Light- Atom Interaction

Phys261 Class Seminar

November 30, 2006

1 General

When radiation field be interacted with an atom, the atom may get quantum jumps between its states either by absorbing or emitting.

Our final objective in this presentation is to discuss how the field interacts with the atomic states and emitts radiation.

First we shall discuss the **Fermi Golden Rule** which can be derived using the **Time Depedent Perturbation Theory**.

Then, we will discuss the **Spontaneous and stimulated Emission** by excited an atom.

This presentation is based on the lecture notes on Light-Atom Interaction provided by Prof. Ladislav during his lectures.

2 Perturbation Theory

Perturbation is a small disturbace in a quantum mechanical system which induces a change in the total Hamiltonian operator for Time Dependent Schrödinger Equation (TDSE). A perturbation can be time-dependent as well as time-independent. The total time-dependent Hamiltonian in the perturbing system can be written as

$$H(t) \; = \; H_0 \; + \; H'(t) \tag{1}$$

Here, $H^1(t)$ is a time dependent perturbation.

Without perturbation, i.e, $H'(t)$ is not present, the energy eigenvalue of the eigenstate $\mid \phi_k \rangle$ is given by

$$H_0 \mid \phi_k \rangle = E_k \mid \phi_k \rangle \tag{2}$$

The energy eigen solutions can be,

$$\alpha_k(t) = e^{-iE_k t/\hbar} \tag{3}$$

and the probability P_a of the system to find in state a is

$$p_a(t) = |\alpha_a(t)|^2 \tag{4}$$

This probability must be either 1 or 0 in an unperturbed case. This can be written in delta function notation,

$$|\alpha_k(t)| = \delta_{ka} \tag{5}$$

such that for $k \neq$, then δ_{ka} vanishes. Any unknown wavefunction $\mid \psi(t) \rangle$ can be written as

$$\mid \psi(t) \rangle = \sum \alpha_i(t) \mid \phi_i \rangle \tag{6}$$

The term $\alpha_i(t)$ are expansion coefficients of the wavefunction given by (6).

This wave function (6) when inserted in the following time dependent Schrödinger Equation, produce a set of coupled differential equations.

$$i \frac{\partial}{\partial \mathrm{t}} \mid \psi(t) \rangle = \hat{H}(t) \ \mid \psi(t) \rangle \tag{7}$$

This is shown below by inserting $\alpha_k(t) = c_k(t)e^{-iE_k t/\hbar}$ for states a and k.

$$i\frac{d}{dt}c_k(t) = H_{ka}(t)\exp(i\frac{(E_k - E_a)t}{\hbar}) \tag{8}$$

Approximation in perturbation theory is similar to an iteractive process. If the transitional probability from between two states is

$$P_{ba}(t) = |c_b^{(1)}(t)|^2 \tag{9}$$

calcualtions show that if,

$$c_b^{(1)}(t) = \frac{1}{i\hbar}\int_{t_0}^{t} H' e^{i\omega t'} dt' \tag{10}$$

then, (9) will be

$$P_{ba}(t) = \frac{1}{\hbar^2}|H'_{ba}|^2 F(t,\omega) \tag{11}$$

For larger t, the function $F(\omega, t)$ approaches to

$$F(t,\omega) = \frac{\sin^2 \frac{\omega t}{2}}{(\frac{\omega}{2})^2} \longrightarrow 2\pi t\delta(\omega) \tag{12}$$

this function (12) shows the shape of the Dirac delta-function. The density of final states can be found by integrating $d\omega$

$$\begin{aligned}\int \delta(\omega)d\omega &= \int \delta(E)dE \\ \delta(\omega) &= \delta(E)\cdot\frac{dE}{d\omega} \\ &= \hbar\delta(E-E_b)\end{aligned} \tag{13}$$

We get a final result, that is Fermi Golden rule.

3 Fermi Golden Rule

Fermi Golden rule is a way to calculate the transition rate from one quantum state of an atom to another state due to perturbation. This gives the transistion probability or decay probability.

