PHYS 261

Physical Optics

Part 2
Lecture notes

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Wave propagation, diffraction, and radiation

0.1 Boundary Value Problems

0.1.1 Angular-spectrum representations

- Consider a 3D scalar wave field $\hat{u}(\mathbf{r},t)$ in a linear, homogeneous, non-dispersive, and isotropic medium.
- In source free regions of space, $\hat{u}(\mathbf{r},t)$ is a solution of the scalar wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\hat{u}(\mathbf{r},t) = 0 \tag{0.1.1}$$

  where c is the phase velocity in the medium.

- To study monochromatic or quasi-monochromatic phenomena we Fourier decompose $\hat{u}(\mathbf{r},t)$:

$$\hat{u}(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(\mathbf{r},\omega)e^{-i\omega t}d\omega. \tag{0.1.2}$$

- Substitution of (0.1.2) in (0.1.1) shows that $u(\mathbf{r},\omega)$ satisfies the Helmholtz equation:

$$\left(\nabla^2 + k^2\right)u(\mathbf{r},\omega) = 0 \quad ; \quad k = \frac{\omega}{c} \tag{0.1.3}$$

- Let the source lie in the half-space $z < 0$, and let the field be known in the plane $z = 0$ (Fig. 1).

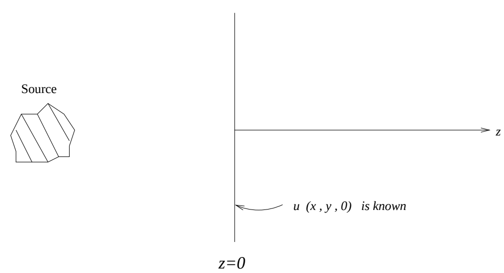

Figure 1: A source in the half-space $z < 0$ radiates a field that is assumed to be known in the plane $z = 0$. The field in the half-space $z > 0$ is to be determined.

- Hereafter we consider only one single Fourier component of the field and write for simplicity $u(\mathbf{r})$ instead of $u(\mathbf{r},\omega)$.
- Through Fourier decomposition of $u(\mathbf{r})$ with respect to x and y we have

$$u(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} \tilde{u}(k_x,k_y;z)e^{i(k_x x+k_y y)}dk_x dk_y. \tag{0.1.4}$$

- It can readily be shown that

$$\tilde{u}(k_x, k_y; z) = U(k_x, k_y)e^{ik_z z} \tag{0.1.5}$$

so that he field in the half-space $z > 0$ is given by

$$u(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} U(k_x, k_y)e^{i\mathbf{k}\cdot\mathbf{r}} dk_x dk_y, \tag{0.1.6}$$

$$\mathbf{k} \cdot \mathbf{r} = k_x x + k_y y + k_z z \tag{0.1.7}$$

$$k_z = \begin{cases} \sqrt{k^2 - k_x^2 - k_y^2} & \text{for } k^2 \geq k_x^2 + k_y^2 \\ i\sqrt{k_x^2 + k_y^2 - k^2} & \text{for } k^2 < k_x^2 + k_y^2. \end{cases} \tag{0.1.8}$$

- Equation (0.1.6) is called an angular-spectrum representation, since the field $u(\mathbf{r})$ is given as a sum of plane waves $\exp(i\mathbf{k} \cdot \mathbf{r})$ that propagate in various directions $\hat{\mathbf{s}}$, where $\hat{\mathbf{s}} = \mathbf{k}/k$.

- When $k^2 \geq k_x^2 + k_y^2$, then $\exp(i\mathbf{k} \cdot \mathbf{r})$ is a homogeneous plane wave.

- When $k^2 < k_x^2 + k_y^2$, then $\exp(i\mathbf{k} \cdot \mathbf{r})$ is an *inhomogeneous* or *evanescent* plane wave.

- An evanescent plane wave propagates in a direction normal to the z axis and decays exponentially with increasing z.

- The amplitude $U(k_x, k_y)$ of each individual plane wave in (0.1.6), called the *angular spectrum*, can be determined by setting $z = 0$ in (0.1.6):

$$U(k_x, k_y) = \mathcal{F}\{u(x, y, 0)\} = \iint_{-\infty}^{\infty} u(x, y, 0)e^{-i(k_x x + k_y y)} dx dy. \tag{0.1.9}$$

- Thus, the angular spectrum $U(k_x, k_y)$ is the Fourier transform of the field in the plane $z = 0$.

- If we know the normal derivative $\frac{\partial u}{\partial z}$ of the field in the plane $z = 0$ (Fig. 2) instead of u, then the field in the half-space $z > 0$ is given by

$$u(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} \frac{U'(k_x, k_y)}{ik_z} e^{i\mathbf{k}\cdot\mathbf{r}} dk_x dk_y, \tag{0.1.10}$$

where $U'(k_x, k_y)$ is the Fourier transform of $\frac{\partial u}{\partial z}$ in the plane $z = 0$.

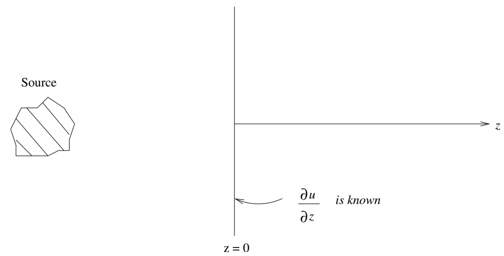

Figure 2: A source in the half-space $z < 0$ radiates a field whose normal derivative is assumed to be known in the plane $z = 0$. The field in the half-space $z > 0$ is to be determined.

- To distinguish between these two solutions, we denote them by u_I and u_{II}:

$$u_I(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} U(k_x, k_y) e^{i\mathbf{k}\cdot\mathbf{r}} dk_x dk_y, \tag{0.1.11}$$

$$U(k_x, k_y) = \iint_{-\infty}^{\infty} u(x, y, 0) e^{-i(k_x x + k_y y)} dx dy, \tag{0.1.12}$$

$$u_{II}(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} \frac{U'(k_x, k_y)}{ik_z} e^{i\mathbf{k}\cdot\mathbf{r}} dk_x dk_y, \tag{0.1.13}$$

$$U'(k_x, k_y) = \iint_{-\infty}^{\infty} \left[\frac{\partial u(\mathbf{r})}{\partial z}\right]_{z=0} e^{-i(k_x x + k_y y)} dx dy. \tag{0.1.14}$$

- Note that we have made no approximations, so that both solutions are exact. In other words, u_I and u_{II} are identical.

0.1.2 Rayleigh-Sommerfeld's and Kirchhoff's diffraction integrals

- Using the convolution theorem for two-dimensional Fourier transform pairs, we find

$$u_I(\mathbf{r}) = \iint_{-\infty}^{\infty} u(x', y', 0) h(x - x', y - y') dx' dy' \tag{0.1.15}$$

where

$$h(x, y, z) = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{r}} dk_x dk_y = -\frac{1}{2\pi}\frac{\partial}{\partial z}\left\{\frac{i}{2\pi}\iint_{-\infty}^{\infty} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k_z} dk_x dk_y\right\}. \tag{0.1.16}$$

- Weyl's plane-wave expansion of a spherical wave is given by

$$\frac{e^{ikr}}{r} = \frac{i}{2\pi}\iint\limits_{-\infty}^{\infty} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k_z} dk_x dk_y. \tag{0.1.17}$$

- Thus, (0.1.15) gives Rayleigh-Sommerfeld's first diffraction integral:

$$u_I(\mathbf{r}) = -\frac{1}{2\pi}\iint\limits_{-\infty}^{\infty} u(x', y', 0)\frac{\partial}{\partial z}\left(\frac{e^{ikR}}{R}\right) dx'dy' \tag{0.1.18}$$

where

$$R = [(x - x')^2 + (y - y')^2 + z_2]^{1/2}. \tag{0.1.19}$$

- Similarly, Rayleigh-Sommerfeld's second diffraction integral is given by

$$u_{II}(\mathbf{r}) = -\frac{1}{2\pi}\iint\limits_{-\infty}^{\infty} \left[\frac{\partial u(x', y', z)}{\partial z}\right]_{z=0} \frac{e^{ikR}}{R} dx'dy'. \tag{0.1.20}$$

- Since we have made no approximations, both (0.1.18) and (0.1.20) are *exact* solutions.

- *Kirchhoff's* diffraction integral is half the sum of the two Rayleigh-Sommerfeld integrals, i.e.

$$u_K(\mathbf{r}) = \frac{1}{2}[u_I(\mathbf{r}) + u_{II}(\mathbf{r})]. \tag{0.1.21}$$

0.2 Diffraction problems

0.2.1 Fresnel and Fraunhofer diffraction

- Consider a field u^i that is generated by sources in the half-space $z < 0$, and that propagates towards an aperture in the plane $z = 0$ (see Fig. 3).

- To determine the diffracted field in the half-space $z > 0$ we use Rayleigh-Sommerfeld's first diffraction integral and a variant of the Kirchhoff approximation.

- Our variant of the Kirchhoff approximation implies that we replace the actual field in the plane $z = 0^+$ by the incident field inside the aperture and by zero outside the aperture.

