Prerequisites in ALGEBRAIC TOPOLOGY
the Nordfjordeid summer school on motivic homotopy theory

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Preface

The intention of this note is **NOT** to write an introductory textbook in algebraic topology!! Many excellent sources exist, let me only point to Hatcher's book [7] which is available online.

On the contrary, these notes (and the resulting lectures at the summer school on motivic homotopy theory) attempt to give a quick overview of the parts of algebraic topology, and in particular homotopy theory, which are needed in order to appreciate that side of motivic homotopy theory.

This means that we have to study spaces and spectra, but in a way which allows for new applications and interpretations. Unfortunately, this point of view is not predominant in most textbooks, and so even students with a first course algebraic topology might be hard put when exposed to this material without some background. In particular, we will use simplicial techniques. Good books on basic simplicial stuff include [6] and [17]. Good books on general model category theory include [21] (the original), [4], [9], and [8].

The first chapter gives a quick presentation of the classical situations where homotopy theory is much used. In the second chapter we make a more thorough study of the key example: simplicial sets. The reason I have chosen to use so much time on this particular example is twofold. Firstly, some of the results were chosen since they were going to be used later in Voevodsky's lectures. Secondly, some of the results were chosen since they are typical models for the kind of arguments that are used over and over again in this theory.

Then a short and inadequate presentation of model category theory appears (this actually was even less complete in the lectures since I was pressed for time at this point). Since spectra are so important to the theory and the set-up uses many of the general ideas of model categories, they close the chapter.

The fourth and last chapter gives one approach to motivic homotopy theory. We give a quick presentation of the category of motivic spaces and their spectra from a functorial point of view. Those not caring overly much for coherent smash-products can stay to the simpler theory, also explained. I stress that this is but one of many possible approaches, and is definitely colored by my own preferences. No proofs are provided, and the reader is referred to [3] for this particular approach, or to [12] for another using symmetric spectra (also discussed in [22]).

Prerequisites. The reader is assumed to be familiar with the basic aspects of point-set topology. For instance chapter 2 and 3 in [18] will be (more than) enough. Categorical language is used freely, and some readers will find comfort in having a copy of [16] within easy reach.

Caution. The sketch proofs spread around in these notes are only just that. Although they may seem to be worded like complete proofs, there may be claims put forth which in reality can be hard to establish. The intention has not been to give full proofs, but rather to expose the reader to the idea and methods useful for proving results of this type.

0.1 Notational quirks

- $\mathbf{N}$: the monoid of natural numbers (contains zero).

- $\mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R} \subseteq \mathbf{C}$: the rings of integers, rationals, reals and complex numbers.

- $\mathcal{E}ns$: the category of sets.
- $\mathcal{A}b$: the category of abelian groups
- If $\mathcal{C}$ is a category and c and d are objects in $\mathcal{C}$, then $\mathcal{C}(c, d)$ is the set of morphisms in $\mathcal{C}$ from c to d.
- If $\mathcal{C}$ is a category then the ***opposite category*** $\mathcal{C}^{\mathrm{op}}$ is the category with the same objects, but all arrows reversed.
- If $f\colon a \to c$ and $f\colon b \to c$ are two maps in a category with coproducts, then the natural map $a \coprod b \to c$ is called $f + g$.

Contents

Chapter I

Basic properties

In this chapter we present our basic actors: topological spaces, simplicial sets, simplicial abelian groups, spectra, and chain complexes. We concentrate on the formal structures and the connections between them, and postpone most technicalities.

The category of topological spaces serves as a reference category: it is here the notion of homotopy appears, and many results and constructions are most naturally understood in this context. However, both from a technical point of view and from the point of view of extending the techniques into algebraic geometry, it is better to work in the combinatorial alternatives. The reason algebraic topologists are free to choose combinatorial models for topological spaces is that algebraic topology is only concerned with those phenomena that can be detected by mapping from certain model spaces. These model spaces are typically discs or simple things you can make by gluing discs together (like spheres), and so it turns out that the spaces that "really matter" are those that **can** be made out of gluing discs together. Simplicial sets are just a wonderful way of doing the bookkeeping for all the gluings in such an approach.

After having presented the few facts we need about topological spaces through the eyes of algebraic topology, we move quickly to simplicial sets and abelian groups.

The notion of spectra appears classically through homology theories. It is a practical way of expressing the "linearity" of certain invariants, and will reappear again when we start to talk about motivic homology.

1 Topological spaces

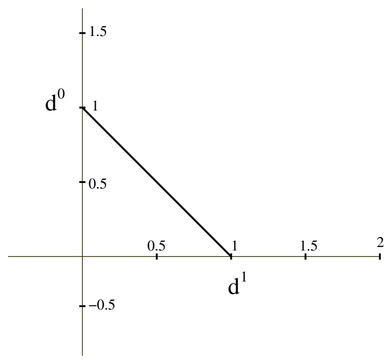
1.1 Singular homology

As a motivation for much that is to come, we recall the definition of the singular homology of a topological space.

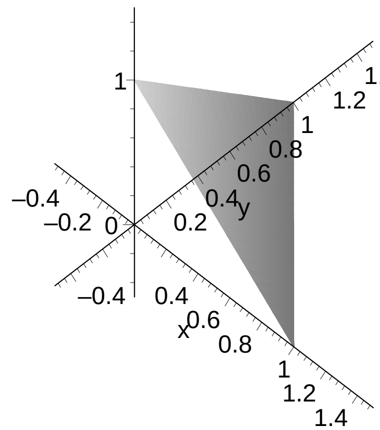
Definition 1.1.1 Let n be a nonnegative integer. The *standard topological n-simplex*, Δ^n, is the subspace

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbf{R}^{n+1} \;\middle|\; \begin{array}{c} \sum_{i=0}^{n} t = 1, \text{ and} \\ t_j \in [0,1] \text{ for all } 0 \le j \le n \} \end{array} \right\}$$

of $\mathbf{R}^{n+1}$.



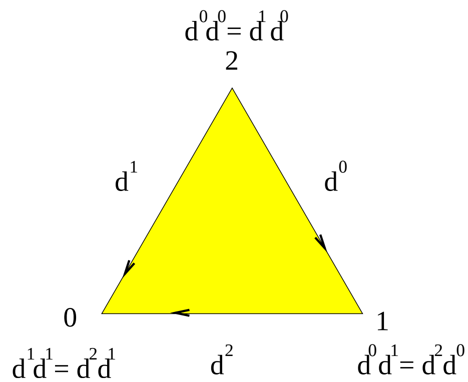
The standard topological 1-simplex $\Delta^1 \subseteq \mathbf{R}^2$. The image of d^0 is the point $(0,1)$, the image of d^1 is $(1,0)$.



The standard topological 2-simplex $\Delta^2 \subseteq \mathbf{R}^3$.

For each integer i with $0 \leq i \leq n+1$ the map $(t_0, \dots, t_n) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_n)$ induces an inclusion which we call the ith *face map*

$$d^i \colon \Delta^n \to \Delta^{n+1}.$$



The standard topological 2-simplex Δ^2 seen head on. The images of the d^j's and $d^i d^j$'s are the indicated parts of the boundary. Note that there are relations between the various ways of including 0-simplices (the number attached to each 0-simplex indicates the axis which passes through this point). The arrows are only for future reference.

For a recollection of the basics on chain complexes, see section 4.1 below. For now, it is enough to recall that a *chain complex* is a sequence

$$(C_*, d) = \left\{ \dots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \dots \right\}$$

of abelian groups and linear maps such that for all n we have that $d_n d_{n+1} = 0$ (the index runs over the integers). The *homology* of (C_*, d) measures its failure to be exact:

$$H_n(C_*, d) = \frac{ker\{d_n \colon C_n \to C_{n-1}\}}{im\{d_{n+1} \colon C_{n+1} \to C_n\}}.$$

Definition 1.1.2 Let X be a topological space. If $n \geq 0$, a *singular n-simplex* is a continuous map $\Delta^n \to X$ from the standard topological n-simplex to X. Let $C_n(X)$ be the free abelian group generated by the singular n-simplices. Define

$$d_n\colon C_n(X) \to C_{n-1}(X)$$

on the generators by letting

$$d_n(\sigma) = \sum_{i=0}^{n}(-1)^i \sigma d^i$$

(we see that $\sigma d^i : \Delta^{n-1} \to \Delta^n \to X$ **is** a singular $n-1$-simplex). Note that $dd = 0$. This defines the *singular chain complex* $(C_*(X), d)$, whose homology is called the *singular homology* of X, and is denoted $H_*(X)$.

1.2 Weak equivalences

Definition 1.2.1 Let I be the unit interval, and let X and Y be topological spaces. For $t \in I$ let $i_t\colon X \to X \times I$ be the inclusion given by $i_t(x) = (x, t)$. Two maps $f_0, f_1\colon X \to Y$ are *homotopic* if there is a map $F\colon X \times I \to Y$ such that $f_t = Fi_t$ for $t = 0, 1$. The map F is called a *homotopy* from f_0 to f_1 and we write $f_0 \sim f_1$.

