Charge in Electric and Magnetic Fields

Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) - \frac{\partial L}{\partial r_i} = 0 \qquad i = 1, 2, ...$$

For conservative systems, i.e. with usual forces from potential energy

$$L(r_i, \dot{r}_i, t) = T(\dot{r}_i) - V(r_i)$$

but the Lorentz force depends on velocity. Lagrange function $L(r_i, \dot{r}_i, t)$ must be modified.

Lorentz force

$$\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\left[\mathbf{v} \times \mathbf{B}\right]\right)$$

With Φ and $\mathbf{A}$ the scalar and vector potentials

$$\mathbf{E} = -\nabla\Phi + -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \qquad\qquad \mathbf{B} = \nabla \times \mathbf{A}$$

the Lorentz force becomes

$$\mathbf{F} = q\left(-\nabla\Phi + -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \frac{1}{c}\mathbf{v} \times \left[\nabla \times \mathbf{A}\right]\right)$$

with $\Phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ describing the fields

With only the scalar potential $\Phi(\mathbf{r},t)$ the Lagrange function would be

$$L(\mathbf{r},\dot{\mathbf{r}},t)=\frac{m}{2}\,\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}-q\Phi(\mathbf{r},t) \quad = \quad \frac{1}{2}\,m\,\mathbf{v}^2-q\Phi(\mathbf{r},t)$$

and the Lagrange equation would lead to the electrostatic

$$m\ddot{\mathbf{r}}=-q\nabla\Phi$$

It can be shown that the Newton equation with the electromagnetic Lorentz force

$$m\ddot{\mathbf{r}}=q\left(-\nabla\Phi+-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}+\frac{1}{c}\mathbf{v}\times[\nabla\times\mathbf{A}]\right)$$

can be derived from a surprisingly simple Lagrange function (see below)

$$L(\mathbf{r},\dot{\mathbf{r}},t)= \quad \frac{1}{2}\,m\,\mathbf{v}^2-q\Phi(\mathbf{r},t)+\frac{q}{c}\mathbf{v}\cdot\mathbf{A}$$

when inserted into the three Lagrange equations ($r_1\to x$, $r_2\to y$, $r_3\to z$)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{r}_i}\right)-\frac{\partial L}{\partial r_i}=0 \qquad i=1,2,3$$

This is because the term $\mathbf{v}\times[\nabla\times\mathbf{A}]$ can be expressed without vector products.

Transforming the term $\mathbf{v} \times [\nabla \times \mathbf{A}]$

This contains time derivatives as well as the x, y, z derivatives

$$\mathbf{v} \times [\nabla \times \mathbf{A}] \quad \longrightarrow \quad \dot{\mathbf{r}} \times [\nabla \times \mathbf{A}]$$

consider first the total time derivative $\dot{\mathbf{A}}$,

$$\dot{\mathbf{A}} = \frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial \mathbf{A}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{A}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{A}}{\partial z}\frac{dz}{dt} = \frac{\partial \mathbf{A}}{\partial t} + v_x\frac{\partial \mathbf{A}}{\partial x} + v_y\frac{\partial \mathbf{A}}{\partial y} + v_z\frac{\partial \mathbf{A}}{\partial z}$$

Take now its x-component and re-arrange

$$\frac{dA_x}{dt} - \frac{\partial A_x}{\partial t} = v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z}$$

Take now x-component of $\mathbf{v} \times [\nabla \times \mathbf{A}]$

$$\{\mathbf{v} \times [\nabla \times \mathbf{A}]\}_x = v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

and now re-arrange - adding and subtracting (and compare with above)

$$v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x} \qquad - v_x\frac{\partial A_x}{\partial x} - v_y\frac{\partial A_x}{\partial y} - v_z\frac{\partial A_x}{\partial z}$$

Comparing those expressions we can replace the second term

$$\{\mathbf{v} \times [\nabla \times \mathbf{A}]\}_x = \frac{\partial}{\partial x}(v_x A_x + v_y A_y + v_z A_z) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}$$

and now it can be written for all components as

$$\mathbf{v} \times [\nabla \times \mathbf{A}] = \underline{\nabla (\mathbf{v} \cdot \mathbf{A}) - \frac{d\mathbf{A}}{dt} + \frac{\partial \mathbf{A}}{\partial t}}$$

H obtained from

$$p \neq m\dot{x}$$

$$\vec{p} = m\dot{x} + \frac{q}{c}\vec{A}$$

$$H = \frac{p^2}{2m} + e\Phi$$

H depends on $\vec{p} \cdot \vec{A}$ type terms