Part 1

$$e^{i\vec{K}\cdot\vec{r}} = 4\pi \sum_{LM} i^L j_L(Kr) Y^*_{LM}(\hat{K}) Y_{LM}(\hat{r}) \tag{1}$$

Part 2

$$\frac{1}{|\vec{r_j} - \vec{R}(t)|} = \sum_{LM} \frac{4\pi}{2L+1} \frac{r_<^L}{r_>^{L+1}} Y^\star_{LM}(\hat{R}) Y_{LM}(\hat{r}) \tag{2}$$

Part 3

$$V_{fi} =< \Phi_f(\vec{r})| \sum_{LM} \frac{4\pi}{2L+1} \frac{r_<^L}{r_>^{L+1}} Y^\star_{LM}(\hat{R}) Y_{LM}(\hat{r}) |\Phi_i(\vec{r}) > \tag{3}$$

where

$$r_< = r, \quad r_> = R \quad \text{for} \quad |\vec{r}| \; < \; |\vec{R}|$$

$$r_< = R, \quad r_> = r \quad \text{for} \quad |\vec{r}| \; > \; |\vec{R}|$$

then:

$$V_{fi} =< R_f^\star(\vec{r}) Y_{l_f m_f}^\star(\hat{r}) | \sum_{LM} \frac{4\pi}{2L+1} \frac{r_<^L}{r_>^{L+1}} Y_{LM}^\star(\hat{R}) Y_{LM}(\hat{r}) | R_i(\vec{r}) Y_{l_i m_i}(\hat{r}) > \tag{4}$$

$$V_{fi} = \sum_{LM} \frac{4\pi}{2L+1} \int_0^R r^2 dr \frac{r^L}{R^{L+1}} R_f^\star(\vec{r}) [Y_{l_f m_f}^\star(\hat{r}) Y_{LM}^\star(\hat{R}) Y_{LM}(\hat{r})] R_i(\vec{r}) Y_{l_i m_i}(\hat{r}) +$$
$$\int_R^\infty r^2 dr \frac{R^L}{r^{L+1}} R_f^\star(\vec{r}) [Y_{l_f m_f}^\star(\hat{r}) Y_{LM}(\hat{r}) Y_{LM}^\star(\hat{R})] R_i(\vec{r}) Y_{l_i m_i}(\hat{r}) \tag{5}$$

This can be described by a simple notation

$$V_{fi} = \sum_{LM} \frac{4\pi}{2L+1} [G_{fi}^L(R(t))][Y_{LM}^\star(\hat{R})][C^L] \tag{6}$$

where $G_{fi}^L(R(t))$ is called The G-function and C^L is composed of Clebsch-Gordan coefficients. The matrix element is different from zero only if:

$$M = m_i + m_f$$

and

$$|l_i - l_f| \leq L \leq |l_i + l_f|$$

on

$$l_f + L + l_i$$

is even.

As can be seen above, we denote

$$G^L_{fi}[R(t)] = \int_0^\infty R^\star_f(\vec{r}) \frac{r^L <}{r^{L+1} >} R_i(\vec{r}) r^2 dr =$$

$$\frac{1}{R^{L+1}} \int_0^R r^L r^2 dr R^\star_f(\vec{r}) R_i(\vec{r}) + R^L \int_R^\infty \frac{1}{r^{L+1}} r^2 dr R^\star_f(\vec{r}) R_i(\vec{r}) \tag{7}$$

$R_i(\vec{r})$ and $R_f(\vec{r})$ are the radial wave functions for initial and final states.

The integration over the angular parts, which is the integral over three spherical harmonics, gives

$$C^L = \int Y^\star_{l_f m_f}(\theta, \varphi) Y_{LM}(\theta, \varphi) Y_{l_i m_i}(\theta, \varphi) d\Omega \tag{8}$$

$$(-1)^{m_f}\left[\frac{(2l_f+1)(2L+1)(2l_i+1)}{4\pi}\right]^{\frac{1}{2}}$$

$$\begin{pmatrix} l_f & L & l_i \\ -m_f & M & m_i \end{pmatrix}\begin{pmatrix} l_f & L & l_i \\ 0 & 0 & 0 \end{pmatrix}$$

This is known as Gaunts formula, and the numerical values $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ are Wigner-3j symbols, up to a factor equal to Clebsch-Gordan coefficients.