The final result is given by this formula:

$$\begin{aligned}P_{ba}(t) &= \frac{1}{\hbar}\int_{E_b-\eta}^{E_b+\eta} 2\pi t|H'_{ba}|^2\delta(E-E_b)\rho(E)dE \\ &= \frac{2\pi}{\hbar}t|<b|H'|a>|^2\rho(E_b)\end{aligned} \tag{14}$$

This shows that the transition probability is a linear function of time.
The time derivative of (14) gives the probability rate:

$$\frac{dP_{ba}}{dt} = W_{ba} = \frac{2\pi}{\hbar}|<b|H'|a>|^2\rho(E_b) \tag{15}$$

This femous result is known as **Fermi Golden Rule**. The term, $\frac{2\pi}{\hbar}|<b|H'|a>|^2$ is interaction matrix and the term the term $rho(E_b)$ is the density of final state.
The transition probability, according to (15) is proportional to final state density, provided that the coupling matrix remain constant.

4 Emission of Radiation by Excited Atom

The time dependent perturbation changes the transition probabilty for the emission of a single photon as shown by Fermi Golden Rule. Let $|i\rangle$ be the initial state and $|f\rangle$ be the final state of an atom, then the transition probability is

$$W_{i\to f}(\vec{k},\lambda) = \frac{2\pi}{\hbar}\left|\langle f|H_p|i\rangle\right|^2 \rho(E_f) \tag{16}$$

where $\hbar\vec{k}$ is the momentum of radiation and λ denotes the polarization. H_p is the perturbing Hamiltonian, which in this situation is

$$H_p = -\frac{e}{mc}\vec{p}\cdot\vec{A} \tag{17}$$

The vector potential $\vec{A}$ is given by

$$\vec{A}(\vec{r},t) = \sum_{\vec{k}\lambda}\sqrt{\frac{2\pi\hbar c^2}{V\omega_k}}\widehat{e}_{\vec{k}\lambda}\left(a_{\vec{k}\lambda}e^{i(\vec{k}\cdot\vec{r}-\omega t)} + a^{\dagger}_{\vec{k}\lambda}e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\right) \tag{18}$$

The eq.(18) gives all possible values of the propagation vector $\vec{k}$ and two possible polarizations $\lambda = 1, 2$. The factor

$$\sqrt{\frac{2\pi\hbar c^2}{V\omega_k}}$$

is a normalization factor. and operators $a_{\vec{k}\lambda}$ and $a^{\dagger}_{\vec{k}\lambda}$ are called **annihilator** and **creater** respectively.

When inserted (18) into the (4), we the perturbing hamiltonian:

$$H_p = -\frac{e}{mc}\sum_{\vec{k}\lambda}\sqrt{\frac{2\pi\hbar c^2}{V\omega_k}}\vec{p}\cdot\widehat{e}_{\vec{k}\lambda}\left(a_{\vec{k}\lambda}e^{i(\vec{k}\cdot\vec{r}-\omega t)} + a^{\dagger}_{\vec{k}\lambda}e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\right) \tag{19}$$

The unperturbed wavefunctions before and after the emission of one photon is

$$|i\rangle = |a\rangle\left|\cdots n_{\vec{k}\lambda}\cdots\right\rangle \tag{20}$$

$$|f\rangle = |b\rangle\left|\cdots n_{\vec{k}\lambda}+1\cdots\right\rangle \tag{21}$$

The energy difference is

$$E_f - E_i = \left(E_b + n_{\vec{k}\lambda}\hbar\omega_k + \hbar\omega_k\right) - \left(E_a + n_{\vec{k}\lambda}\hbar\omega_k\right) = E_b - E_a + \hbar\omega_k \tag{22}$$

