mathcal{R}^{TE}|, \tag{9.3.69}$$

where $|\mathcal{R}^{TM} - \mathcal{R}^{TE}|$ is called the polarised part of the reflected light.

Similarly, we find for the transmitted light

$$\overline{\mathcal{T}} = \frac{1}{2}(\mathcal{T}^{TM} + \mathcal{T}^{TE}) \quad ; \quad P^t = \frac{1}{\overline{\mathcal{T}}} \frac{1}{2} |\mathcal{T}^{TM} - \mathcal{T}^{TE}|. \tag{9.3.70}$$

9.3.4 Rotation of the plane of polarisation upon reflection and refraction

Note that if the incident light is linearly polarised, then also the reflected and the transmitted light will be linearly polarised, since the phases only change by 0 or π. This follows from the fact that the reflection and transmission coefficients are real quantities [cf. (9.3.33)-(9.3.36)]. But the planes of polarisation for the reflected and the transmitted light are rotated in opposite directions relative to the polarisation plane of the incident light. The angles α^i, α^r, and α^t that the planes of polarisation of the incident, reflected, and transmitted light form with the plane of incidence, are given by [cf. Fig. 9.5]

$$\tan \alpha^i = \frac{E^{TEi}}{E^{TMi}}, \tag{9.3.71}$$

$$\tan \alpha^r = \frac{E^{TEr}}{E^{TMr}} = \frac{\frac{E^{TEr}}{E^{TEi}}}{\frac{E^{TMr}}{E^{TMi}}} \frac{E^{TEi}}{E^{TMi}} = \frac{R^{TE}}{R^{TM}} \tan \alpha^i, \tag{9.3.72}$$

$$\tan \alpha^t = \frac{E^{TEt}}{E^{TMt}} = \frac{\frac{E^{TEt}}{E^{TEi}}}{\frac{E^{TMt}}{E^{TMi}}} \frac{E^{TEi}}{E^{TMi}} = \frac{T^{TE}}{T^{TM}} \tan \alpha^i. \tag{9.3.73}$$

By use of the Fresnel formulas (9.3.35)-(9.3.36) we can write

$$\tan \alpha^r = -\frac{\cos(\theta^i - \theta^t)}{\cos(\theta^i + \theta^t)} \tan \alpha^i, \tag{9.3.74}$$

$$\tan \alpha^t = \cos(\theta^i - \theta^t) \tan \alpha^i. \tag{9.3.75}$$

Since $0 \leq \theta^i \leq \frac{\pi}{2}$ and $0 \leq \theta^t \leq \frac{\pi}{2}$, we get

$$|\tan \alpha^r| \geq |\tan \alpha^i|, \tag{9.3.76}$$

$$|\tan \alpha^t| \leq |\tan \alpha^i|. \tag{9.3.77}$$

In (9.3.76) the equality sign applies at normal incidence ($\theta^i = \theta^t = 0$) and at grazing incidence ($\theta^i = \frac{\pi}{2}$), whereas in (9.3.77) the equality sign applies only at normal incidence. These two inequalities imply that upon reflection the plane of polarisation is rotated *away from* the plane of incidence, whereas upon transmission it is rotated *towards* the plane of incidence. Note that when $\theta^i = \theta^{iB}$, so that $\theta^{iB} + \theta^{tB} = \frac{\pi}{2}$, then $\tan\alpha^r = \infty$. Thus, we have $\alpha^r = \frac{\pi}{2}$ in accordance with Brewster's law.

9.3.5 Total reflection

Snell's law (9.1.20) can be written in the form

$$\sin\theta^t = \frac{\sin\theta^i}{n} \quad ; \quad n = \frac{n_2}{n_1} = \sqrt{\frac{\varepsilon_2\mu_2}{\varepsilon_1\mu_1}}. \tag{9.3.78}$$

Hence, it follows that if $n < 1$, then we get $\sin\theta^t = 1$ when $\theta^i = \theta^{ic}$, where

$$\sin\theta^{ic} = n. \tag{9.3.79}$$

This implies that when $\theta^i = \theta^{ic}$, we get $\theta^t = \frac{\pi}{2}$, so that the transmitted light propagates along the interface. If $\theta^i \geq \theta^{ic}$, we have total reflection, i.e. no light will pass into the other medium. All light is then reflected. There exists a field in the other medium, but there is no energy transport through the interface. When $\theta^i > \theta^{ic}$, then $\sin\theta^t > 1$, which means that θ^t is *complex*. We have from (9.3.78)

$$\cos\theta^t = \pm\sqrt{1-\sin^2\theta^t} = \pm i\sqrt{\frac{\sin^2\theta^i}{n^2} - 1} = \frac{\pm i\sqrt{\sin^2\theta^i - n^2}}{n}. \tag{9.3.80}$$