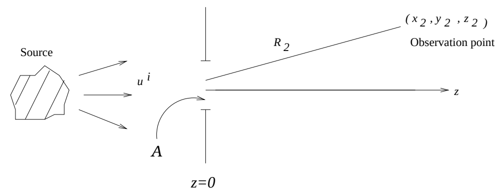

Figure 3: A source in the half-space $z < 0$ radiates a field u^i that is diffracted through an aperture $\mathcal{A}$ in the plane $z = 0$. The field in the observation point (x_2, y_2, z_2) in the half-space $z > 0$ is to be determined.

- Thus, we have

$$u_I(x_2, y_2, z_2 > 0) = \frac{-1}{2\pi} \iint\limits_{\mathcal{A}} u^i(x, y, 0) \frac{\partial}{\partial z_2}\left(\frac{e^{ikR_2}}{R_2}\right) dxdy$$

$$= -\frac{1}{2\pi} \iint\limits_{-\infty}^{\infty} t(x, y) u^i(x, y, 0) \frac{\partial}{\partial z_2}\left(\frac{e^{ikR_2}}{R_2}\right) dxdy \tag{0.2.1}$$

  where $\mathcal{A}$ is the aperture area, and $t(x, y)$ has the value 1 inside the aperture and the value 0 outside the aperture.

- R_2 is the distance from an integration point $(x, y, 0)$ in the aperture plane to the observation point (x_2, y_2, z_2):

$$R_2 = \sqrt{(x - x_2)^2 + (y - y_2)^2 + z_2^2}. \tag{0.2.2}$$

- Carrying out the differentiation with respect to z_2 in (0.2.1), we find

$$\frac{\partial}{\partial z_2}\left(\frac{e^{ikR_2}}{R_2}\right) == \frac{z_2}{R_2} ik \frac{e^{ikR_2}}{R_2}\left(1 + \frac{i}{kR_2}\right). \tag{0.2.3}$$

- Assuming that $kR_2 >> 1$, and introducing the paraxial approximation

$$\frac{z_2}{R_2} \simeq 1 \tag{0.2.4}$$

  we have

$$\frac{\partial}{\partial z_2}\left(\frac{e^{ikR_2}}{R_2}\right) \simeq ik \frac{e^{ikR_2}}{z_2}. \tag{0.2.5}$$

- Note that we have used the paraxial approximation only in the amplitude factor on the right-hand side of (0.2.3).

0.2.2 Fresnel diffraction

- Since $\exp(ikR_2)$ is a rapidly oscillating function compared with R_2, we introduce the Fresnel approximation:

$$R_2 = z_2\sqrt{1 + \frac{(x-x_2)^2 + (y-y_2)^2}{z_2^2}} \simeq z_2 \left\{1 + \frac{1}{2}\frac{(x-x_2)^2 + (y-y_2)^2}{z_2^2}\right\} \tag{0.2.6}$$

which requires that

$$\frac{(x-x_2)^2 + (y-y_2)^2}{z_2^2} << 1. \tag{0.2.7}$$

- Using both the Fresnel and the paraxial approximations, we obtain *Fresnel diffraction*:

$$u_I = \frac{C'}{i\lambda z_2}\iint_{-\infty}^{\infty} u^i(x,y,0)t(x,y)\exp\left[ik\left(\frac{x^2+y^2}{2z_2} - \frac{x_2 x + y_2 y}{z_2}\right)\right]dxdy \tag{0.2.8}$$

where

$$C' = e^{ik\Phi} \quad ; \quad \Phi = z_2 + \frac{x_2^2 + y_2^2}{2z_2}. \tag{0.2.9}$$

0.2.3 Fraunhofer diffraction

- Let the observation distance z_2 be so large that we may set the factor $\exp[ik(x^2+y^2)/2z_2]$ in (0.2.8) equal to 1.
- Then we have *Fraunhofer diffraction*, which requires that

$$\frac{k(x^2+y^2)}{2z_2} << 1. \tag{0.2.10}$$

- From (0.2.8) it follows that the diffracted field becomes equal to the Fourier transform of the field in the aperture:

$$u_I = \frac{C'}{i\lambda z_2}A(k_x,k_y) \quad ; \quad k_x = \frac{kx_2}{z_2} \quad ; \quad k_y = \frac{ky_2}{z_2} \tag{0.2.11}$$

where

$$a(x,y) = u(x,y,0)t(x,y) \tag{0.2.12}$$

$$A(k_x,k_y) = \mathcal{F}\{a(x,y)\} = \iint_{-\infty}^{\infty} a(x,y)e^{-i(k_x x + k_y y)}dxdy. \tag{0.2.13}$$

0.2.4 Circular aperture

- Consider a plane wave u^i that is normally incident upon a circular aperture of radius a in the plane $z = 0$ (Fig. 4).
- Thus, $u^i = e^{ikz}$, $u^i(x, y, 0) = 1$, and (see Fig. 4)

$$t(x,y) = \begin{cases} 1 & \text{for } x^2 + y^2 \leq a^2 \\ 0 & \text{for } x^2 + y^2 > a^2. \end{cases} \tag{0.2.14}$$

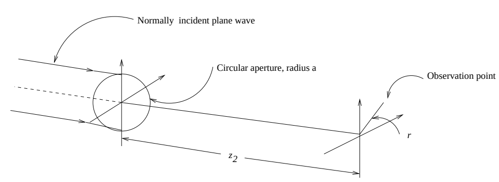

Figure 4: A plane wave is normally incident upon a circular aperture of radius a. The diffracted field is observed in a plane parallel to the aperture at a distance z_2 from it and at a distance r from the aperture axis.

Fresnel diffraction. Introducing dimensionless co-ordinates u and v defined by

$$v = k\left(\frac{a}{z_2}\right)r = \frac{2\pi}{\lambda}\left(\frac{a}{z_2}\right)r \quad ; \quad u = k\frac{a^2}{z_2} = \frac{2\pi}{\lambda}\left(\frac{a}{z_2}\right)^2 z_2 \tag{0.2.15}$$

we obtain in the Fresnel approximation:

$$u_I = -2iC'\frac{\pi a^2}{\lambda z_2}\int_0^1 J_0(vt)e^{i\frac{1}{2}ut^2}tdt \tag{0.2.16}$$

where the zeroth-order Bessel function is given by

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{\pm ix\cos(\phi-\beta)}d\phi. \tag{0.2.17}$$

Fraunhofer diffraction. As $z_2 \to \infty$, u in (0.2.15) approaches zero, so that (0.2.16) gives

$$u_I = 2C\int_0^1 J_0(vt)tdt = C\frac{2J_1(v)}{v} \quad ; \quad C = \frac{\pi a^2}{i\lambda z_2}\exp\left\{ik\left(z_2 + \frac{r^2}{2z_2}\right)\right\}. \tag{0.2.18}$$

- Then the intensity distribution becomes

$$I = |u_I|^2 = I_0 \left(\frac{2J_1(v)}{v}\right)^2 \quad ; \quad I_0 = \left(\frac{\pi a^2}{\lambda z_2}\right)^2. \tag{0.2.19}$$

- This intensity distribution is called the Airy diffraction pattern.

- The first zero of $J_1(x)$ occurs when $x = 3.83$, so that the diameter of the Airy pattern is determined by

$$v = v_0 = \frac{2\pi}{\lambda}\frac{a}{z_2} r_0 = 3.83$$

or

$$D = 2r_0 = \frac{3.83}{\pi}\frac{z_2}{a}\lambda = 1.22\left(\frac{z_2}{a}\right)\lambda. \tag{0.2.20}$$

- Introducing the f-number, defined by $F = \frac{z_2}{2a}$, we get

$$D = 2.44F\lambda. \tag{0.2.21}$$

- Note that the diameter D of the Airy disc is *inversely* proportional to a.

Axial field. At axial observation points $r = 0$, we have in the Fresnel aprrox-imation:

$$u_I(u, 0) = 2\,\exp\left\{i\left[kz_2 + \frac{u}{4} - \frac{\pi}{2}\right]\right\}\sin(\frac{u}{4}) \quad ; \quad I = 4\,\sin^2(\frac{u}{4}). \tag{0.2.22}$$

- In the limit of Fraunhofer diffraction ($u \to 0$), we have:

$$u_I(0, 0) = \frac{\pi a^2}{\lambda z_2}\exp\left\{i\left[kz_2 - \frac{\pi}{2}\right]\right\} \quad ; \quad I(0, 0) = \left(\frac{\pi a^2}{\lambda z_2}\right)^2, \tag{0.2.23}$$

in agreement with the results obtained from (0.2.18) and (0.2.19) in the limit as $v \to 0$.

- As $z_2 \to 0$, the axial intensity oscillates very rapidly. To see this, we note that

$$|\Delta u| = 2\pi\left(\frac{a}{z_2}\right)^2\frac{|\Delta z_2|}{\lambda}. \tag{0.2.24}$$

- Thus, when $z_2 = a$, a change in $|\Delta z_2|$ of 4λ produces a full cycle of $\sin(u/4)$.

- When $z_2 = 0.01a$, a change in Δz_2 of $4\lambda \times 10^{-4}$ produces a full cycle of $\sin(u/4)$.

