Charge in Electric and Magnetic Fields

Based on

Bransden and Joachain: Physics of Atoms and Molecules

and

Goldstein: Classical Mechanics

With 06.11.2014 notes **09.11.2013**

Lagrange equations

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$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) - \frac{\partial L}{\partial r_i} = 0 \qquad i = 1, 2, ...$$

For conservative systems, i.e. with usual forces from potential energy

$$L(r_i, \dot{r}_i, t) = T(\dot{r}_i) - V(r_i)$$

but the Lorentz force depends on velocity. Lagrange function $L(r_i, \dot{r}_i, t)$ must be modified.

Lorentz force

$$\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}[\mathbf{v} \times \mathbf{B}]\right)$$

electrostatic ϕ

With Φ and $\mathbf{A}$ the scalar and vector potentials

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \qquad\qquad \mathbf{B} = \nabla \times \mathbf{A}$$

vector potential A

the Lorentz force becomes

$$\mathbf{F} = q\left(-\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \frac{1}{c}\mathbf{v} \times [\nabla \times \mathbf{A}]\right)$$

with $\Phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ describing the fields

With only the scalar potential $\Phi(\mathbf{r}, t)$ the Lagrange function would be

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{m}{2}\, \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - q\Phi(\mathbf{r}, t) \quad = \quad \frac{1}{2}\, m\, \mathbf{v}^2 - q\Phi(\mathbf{r}, t)$$

and the Lagrange equation would lead to the electrostatic

$$m\ddot{\mathbf{r}} = -q\nabla\Phi$$

It can be shown that the Newton equation with the electromagnetic Lorentz force

$$m\ddot{\mathbf{r}} = q\left(-\nabla\Phi + -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \frac{1}{c}\mathbf{v} \times [\nabla \times \mathbf{A}]\right)$$

can be derived from a surprisingly simple Lagrange function (see below)

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \quad \frac{1}{2}\, m\, \mathbf{v}^2 - q\Phi(\mathbf{r}, t) + \frac{q}{c}\mathbf{v} \cdot \mathbf{A}(\mathbf{r}, t)$$

when inserted into the three Lagrange equations ($r_1 \to x$, $r_2 \to y$, $r_3 \to z$)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) - \frac{\partial L}{\partial r_i} = 0 \qquad i = 1, 2, 3$$

This is because the term $\mathbf{v} \times [\nabla \times \mathbf{A}]$ can be expressed without vector products.

Transforming the term $\mathbf{v} \times [\nabla \times \mathbf{A}]$

This contains time derivatives as well as the x, y, z derivatives

$$\mathbf{v} \times [\nabla \times \mathbf{A}] \quad \longrightarrow \quad \dot{\mathbf{r}} \times [\nabla \times \mathbf{A}]$$

consider first the total time derivative $\dot{\mathbf{A}}$,

$$\dot{\mathbf{A}} = \frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial \mathbf{A}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{A}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{A}}{\partial z}\frac{dz}{dt} = \frac{\partial \mathbf{A}}{\partial t} + v_x\frac{\partial \mathbf{A}}{\partial x} + v_y\frac{\partial \mathbf{A}}{\partial y} + v_z\frac{\partial \mathbf{A}}{\partial z}$$

Take now its x-component and re-arrange

$$\frac{dA_x}{dt} - \frac{\partial A_x}{\partial t} = v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z}$$

Take now x-component of $\mathbf{v} \times [\nabla \times \mathbf{A}]$

$$\{\mathbf{v} \times [\nabla \times \mathbf{A}]\}_x = v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

and now re-arrange - adding and subtracting (and compare with above)

$$v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x} \qquad - v_x\frac{\partial A_x}{\partial x} - v_y\frac{\partial A_x}{\partial y} - v_z\frac{\partial A_x}{\partial z}$$

Comparing those expressions we can replace the second term

$$\{\mathbf{v} \times [\nabla \times \mathbf{A}]\}_x = \frac{\partial}{\partial x}\left(v_x A_x + v_y A_y + v_z A_z\right) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}$$

and now it can be written for all components as

$$\mathbf{v} \times [\nabla \times \mathbf{A}] = \nabla\left(\mathbf{v} \cdot \mathbf{A}\right) - \frac{d\mathbf{A}}{dt} + \frac{\partial \mathbf{A}}{\partial t}$$

Now we insert this expression into the Lorentz force

$$\mathbf{F} = q\left(-\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \frac{1}{c}\mathbf{v} \times [\nabla \times \mathbf{A}]\right)$$

So that the Lorentz force becomes

$$\mathbf{F} = q\left(-\nabla\Phi - \frac{1}{c}\frac{d\mathbf{A}}{dt} + \frac{1}{c}\nabla\left(\mathbf{v} \cdot \mathbf{A}\right)\right)$$

This can be also written as

$$\mathbf{F} = q\left(-\nabla\Phi - \frac{1}{c}\frac{d}{dt}\nabla_{\mathbf{v}}\left(\mathbf{v} \cdot \mathbf{A}\right) + \frac{1}{c}\nabla\left(\mathbf{v} \cdot \mathbf{A}\right)\right)$$

Lagrange equations in vector form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) - \frac{\partial L}{\partial r_i} = 0 \qquad i = 1, 2, \ldots . \qquad\qquad \frac{d}{dt}\left(\nabla_{\dot{\mathbf{r}}} L\right) - \left(\nabla_{\mathbf{r}} L\right) = 0$$

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2}\, m\, \dot{\mathbf{r}}^2 - q\Phi(\mathbf{r}, t) + \frac{q}{c}\, \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

$$\nabla_{\dot{\mathbf{r}}} L = \underbrace{m\, \dot{\mathbf{r}} + \frac{q}{c}\, \mathbf{A}}_{\vec{p}} \qquad\qquad \nabla_{\mathbf{r}} L = -q\nabla_{\mathbf{r}}\Phi + \frac{q}{c}\, \nabla_{\mathbf{r}}\left(\dot{\mathbf{r}} \cdot \mathbf{A}\right)$$

p usual $\quad m\dot{r}$

$H = \vec{p}^{\,2} \ldots$

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Hamiltonian formalism results from a Legendre transform from x_i and its derivative $\dot{x}_i$ to a pair of conjugated variables x_i and p_i where

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$$p_i = \frac{\partial L}{\partial \dot{r}_i} \qquad\qquad \mathbf{p} = \nabla_{\dot{\mathbf{r}}} L$$

$\cancel{\vec{p} = m\vec{v}}$

when $L(x_i, \dot{x}_i, t)$ is replaced by $H(x_i, \dot{p}_i, t)$ with

$$H(x_i, \dot{p}_i, t) = \sum(\dot{x}_i p_i) - L(x_i, \dot{x}_i, t) \qquad\qquad H = \dot{\mathbf{r}} \cdot \mathbf{p} - L$$

We have from above

$$L = \frac{1}{2}\, m\, \dot{\mathbf{r}}^2 - q\Phi + \frac{q}{c}\, \dot{\mathbf{r}} \cdot \mathbf{A} \qquad\qquad \mathbf{p} = \nabla_{\dot{\mathbf{r}}} L \qquad\qquad \mathbf{p} = m\, \dot{\mathbf{r}} + \frac{q}{c}\, \mathbf{A}$$

FOR SIMPLE KI.

$L = \frac{1}{2} m \vec{v}^2 \qquad \nabla_{\vec{v}} L = m\vec{v} \longrightarrow p = m\vec{v}$

NOT GENERAL

$H = \left(p - \frac{q}{c} A\right)^2$