The lower sign in (9.3.80) must be discarded. Otherwise the field in medium 2 would grow exponentially with increasing distance from the interface. The electric field in medium 2 is

$$\mathbf{E}^{pt} = T^p E^{pi}\hat{\mathbf{e}}^{pt} e^{i(\mathbf{k}^t\cdot\mathbf{r}-\omega t)} \quad (p = TE, TM), \tag{9.3.81}$$

where

$$\mathbf{k}^t \cdot \mathbf{r} = k_x x + k_y y + k_{z2} z, \tag{9.3.82}$$

with (cf. Fig. 9.1 and (9.3.80) with upper sign)

$$k_{z2} = k_2\cos\theta^t = ik_2\frac{1}{n}\sqrt{\sin^2\theta^i - n^2} \quad ; \quad n = \frac{n_2}{n_1}, \tag{9.3.83}$$

so that

$$e^{i\mathbf{k}^t\cdot r} = e^{i(k_x x + k_y y)} e^{-|k_{z2}|z} \quad ; \quad |k_{z2}| = \frac{k_2}{n}\sqrt{\sin^2\theta^i - n^2}\,. \tag{9.3.84}$$

We see that $\mathbf{E}^{pt}$ represents a wave that propagates along the interface and is exponentially damped with the distance z into medium 2.

From $\nabla\cdot\mathbf{D}^t = \varepsilon_2\nabla\cdot\mathbf{E}^t = 0$ it follows that

$$\mathbf{k}^t \cdot \mathbf{E}^t = 0, \tag{9.3.85}$$

which gives

$$E_z^t = -\frac{(k_x E_x^t + k_y E_y^t)}{k_{z2}}. \tag{9.3.86}$$

If we let the plane of incidence coincide with the xz plane, we have (cf. Fig. 9.7)

$$k_x = -k_1\sin\theta^i \quad ; \quad k_y = 0, \tag{9.3.87}$$

$$E_y^t = E^{TEt} e^{i(k_x x - \omega t)} e^{-|k_{z2}|z} = T^{TE} E^{TEi} e^{i(k_x x-\omega t)} e^{-|k_{z2}|z}, \tag{9.3.88}$$

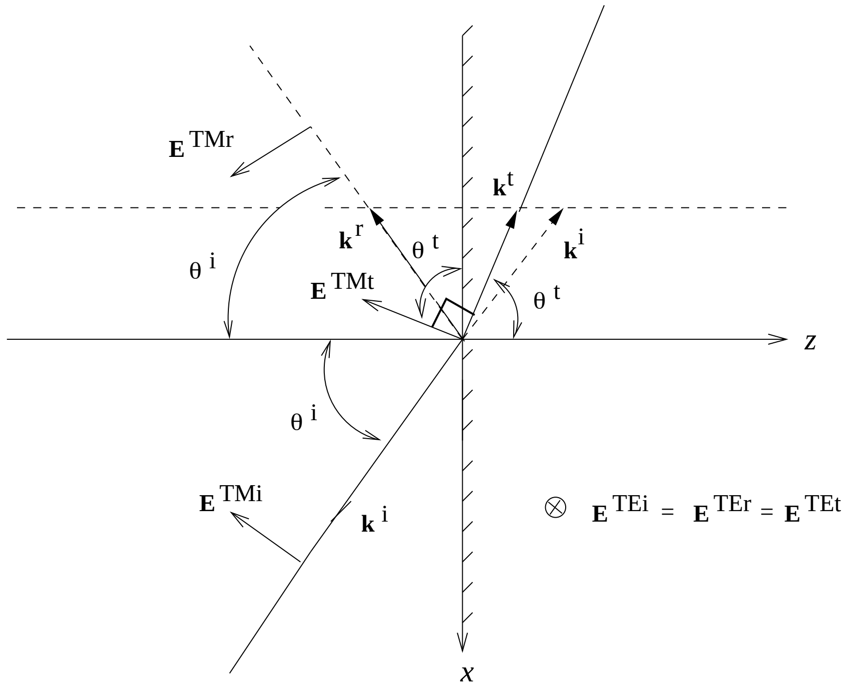

Figure 9.7: Illustration of the refraction of a plane wave into an optically thinner medium, so that $\theta^i < \theta^t$. When $\theta^i \to \theta^{ic}$, then $\theta^t \to \pi/2$, and we get total reflection.

$$E_x^t = -E^{TMt}\cos\theta^t e^{i(k_x x-\omega t)}e^{-|k_{z2}|} = -T^{TM}E^{TMi}\cos\theta^t e^{i(k_x x-\omega t)}e^{-|k_{z2}|z}, \tag{9.3.89}$$

$$E_z^t = -\frac{k_x}{k_{z2}}E_x^t = \frac{k_x}{k_2}T^{TM}E^{TMi}e^{i(k_x x-\omega t)}e^{-|k_{z2}|z}. \tag{9.3.90}$$

From these expressions for the components of $\mathbf{E}^t$ and corresponding expressions for the components of $\mathbf{H}^t$ one can show (Exercise 11) that the time average of the z component of the Poynting vector is zero, which implies that there is no energy transport through the interface, as asserted earlier.