0.2.5 Rectangular aperture

- Consider a plane wave that is normally incident upon a rectangular aperture in the plane $z = 0$ with midpoint at $x = y = 0$ and with sides $2a$ and $2b$ in the x and y direction, respectively.
- Then (0.2.8) gives

$$u_I = \frac{C'}{i\lambda z_2} \int_{-a}^{a} \exp\left\{ik\left[\frac{x^2}{2z_2} - \frac{x_2}{z_2}x\right]\right\} dx \int_{-b}^{b} \exp\left\{ik\left[\frac{y^2}{2z_2} - \frac{y_2}{z_2}y\right]\right\} dy. \quad (0.2.25)$$

Fraunhofer diffraction. Let us assume that

$$\frac{ka^2}{2z_2} << 1 \quad \text{og} \quad \frac{kb^2}{2z_2} << 1. \quad (0.2.26)$$

- Then (0.2.25) gives

$$u_I = \frac{C'}{i\lambda z_2} 4ab\, \mathrm{sinc}(v_a)\, \mathrm{sinc}(v_b) \quad ; \quad v_a = k\frac{a}{z_2}x_2 \quad ; \quad v_b = k\frac{b}{z_2}y_2. \quad (0.2.27)$$

- The intensity of the diffraction pattern becomes

$$I = |u_I|^2 = \left(\frac{4ab}{\lambda z_2}\right)^2 \mathrm{sinc}^2(v_a)\, \mathrm{sinc}^2(v_b). \quad (0.2.28)$$

- Since $\mathrm{sinc}(x)$ has its first zeros at $x = \pm\pi$, the extent of the diffraction pattern between the two first zeros in the x or y direction becomes

$$D_x = \frac{z_2}{a}\lambda \quad ; \quad D_y = \frac{z_2}{b}\lambda. \quad (0.2.29)$$

- For a square aperture ($a = b$) we get

$$D_x = D_y = \frac{z_2}{a}\lambda \quad (0.2.30)$$

which is seen to be a little less than the corresponding extent of the Airy disc for a circular aperture of radius a. In the latter case we have according to (0.2.20)

$$D = 1.22\frac{z_2}{a}\lambda. \quad (0.2.31)$$

Fresnel diffraction. When the Fraunhofer condition (0.2.26) is not satisfied, we have Fresnel diffraction.

- Then we write u_I in the form

$$u_I = \frac{C'}{i\lambda z_2} I_x I_y \quad ; \quad C' = e^{ik[z_2 + \frac{x_2^2+y_2^2}{2z_2}]}. \tag{0.2.32}$$

where I_x is given by

$$I_x = \sqrt{\frac{\lambda z_2}{\pi}} e^{-ik\frac{x_2^2}{2z_2}} \left\{ \int_0^{\alpha_a^+} [\cos(\alpha^2) + i\sin(\alpha^2)] d\alpha - \int_0^{\alpha_a^-} [\cos(\alpha^2) + i\sin(\alpha^2)] d\alpha \right\} \tag{0.2.33}$$

with

$$\alpha_a^\pm = \sqrt{\frac{\pi}{\lambda z_2}} (\pm a - x_2). \tag{0.2.34}$$

- The Fresnel integrals $C(u)$ and $S(u)$ are defined as

$$C(u) = \sqrt{\frac{2}{\pi}} \int_0^u \cos(t^2) dt \quad ; \quad S(u) = \sqrt{\frac{2}{\pi}} \int_0^u \sin(t^2) dt. \tag{0.2.35}$$

- Thus, (0.2.33) can be expressed in terms of C and S in the following manner

$$I_x = \sqrt{\frac{\lambda z_2}{2}} e^{-ik\frac{x_2^2}{2z_2}} \left\{ C(\alpha_a^+) - C(\alpha_a^-) + i[S(\alpha_a^+) - S(\alpha_a^-)] \right\}. \tag{0.2.36}$$

- Similarly, we obtain for the integral I_y:

$$I_y = \sqrt{\frac{\lambda z_2}{2}} e^{-ik\frac{y_2^2}{2z_2}} \left\{ C(\alpha_b^+) - C(\alpha_b^-) + i[S(\alpha_b^+) - S(\alpha_b^-)] \right\}, \tag{0.2.37}$$

where

$$\alpha_b^\pm = \sqrt{\frac{\pi}{\lambda z_2}} (\pm b - x_2). \tag{0.2.38}$$

- According to (0.2.32), the diffracted field for Fresnel diffraction through a rectangular aperture is given by

$$u_I = I_x I_y \frac{C'}{i\lambda z_2} \quad ; \quad C' = e^{ik[z_2 + \frac{x_2^2+y_2^2}{2z_2}]}. \tag{0.2.39}$$

- Using (0.2.36) and (0.2.37), we get

$$u_I = \frac{1}{2i} e^{ikz_2} \{[C(\alpha_a^+) - C(\alpha_a^-)] + i[S(\alpha_a^+) - S(\alpha_a^-)]\} \times \{[C(\alpha_b^+) - C(\alpha_b^-)] + i[S(\alpha_b^+) - S(\alpha_b^-)]\}. \quad (0.2.40)$$

- The intensity distribution for Fresnel diffraction through a rectangular aperture becomes:

$$I = |u_I|^2 \quad (0.2.41)$$

where u_I is given in (0.2.40).

- For an infinitely large aperture:

$$\alpha_a^\pm \to \pm\infty \quad ; \quad \alpha_b^\pm \to \pm\infty \quad (0.2.42)$$

and hence

$$u_I = \frac{1}{2i} e^{ikz_2}(1+i)(1+i) = \frac{1}{2i} e^{ikz_2}(1+2i-1) = e^{ikz_2} \quad (0.2.43)$$

as expected.

0.2.6 Half-plane

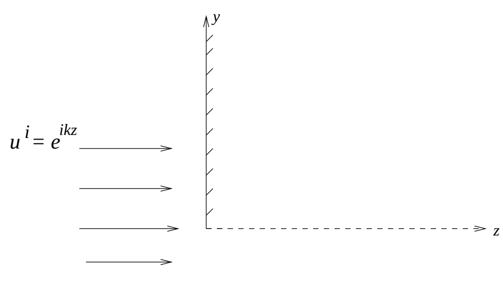

Figure 5: Diffraction of a plane wave that is normally incident upon the half-plane $z = 0$, $y \geq 0$.

- Consider a plane wave that is normally incident upon a half-plane, as illustrated in Fig. 5.
- Then the diffracted field and intensity follow from (0.2.40) and (0.2.41) by setting

  1. $a = \infty$, so that $\alpha_a^\pm = \pm\infty$

2. $b \to \infty$ in α_b^-, so that $\alpha_b^- = -\infty$
3. $b \to 0$ in α_b^+, so that $\alpha_b^+ \to -\sqrt{\frac{kz_2}{2}}\frac{y_2}{z_2} = -A\,y_2 \quad ; \quad A = \sqrt{\frac{\pi}{\lambda z_2}}$.

- Thus we get

$$u_I = \frac{e^{ikz_2}}{2i}\{1+i\}\left\{C(-Ay_2) + \frac{1}{2} + i\left[S(-Ay_2) + \frac{1}{2}\right]\right\} \tag{0.2.44}$$

$$I = |u_I|^2 = \frac{1}{2}\left\{\left[C(-Ay_2) + \frac{1}{2}\right]^2 + \left[S(-Ay_2) + \frac{1}{2}\right]^2\right\} \tag{0.2.45}$$

where

$$A = \sqrt{\frac{k}{2z_2}} = \sqrt{\frac{\pi}{\lambda z_2}}. \tag{0.2.46}$$

- When $y_2 \to +\infty$, $C = S = -\frac{1}{2}$, and hence $u_I = 0$, $I = 0$.
- When $y_2 \to -\infty$, $C = S = +\frac{1}{2}$, and hence $u_I = \exp(ikz_2)$, $I = 1$.
- When $y_2 = 0$, $C = S = 0$, and hence $u_I = \frac{1}{2}\exp(ikz_2)$, $I = \frac{1}{4}$.
- For large absolute values of the argument u we have

$$C(u) \sim \frac{1}{2}\mathrm{sgn}(u) + \sin(u^2)/\sqrt{2\pi}u \tag{0.2.47}$$

$$S(u) \sim \frac{1}{2}\mathrm{sgn}(u) - \cos(u^2)/\sqrt{2\pi}u. \tag{0.2.48}$$

- Thus, for $|u| = A|y_2| >> 1$, the diffracted field becomes

$$u_I = \frac{e^{ikz_2}}{2}\left\{1 - \mathrm{sgn}(y_2) + \frac{e^{i(A^2y_2^2+\pi/4)}}{\sqrt{\pi}Ay_2}\right\} \tag{0.2.49}$$

and the corresponding intensity becomes

$$I = \frac{1}{4}\left\{(1-\mathrm{sgn}(y_2))^2 + \frac{1}{\pi A^2 y_2^2} + 2(1-\mathrm{sgn}(y_2))\frac{\cos(A^2y_2^2 + \frac{\pi}{4})}{\sqrt{\pi}Ay_2}\right\}. \tag{0.2.50}$$

Recalling that $A = \sqrt{\pi/\lambda z_2}$, may rewrite the result in (0.2.50) as follows

$$I = \begin{cases} \frac{\lambda z_2}{4\pi^2 y_2^2} & \text{for} \quad y_2 > 3\sqrt{\lambda z_2/\pi} \\ \frac{1}{4} & \text{for} \quad y_2 = 0 \\ 1 + \frac{\lambda z_2}{4\pi^2 y_2^2} - \frac{\sqrt{\lambda z_2}}{\pi y_2}\cos\left\{\frac{\pi}{\lambda z_2}y_2^2 + \frac{\pi}{4}\right\} & \text{for } y_2 < -3\sqrt{\lambda z_2/\pi}. \end{cases} \tag{0.2.51}$$

- The result in (0.2.51) shows that
  - In the geometrical shadow behind the half-plane ($y_2 > 3\sqrt{\lambda z_2/\pi}$) the intensity decays monotonically to zero.
  - At the geometrical shadow boundary ($y_2 = 0$) the intensity is one quarter of the intensity of the incident wave.
  - In the geometrically lit region ($y_2 < -3\sqrt{\lambda z_2/\pi}$) the intensity oscillates around the intensity value of 1 of the incident wave (when disregarding the term $\lambda z_2/4\pi^2 y_2^2$, which is small compared to 1).
  - The period of the oscillation in the geometrically lit region ($y_2 < -3\sqrt{\lambda z_2/\pi}$) decreases as $|y_2|$ increases due to the term $(\pi/\lambda z_2)y_2^2$ in the argument of the cosine.
  - The amplitude of the oscillation in the geometrically lit region ($y_2 < -3\sqrt{\lambda z_2/\pi}$) decreases as $|y_2|$ increases due to the factor $\sqrt{\lambda z_2}/\pi y_2$ in front of the cosine term.