A map $f\colon X \to Y$ is a *homotopy equivalence* if there is a map $g\colon Y \to X$ such that $fg \sim 1_Y$ and $gf \sim 1_X$.

Homotopy is an equivalence relation.

We write

$$[X, Y]$$

for the set of homotopy classes of maps from X to Y. If X and Y are pointed (the base point is called $*$ regardless of home) topological spaces we define *pointed homotopy* by insisting that $* \times I \subseteq X \times I$ is sent to $*$. We write

$$[X, Y]_*$$

for the set of **pointed** homotopy classes of maps.

Definition 1.2.2 Let X be a pointed topological space and q a nonnegative integer. Then

$$\pi_q(X) = [S^q, X]_*$$

where the q-sphere $S^q = \{x \in \mathbf{R}^{q+1} \mid |x| = 1\}$ is pointed at $(1, 0, \dots, 0)$.

The set of path components is exactly $\pi_0(X)$. It is a standard fact that $\pi_q(X)$ is a group (by pinching spheres) for $q \geq 1$ and an abelian group for $q > 1$. They are called the *homotopy groups* of X.

Definition 1.2.3 Let X and Y be topological spaces. A *weak equivalence* is a map $f\colon X \to Y$ such that for any choice of base point f induces an isomorphism

$$\pi_*(X) \to \pi_*(Y)$$

(i.e., it induces an isomorphism $\pi_q(X) \to \pi_q(Y)$ for **all** nonnegative q - and all choices of base-points).

Fact 1.2.4 If $X \to Y$ is a weak equivalence, then the induced map $H_*(X) \to H_*(Y)$ is an isomorphism too.

A homotopy equivalence is a weak equivalence, but not necessarily conversely. It turns out that for all "reasonable" spaces these notions coincide.

One can show that it makes sense to formally "invert" the weak equivalences. If you do that, you get a category $HoTop$ called the *homotopy category.* We will give an explicit construction later.

1.3 Mapping spaces

The set of continuous functions $A \to X$ can be given various topologies, but for our purposes the following one is the most convenient one.

Definition 1.3.1 Let A and X be topological spaces. Given a compact subset K of A and an open subset U of X, let

$$W(K,U) = \{f \mid f \text{ is a continuous function } A \to X \text{ s.t. } f(K) \subseteq U\}$$

The *compact-open* topology on the set of continuous functions from A to X is the topology given by unions of finite intersections of $W(K,U)$'s when K vary over compact subsets of A and U over open subsets of X. The resulting topological space is denoted X^A.

Fact 1.3.2 (see e.g. [18, 46.11]) If A is a locally compact Hausdorff space, and $B, X \in \mathcal{T}op$, then there is a natural bijection

$$\mathcal{T}op(A \times B, X) \cong \mathcal{T}op\left(B, X^A\right).$$

(recall that $\mathcal{T}op(-,-)$ is the set of all continuous functions).

Conditions like "locally compact" cause havoc in the theory, and is part of the reason why we are searching for combinatorial substitutes for topological spaces. I will try to suppress these issues. For a quick list of relevant facts and references I recommend [9, p. 58ff].

Definition 1.3.3 Let X be a pointed topological space. The *loop space* ΩX on X is the space of **pointed** maps from the standard circle to X.

2 Simplicial sets

For homotopy theory it suffices to focus on "reasonable" spaces. The so-called CW-complexes are examples of reasonable spaces, but they do not form a nice category. A good substitute, which is often convenient, is simplicial sets. The idea is to build spaces out of standard topological simplices. The only problem is to construct a good category out of the simplices allowing for all the necessary gluings.

Any algebraic geometer would suggest the following: take the "category of standard simplices" and consider the (pre)sheaves on this category. This is exactly what we do, except that we model the simplices by means of finite ordered sets. The connection to topological spaces will be discussed in 2.2 below. Hang on:

2.1 The category Δ

Definition 2.1.1 Let Δ be the category consisting of the finite ordered sets

$$[n] = \{0 < 1 < 2 < \cdots < n\}$$

for nonnegative integers n, and order-preserving (a.k.a. nondecreasing or weakly monotone) functions.

Definition 2.1.2 A *simplicial set* is a functor

$$\Delta^{\mathrm{op}} \to \mathcal{E}ns$$

where $\mathcal{E}ns$ is the category of sets. A *map of simplicial sets* (or *simplicial map*) is a natural transformation. The category of simplicial sets is denoted $\mathcal{S}$.

More generally, a *simplicial object* in a category $\mathcal{C}$ is a functor $X\colon \Delta^{\mathrm{op}} \to \mathcal{C}$.

If X is a simplicial set we usually we write X_n instead of $X([n])$, and the elements of X_n are referred to as *n-simplices*.

Example 2.1.3 Let n be a nonnegative integer. The *standard (simplicial) n-simplex* $\Delta[n] \in \mathcal{S}$ is given by

$$[q] \mapsto \Delta([q],[n]) = \{\text{order preserving functions } [q] \to [n]\}.$$

Note that the $\Delta[n]$ are nothing but the values of the Yoneda map

$$\Delta \to \mathcal{S}, \qquad [n] \mapsto \Delta[n] = \Delta(-,[n])$$

Note 2.1.4 By Yoneda's lemma we have a natural isomorphism between X_n and the set of simplicial maps $\Delta[n] \to X$.

Exercise 2.1.5 Let $k \geq 0$. Show that $\Delta[0]_k = \Delta([k],[0])$ has one element and that $\Delta[1]_k = \Delta([k],[1])$ has $k+2$ elements. Show that the two maps $[0] \to [1] \in \Delta$ induce injections $\Delta[0] \to \Delta[1]$, and that the union of their images form a simplicial subset $\partial\Delta[1] \subset \Delta[1]$ with two simplices in every dimension. How many k-simplices does the *(simplicial) circle* $S^1 = \Delta[1]/\partial\Delta[1]$ have?

Exercise 2.1.6 Show that there are $\binom{n+1}{k+1}$ **injective** order preserving functions $[k] \to [n]$ (considered as elements in $\Delta[n]_k$ these are called "non-degenerate k-simplices" in $\Delta[n]$.)

A *degenerate* k-simplex in $X \in \mathcal{S}$ is an element $x \in X_k$ such that $x = \phi_* y$ for some $y \in X_n$ and non-injective $\phi\colon [k] \to [n]$.

2.1.7 For the record: Δ described by "generators and relations"

In particular, for $0 \leq i \leq n$ we have the maps

$$d^i\colon [n-1] \to [n], \qquad d^i(j) = \begin{cases} j & j < i \\ j+1 & i \leq j \end{cases} \qquad \text{"skips } i\text{"}$$

$$s^i\colon [n+1] \to [n], \qquad s^i(j) = \begin{cases} j & j \leq i \\ j-1 & i < j \end{cases} \qquad \text{"hits } i \text{ twice".}$$

Every map in Δ has a factorization in terms of these maps. Let $\phi \in \Delta([n],[m])$. Let $\{i_1 < i_2 < \cdots < i_k\} = [m] - im(\phi)$, and $\{j_1 < j_2 < \cdots < j_l\} = \{j \in [n] | \phi(j) = \phi(j+1)\}$. Then

$$\phi(j) = d^{i_k} d^{i_{k-1}} \cdots d^{i_1} s^{j_1} s^{j_2} \cdots s^{j_l}(j).$$

This factorization is unique, and hence we could describe Δ as being generated by the maps d^i and s^i subject to the "cosimplicial identities" :

$$d^j d^i = d^i d^{j-1} \quad \text{for } i < j$$

$$s^j s^i = s^{i-1} s^j \quad \text{for } i > j$$

and

$$s^j d^i = \begin{cases} d^i s^{j-1} & \text{for } i < j \\ id & \text{for } i = j, j+1 \\ d^{i-1} s^j & \text{for } i > j+1 \end{cases}.$$

If X is a simplicial set, we let X_n be the image of $[n]$, and for a map $\phi \in \Delta$ we will often write ϕ^* for $X(\phi)$. For the particular maps d^i and s^i, we write simply d_i and s_i for $X(d^i)$ and $X(s^i)$, and call them *face* and *degeneracy maps*. Note that the face and degeneracy maps satisfy the "simplicial identities" which are the duals of the cosimplicial identities.

Hence a simplicial set is often defined in the literature to be a sequence of sets X_n and maps d_i and s_i

$$\begin{tikzcd} X_0 \arrow[r, "s_0"] & X_1 \arrow[l, shift right=2, "d_0"'] \arrow[l, shift left=2, "d_1"] \arrow[r, shift right=1, "s_0"] \arrow[r, shift left=1, "s_1"] & X_2 \arrow[l, shift right=3, "d_0"'] \arrow[l, "d_1"] \arrow[l, shift left=3, "d_2"] & \cdots \end{tikzcd}$$

satisfying the simplicial identities.