The reflection coefficients in (9.3.35)-(9.3.36) can be written as follows

$$R^{TM} = \frac{\sin\theta^i\cos\theta^i - \sin\theta^t\cos\theta^t}{\sin\theta^i\cos\theta^i + \sin\theta^t\cos\theta^t}\ , \tag{9.3.91}$$

$$R^{TE} = -\frac{\sin\theta^i\cos\theta^t - \sin\theta^t\cos\theta^i}{\sin\theta^i\cos\theta^t + \sin\theta^t\cos\theta^i}\ . \tag{9.3.92}$$

By combining Snell's law (9.3.78) and (9.3.80) with the upper sign with (9.3.91)-(9.3.92), we get

$$R^{TM} = \frac{n^2\cos\theta^i - i\sqrt{\sin^2\theta^i - n^2}}{n^2\cos\theta^i + i\sqrt{\sin^2\theta^i - n^2}}, \tag{9.3.93}$$

$$R^{TE} = \frac{\cos\theta^i - i\sqrt{\sin^2\theta^i - n^2}}{\cos\theta^i + i\sqrt{\sin^2\theta^i - n^2}}. \tag{9.3.94}$$

Since both reflection coefficients are of the form z/z^*, where z is a complex number, it follows that

$$|R^{TM}| = |R^{TE}| = 1, \tag{9.3.95}$$

which shows that for each polarisation the intensity of the totally reflected light is equal to the intensity of the incident light.

But the phase is altered upon total reflection. Letting

$$R^p = \frac{E^{pr}}{E^{pi}} = e^{i\delta^p} = \frac{z^p}{z^{p*}} = e^{2i\alpha^p} \quad (p = TE, TM), \tag{9.3.96}$$

where [cf. (9.3.93)-(9.3.94)]

$$z^{TM} = n^2 \cos\theta^i - i\sqrt{\sin^2\theta^i - n^2} = |z^{TM}|e^{i\alpha^{TM}}, \tag{9.3.97}$$

$$z^{TE} = \cos\theta^i - i\sqrt{\sin^2\theta^i - n^2} = |z^{TE}|e^{i\alpha^{TE}}, \tag{9.3.98}$$

we find

$$\tan\alpha^{TM} = \tan\left(\frac{1}{2}\delta^{TM}\right) = -\frac{\sqrt{\sin^2\theta^i - n^2}}{n^2\cos\theta^i}, \tag{9.3.99}$$

$$\tan\alpha^{TE} = \tan\left(\frac{1}{2}\delta^{TE}\right) = -\frac{\sqrt{\sin^2\theta^i - n^2}}{\cos\theta^i}. \tag{9.3.100}$$

The relative phase difference

$$\delta = \delta^{TE} - \delta^{TM}, \tag{9.3.101}$$

is determined by

$$\tan\left(\frac{1}{2}\delta\right) = \frac{\tan\left(\frac{1}{2}\delta^{TE}\right) - \tan\left(\frac{1}{2}\delta^{TM}\right)}{1 + \tan\left(\frac{1}{2}\delta^{TE}\right)\tan\left(\frac{1}{2}\delta^{TM}\right)}, \tag{9.3.102}$$

which upon substitution from (9.3.99)-(9.3.100) gives

$$\tan\left(\frac{1}{2}\delta\right) = \frac{\cos\theta^i\sqrt{\sin^2\theta^i - n^2}}{\sin^2\theta^i}. \tag{9.3.103}$$

We see that $\delta = 0$ for $\theta^i = \frac{\pi}{2}$ (grazing incidence) and $\theta^i = \theta^{ic}$ (critical angle of incidence). Between these two values there is an angle of incidence $\theta^i = \theta^{im}$ which gives a maximum phase difference $\delta = \delta^m$, where θ^{im} is determined by

$$\left.\frac{d\delta}{d\theta_i}\right|_{\theta^{im}} = 0. \tag{9.3.104}$$

From (9.3.104) we find

$$\sin^2\theta^{im} = \frac{2n^2}{1+n^2}, \tag{9.3.105}$$

which upon substitution in (9.3.103) gives (Exercise 10)

$$\tan\left(\frac{1}{2}\delta^m\right) = \frac{1-n^2}{2n}. \tag{9.3.106}$$

If the phase difference δ is equal to $\pm\frac{\pi}{2}$ and in addition $E^{TMi} = E^{TEi}$, the totally reflected light will be circularly polarised. By choosing the angle α^i between the polarisation plane and the plane of incidence equal to 45°, we make E^{TMi} equal to E^{TEi}. In order to obtain $\delta = \frac{\pi}{2}$ we must have $\frac{\delta^m}{2} \geq \frac{\pi}{4}$. This means that $\tan\left(\frac{\delta^m}{2}\right) \geq \tan\left(\frac{\pi}{4}\right) = 1$, which according to (9.3.106) implies that

$$n^2 + 2n - 1 \leq 0. \tag{9.3.107}$$

By completing the square on the left-hand side of (9.3.107), we find that

$$n \leq \sqrt{2} - 1 \quad ; \quad \frac{1}{n} = \frac{n_1}{n_2} \geq \sqrt{2} + 1 = 2.41. \qquad (9.3.108)$$

Thus, $\frac{n_1}{n_2}$ must exceed 2.41 in order that we shall obtain a phase difference of $\frac{\pi}{2}$ in one single reflection.