0.3 Exact solution for diffraction by a half-plane

0.3.1 Exact solution

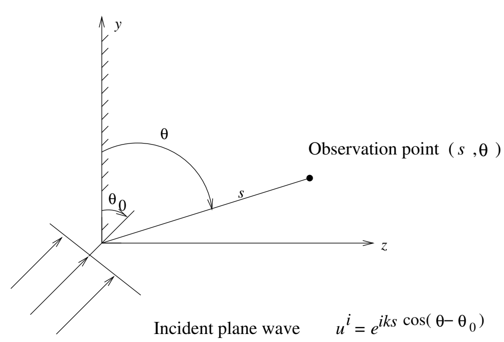

Figure 6: Diffraction of a plane wave by the half-plane $z = 0$, $y \geq 0$. The propagation direction of the incident plane wave forms an angle θ_0 with the positive y axis.

- Let a plane wave be incident upon a half-plane, and let the wave vector of the incident wave be normal to the edge of the half-plane, but not necessarily normal to the half-plane itself (see Fig. 6).
- The exact solution for the diffracted field is given by

$$\left\{ \begin{array}{c} u^s \\ u^h \end{array} \right\} = F(\xi^i)u^i \mp F(\xi^r)u^r \tag{0.3.1}$$

where

$$\left\{ \begin{array}{c} \xi^i \\ \xi^r \end{array} \right\} = \mp\sqrt{2ks}\,\sin\frac{1}{2}(\theta \mp \theta_0). \tag{0.3.2}$$

- Here u^i and u^r are respectively the incident and the reflected plane wave:

$$\left\{ \begin{array}{c} u^i \\ u^r \end{array} \right\} = \left\{ \begin{array}{c} e^{i\mathbf{k}^i\cdot\mathbf{r}} \\ e^{i\mathbf{k}^r\cdot\mathbf{r}} \end{array} \right\} = e^{iks\cos(\theta\mp\theta_0)}. \tag{0.3.3}$$

- θ and θ_0 are the angles between the positive y axis and the directions of incidence and observation, respectively (Fig. 6).
- The function $F(x)$ in (0.3.1) is a generalised, complex Fresnel integral defined as

$$F(x) = \frac{e^{-i\pi/4}}{\sqrt{\pi}} \int_x^\infty e^{it^2} dt. \tag{0.3.4}$$

- The solutions u^s and u^h apply to "soft" and "hard" boundary conditions:

$$u^s = 0 \quad \text{for } z = 0 \quad \text{and } y \geq 0 \tag{0.3.5}$$

$$\frac{\partial u^h}{\partial z} = 0 \quad \text{for } z = 0 \quad \text{and } y \geq 0. \tag{0.3.6}$$

- In both cases the half-plane is a perfect reflector in the sense that the absolute value of the reflection coefficient is equal to 1, i.e. $|R^s| = |R^h| = 1$.
- For "soft" boundary condition we have a phase shift of π upon reflection: $R^s = -R^h = -1$.
- We can express $F(x)$ in terms of the real Fresnel integrals:

$$F(x) = \frac{1}{2}\{1 - C(x) - S(x) + i[C(x) - S(x)]\}. \tag{0.3.7}$$

- Since $C(0) = S(0) = 0$ and $C(\pm\infty) = S(\pm\infty) = \pm\frac{1}{2}$, it follows that

$$F(-\infty) = 1 \quad ; \quad F(0) = \frac{1}{2} \quad ; \quad F(\infty) = 0. \tag{0.3.8}$$

- Further, we have

$$|F(x)|^2 = \frac{1}{2}\left\{\left[C(x) - \frac{1}{2}\right]^2 + \left[S(x) - \frac{1}{2}\right]^2\right\}. \tag{0.3.9}$$

- The arguments ξ^i and ξ^r of the generalised Fresnel integrals in (0.3.1) are called *detour parameters*.
- For the special case of *normal* incidence upon the half-plane, $\theta_0 = \pi/2$, we have

$$\left(\xi^i\right)^2 = ks(1 - \sin\theta) = k(s - z) \qquad (0.3.10)$$

$$\left(\xi^r\right)^2 = ks(1 + \sin\theta) = k[s - (-z)]. \qquad (0.3.11)$$

- From Fig. 7 and (0.3.10) and (0.3.11) we see that the square of the detour parameter, i.e. $(\xi^i)^2$ (or $(\xi^r)^2$), is equal to the difference between the phase measured along the diffracted ray and the phase measured along the direct incident (or reflected) ray.
- Note that:
  - $\xi^i > 0$ when the observation point lies in the shadow zone of the incident wave, i.e. when $\theta < \frac{\pi}{2}$.
  - $\xi^r > 0$ when the observation point lies in the shadow zone of the reflected wave, i.e. when $\theta < \frac{3\pi}{2}$.
- The intensity of the diffracted field is given by

$$\begin{aligned} \left\{ \begin{array}{c} I^s \\ I^h \end{array} \right\} &= \left\{ \begin{array}{c} |u^s|^2 \\ |u^h|^2 \end{array} \right\} = |F(\xi^i)u^i \mp F(\xi^r)u^r|^2 \\ &= |F(\xi^i)|^2 + |F(\xi^r)|^2 \mp 2\mathrm{Re}\left[F(\xi^i)u^i F^*(\xi^r)(u^r)^*\right]. \qquad (0.3.12) \end{aligned}$$

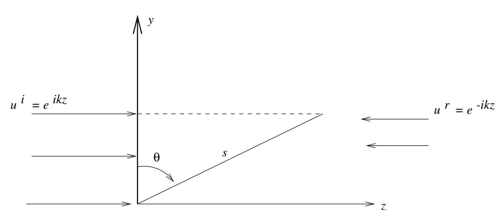

Figure 7: Diffraction of a plane wave that is normally incident upon the half-plane $z = 0$, $y \geq 0$. The incident wave propagates in the z direction.

0.3.2 Comparison with the Kirchhoff solution

- Consider now the case in which $z > 0$ and $|\xi^r| >> 1$. Then $|F(\xi^r)| << |F(\xi^i)|$, so that we obtain from (0.3.12)

$$\left\{ \begin{array}{c} I^s \\ I^h \end{array} \right\} \approx |F(\xi^i)|^2. \tag{0.3.13}$$

- Further, let the observation point be close to the shadow boundary of the incident wave, so that

$$s = \sqrt{y_2^2 + z_2^2} \simeq z_2\sqrt{1 + \left(\frac{y_2}{z_2}\right)^2} \simeq z_2 + \frac{1}{2}\frac{y_2^2}{z_2} \tag{0.3.14}$$

$$(\xi^i)^2 = k(s - z_2) \simeq \frac{k}{2z_2}y_2^2 = A^2 y_2^2 \quad ; \quad A = \sqrt{\frac{k}{2z_2}} \quad ; \quad \xi^i = Ay_2. \tag{0.3.15}$$

- Using (0.3.9), $C(x) = -C(-x)$, and $S(x) = -S(-x)$, we get

$$\left\{ \begin{array}{c} I^s \\ I^h \end{array} \right\} \simeq \frac{1}{2}\left\{\left[C(-Ay_2) + \frac{1}{2}\right]^2 + \left[S(-Ay_2) + \frac{1}{2}\right]^2\right\}. \tag{0.3.16}$$

- Comparison of the exact intensity in (0.3.16) with the corresponding intensity obtained in the Kirchhoff approximation [see (0.2.45)], shows that the two results are equal.

- Thus, when the observation point lies near the shadow boundary of the incident wave ($\theta \simeq \frac{\pi}{2}$) and $\xi^r = \sqrt{2ks}\sin\frac{1}{2}(\theta + \theta_0) >> 1$ or $\sqrt{2ks} >> 1$, the two exact solutions and the approximate Kirchhoff solution give the same intensity.

0.4 Focusing and imaging

- Consider the imaging system illustrated in Fig. 8, where an on-axis object point emits a diverging spherical wave, which is transformed by a lens into a converging spherical wave with focus or image point at $(0, 0, z_1)$.

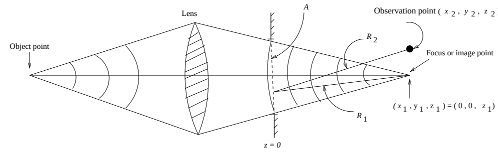

Figure 8: Imaging system. An object point on the axis emits a diverging spherical wave that is transformed by a lens into a converging spherical wave with image point at (0, 0, z_1).