2.2 Simplicial sets vs. topological spaces

We mentioned that Δ was modeled on simplices. This is manifested in the functor $\Delta \to \mathcal{T}op$ given by $[n] \mapsto \Delta^n$ and sending $\phi\colon [n] \to [m] \in \Delta$ to $\phi_*\colon \Delta^n \to \Delta^m$ given by

$$\phi_*(x_0, \dots, x_n) = (\sum_{j\in\phi^{-1}(0)} x_j, \dots, \sum_{j\in\phi^{-1}(m)} x_j)$$

(the face and degeneracies are thus

$$\begin{aligned} d^i(x_0, \dots, x_{n-1}) =&(x_0, \dots, x_i, 0, x_{i+1}, \dots, x_{n-1}) \\ s^i(x_0, \dots, x_{n+1}) =&(x_0, \dots, x_{i-1}, x_i + x_{i+1}, x_{i+2}, \dots, x_{n+1}). \end{aligned}$$

Note that the formula "$dd = 0$" for the singular chain complex 1.1.2 follows from the cosimplicial identities).

Exercise 2.2.1 Prove that $[n] \mapsto \Delta^n$ **is** a functor.

By the way, a functor from Δ to some category is called a *cosimplicial object* in that category. There is a pair of functors

$$\mathcal{T}op \underset{\text{sing}}{\overset{|-|}{\leftrightarrows}} \mathcal{S}$$

defined as follows.

Definition 2.2.2 For $Y \in \mathcal{T}op$, the *singular complex* is defined as

$$\operatorname{sing} Y = \{[n] \mapsto \mathcal{T}op(\Delta^n, Y)\}$$

(the set of unbased continuous functions from the topological standard n-simplex to Y). As $[n] \mapsto \Delta^n$ is a cosimplicial space, this becomes a simplicial set.

Exercise 2.2.3 Elaborate: why is $\operatorname{sing} Y$ a simplicial set, and why is sing a functor?

Definition 2.2.4 For $X \in \mathcal{S}$, the *realization* of X is defined as the quotient space

$$|X| = \left(\coprod_n X_n \times \Delta^n\right) / \sim$$

where if $(x, u) \in X_m \times \Delta^n$ and $\phi\colon [n] \to [m] \in \Delta$ we identify the points $(\phi^* x, u) \in X_n \times \Delta^n$ and $(x, \phi_* u) \in X_m \times \Delta^m$.

Example 2.2.5 Several familiar topological spaces are realizations of simplicial sets. Here are some examples:

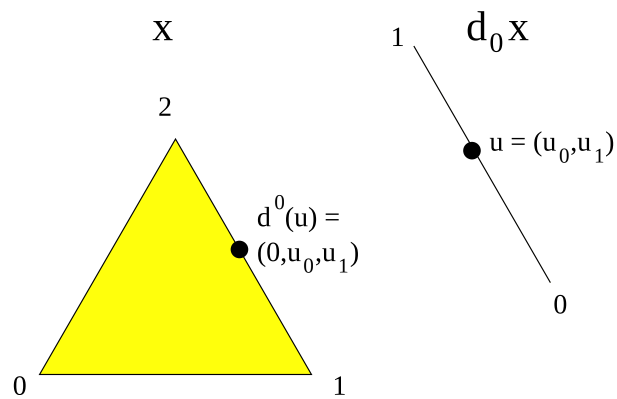
- The standard topological n-simplex: $\Delta^n \cong |\Delta[n]|$.
- Let $\partial\Delta[n]$ be the sub-simplicial set of $\Delta[n]$ generated by all the *faces* (i.e., the images of the injections $\Delta[n-1] \to \Delta[n]$ induced by the projections $d^i\colon [n] \to [n-1] \in \Delta$, $i = 0, \dots, n$, cf. 2.1.7). Then $|\partial\Delta[n]|$ is homeomorphic to the $n-1$-sphere.

  In particular, the realization of the (simplicial) circle S^1 of exercise 2.1.5 is (homeomorphic to) the usual (topological) circle.

Exercise 2.2.6 In the previous example: what does "generate" mean precisely? Prove that $|\partial\Delta[n]|$ is homeomorphic to the $(n-1)$-sphere.

Note 2.2.7 The picture for $|X|$ is as follows: for each m-simplex $x \in X_m$, you insert a topological m-simplex Δ^m. The maps in Δ keep track of how these simplices should be glued together.

For instance $(d_i x, u) \sim (x, d^i u)$ tells you that the Δ^{m-1} associated with $d_i x \in X_{m-1}$ should be glued to the ith face of the Δ^m associated with $x \in X_m$.



The realization functor glues simplices together. Here $(x, d^0 u) \sim (d_0 x, u)$ tells us that we should identify the two black dots.

It is confusing at first to understand the rôle of the surjective maps in Δ. They dictate that once you have a simplex, there will be higher dimensional simplices which are to be identified with it after realization. Often one can do without them (e.g., the singular homology only saw the d_is), but trying to do so in general leads to a much more complex theory. So we just will have to live with facts like that the "interval" $\Delta[1]$ has three 1-simplices: one degenerate for each end point and one spanning the interval itself.

Fact 2.2.8 If X is a simplicial set, then $|X|$ is a compactly generated Hausdorff space, but it is usually not locally compact.

Lemma 2.2.9 *The realization functor is left adjoint to the singular functor.*

Sketch proof: Check it by hand as an exercise: Explicitly, you have to show that there is a natural bijection of morphism sets

$$\mathcal{T}op(|X|, Y) \cong \mathcal{S}(X, \operatorname{sing} Y).$$

This bijection is induced by the "adjunction maps"

$$\begin{array}{ccc} X & \to & \operatorname{sing}|X| \\ x \in X_n & \mapsto & (\Delta^n \overset{u \mapsto (x,u)}{\to} X_n \times \Delta^n \to |X|) \in \operatorname{sing}|X|_n \end{array}$$

and

$$\begin{array}{ccc} |\operatorname{sing} Y| & \to & Y \\ (y, u) \in \operatorname{sing}(Y)_n \times \Delta^n & \mapsto & y(u) \in Y. \end{array}$$

Note 2.2.10 That the realization is the left adjoint implies by general nonsense that it preserves all colimits ($\mathcal{S}$ has all (small) (co)limits: just take the (co)limit in each degree separately and you get the right answer).

It is a very useful fact that the realization often preserves finite limits (when interpreted in "k-spaces" this is always true, see e.g., [9, 3.2.4]).

2.3 Weak equivalences

Definition 2.3.1 A map $f\colon X \to Y$ of simplicial sets is a *weak equivalence* if $|f|\colon |X| \to |Y|$ is a weak equivalence in $\mathcal{T}op$.

Just as in $\mathcal{T}op$ it makes sense to invert all weak equivalences in $\mathcal{S}$, and the resulting category, $Ho\mathcal{S}$, is called the homotopy category of $\mathcal{S}$. In the simplicial case I can be quite specific:

$$ob\, Ho\mathcal{S} = ob\, \mathcal{S},$$

and if $X, Y \in ob\, \mathcal{S} = ob\, Ho\mathcal{S}$, then

$$Ho\mathcal{S}(X,Y) \cong [|X|, |Y|].$$

This could be a perfectly fine definition. The important point is that the answer is relevant.

Theorem 2.3.2 *(see any book on simplicial sets). The maps of adjunction $X \to \operatorname{sing}|X|$ and $|\operatorname{sing} Y| \to Y$ are both weak equivalences.*

The singular/realization adjoint pair induces an equivalence between the homotopy categories of topological spaces and simplicial sets.

Note 2.3.3 We will later see that the correspondence between $\mathcal{S}$ and $\mathcal{T}op$ is even better than just something inducing an equivalence on the homotopy category level (it is something called a Quillen equivalence, see III.1.3.1).

3 Some constructions in $\mathcal{S}$

One of the good things about the category of simplicial sets is that it offers no categorical surprises (as opposed to $\mathcal{T}op$ which is just a mess). Limits and colimits are calculated in every degree separately, e.g.,

$$(X \coprod Y)_q = X_q \coprod Y_q, \text{ and } (X \times Y)_q = X_q \times Y_q.$$

The *mapping space*

$$\underline{\mathcal{S}}(X,Y) = \{[q] \mapsto \mathcal{S}(X \times \Delta[q], Y)\}$$

satisfies the exponential formula

$$\underline{\mathcal{S}}(X, \underline{\mathcal{S}}(Y,Z)) \cong \underline{\mathcal{S}}(Y \times X, Z)$$

on the nose.

Exercise 3.0.1 Prove this.