Comparing the last part of the eq.(16) with the eq.(22), we find out that $\delta(E_f - E_i$ emphasis on the energy conservation in the transition processes. Inserting the final and initial states and the Hamiltonian into the matrix element, we obtain

$$\langle f \left| H_p \right| i \rangle = -\frac{e}{mc}\sqrt{\frac{2\pi\hbar c^2}{V\omega_k}}\left\langle b \left| \vec{p}\cdot\hat{e}_{\vec{k}\lambda}e^{-i\vec{k}\cdot\vec{r}} \right| a \right\rangle \sqrt{n_{\vec{k}\lambda}+1} \tag{23}$$

By considering the eq.(23) we can transform the Golden Rule formula (16) into the form

$$W_{i\to f}(\vec{k},\lambda) = \frac{2\pi}{\hbar}\left(\frac{e}{mc}\right)^2\left(\frac{2\pi\hbar c^2}{V\omega_k}\right)(n_{\vec{k}\lambda}+1)\left|\left\langle b \left| \vec{p}\cdot\hat{e}_{\vec{k}\lambda}e^{-i\vec{k}\cdot\vec{r}} \right| a \right\rangle\right|^2 \rho(E_f) \tag{24}$$

From the equation (24), we see that; one term proportional to the number of photons in the field $n_{\vec{k}\lambda}$and the other term independent of the field. The first term is for **stimulated emission** while the other term is responsible for **spontaneous emission**. The lifetime of the excited state, against this emission is given by

$$\left(\frac{1}{\tau}\right)_{a\to b} = \sum_{\vec{k}\lambda} W_{i\to f}(\vec{k},\lambda) \tag{25}$$

5 Einstein Coefficients

Consider an enclosure containing atoms and field in an equilibrium state at T *deg.* K, and let a and b denote two non-degenerate atomic states, with energy E_a and E_b such that E_b is greater than E_a . We denote by ρ the energy density of the radiation at the angular frequency $\omega = (E_b - E_a)/\hbar$.

The number of atoms making the transition from $a \to b$ per unit time by absorbing radiation, ΔN_b ,is proportional to the total number N_a of atoms in the state a and to the energy density ρ.

$$\Delta N_b = B_\uparrow N_a \rho \tag{26}$$

Figure 1:

$B_\uparrow$ is called the *Einstein Coefficient* for the absorption. ρ is defined as the energy density in the unit $\rho = I/c = \frac{energy}{volume}$

And the number of atoms making the transition $b \to a$ per unit time be ΔN_a , is the sum of the number of spontaneous transition per unit time, which is independent of ρ .Thus

Figure 2:

$$\Delta N_a = A_\downarrow N_b + B_\downarrow N_b \rho \tag{27}$$

$A_\downarrow$ is the *Einstein Coefficient* for spontaneous emission. So, atoms can do stimulated emission by the field present. At equilibrium,

$$(\Delta N_{b\uparrow}) + (\Delta N_{b\downarrow}) = 0 \Longrightarrow (\Delta N_{b\uparrow}) = (\Delta N_{b\downarrow}) \tag{28}$$

leads to

$$\frac{N_a}{N_b} = \frac{A + B_\downarrow\ \rho}{B_\uparrow\ \rho} \tag{29}$$

and at thermal equilibrium it's given

$$\frac{N_a}{N_b} = e^{-(E_a - E_b)/kT} = e^{\Delta E/kT} = e^{\hbar\omega/kT} \tag{30}$$

Comparing (29) and (30) in a equilibrium situation, the energy density $\rho(\omega)$ can be found as

$$\rho = \frac{A_\downarrow}{e^{\hbar\omega/kT} B_\uparrow - B_\downarrow} \tag{31}$$

Again comparing with Plack's distribution law,

$$\rho(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/kT} - 1} \tag{32}$$

We conclude a condition that

$$B_\downarrow = B_\uparrow$$

and

$$\frac{A_\downarrow}{B_\downarrow} = \frac{\hbar\omega^3}{\pi^2 c^3} \cdot \Delta\omega \tag{33}$$