0.4.1 Diffracted field in the focal area

- The converging spherical wave passes through an aperture in the plane $z = 0$.
- The diffracted field in the focal area of the lens is obtained by using Rayleigh-Sommerfeld's first diffraction integral and the Kirchhoff approximation. With $kR_2 >> 1$, we have

$$u_I \simeq \frac{1}{i\lambda} \iint\limits_{\mathcal{A}} u^i \frac{z_2}{R_2} \frac{e^{ikR_2}}{R_2} dxdy \tag{0.4.1}$$

where

$$R_2 = \sqrt{(x - x_2)^2 + (y - y_2)^2 + z_2^2}. \tag{0.4.2}$$

- The field u^i that is incident upon the aperture in Fig. 8, is a converging spherical wave:

$$u^i = \frac{e^{-ikR_1}}{R_1} \tag{0.4.3}$$

where R_1 is the distance from the focal point or image point (x_1, y_1, z_1) to the integration point $(x, y, 0)$:

$$R_1 = \sqrt{(x - x_1)^2 + (y - y_1)^2 + z_1^2}. \tag{0.4.4}$$

- Restricting our attention to paraxial geometries, using the Fresnel approximation, and letting the image point (focus) lie on the z axis, so that $x_1 = y_1 = 0$, we have

$$u_I \simeq \frac{C}{i\lambda z_1 z_2} \iint\limits_{\mathcal{A}} \exp\left\{-ik\frac{xx_2 + yy_2}{z_2}\right\} \exp\left\{i\frac{k}{2}\left(\frac{1}{z_2} - \frac{1}{z_1}\right)(x^2 + y^2)\right\} dxdy \tag{0.4.5}$$

where

$$C = \exp\left\{ik\left[z_2 - z_1 + \frac{x_2^2 + y_2^2}{2z_2}\right]\right\}. \tag{0.4.6}$$

0.4.2 Circular aperture

- We introduce polar co-ordinates to obtain from (0.4.5)

$$u_I \simeq \frac{C}{i\lambda}\frac{a^2}{z_1 z_2} 2\pi \int_0^1 J_0(v't) \exp\left\{-i\frac{1}{2}u't^2\right\} tdt \tag{0.4.7}$$

where

$$v' = v\frac{z_1}{z_2} \quad ; \quad v = k\frac{a}{z_1}r \tag{0.4.8}$$

$$u' = u\frac{z_1}{z_2} \quad ; \quad u = k\left(\frac{a}{z_1}\right)^2 \tilde{z} \quad ; \quad \tilde{z} = z_2 - z_1. \tag{0.4.9}$$

0.4.3 Classical theory

- The *classical* theory of focusing is based on the assumption that the distance from the aperture to the focus is infinitely large.
- Hence $z_1/z_2 \simeq 1$, $u' \simeq u$, and $v' \simeq v$, so that (0.4.7) gives

$$I = |u_I|^2 = I_0 \left| 2\int_0^1 J_0(vt)e^{-i\frac{1}{2}ut^2} t dt \right|^2 \tag{0.4.10}$$

where $I_0 = \left(\frac{\pi a^2}{\lambda z_1^2}\right)^2$ is the intensity in the focal point $u = v = 0$.

- According to the classical theory, the diffraction pattern is *symmetric* about the focal plane:

$$I(u,v) = I(-u,v) \quad ; \quad u = k\left(\frac{a}{z_1}\right)^2 \tilde{z}. \tag{0.4.11}$$

- Along the axis $v = 0$, (0.4.10) gives

$$I(u,0) = I_0\left(\frac{\sin(u/4)}{u/4}\right)^2 = I_0 \operatorname{sinc}^2(u/4). \tag{0.4.12}$$

0.4.4 Focal shift

The assumptions upon which the classical theory is based are not satisfied at low Fresnel numbers N, defined by

$$N = \frac{a^2}{\lambda z_1}.$$

- When $N \sim 1$, one can see large deviations between observations and results of the classical theory.
- Then we must return to (0.4.7), which gives

$$I(u',v') = |u_I|^2 = I_0\left(\frac{z_1}{z_2}\right)^2 \left| 2\int_0^1 J_0(v't)e^{-i\frac{1}{2}u't^2} t dt \right|^2. \tag{0.4.13}$$

- From (0.4.9) we have

$$u' = 2\pi N \frac{z_2 - z_1}{z_2} = 2\pi N \left(1 - \frac{z_1}{z_2}\right) \quad ; \quad \frac{z_1}{z_2} = 1 - \frac{u'}{2\pi N}. \tag{0.4.14}$$

so that (0.4.13) becomes

$$I(u', v') = I_0 \left(1 - \frac{u'}{2\pi N}\right)^2 \left|2 \int_0^1 J_0(v't) e^{-i\frac{1}{2}u't^2} t dt\right|^2, \tag{0.4.15}$$

- Still we have a kind of symmetry:

$$\left(1 + \frac{u'}{2\pi N}\right)^2 I(u', v') = I(-u', v') \left(1 - \frac{u'}{2\pi N}\right)^2. \tag{0.4.16}$$

- But there is *no* symmetry about the focal plane $\tilde{z} = 0$, since there is a nonlinear relation between u' and $\tilde{z}$.
- When $\frac{z_2}{z_1} \simeq 1$, we have approximate symmetry about the focal plane.
- Include focal shift figures!!

0.4.5 Aberrations

According to the classical theory, the intensity in the focal area of a perfect imaging syatem is given by (0.4.10), which can be written

$$I = I_0 \frac{1}{\pi^2} \left| \int_0^{2\pi} \int_0^1 e^{-i\left[vt\cos(\phi-\beta) + \frac{1}{2}ut^2\right]} t dt d\phi \right|^2. \tag{0.4.17}$$

- If the imaging system is not perfect, we introduce an aberration function $\phi_0(t, \beta)$, which describes the deviations of the converging wave front will have deviations from spherical shape.
- Then the intensity in (0.4.17) becomes

$$I = I_0 \frac{1}{\pi^2} \left| \int_0^{2\pi} \int_0^1 e^{i\left[k\phi_0(t,\phi) - vt\cos(\phi-\beta) - \frac{1}{2}ut^2\right]} t dt d\phi \right|^2. \tag{0.4.18}$$

- If the object point lies on the optical axis, we only have spherical aberrations of various orders. For first-order spherical aberration the aberration function is given by

$$\phi_0 = \delta_1 \lambda t^4 \tag{0.4.19}$$

where δ_1 is the deviation of the wave front from spherical shape at the edge of the aperture, measured in wavelengths.

- As the object point moves away from the axis, coma is the first off-axis aberration to appear. First-order coma is given by

$$\phi_0 = \delta_2 \lambda t^3 \cos\phi, \tag{0.4.20}$$

- As the object point moves sufficiently far from the axis, astigmatism starts to play a role. For pure first-order astigmatism the aberration function is

$$\phi_0 = \delta_3 \lambda t^2 \cos^2\phi. \tag{0.4.21}$$

0.5 Radiation problems

0.5.1 Field radiated by a localised source

- Our task is to determine the field radiated by a given time-harmonic source $s(\mathbf{r})$, so that the Helmholz equation becomes

$$(\nabla^2 + k^2)u(\mathbf{r}) = s(\mathbf{r}). \tag{0.5.22}$$

- We define a three-dimensional Fourier transform pair:

$$a(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^3 \iiint_{-\infty}^{\infty} A(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3\mathbf{k}, \tag{0.5.23}$$

$$A(\mathbf{k}) = \iiint_{-\infty}^{\infty} a(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^3\mathbf{r}. \tag{0.5.24}$$

- Expressing both $u(\mathbf{r})$ and $s(\mathbf{r})$ as Fourier integrals, we find upon substitution in (0.5.22)

$$\iiint_{-\infty}^{\infty} \left[\left(-k_3^2 + k^2\right) U(\mathbf{k}) - S(\mathbf{k})\right] e^{i\mathbf{k}\cdot\mathbf{r}} d^3\mathbf{k} = 0 \tag{0.5.25}$$

where $k_3^2 = k_x^2 + k_y^2 + k_z^2$.

- The uniqueness of Fourier integrals gives

$$U(\mathbf{k}) = -\frac{S(\mathbf{k})}{k_3^2 - k^2}. \tag{0.5.26}$$

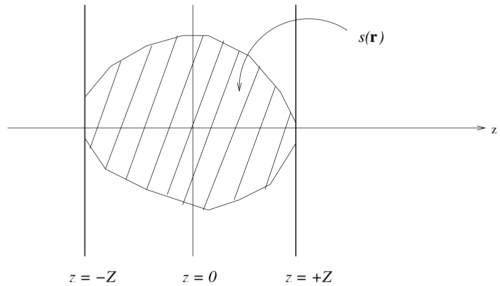

Figure 9: Radiation by a localised source that vanishes for $|z| > Z$.