Note that $\mathcal{S}(X,Y) = \underline{\mathcal{S}}(X,Y)_0$, and that we have a (n associative) "composition"

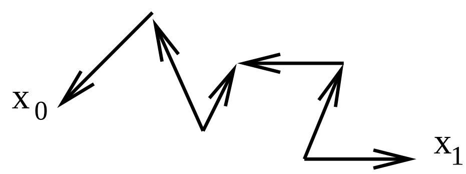
$$\underline{\mathcal{S}}(Y,Z) \times \underline{\mathcal{S}}(X,Y) \to \underline{\mathcal{S}}(X,Z)$$

induced by sending $g\colon Y \times \Delta[q] \to Z$ and $f\colon X \times \Delta[q] \to Y$ to the composite

$$X \times \Delta[q] \xrightarrow{id \times \text{diagonal}} X \times \Delta[q] \times \Delta[q] \xrightarrow{f \times id} Y \times \Delta[q] \xrightarrow{g} Z$$

3.0.2 Path components and homotopies

Let X be a simplicial set, and let $x_0, x_1 \in X_0$ be two zero simplices (a.k.a.. *points* or *vertices*). If there is a 1-simplex $x \in X_1$ such that $d_0 x = x_0$ and $d_1 x = x_1$ we say that x is a *path* from x_0 to x_1. Being connected by a path is neither symmetric nor transitive, but we may anyhow consider the equivalence relation generated by the path relation. We call the set of equivalence classes the set of *path components* and write $\pi_0 X$. We see that $\pi_0 X$ is naturally isomorphic to $\pi_0|X|$.



Two points are in the same path components if they can be joined by a finite "chain of paths".

We will later also define the higher homotopy groups in a combinatorial fashion, but whatever definition you use, it should be naturally isomorphic to $\pi_*|-|$.

Definition 3.0.3 Let $X, Y \in \mathcal{S}$, and let f_0 and f_1 be two maps from X to Y. A *homotopy* from f_0 to f_1 is a path in $\underline{\mathcal{S}}(X,Y)$ from f_0 to f_1.

In other words, a homotopy is a map $H\colon X \times \Delta[1] \to Y$ such that the composites

$$X \cong X \times \Delta[0] \xrightarrow{id \times d_i} X \times \Delta[1] \xrightarrow{H} Y, \qquad i = 0, 1$$

are f_1 and f_0.

Since $|X \times \Delta[1]| \cong |X| \times |\Delta[1]|$, we see that the realization of a homotopy is a homotopy in $\mathcal{T}op$.

Definition 3.0.4 A map $f\colon X \to Y \in \mathcal{S}$ is a *(simplicial) homotopy equivalence* if there is a map $g\colon Y \to X$ such that both composites $f \circ g$ and $g \circ f$ are homotopic to the identity.

4 Simplicial abelian groups

Let $\mathcal{A}b$ be the category of abelian groups. Since $\mathcal{A}b$ has all (co)limits, so has the category $\mathcal{A} = s\mathcal{A}b$ of simplicial abelian groups. There are free/forgetful functors

$$\mathcal{A}b \underset{U}{\overset{\mathbf{Z}[-]}{\leftrightarrows}} \mathcal{E}ns$$

where $\mathbf{Z}[X]$ is the free abelian group on the set X and UM is the underlying set of the abelian group M. Applying this in every degree we get an adjoint pair

$$\mathcal{A} \underset{U}{\overset{\mathbf{Z}[-]}{\leftrightarrows}} \mathcal{S}.$$

Note 4.0.5 The category $\mathcal{A} = s\mathcal{A}b$ inherits structure from $\mathcal{S}$ via this adjoint pair. For instance $\mathcal{A}$ has its own "internal" hom-object via

$$\underline{\mathcal{A}}(M,N)_q = \mathcal{A}(M \otimes \mathbf{Z}[\Delta[q]], N),$$

and so a natural notion of homotopies of maps.

4.1 Simplicial abelian groups vs. chain complexes

As usual, a chain complex is a sequence

$$C_* = \{\cdots \leftarrow C_{q-1} \leftarrow C_q \leftarrow C_{q+1} \leftarrow \dots\}$$

such that any composite is zero, and a map of chain complexes $f_*\colon C_* \to D_*$ is a collection of maps $f_q\colon C_q \to D_q$ such that the diagrams

$$\begin{array}{ccc} C_q & \xrightarrow{f_q} & D_q \\ \downarrow & & \downarrow \\ C_{q-1} & \xrightarrow{f_{q-1}} & D_{q-1} \end{array}$$

commute. We let Ch be the category of chain complexes, and $Ch^{\geq 0}$ be the full subcategory of chain complexes C_* such that $C_q = 0$ if $q < 0$.

If C_* is a chain complex, we let

$$\begin{aligned} Z_qC &= \ker\{C_q \to C_{q-1}\} \text{ (cycles)}, \\ B_qC &= \operatorname{im}\{C_{q+1} \to C_q\} \text{ (boundaries) and} \\ H_qC_* &= Z_qC/B_qC \text{ (homology)}. \end{aligned}$$

A simplicial abelian group M gives rise to a chain complex called the *Moore complex*:

$$C_*(M) = \{M_0 \xleftarrow{d_0 - d_1} M_1 \xleftarrow{d_0 - d_1 + d_2} M_2 \xleftarrow{d_0 - d_1 + d_2 - d_3} \dots\}.$$

Note that the singular homology of a topological space Y 1.1.2 was defined as

$$H_*(Y) = H_*(C_*\mathbf{Z}[\operatorname{sing} Y]).$$

More generally, for a simplicial set X we define

$$H_*(X) = H_*(C_*\mathbf{Z}[X]).$$

Fact 4.1.1 Let M be a simplicial abelian group. Then there is a natural isomorphism

$$\pi_*(|U(M)|) \cong H_*(C_*(M)).$$

The map $X \cong 1 \cdot X \subseteq U\mathbf{Z}[X]$ induces what is called the *Hurewicz map* $\pi_*(|X|) \to H_*(|X|)$ on homotopy groups [7, 4.2].

4.2 The normalized chain complex

If M is a simplicial abelian group, then the normalized chain complex $C^{\mathrm{norm}}_*(M)$ (which is usually called N_*M, an option unpalatable to us since this notation will be occupied by the nerve) is the chain complex given by

$$C^{\mathrm{norm}}_q(M) = \bigcap_{i=0}^{q-1} \ker\{d_i\colon M_q \to M_{q-1}\}$$

and boundary map $C^{\mathrm{norm}}_q M \to C^{\mathrm{norm}}_{q-1} M$ given by the remaining face map d_q.

Fact 4.2.1 (Dold-Kan) The normalized chain complex gives an equivalence of categories

$$C^{\mathrm{norm}}\colon \mathcal{A} \to Ch^{\geq 0}.$$

Furthermore, this equivalence sends homotopies to chain homotopies and vice versa for the adjoint.

The inclusion of the normalized complex into the Moore complex $C^{\mathrm{norm}}_*(M) \subseteq C_*(M)$ is a homotopy equivalence (see e.g., [6, III.2]).

5 The pointed case

Most of what have been said so far carries over to the pointed setting (a pointed simplicial set is by definition a simplicial pointed set in case you wondered). Just a tad of notational stuff.

If X and Y are pointed sets, then the *wedge*

$$X \vee Y$$

is what you get if you take the disjoint union of X and Y and identify their basepoints. Alternatively (and more concretely) you may think of $X \vee Y$ as the subset of $X \times Y$ where (at least) one of the coordinates is the base point. The quotient

$$X \wedge Y = (X \times Y)/(X \vee Y)$$

is called the *smash.* The *suspension* of $X \in \mathcal{S}_*$ is just another word for the smash $S^1 \wedge X$ where $S^1 = \Delta[1]/\partial\Delta[1]$ is the simplicial circle. We define the higher spheres by

$$S^n = S^1 \wedge S^{n-1},$$

getting the isomorphisms $S^n \wedge S^m \cong S^{m+n}$. By the way, $S^0 = \{0,1\} = \partial\Delta[1]$ pointed in 0.

If Z is an (unpointed) set, then

$$Z_+$$

is Z to which we have added a base point.

The free/forgetful pair connecting abelian groups and sets factors through the pointed sets (abelian groups are pointed in zero)

$$\mathcal{A}b \overset{\tilde{\mathbf{Z}}[-]}{\leftrightarrows} \mathcal{E}ns_* \overset{X \mapsto X_+}{\leftrightarrows} \mathcal{E}ns$$

where $\tilde{\mathbf{Z}}[X] = \mathbf{Z}[X]/\mathbf{Z}[*]$, $\mathcal{E}ns_*$ is the category of pointed sets, and the arrows pointing to the right are the forgetful functors (on the bottom, following the convention that right adjoints are written below the left adjoint).

There are many (natural!) relations between these constructions, such as

1. $(Z \times S)_+ \cong Z_+ \wedge S_+$,
2. $X \wedge (Y_1 \vee Y_2) \cong (X \wedge Y_1) \vee (X \wedge Y_2)$,
3. $S^0 \wedge X \cong X$,
4. $\tilde{\mathbf{Z}}[X \vee Y] \cong \tilde{\mathbf{Z}}[X] \oplus \tilde{\mathbf{Z}}[Y]$,
5. $\tilde{\mathbf{Z}}[X \wedge Y] \cong \tilde{\mathbf{Z}}[X] \otimes \tilde{\mathbf{Z}}[Y]$
6. if $A \subseteq X$, then $0 \to \tilde{\mathbf{Z}}[A] \to \tilde{\mathbf{Z}}[X] \to \tilde{\mathbf{Z}}[X/A] \to 0$ is exact.

