**The concept of Group velocity**

We start with a pulse which is a superposition of plane waves. The medium is such that there is a general relation between wavenumber and frequency

$$\omega = \omega(k)$$

i.e. the medium is called dispersive. The time developement of the pulse is thus

$$\psi(x,t) = \int \mathrm{dk}\, \tilde{\psi}(k) e^{i[\mathrm{kx} - \omega(k)t]}$$

There is also a question about $2\pi$ factor. Here it is swallowed by $\tilde{\psi}(k)$.

(Shapes of pulses: broad frequency spectrum, sharp time; narrow frequency, long duration, close to monochromatic - or clean tone). Remember by flute and guitar (which is which?).

The spectrum of the pulse is such that the it is peaked at a certain value $k_0$, such that

$$\max \tilde{\psi}(k) = \tilde{\psi}(k_0)$$

If the spectral distribution $\tilde{\psi}(k)$ is sufficiently narrow in $k$ and limited to narrow region arround $k_0$, we can use a linear approximation for $\omega(k)$

$$\omega(k) = \omega(k_0) + \frac{d\omega}{\mathrm{dk}}(k - k_0) = \omega_0 + \frac{d\omega}{\mathrm{dk}}(k - k_0)$$

By simply setting t=0

$$\psi(x,0) = \int \mathrm{dk}\, \tilde{\psi}(k) e^{i[\mathrm{kx}]}$$

This can easily be transformed to

$$e^{i[k_0 x]} \int_{-\infty}^{\infty} dk \tilde{\psi}(k) e^{i[(k-k_0)x]} \longrightarrow e^{i[k_0 x]} \int_{-\infty}^{\infty} dk' \tilde{\psi}(k' + k_0) e^{i[k' x]}$$

We can define the envelope (or modulation) as

$$M(x) = e^{-i[k_0 x]} \psi(x,0) = \int_{-\infty}^{\infty} \mathrm{dk}'\, \tilde{\psi}(k' + k_0) e^{i[k' x]}$$

The inverse relation (or definition) for the wavenumber function:

$$\tilde{\psi}(k) = \frac{1}{2\pi} \int \mathrm{dx}\, \psi(x,0) e^{-i[\mathrm{kx}]}$$

Now we go back to

$$\psi(x,t) = \int \mathrm{dk}\, \tilde{\psi}(k) e^{i[\mathrm{kx} - \omega(k)t]}$$

and transform it like this

$$\int \mathrm{dk}\, \tilde{\psi}(k) e^{i[k_0 x]} e^{i(k-k_0)x} e^{-i\omega_0 t]} e^{-i[\frac{d\omega}{\mathrm{dk}}(k-k_0)t]}$$

after rearrangement

$$e^{i[k_0 x]} e^{-i\omega_0 t} \int \mathrm{dk}\, \tilde{\psi}(k) e^{i(k-k_0)x} e^{-i[\frac{d\omega}{\mathrm{dk}}(k-k_0)t]}$$

with anticipated change of variables

$$(k - k_0) = k^{'}$$

We rewrite this as

$$e^{i[k_0 x - \omega_0 t]} \int \mathrm{dk}\, \tilde{\psi}(k) e^{i[(k-k_0)x - \frac{d\omega}{\mathrm{dk}}(k-k_0)t]}$$

and then obtain

$$e^{i[k_0 x - \omega_0 t]} \int \mathrm{dk}^{'}\, \tilde{\psi}(k' + k_0) e^{i[k' x - k' \frac{d\omega}{\mathrm{dk}} t]}$$

or rearranged as

$$\psi(x,t) \approx e^{i[k_0 x - \omega_0 t]} \int \mathrm{dk}^{'}\, \tilde{\psi}(k' + k_0) e^{\mathrm{ik}^{'}[x - \frac{d\omega}{\mathrm{dk}} t]}$$

With the definition of the envelope above

$$M(x) = \int_{-\infty}^{\infty} \mathrm{dk}^{'}\, \tilde{\psi}(k' + k_0) e^{i[k' x]}$$

the $\psi(x,t)$becomes

$$\psi(x,t) \approx e^{i[k_0 x - \omega_0 t]} M\left(x - \frac{d\omega}{\mathrm{dk}} t\right)$$

We thus see that provided that the integral over $k^{'}$ is peaked close enough to $k_0$ so that the linear approximation holds, the wave propagates as a plane wave $e^{i[k_0 x - \omega_0 t]}$modulated by the envelope $M\left(x - \frac{d\omega}{\mathrm{dk}} t\right)$ which moves with velocity $v_g = \frac{d\omega}{\mathrm{dk}}$.

Note that if $\omega(k) = \mathrm{ck}$, then the approximation is in fact an exact expression and the group velocity is the same as the phase velocity c.

In general, the approximation does not display the fact that the envelope $M\left(x\right)$ is in fact a function of time, $M\left(x,t\right)$. Generally, a pulse broadens in the dispersive medium.