- Thus, the Fourier representation of $u(\mathbf{r})$ becomes [cf. (0.5.23)]

$$u(\mathbf{r}) = -\left(\frac{1}{2\pi}\right)^3 \iiint_{-\infty}^{\infty} \frac{S(\mathbf{k})}{k_3^2 - k^2} e^{i\mathbf{k}\cdot\mathbf{r}} d^3\mathbf{k}. \tag{0.5.27}$$

- We let the source be confined to the slab $|z| < Z$ (Fig. 9).
- We perform the k_z integration in (0.5.27) by using the calculus of residues and close the contour of integration in the upper half of the complex k_z plane for $z > Z$ and in the lower part of the k_z plane for $z < -Z$.
- Since the source is confined to the slab $|z| < Z$, the integral along that part of the integration path which lies on a semi-circle with infinite radius in the upper half-plane when $z > Z$ or in the lower half-plane when $z < -Z$, will *not* contribute to the integral over the closed integration path.
- Thus, the k_z integral in (0.5.27) is equal to $2\pi i$ times the sum of the residus of the poles in the upper half-plane when $z > Z$ and equal to $-2\pi i$ times the sum of the residues of the poles in the lower half-plane when $z < -Z$.
- Since $S(\mathbf{k})$ is an entire function of k_z, the only singularities in (0.5.26) are the poles contained in the factor $\frac{1}{k_3^2 - k^2}$.
- Carrying out the k_z integration in the manner just explained, we find

$$u^{\pm}(\mathbf{r}) = -\frac{i}{2}\left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} \frac{S(\mathbf{k}^{\pm})}{k_z} e^{i\mathbf{k}^{\pm}\cdot\mathbf{r}} dk_x dk_y \tag{0.5.28}$$

where

$$\mathbf{k}^{\pm} = k_x\hat{\mathbf{e}}_x + k_y\hat{\mathbf{e}}_y \pm k_z\hat{\mathbf{e}}_z \tag{0.5.29}$$

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2} \quad ; \quad \mathrm{Im}(k_z) \geq 0. \tag{0.5.30}$$

- Here u^+ represents u in the half-space $z > Z$, and u^- represents u in the half-space $z < -Z$.

0.5.2 Field due to a point source - Green's function

- Consider the field radiated by a point source located at the origin. Then the source is

$$s(\mathbf{r}) = \delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$$

and hence

$$S(\mathbf{k}) = \iiint_{-\infty}^{\infty} s(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}d^3\mathbf{r} = \iiint_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)e^{-i\mathbf{k}\cdot\mathbf{r}}dxdydz = 1. \tag{0.5.31}$$

- Equation (0.5.28) now gives

$$u^{\pm}(\mathbf{r}) = -\frac{1}{4\pi}\left(\frac{i}{2\pi}\iint_{-\infty}^{\infty}\frac{e^{i(k_x x+k_y y+k_z|z|)}}{k_z}dk_x dk_y\right). \tag{0.5.32}$$

- The expression inside the parenthesis is Weyl's plane-wave expansion of a spherical wave. Thus

$$u(\mathbf{r}) = u^+(\mathbf{r}) = u^-(\mathbf{r}) = -\frac{1}{4\pi}\frac{e^{ikr}}{r} \quad ; \quad r = \sqrt{x^2+y^2+z^2}. \tag{0.5.33}$$

- This particular solution is called the Green's function. For wave propagation in three dimensions we thus have

$$(\nabla^2 + k^2)G(\mathbf{r}) = \delta(x)\delta(y)\delta(z) \tag{0.5.34}$$

where

$$G = -\frac{1}{4\pi}\frac{e^{ikr}}{r}. \tag{0.5.35}$$

- In terms of the Gren's function, the field radiated by a source s($\mathbf{r}$) can be expressed as

$$u(\mathbf{r}) = \iiint_{-\infty}^{\infty} s(\mathbf{r}')G(\mathbf{r}-\mathbf{r}')d^3\mathbf{r}' \tag{0.5.36}$$

where

$$G(\mathbf{r}-\mathbf{r}') = -\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{0.5.37}$$

- The physical interpretation of (0.5.36) is that the field consists of a sum (integral) of point-source solutions $G(\mathbf{r}-\mathbf{r}')$ that are weighted by the source $s(\mathbf{r})$.

0.5.3 Two-dimensional wave propagation

- If the source does *not* vary with y, so that

$$s(\mathbf{r}) = s(x,z) = s(\mathbf{r}_2) \quad ; \quad \mathbf{r}_2 = x\hat{\mathbf{e}}_x + z\hat{\mathbf{e}}_z \tag{0.5.38}$$

we obtain in the same manner as in the 3D case

$$u^{\pm}(\mathbf{r}_2) = -\left(\frac{1}{2\pi}\right)^2 2\pi i \int_{-\infty}^{\infty} \frac{S(\mathbf{k}_2^{\pm})}{2k_{z2}} e^{i\mathbf{k}_2^{\pm}\cdot\mathbf{r}_2} dk_x \tag{0.5.39}$$

where

$$\mathbf{k}_2^{\pm} = k_x\hat{\mathbf{e}}_x \pm k_{z2}\hat{\mathbf{e}}_z \quad ; \quad k_{z2} = \sqrt{k^2 - k_x^2} \quad ; \quad \mathrm{Im}(k_{z2}) \geq 0. \tag{0.5.40}$$

- Here the upper sign applies for $z > Z$ and the lower sign applies for $z < -Z$.

0.5.4 Field radiated by a line source - Green's function

- For a line source located at the origin, the source is given by

$$s(\mathbf{r}_2) = \delta(x)\delta(z) \tag{0.5.41}$$

so that

$$S(\mathbf{k}_2) = 1. \tag{0.5.42}$$

- Thus, (0.5.39) gives

$$u^{+}(\mathbf{r}_2) = u^{-}(\mathbf{r}_2) = u(\mathbf{r}_2) = -\frac{i}{4\pi}\int_{-\infty}^{\infty} \frac{e^{i(k_x x + k_{z2}|z|)}}{k_{z2}} dk_x. \tag{0.5.43}$$

- Using the plane-wave expansion for the zeroth-order Hankel function of the first kind:

$$H_0^{(1)}(k|\mathbf{r}_2|) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{i(k_x x + k_{z2}|z|)}}{k_{z2}} dk_x \tag{0.5.44}$$

we have

$$u(\mathbf{r}_2) = -\frac{i}{4} H_0^{(1)}(k|\mathbf{r}_2|). \tag{0.5.45}$$

- Similarly as in the three-dimensional case, we call this solution the Green's function. Thus, for two-dimensional wave propagation we have

$$(\nabla_2^2 + k^2)G(\mathbf{r}_2) = \delta(x)\delta(y), \tag{0.5.46}$$

where

$$G(\mathbf{r}_2) = -\frac{i}{4}H_0^{(1)}(k|\mathbf{r}_2|) \quad ; \quad |\mathbf{r}_2| = \sqrt{x^2 + z^2}. \tag{0.5.47}$$

- In terms of the 2D Green's function, the field radiated by a line source becomes:

$$u(\mathbf{r}_2) = \iint_{-\infty}^{\infty} s(\mathbf{r}_2')G(\mathbf{r}_2 - \mathbf{r}_2')d^2\mathbf{r}_2' \tag{0.5.48}$$

where

$$G(\mathbf{r}_2 - \mathbf{r}_2') = -\frac{i}{4}H_0^{(1)}(k|\mathbf{r}_2 - \mathbf{r}_2'|). \tag{0.5.49}$$

0.6 Electromagnetic radiation problems

0.6.1 Field radiated by localised source

- Maxwell's equations in Gaussian units are given by:

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = 4\pi\rho(\mathbf{r}, t) \tag{0.6.1}$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{1}{c}\dot{\mathbf{B}}(\mathbf{r}, t) \tag{0.6.2}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0 \tag{0.6.3}$$

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = \frac{\mu}{c}\dot{\mathbf{D}}(\mathbf{r}, t) + \frac{4\pi\mu}{c}\mathbf{J}(\mathbf{r}, t). \tag{0.6.4}$$

- The total current density consists of two terms:

$$\mathbf{J}(\mathbf{r}, t) = \mathbf{J}_0(\mathbf{r}, t) + \sigma\mathbf{E}(\mathbf{r}, t), \tag{0.6.5}$$

- In addition we have the continuity equation

$$\dot{\rho}(\mathbf{r}, t) = -\nabla \cdot \mathbf{J}(\mathbf{r}, t) = -\nabla \cdot [\mathbf{J}_0(\mathbf{r}, t) + \sigma\mathbf{E}(\mathbf{r}, t)]. \tag{0.6.6}$$

- If the source is time-harmonic, so that

$$\mathbf{J}_0(\mathbf{r},t) = \mathrm{Re}\left\{\hat{\mathbf{J}}_0(\mathbf{r},\omega)e^{-i\omega t}\right\} \tag{0.6.7}$$

the radiated field will be time-harmonic as well.