Exercise 5.0.1 Write up a couple more relations like the ones above and prove all of them. Note that the short exact sequence in 6 splits (if you believe in the axiom of choice); but this splitting will **not** be natural (regardless of faith).

Performing these constructions degreewise, we get the corresponding constructions for pointed simplicial sets.

Reduced homology is given as the homotopy groups of $\tilde{\mathbf{Z}}[X]$, the wedge axiom is reflected in 4, the Künneth theorem in 5, and excision in 6.

Definition 5.0.2 The space $\tilde{\mathbf{Z}}[S^n]$ is called the *nth integral Eilenberg-Mac Lane space*, and has the property that

$$\pi_q \tilde{\mathbf{Z}}[S^n] = \begin{cases} 0 & \text{if } n \neq q \\ \mathbf{Z} & \text{if } n = q \end{cases}.$$

5.0.3 Mapping spaces and homotopies

We have mapping spaces as well, let X and Y be pointed simplicial sets, then $\underline{\mathcal{S}_*}(X,Y)$ is the pointed simplicial set with q simplices

$$\underline{\mathcal{S}_*}(X,Y)_q = \{\text{pointed simplicial maps } X \wedge \Delta[q]_+ \to Y\}.$$

Exercise 5.0.4 Note that $\mathcal{S}_*(X,Y) = \underline{\mathcal{S}_*}(X,Y)_0$, and that we have a "composition"

$$\underline{\mathcal{S}_*}(Y,Z) \wedge \underline{\mathcal{S}_*}(X,Y) \to \underline{\mathcal{S}_*}(X,Z),$$

and an exponential law

$$\underline{\mathcal{S}_*}(X \wedge Y, Z) \cong \underline{\mathcal{S}_*}(Y, \underline{\mathcal{S}_*}(X,Z))$$

If $f_0, f_1 \in \mathcal{S}_*(X,Y)$, a *(pointed) homotopy* between them is a path in $\underline{\mathcal{S}_*}(X,Y)$. In other words, it is a map

$$X \wedge \Delta[1]_+ \to Y$$

restricting to f_0 and f_1 on the boundary of $\Delta[1]$. (Pointed) *homotopy equivalences* are defined in the obvious way.

5.0.5 Loop spaces and cohomology

We note that $\operatorname{sing}\Omega|X| \cong \underline{\mathcal{S}_*}(S^1, \operatorname{sing}|X|)$ (this uses 1.3.2, that the circle is (locally) compact Hausdorff and that realization commutes with finite products if one of the factors is locally compact).

It is an important fact that even though $X \to \operatorname{sing}|X|$ is a weak equivalence,

$$\underline{\mathcal{S}_*}(A,X) \to \underline{\mathcal{S}_*}(A, \operatorname{sing}|X|)$$

may not be a weak equivalence.

Exercise 5.0.6 Prove that $\pi_0\underline{\mathcal{S}_*}(S^1,S^1) \cong S^0$, whereas $\pi_0\underline{\mathcal{S}_*}(S^1, \operatorname{sing}|S^1|) \cong \pi_1|S^1| \cong \mathbf{Z}$.

It is $\underline{\mathcal{S}_*}(A, \operatorname{sing}|X|)$ which has the "right" homotopy type, and we define the *loop space*

$$\Omega X = \underline{\mathcal{S}_*}(S^1, \operatorname{sing}|X|) \cong \operatorname{sing}\Omega|X|.$$

Then loop and suspension are not adjoint, but we still get that a map $S^1 \wedge X \to Y$ induces a map $X \to \Omega Y$ (by using the adjunction on the composite $S^1 \wedge X \to Y \to \operatorname{sing}|Y|$).

Definition 5.0.7 Let $X \in \mathcal{S}$. The nth (reduced, integral) *cohomology* of X is given by the group

$$\tilde{H}^n(X) = \pi_0\underline{\mathcal{S}_*}(X, \tilde{\mathbf{Z}}[S^n])$$

Note 5.0.8 It is a special feature of $\tilde{\mathbf{Z}}[S^n]$ that there is no need for $\operatorname{sing}|-|$ around it (since it is a simplicial abelian group it is "fibrant" in a language to come).

Once this point is properly understood, it is not too difficult to derive the axioms for cohomology from this definition. A direct proof that it agrees with the usual cochain definition (not given here) is an application of the isomorphism of the category of simplicial abelian groups and chain complexes concentrated in non-negative degrees (together with the fact that $\tilde{\mathbf{Z}}[S^n]$ correspond to the chain complex with a single nontrivial group $\mathbf{Z}$ concentrated in degree n).

6 Spectra

6.1 Introduction

Many phenomena and invariants are "stable" in the sense that suspending simply acts as shifting. More to the point: for (generalized) cohomology theories, gluing of spaces can be easily understood through excision, and Brown representability says that any cohomology theory can be realized by mapping into a "spectrum". Spectra are a sensible half-way house between spaces and abelian groups where cohomology theories "live"; here suspension is equivalent to shifting and finite coproducts are equivalent to products. However, the category of spectra is not simply a jumble of invariants with values in abelian groups, it carries the same kind of structure as the category of spaces and is open for analysis through the same type of machinery.

Definition 6.1.1 In algebraic topology (as opposed to algebraic geometry), a *spectrum* is a sequence of simplicial sets

$$E = \{E^0, E^1, E^2, \dots\}$$

together with (structure) maps

$$S^1 \wedge E^k \to E^{k+1}$$

for $k \geq 0$. A map of spectra $f\colon E \to F$ is a sequence of maps $f^k\colon E^k \to F^k$ compatible with the structure maps: the diagrams

$$\begin{tikzcd} S^1 \wedge E^k \arrow[r] \arrow[d, "id_{S^1} \wedge f^k"] & E^{k+1} \arrow[d, "f^{k+1}"] \\ S^1 \wedge F^k \arrow[r] & F^{k+1} \end{tikzcd}$$

commute. We let $\mathcal{S}pt$ be the resulting category of spectra.

This definition is apparently due to Lima [14].

There are some especially important spectra:

Example 6.1.2

1. the *sphere spectrum*

$$\underline{\mathbf{S}} = \{k \mapsto S^k = S^1 \wedge \cdots \wedge S^1\}$$

whose structure maps are the identity.

2. the (integral) *Eilenberg-Mac Lane spectrum*

$$H\mathbf{Z} = \{k \mapsto \tilde{\mathbf{Z}}[S^k]\}$$

whose structure map is induced by the natural map $\tilde{\mathbf{Z}}[X] \wedge Y \to \tilde{\mathbf{Z}}[X \wedge Y]$.

The Eilenberg-Mac Lane spectra are examples of Ω-*spectra*, that is the adjoint of the structure maps give rise to equivalences $E^k \to \Omega E^{k+1}$. In various treatments this property is taken as a part of the definition of spectra. We do not; it takes more categorical effort to make this work. Our approach is a typical example of how modern homotopy theory treat this kind of issues. The Ω-spectra are admittedly the spectra that matter, but many natural constructions on spectra takes us outside the Ω-spectra, and so the approach is to admit all spectra, but allow for equivalences (*stable equivalences*, see below) that are measured by Ω-spectra (in model categorical language to be explained in chapter III, the Ω-spectra are the *fibrant* spectra).

6.2 Relation to simplicial sets

If X is a pointed simplicial set and E is a spectrum, then $E \wedge X$ is the spectrum $n \mapsto E^n \wedge X$, and E^X is the spectrum $n \mapsto \underline{\mathcal{S}_*}(X, E^n)$.

There is a pair of adjoint functors

$$\mathcal{S}pt \underset{R}{\overset{\Sigma^\infty}{\leftrightarrows}} \mathcal{S}_*$$

where

$$\Sigma^\infty X = \{n \mapsto S^n \wedge X\}$$

is the *suspension spectrum* with right adjoint $RE = E^0$ – the *zero'th space.*

Occasionally RE is referred to as the "underlying space" of the spectrum, but this term is also sometimes used for the *underlying infinite loop space* $\Omega^\infty E = \lim_{\overrightarrow{n}} \Omega^n E^n$ (the maps in the colimit are the same as in 6.3.1 below).

Exercise 6.2.1 Prove that R and Σ^∞ are adjoint.

6.3 Stable equivalences

The relevant equivalences giving the right correspondence between cohomology theories and spectra are the stable equivalences:

Definition 6.3.1 Let E be a spectrum. The (stable) *homotopy groups* of E are defined as

$$\pi_q E = \lim_{\overrightarrow{k}} \pi_{q+k} E^k$$

where the colimit is over the maps $\pi_{q+k}E^k \to \pi_{q+k}\Omega E^{k+1} \cong \pi_{q+k+1}E^{k+1}$ for $k > -q$.

Exercise 6.3.2 Make the maps $\pi_{q+k}E^k \to \pi_{q+k}\Omega E^{k+1} \cong \pi_{q+k+1}E^{k+1}$ explicit using the definition in 5.0.5 of Ω in $\mathcal{S}_*$.

Note that the stable homotopy groups define a functor from spectra to $\mathbf{Z}$-graded abelian groups.