- Then each scalar component in (0.6.1)-(0.6.7) can be expressed as follows

$$a(\mathbf{r},t) = \mathrm{Re}\left\{\hat{a}(\mathbf{r},\omega)e^{-i\omega t}\right\}. \tag{0.6.8}$$

- Maxwell's equations (0.6.1)-(0.6.4) become:

$$\nabla \cdot \hat{\mathbf{D}}(\mathbf{r},\omega) = 4\pi\hat{\rho}(\mathbf{r},\omega) \tag{0.6.9}$$

$$\nabla \times \hat{\mathbf{E}}(\mathbf{r},\omega) = \frac{i\omega}{c}\hat{\mathbf{B}}(\mathbf{r},\omega) \tag{0.6.10}$$

$$\nabla \cdot \hat{\mathbf{B}}(\mathbf{r},\omega) = 0 \tag{0.6.11}$$

$$\nabla \times \hat{\mathbf{B}}(\mathbf{r},\omega) = -\frac{i\mu\omega}{c}\hat{\mathbf{D}}(\mathbf{r},\omega) + \frac{4\pi\mu}{c}[\hat{\mathbf{J}}_0(\mathbf{r},\omega) + \sigma\hat{\mathbf{E}}(\mathbf{r},\omega)]. \tag{0.6.12}$$

- As in the scalar case, we introduce Fourier representations:

$$\hat{\mathbf{A}}(\mathbf{r},\omega) = \left(\frac{1}{2\pi}\right)^3 \iiint_{-\infty}^{\infty} \tilde{\mathbf{A}}(\mathbf{k},\omega)e^{i\mathbf{k}\cdot\mathbf{r}}d^3\mathbf{k} \tag{0.6.13}$$

where

$$\tilde{\mathbf{A}}(\mathbf{k},\omega) = \iiint_{-\infty}^{\infty} \hat{\mathbf{A}}(\mathbf{r},\omega)e^{-i\mathbf{k}\cdot\mathbf{r}}d^3\mathbf{r}. \tag{0.6.14}$$

- Substituting (0.6.13) in (0.6.9)-(0.6.12), we obtain algebraic equations for $\tilde{\mathbf{E}}(\mathbf{k},\omega)$ and $\tilde{\mathbf{B}}(\mathbf{k},\omega)$.

- We solve these algebraic equations, substitute the results in (0.6.13), and carry out the k_z integration in the same way as in the scalar case to obtain:

$$\hat{\mathbf{E}}^{\pm}(\mathbf{r},\omega) = \iint_{-\infty}^{\infty} \mathcal{E}(\mathbf{k}^{\pm},\omega)e^{i\mathbf{k}^{\pm}\cdot\mathbf{r}}dk_x dk_y \tag{0.6.15}$$

$$\mathcal{E}(\mathbf{k}^{\pm}) = -\frac{\omega\mu}{2\pi k^2c^2k_z}[k^2\tilde{\mathbf{J}}_0(\mathbf{k}^{\pm}) - \mathbf{k}^{\pm}(\mathbf{k}^{\pm}\cdot\tilde{\mathbf{J}}_0(\mathbf{k}^{\pm}))] = \frac{\omega\mu}{2\pi k^2c^2k_z}\mathbf{k}^{\pm}\times[\mathbf{k}^{\pm}\times\tilde{\mathbf{J}}_0(\mathbf{k}^{\pm})] \tag{0.6.16}$$

$$\hat{\mathbf{B}}^{\pm}(\mathbf{r},\omega) = \iint_{-\infty}^{\infty} \mathcal{B}(\mathbf{k}^{\pm},\omega)e^{i\mathbf{k}^{\pm}\cdot\mathbf{r}}dk_x dk_y \tag{0.6.17}$$

$$\mathcal{B}(\mathbf{k}^{\pm},\omega) = \frac{c}{\omega}\mathbf{k}^{\pm}\times\mathcal{E}(\mathbf{k}^{\pm},\omega) = \frac{-\mu(\mathbf{k}^{\pm}\times\tilde{\mathbf{J}}_0(\mathbf{k}^{\pm}))}{2\pi c k_z}. \tag{0.6.18}$$

0.6.2 Field radiated by a dipole

- Let the source be a dipole located at the origin and polarised along the unit vector $\hat{\mathbf{n}}$. Then we have

$$\mathbf{J}_0(\mathbf{r},t) = \Re\left\{\hat{\mathbf{J}}_0(\mathbf{r},\omega)e^{-i\omega t}\right\} \quad ; \quad \hat{\mathbf{J}}_0(\mathbf{r},\omega) = \hat{\mathbf{n}}I\delta(\mathbf{r}) \tag{0.6.19}$$

so that

$$\tilde{\mathbf{J}}_0(\mathbf{k}) = \hat{\mathbf{n}}I\iiint_{-\infty}^{\infty}\delta(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}d^3\mathbf{r} = \hat{\mathbf{n}}I \tag{0.6.20}$$

where I is the dipole strength.

- Thus, from (0.6.15) - (0.6.18):

$$\mathbf{E}^{\pm} = \frac{\omega\mu I}{2\pi c^2 k^2}\iint_{-\infty}^{\infty}\frac{\mathbf{k}^{\pm}\times(\mathbf{k}^{\pm}\times\hat{\mathbf{n}})}{k_z}e^{i\mathbf{k}^{\pm}\cdot\mathbf{r}}dk_x dk_y \tag{0.6.21}$$

$$\mathbf{B}^{\pm} = \frac{-\mu I}{2\pi c}\iint_{-\infty}^{\infty}\frac{\mathbf{k}^{\pm}\times\hat{\mathbf{n}}}{k_z}e^{i\mathbf{k}^{\pm}\cdot\mathbf{r}}dk_x dk_y. \tag{0.6.22}$$

- Using Weyl's plane-wave expansion of a spherical wave, we can rewrite these expressions as:

$$\mathbf{E} = \frac{i\omega\mu I}{c^2k^2}\nabla\times\nabla\times\left[\hat{\mathbf{n}}\frac{e^{ikr}}{r}\right] \tag{0.6.23}$$

$$\mathbf{B} = \frac{\mu I}{c}\nabla\times\left(\hat{\mathbf{n}}\frac{e^{ikr}}{r}\right). \tag{0.6.24}$$

- Carrying out the differentiations, the expressions for $\mathbf{E}$ and $\mathbf{B}$ become

$$\mathbf{B} = -ik\frac{\mu I}{c}\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\hat{\mathbf{n}}\times\hat{\mathbf{e}}_r \tag{0.6.25}$$

$$\mathbf{E} = -i\frac{\omega\mu I}{c^2}\frac{e^{ikr}}{r}\left\{\left(1+\frac{3i}{kr}-\frac{3}{(kr)^2}\right)\hat{\mathbf{e}}_r(\hat{\mathbf{e}}_r\cdot\hat{\mathbf{n}}) - \left(1+\frac{i}{kr}-\frac{1}{(kr)^2}\right)\hat{\mathbf{n}}\right\}. \tag{0.6.26}$$

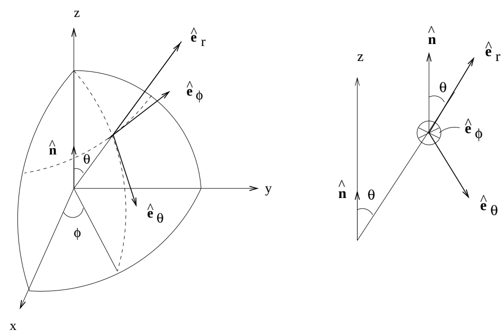

Figure 10: Co-ordinate systems related to the study of the field radiated by a dipole that is placed at the origin and polarised along the z axis.

- With a co-ordinate system such that $\hat{\mathbf{n}}$ points along the z axis, and with spherical co-ordinates r, θ, and ϕ, as shown in Fig. 10, we have

$$\mathbf{E} = E_r\hat{\mathbf{e}}_r + E_\theta\hat{\mathbf{e}}_\theta \quad ; \quad \mathbf{B} = B_\phi\hat{\mathbf{e}}_\phi, \tag{0.6.27}$$

- When $kr >> 1$, we may neglect E_r and the higher-order terms in E_θ and B_ϕ to obtain

$$E_\theta \sim -i\frac{\omega\mu I}{c^2}\frac{e^{ikr}}{r}\sin\theta \tag{0.6.28}$$

$$B_\phi \sim -i\frac{k\mu I}{c}\frac{e^{ikr}}{r}\sin\theta. \tag{0.6.29}$$

- In vacuum, where $\mu = \varepsilon = 1$, $\sigma = 0$, and $k = \frac{\omega}{c}$, the result in the far zone becomes

$$E_\theta \sim B_\phi \sim -i\frac{\omega I}{c^2}\left(\frac{e^{ikr}}{r}\right)\sin\theta. \tag{0.6.30}$$

- Thus, in the far zone $\mathbf{E}$ and $\mathbf{B}$ are of equal size, and they are normal to one another and to $\hat{\mathbf{e}}_r$, which now points in the direction of the Poynting vector.

- This implies that the far field radiated by a dipole behaves locally as a plane wave.

- Note, however, that the amplitude of the field is proportional to $\sin\theta$, which means that the radiated energy is proportional to $\sin^2\theta$. Therefore, the dipole does *not* radiate energy along its own axis.

0.7 Retarded solution of the wave equation

- Consider the inhomogeneous, scalar wave equation in a uniform, non-dispersive medium:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\hat{u}(\mathbf{r},t) = \hat{s}(\mathbf{r},t). \tag{0.7.1}$$

- By Fourier decomposition of the field, we have

$$\hat{u}(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(\mathbf{r},\omega)e^{-i\omega t}d\omega = 2\mathrm{Re}\frac{1}{2\pi}\int_{0}^{\infty} u(\mathbf{r},\omega)e^{-i\omega t}d\omega \tag{0.7.2}$$

which implies that $u(\mathbf{r},\omega)$ satisfies the inhomogeneous Helmholz equation:

$$(\nabla^2 + k^2)u(\mathbf{r},\omega) = s(\mathbf{r},\omega) \quad ; \quad k = \frac{\omega}{c}. \tag{0.7.3}$$

- The solution of (0.7.3) can be written

$$u(\mathbf{r},\omega) = \iiint_{-\infty}^{\infty} s(\mathbf{r},\omega)G(\mathbf{r}-\mathbf{r}',\omega)d^3\mathbf{r}' \quad ; \quad G(\mathbf{r}-\mathbf{r}',\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{-4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{0.7.4}$$

- On substituting (0.7.4) in (0.7.2), we obtain:

$$u(\mathbf{r},t) = -\frac{1}{4\pi}\iiint_{-\infty}^{\infty} \frac{s(\mathbf{r}', t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|)}{\mathbf{r}-\mathbf{r}'}d^3\mathbf{r}'. \tag{0.7.5}$$

- This expression is called the *retarded* solution of the wave equation.

- The physical intepretation of this solution is that the field at the observation point $\mathbf{r}$ at time t consists of a sum of contributions from various source elements $\mathbf{r}'$ that are radiated at the preceding time $t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|$.

- Each contribution has to leave the source element at this earlier time in order to reach the observation point at the given time t.

0.8 Asymptotic diffraction theory

- According to Huygen's principle, the field that is diffracted through an aperture consists of contributions from an infinite number of secondary waves, one from each point in the aperture. Mathematically this sum is expressed as a diffraction integral.

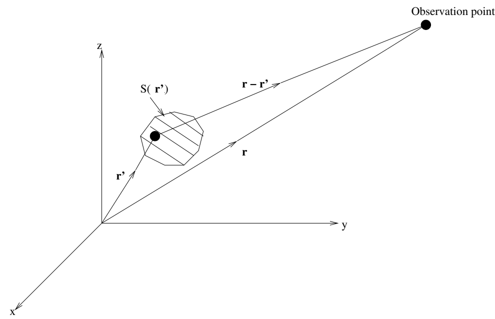

Figure 11: Geometry for interpretation of the retarded solution of the wave equation.

- By evaluating the diffraction integral by means of asymptotic techniques, one can show that only a few of the secondary waves contribute significantly to the diffracted field.

0.8.1 Two-dimensional and three-dimensional diffraction problems

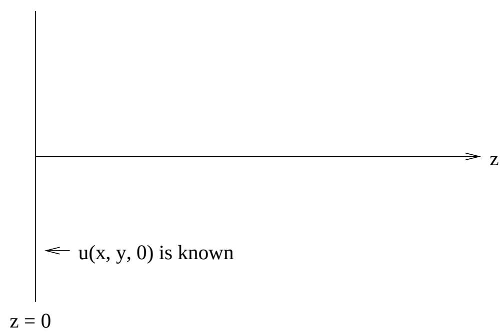

Figure 12: Illustration of a two-dimensional boundary-value problem where the field in the plane $z = 0$ is known and the field in the half-space $z > 0$ is to be determined.

- Consider three-dimensional (3D) and two-dimensional (2D) wave propagation in a homogeneous medium, and let the field be generated by sources in the half-space $z \leq 0$, and let it be known in the plane $z = 0$ (see Fig. 12).
- In the Kirchhoff approximation the solution to 3D and 2D diffraction problems can be expressed as

$$\text{3D:} \quad u_I(x_2, y_2, z_2) = \iint_{\mathcal{A}} g(x, y) e^{ikf(x,y)} dx dy \tag{0.8.1}$$

$$f(x, y) = R_1 + R_2 \;\; ; \;\; g(x, y) = \frac{1}{i\lambda} \frac{z_2}{R_2} \frac{1}{R_1 R_2} \tag{0.8.2}$$

$$R_j = \sqrt{(x - x_j)^2 + (y - y_j)^2 + z_j^2}. \tag{0.8.3}$$

$$\text{2D:} \quad u_I(x_2, z_2) = \int_a^b g(x) e^{ikf(x)} dx, \tag{0.8.4}$$

$$f(x) = R_1 + R_2 \;\; ; \;\; g(x) = \frac{1}{\sqrt{i\lambda}} \frac{z_2}{R_2} \frac{1}{\sqrt{R_1 R_2}}, \tag{0.8.5}$$

$$R_j = \sqrt{(x - x_j)^2 + z_j^2}. \tag{0.8.6}$$

0.8.2 The method of stationary phase for single integrals

- For simplicity we concentrate on two-dimensional diffraction problems, so that the diffracted field can be expressed as:

$$J = \int_{x_1}^{x_2} g(x) e^{ikf(x)} dx. \tag{0.8.7}$$

- When $k|x_2 - x_1|$ is sufficiently large, then $\exp[ikf(x)]$ will oscillate so rapidly compared to $g(x)$ that cancellation occurs except in the immediate neighbourhood of *stationary* points $x = x_s$, where $f'(x_s) = 0$, or in the immediate neighbourhood of either end point $x = x_1$ or $x = x_2$.

- For simplicity we assume that we have *isolated* stationary points and end points.

Isolated, interior stationary point Suppose we have an *isolated, interior* stationary point x_s, so that $x_1 << x_s << x_2$.

- In order to determine the asymptotic contribution to the integral J in (0.8.7) we expand $f(x)$ and $g(x)$ about $x = x_s$, so that

$$\left.\begin{aligned} g(x) &= g_0 + g_1 t + g_2 t^2 + \ldots \\ f(x) &= f_0 + f_2 t^2 + f_3 t^3 + \ldots \end{aligned}\right\} t = x - x_s, \tag{0.8.8}$$

where

$$g_n = \frac{1}{n!} \left.\frac{\partial^n g}{\partial x^n}\right|_{x=x_s} \quad ; \quad f_n = \frac{1}{n!} \left.\frac{\partial^n f}{\partial x^n}\right|_{x=x_s}. \tag{0.8.9}$$

- Further, we write

$$e^{ikf(x)} = e^{ikf_0} e^{ikf_2 t^2} e^{ik\Delta f} \quad ; \quad \Delta f = f_3 t^3 + f_4 t^4 \ldots \tag{0.8.10}$$

and we expand $e^{ik\Delta f}$:

$$e^{ik\Delta f} = 1 + ik\Delta f + \frac{1}{2}(ik\Delta f)^2 + \ldots . \tag{0.8.11}$$

- To the lowest asymptotic order we get the following contribution J_S to the integral J in (0.8.7) from the interior stationary point x_s:

$$J_S = \sqrt{\frac{\pi}{k|f_2|}} g_0 e^{i\left[kf_0 + \frac{\pi}{4}\text{sgn}(f_2)\right]}. \tag{0.8.12}$$

- Higher-order terms in the asymptotic contribution from an isolated, interior stationary point are given by (cf. equations (8.8a)-(8.8g) in Stamnes, 1986)

$$J_S \sim \sqrt{\frac{\pi}{k|f_2|}} e^{i\left(kf_0 + \frac{\pi}{4} - \frac{1}{2}\arg f_2\right)} [Q_0 + Q_2 + Q_4] \tag{0.8.13}$$

where

$$Q_0 = g_0 \tag{0.8.14}$$

$$Q_2 = \frac{i}{kf_2}\left(\frac{1}{2}g_2 - \frac{3}{4}\frac{g_1 f_3 + g_0 f_4}{f_2} + \frac{15}{16}\frac{g_0 f_3^2}{f_2^2}\right) \tag{0.8.15}$$

$$Q_4 = \frac{1}{(kf_2)^2}\left[-\frac{3}{4}g_4 + \frac{15}{8}\frac{A}{f_2} - \frac{105}{32}\frac{B}{f_2^2} + \frac{315}{64}\frac{C}{f_2^3} - \frac{3465}{512}g_0\left(\frac{f_3}{f_2}\right)^4\right]. \tag{0.8.16}$$

- Here the coefficients A, B, and C are given by

$$A = g_3 f_3 + g_2 f_4 + g_1 f_5 + g_0 f_6 \tag{0.8.17}$$

$$B = g_2 f_3^2 + 2g_1 f_3 f_4 + g_0(f_4^2 + 2f_3 f_5) \tag{0.8.18}$$

$$C = g_1 f_3^3 + 3g_0 f_3^2 f_4. \tag{0.8.19}$$

Stationary end point Let the stationary point coincide with the lower end point, so that $x_s = x_1$.

- Then we get the following asymptotic contribution

$$J_{SE} \sim J_{even} + J_{odd} \sim \frac{1}{2}\left(\frac{\pi}{k|f_2|}\right)^{1/2} e^{i(kf_0+\psi)}(Q_0 + Q_1 + Q_2 + Q_3 + Q_4) \quad (0.8.20)$$

where

$$\psi = \frac{\pi}{4} - \frac{1}{2}\text{arg}f_2. \quad (0.8.21)$$

- Here Q_1 and Q_3 are given in equations (8.11b) and (8.11c) in Stamnes (1986).
- Note that to the lowest asymptotic order the contribution from an end point is half as large as that from an interior stationary point.

Non-stationary end point To determine the asymptotic contribution from a non-stationary end point we may follow a similar procedure as in the previous two cases.

- But in this case it is just as simple to use integration by parts.
- To the lowest asymptotic order the contribution from a non-stationary end point $x = x_j$ $(j = 1, 2)$ becomes

$$J_E = (-1)^j \frac{g(x_j)}{ikf'(x_j)} e^{ikf(x_j)}, \quad (0.8.22)$$

where $j = 2$ applies to an upper end point, and where $j = 1$ applies to a lower end point.