Definition 6.3.3 A map of spectra $f\colon E \to F$ is a *stable equivalence* if it induces an isomorphism on stable homotopy groups.

Fact 6.3.4 An important fact about spectra is that the natural map

$$E \vee F \to E \times F$$

is a stable equivalence. This is related to the fact that if $X, Y \in \mathcal{S}$ are such that $\pi_i X = 0$ for $i < n$ and $\pi_j Y = 0$ for $j < m$, then $\pi_k(X \wedge Y) = 0$ for $k < m + n$.

Exercise 6.3.5 In the "fact" above: how do you define $E \vee F$ and $E \times F$ (i.e., what are the structure maps)?

Again it makes sense to invert all stable equivalences, and the resulting category is called the *stable homotopy category,*

$$Ho\mathcal{S}pt$$

(often in the literature you will find the term "the stable category", a term which may cause confusion, and which we will avoid). Facts like 6.3.4 adds up to show that $Ho\mathcal{S}pt$ is an additive category (finite sum=product), and even better, it has a "tensor product" which is somehow derived from the smash on $\mathcal{S}_*$.

6.4 Homology theories

A spectrum E gives rise to a "*(co)homology theory*": if X is a simplicial set, we let

$$E_n(X) = \pi_n(E \wedge X)$$

and

$$E^n(X) = \pi_{-n}E^X.$$

The *stable homotopy group* $\pi_n^S(X)$ of a pointed space X is by definition

$$\underline{\mathbf{S}}_n(X) = \pi_n(\underline{\mathbf{S}} \wedge X) = \varinjlim_k \pi_{n+k}(S^k \wedge X).$$

Theorem 6.4.1 *There are natural isomorphisms*

$$(H\mathbf{Z})_n(X) \cong \tilde{H}_n(X)$$

$$(H\mathbf{Z})^n(X) \cong \tilde{H}^n(X).$$

Sketch proof: Homology part: First note that $\tilde{H}_n(X) \cong \tilde{H}_{n+k}(S^k \wedge X) \cong \pi_{n+k}\tilde{\mathbf{Z}}[S^k \wedge X]$ for all $k \geq 0$. Given this, the natural isomorphism $(H\mathbf{Z})_n(X) \cong \tilde{H}_n(X)$ is given as the colimit (over k) of

$$\pi_{n+k}((H\mathbf{Z} \wedge X)^k) = \pi_{n+k}(\tilde{\mathbf{Z}}[S^k] \wedge X) \to \pi_{n+k}(\tilde{\mathbf{Z}}[S^k \wedge X])$$

which is an isomorphism for $k > n$ (by a "stability result" similar to 6.3.4).

Cohomology part: $\tilde{H}^n(X) = \tilde{H}^k(S^{k-n} \wedge X) = \pi_0\underline{\mathcal{S}_*}(S^{k-n} \wedge X, \tilde{\mathbf{Z}}[S^k]) \cong \pi_{k-n}\underline{\mathcal{S}_*}(X, \tilde{\mathbf{Z}}[S^k])$ for all $k \geq n$.

6.5 Relation to chain complexes

There is a close connection between chain complexes and spectra.

The definition of spectra is reminiscent of how you'd represent arbitrary chain complexes by means of chain complexes concentrated in non-negative dimensions: a chain complex C can be given by a sequence

$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \\
\downarrow d & \downarrow d & \downarrow d & \downarrow d & \\
C_2 & C_1 & C_0 & C_{-1} & \cdots \\
\downarrow d & \downarrow d & \downarrow d & \downarrow d & \\
C_1 & C_0 & C_{-1} & C_{-2} & \cdots \\
\downarrow d & \downarrow d & \downarrow d & \downarrow d & \\
C_0 & C_{-1} & C_{-2} & C_{-3} & \cdots
\end{array}$$

of non-negatively graded chain complexes C^0, C^1,... together with isomorphisms $C^i_j \cong C^{i+1}_{j+1}$. Homology in arbitrary dimensions is then accessible as

$$H_j(C) \cong \varinjlim_n H_{n+j}(C^n).$$

Notice that if we let $\mathbf{Z}[1]$ be the chain complex concentrated in degree 1 with a single $\mathbf{Z}$, then the isomorphism $C^i_j \cong C^{i+1}_{j+1}$ can be reformulated through a map

$$\mathbf{Z}[1] \otimes C^i \to C^{i+1}$$

which is an isomorphism in positive degrees. The tensor product of chain complexes is given by $(C\otimes D)_n = \oplus_{p+q=n} C_p \otimes D_q$ (with the appropriate sign conventions on the differentials), and so $(\mathbf{Z}[1]\otimes C^i)_n \cong C^i_{n-1} \cong C^{i+1}_n$ for $n > 0$.

As a matter of fact, if we replace the category of simplicial sets $\mathcal{S}$, the smash product $\wedge$ and the circle S^1 with

1. the category of simplicial abelian groups $\mathcal{A}$, with degreewise tensor, and $\tilde{\mathbf{Z}}[S^1]$
2. the category of chain complexes concentrated in non-negative degrees $Ch^{\geq 0}$, with tensor of chain complexes and $C^{\text{norm}}\tilde{\mathbf{Z}}[S^1] = \mathbf{Z}[1]$ or
3. the category of chain complexes Ch, with tensor of chain complexes and $\mathbf{Z}[1]$

word for word in the definition of spectra we get categories $\mathcal{S}pt(\mathcal{A})$, $\mathcal{S}pt(Ch^{\geq 0})$ and $\mathcal{S}pt(Ch)$ which for all practical purposes play the rôle of chain complexes, and which is related to spectra by the usual free/forgetful functor connecting abelian groups and sets.

The relation $\mathcal{S}pt$ to Ch is as follows

$$\mathcal{S}pt \overset{\tilde{\mathbf{Z}}[-]}{\leftrightarrows} \mathcal{S}pt(\mathcal{A}) \underset{C^{\text{norm}}}{\leftrightarrows} \mathcal{S}pt(Ch^{\geq 0}) \underset{\text{include}}{\overset{\text{truncate}}{\leftrightarrows}} \mathcal{S}pt(Ch) \underset{R}{\leftrightarrows} Ch$$

The maps to the left of $\mathcal{S}pt(\mathcal{A})$ all induce equivalences on the associated homotopy categories.

Chapter II

Deeper structure: simplicial sets

In this chapter we will develop some further properties necessary to understand simplicial sets. In order to control the weak equivalences we introduce two classes of maps: fibrations and cofibrations. These maps formalize "obstruction theory", or rather they tell us when existence of liftings can be expected. This is intimately connected with the fact that weak equivalences are not isomorphisms, although they become so in the homotopy category.

So a natural question could be: for what kind of weak equivalences $X \xrightarrow{\sim} Y$ can we expect to find "liftings" for each diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\text{injective}"'] & X \arrow[d, "\simeq"] \\ B \arrow[r] & Y \end{tikzcd}$$

i.e., maps $B \to X$ you can insert in the diagram without destroying the commutativity? This is reminiscent of Tietze's extension theorem in point set topology which states that if $X = [0,1] \xrightarrow{\sim} * = Y$ then extensions exist for all closed inclusions $A \subseteq B$ where B is normal. What makes topological spaces different from simplicial sets is that "cofibrations" (more about these later) in $\mathcal{T}op$ are rather complicated, making qualifications such as "closed" and "normal" necessary. By contrast a "cofibration" of simplicial sets is simply an inclusion.

By choosing $A \subseteq B$ to be $\emptyset \subseteq Y$ we see that such an equivalence $X \xrightarrow{\sim} Y$ would have a splitting $Y \to X$, and by another choice of $A \to B$ (hint: try to lift a "trivial homotopy") we can show that $X \xrightarrow{\sim} Y \to X$ is homotopic to the identity. So in particular $X \to Y$ is a homotopy equivalence, and $Y \to X$ realizes its homotopy inverse.

A systematizing fact about simplicial sets is that they can be built by gluing simplices along their boundary, so the inclusions $\partial\Delta[n] \subseteq \Delta[n]$ play a prominent rôle. Here $\partial\Delta[n]$ is the subcomplex of $\Delta[n]$ generated by the faces d^i.

The inclusions that are also weak equivalences also have their "building blocks" discussed later, and the fibrations alluded to above are the maps $X \to Y$ having the property that for every diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

where $A \to B$ is both an inclusion and a weak equivalence we have a lifting $B \to X$. The simplicial sets X having the property that the canonical map $X \to *$ is a fibration are called *fibrant*, and play a preferred rôle. Homotopy theory for fibrant objects is not hampered by problems such as weak equivalences not having homotopy inverses.

Any simplicial set is equivalent to a fibrant simplicial set, so why not just stay with the fibrant ones? The reason is that whereas $\mathcal{S}$ has good and transparent categorical properties, the subcategory of fibrant objects is very bad. Many of the constructions we use will take you out of fibrant objects. So the solution is to stick to $\mathcal{S}$, but have the (co)fibrations and weak equivalences as part of your data.

It all ends up in the statement that $\mathcal{S}$ is a "model category" (see chapter III) and the realization functor is a "Quillen equivalence" $\mathcal{T}op \to \mathcal{S}$. From this technical point of view (which we will develop in the next chapter) the thing which is special about $\mathcal{S}$ (apart from its categorical simplicity) is that all objects are "cofibrant", and properties relating to the fact that π_* commutes with colimits over $\mathbf{N}$.

Before we start with the technical issues we allow ourselves to give a variation of an important point from the first chapter.

0.1 Realization as an extension through presheaves

Another way of presenting the realization/singular adjoint pair, more reminiscent of the idea of simplicial sets as presheaves is the following. Consider the topological standard simplices as a functor $\Delta \to \mathcal{T}op$, and extend it to simplicial sets through the Yoneda functor

$$h\colon \Delta \to \text{Presheaves on } \Delta = \mathcal{S}, \qquad [n] \mapsto \Delta[n] = \Delta(-, [n]).$$

The realization is the "filler" (the precise notion is "left Kan extension" according to most working mathematicians) in

$$\begin{tikzcd} \Delta \arrow[r, "{[n]\mapsto\Delta^n}"] \arrow[d, "h"'] & \mathcal{T}op \\ \mathcal{S} \arrow[ur, dashed] & \end{tikzcd}$$

making the diagram commutative up to natural isomorphism (the above mentioned working mathematicians would also write things like $|X| = \int^{[n]} X_n \times \Delta^n$ and claim that the string of symbols $\mathcal{T}op(\int^{[n]} X_n \times \Delta^n, Y) \cong \int_{[n]} \mathcal{T}op(X_n, Y^{\Delta^n}) \cong \int_{[n]} \mathcal{E}ns(X_n, \mathcal{T}op(\Delta^n, Y))$ is another way of saying that the realization and singular functors are adjoint).

0.1.1 Categories and simplicial sets

The ideas involved in the singular/realization pair connecting topological spaces and simplicial sets is quite general, and we will see other examples later. For now we are content with giving just one, namely the connection between small categories and simplicial sets.

The category Δ may be considered as a subcategory of the category $\mathcal{C}at$ of small categories, by viewing $[q] \in \Delta$ as the category $\{0 \leftarrow 1 \leftarrow \cdots \leftarrow q\}$. Order preserving functions correspond to functors, and so we have an "inclusion" $\Delta \hookrightarrow \mathcal{C}at$.

In analogy with the singular functor, the Yoneda functor $h\colon \Delta \to \mathcal{S}$ extends to the *nerve*

$$N\colon \mathcal{C}at \to \mathcal{S}.$$

For a given category $\mathcal{C}$ the q simplices $N_q\mathcal{C}$ of the nerve is the set of functors $[q] \to \mathcal{C}$, or in other words, the set of all composable arrows

$$c_0 \leftarrow c_1 \leftarrow \cdots \leftarrow c_q$$

in $\mathcal{C}$. The nerve embeds $\mathcal{C}at$ as a full subcategory of $\mathcal{S}$.

The left adjoint to the nerve – corresponding to the realization – is less important, but is constructed as before as the filler in

$$\begin{tikzcd} \Delta \arrow[r, hook] \arrow[d, "h"'] & \mathcal{C}at \\ \mathcal{S} \arrow[ur, dashed] & \end{tikzcd}$$

Exercise 0.1.2 (nerves, natural transformations and homotopies). Let $f_0, f_1 : \mathcal{C} \to \mathcal{D}$ be functors between small categories. Prove that there is a natural transformation from f_0 to f_1 if and only if there is a homotopy from Nf_0 to Nf_1. (Hint: $N[1] = \Delta[1]$.)

1 (Co)fibrations

1.1 Simplicial sets are built out of simplices

Let us try to make some content to the title of this subsection.

I could mean the standard fact that "presheaves are colimits of representables": If X is a simplicial set, let the *simplex category* ΔX be the category of representable objects over X (i.e., the objects of ΔX are maps $\Delta[n] \to X$, and a morphism is a commutative diagram

$$\begin{tikzcd}
\Delta[n] \arrow[d] \arrow[dr] & \\
 & X \\
\Delta[n'] \arrow[ur] &
\end{tikzcd}$$

where - by Yoneda - the vertical map is induced by a map $[n] \to [n'] \in \Delta$). We have a functor $\Delta X \to \mathcal{S}$ sending $\Delta[n] \to X$ to $\Delta[n]$, and a natural isomorphism from the colimit of this functor to X.

However, what I really have in mind when writing the title is the following fact:

Lemma 1.1.1 *Let $K \subseteq L$ be an inclusion of simplicial sets. Then there is a functorial factorization*

$$K = K(-1) \subseteq K(0) \subseteq K(1) \subseteq \cdots \subseteq \bigcup_{n \geq -1} K(n) = L$$

such that $K(i-1) \subseteq K(i)$ is gotten by "attaching cells", i.e., it fits in a pushout diagram

$$\begin{tikzcd}
\coprod \partial\Delta[i] \arrow[r, "\subseteq"] \arrow[d] & \coprod \partial\Delta[i] \arrow[d] \\
K(i-1) \arrow[r, "\subseteq"] & K(i)
\end{tikzcd}$$

Sketch proof: By induction, we define $K(i)$ such that $K_n \subseteq L_n$ factors as $K_n \subseteq K(i)_n = L_n$ for all $n \leq i$ as follows. Assume we have constructed $K(i-1)$, and consider the complement $L_i \setminus K(i-1)_i$. By the Yoneda lemma we may consider each element in $L_i \setminus K(i-1)_i$ as a map $\Delta[i] \to L$, and by the assumption on $K(i-1)$ the composite $\partial\Delta[i] \subseteq \Delta[i] \to L$ factors through $K(i-1) \subseteq L$.

Now define $K(i)$ by means of the pushout in the statement of the lemma where the coproduct is taken over $L_i \setminus K(i-1)_i$. Finally one must check that the canonical map $K(i) \to L$ is an injection (and so can be chosen to be an inclusion).

1.2 Lifting properties and factorizations

In this section we will meet our first argument involving lifting properties. It is important because we look at an example which displays some standard methods, and forms the cornerstone of homotopical algebra.

In homotopy theory one of the important issues are when we can lift or extend maps. For instance, if a map $f\colon \partial\Delta[n] \to X$ in $\mathcal{S}$ can be extended to a map $\Delta[n] \to X$, then $|f|$ does not contribute to the homotopy of $|X|$. More generally, given a (commutative) diagrams of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

one may ask whether there is a map $s\colon B \to X$ making the following diagram commutative

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, "s"] & Y \end{tikzcd} .$$

We call such a map s a *lifting* of the original diagram. For instance, the class of maps $f\colon X \to Y$ having the property that all diagrams of the form

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, "\text{incl.}"] & X \arrow[d, "f"] \\ \Delta \arrow[r] & Y \end{tikzcd}$$

for $n \geq 0$ have liftings, are called ***trivial fibrations*** (not to be confused with product fibrations: I am sorry about the unfortunate terminology). We refer to the defining property of trivial fibrations as "the maps having the ***right lifting property*** with respect to the inclusions $\partial\Delta[n] \subseteq \Delta[n]$", and we will often display trivial fibrations by decorated arrows: $\overset{\sim}{\twoheadrightarrow}$.

Note 1.2.1 In view of 1.1.1, trivial fibration may be alternatively classified as the maps having the right lifting property with respect to arbitrary injections of simplicial sets.

Theorem 1.2.2 *Any map $f\colon X \to Y \in \mathcal{S}$ may be factored as an inclusion followed by a trivial fibration*

$$X \hookrightarrow Z \overset{\sim}{\twoheadrightarrow} Y .$$

Furthermore, this factorization may be chosen functorially.

Sketch proof: Consider the set D_1 of diagrams of the form

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, "\text{incl.}"'] & X \arrow[d, "f"] \\ \Delta[n] \arrow[r] & Y \end{tikzcd} .$$

Let X_1 be what you get if you "fill all these holes", i.e., the pushout

$$\begin{tikzcd} \coprod_{d\in D_1} \partial\Delta[n_d] \arrow[r] \arrow[d, "\coprod_{d\in D_1} \text{incl.}"'] & X \arrow[d] \\ \coprod_{d\in D_1} \Delta[n_d] \arrow[r] & X_1 \end{tikzcd} .$$

Note that $X \to X_1$ is an inclusion. By the universal property of the pushout, f factors through $X \to X_1$, and we can play the game over again, this time to the induced map $X_1 \to Y$. The upshot is a chain

$$X = X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$$

whose colimit (union) we call X_∞. We let

$$\begin{tikzcd} X \arrow[r, "\iota_f"] & Z_f \arrow[r, two heads, "\phi_f", "\sim"'] & Y \end{tikzcd}$$

be

$$X \to X_\infty \to Y$$

Note that $X \to X_\infty$ is an inclusion. The important thing is that $X_\infty \to Y$ is a trivial fibration.

To see this, we need to observe that $\partial\Delta[n]$ is *small* (much more about small objects later), which for our current purposes implies that a map $\partial\Delta[n] \to X_\infty$ must actually factor through $X_m \subseteq X_\infty$ for some (possibly very big) integer m.

So if we have a square of the sort

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, "\text{incl.}"] & X_\infty \arrow[d, "\phi_f"] \\ \Delta[n] \arrow[r] & Y \end{tikzcd}$$

and ask for a lifting, let m be such that $\partial\Delta[n] \to X_\infty$ factors through $X_m \to X_\infty$ and notice that by the very construction of X_{m+1} we have a commutative diagram

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, "\text{incl.}"] & X_m \arrow[d] \arrow[dr] & \\ \Delta[n] \arrow[r] \arrow[drr] & X_{m+1} \arrow[r] \arrow[dr] & X_\infty \arrow[d, "\phi_f"] \\ & & Y \end{tikzcd} .$$

But since $X_{m+1} \to Y$ factors through X_∞ we have our desired lifting.

1.3 Small objects

The proof above exemplified what is known as the *small object argument.*

The simplest case is well known for sets.

Fact 1.3.1 (see e.g., [16]: "filtered colimits commute with finite limits") If $X_0 \to X_1 \to \dots$ is a sequence of sets, and A a finite set, then the natural function

$$\lim_{\to} \left(X_i^A\right) \to \left(\lim_{\to} X_i\right)^A$$

is a bijection.

The full-fledged version used in homotopy theory these days often involves a lot of big cardinals, but this is more than enough for our present purposes.

A *finite* simplicial set is a simplicial set with only finitely many non-degenerate simplices. Examples include $\Delta[n]$ and all its sub simplicial sets.

Proposition 1.3.2 *Let A be finite simplicial set and $X(0) \to X(1) \to \dots$ a sequence of simplicial sets. Then the natural map*

$$\lim_{\to} \mathcal{S}(A, X(i)) \to \mathcal{S}(A, \lim_{\to} X(i))$$

is a bijection.

Sketch proof: Let us for simplicity assume that all the maps $X(i) \to X(i+1)$ are inclusions (that is all we needed in 1.2.2 anyhow). Then the natural map is perforce an inclusion, and we only need to show surjectivity.

Let $f: A \to \lim_{\to} X(i) = \bigcup X(i) \in \mathcal{S}$. Every k-simplex in A must necessarily map to $X(i)_k$ for some i depending on the simplex, but since there are only finitely may non-degenerate simplices we may choose a specific i such that they all map to $X(i)$.

If $x \in A_n$ is any simplex, there is a unique non-degenerate simplex y such that $x = \phi^* y$ for some surjective $\phi \in \Delta$, and so $f(x) = f(\phi^* y) = \phi^* f(y)$ must also be in $X(i)$ (since $X(i) \subseteq \bigcup X(j)$ is a simplicial map).

Remark 1.3.3 *We have the small object argument for any $A \in \mathcal{S}$, provided the cardinality of the indexing of the colimit is sufficiently big.*

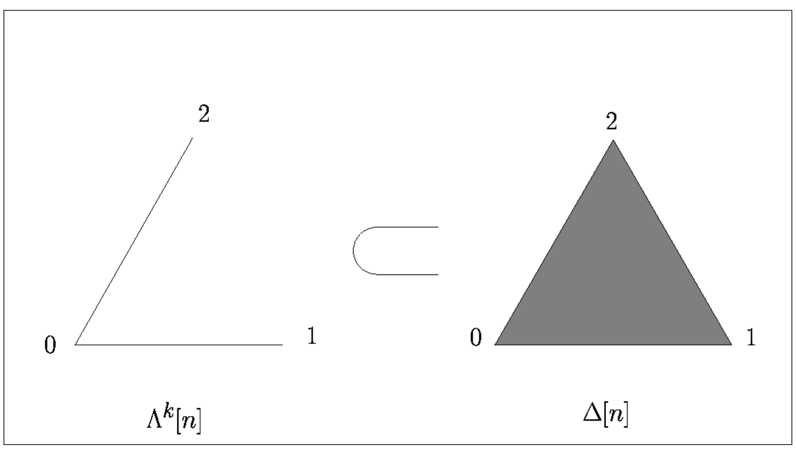
1.4 Fibrations

There is **one** aspect which is more awkward in simplicial sets than in topological spaces, and we will try to explain this, and at the same time give a definition of fibrations.

Definition 1.4.1 Let n and k be integers satisfying $0 \leq k \leq n > 0$. The k^{th}-horn

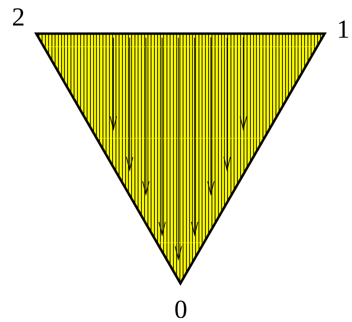
$$\Lambda^k[n] \subseteq \Delta[n]$$

be the sub-simplicial set generated by all faces of $\Delta[n]$ but the k^{th}.



The inclusion of the horn, here illustrated with $n = 2$ and $k = 0$. The 0^{th} horn is generated by all faces but the 0^{th}, which is the face opposite to the 0^{th}-vertex.

Obviously, the inclusion of the horn into the standard simplex is a weak equivalence. If you realize, you get that $|\Lambda^k[n]| \subseteq |\Delta[n]|$ is a deformation retract:



The inclusion $|\Lambda^0[2]| \subseteq |\Delta[2]|$ is a deformation retract by the retraction illustrated.

However, there is no retraction before realizing. This is the only bad thing about simplicial sets, and unfortunately there is no pain-free cure.

However, there is a large class of simplicial sets – called *fibrant* – that do not have this problem. Fortunately, every simplicial set is weakly equivalent to a fibrant simplicial set:

Definition 1.4.2 A map $f\colon X \to Y \in \mathbf{S}$ is a *fibration* if it has the right lifting property with respect to all the inclusions $\Lambda^k[n] \subseteq \Delta[n]$ for $0 \le k \le n > 0$.

If $X \to *$ is a fibration, we say that X is *fibrant*.

Example 1.4.3

1. If Y is a topological space, then $\operatorname{sing} Y$ is fibrant.
2. If M is a simplicial group, then the underlying simplicial set UM is fibrant (essentially because you have inverses: see e.g., [6, I.3.4]).

Exercise 1.4.4 Prove that if $Y \in \mathcal{T}op$, then $\operatorname{sing} Y$ is fibrant.

Exercise 1.4.5 Note that a trivial fibration actually is a fibration.

Fact 1.4.6 One may show that being a trivial fibration is equivalent to being both a fibration and a weak equivalence.

Fact 1.4.7 Our definition of fibrations is equivalent to saying that a fibration is a map which has the right lifting property with respect to injections that are weak equivalences. To see this one has to show that inclusions that are weak equivalences can be built out of the filling of horns (more precisely, they are retracts of injections $X(0) \to \bigcup X(i)$ where each $X(i-1) \to X(i)$ are pushouts of disjoint unions of $\Lambda^k[n] \subseteq \Delta[n]$'s).

Fact 1.4.8 Using the small object argument again we get that given any map $f\colon X \to Y$ there exists a (functorial) factorization

$$X \underset{\sim}{\overset{\iota_f}{\hookrightarrow}} Z_f \overset{\phi_f}{\twoheadrightarrow} Y$$

of f into a inclusion that is a weak equivalence followed by a fibration.

Note 1.4.9 In the literature fibrations of simplicial sets are often referred to as *Kan fibrations* and fibrant simplicial sets as *Kan complexes*.

Proposition 1.4.10 *If $i\colon A \subseteq B$ and $f\colon X \to Y$ is a fibration, then the canonical map*

$$(i,p)_*\colon \underline{S}(B,X) \to \underline{S}(B,Y) \times_{\underline{S}(A,Y)} \underline{S}(A,X)$$

is a fibration. If either i or f are weak equivalences then so is $(i,p)_*$.

Sketch proof: We only prove the case $A=\emptyset$ and $Y=*$ since this is all we need just now. In that case it reduces to showing that if X is fibrant, then so is $\underline{\mathcal{S}}(B,X)$, and if in addition $X\to *$ is a weak equivalence, so is $\underline{\mathcal{S}}(B,X)\to *$.

Consider a diagram of the sort

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \underline{\mathcal{S}}(B,X) \arrow[d] \\ L \arrow[r] & * \end{tikzcd}.$$

Asking about liftings is the same as asking whether the map

$$\mathcal{S}(L,\underline{\mathcal{S}}(B,X))\to\mathcal{S}(K,\underline{\mathcal{S}}(B,X))$$

is surjective, which is the same as asking whether

$$\mathcal{S}(B\wedge L,X)\to\mathcal{S}(B\wedge K,X)$$

is surjective, which is the same as asking about liftings in the diagram below

$$\begin{tikzcd} K\wedge B \arrow[r] \arrow[d] & X \arrow[d] \\ L\wedge B \arrow[r] & * \end{tikzcd}.$$

But if $K\to L$ is injective or a weak equivalence, then so is $K\wedge B\to L\wedge B$ (injectivity is obvious, and weak equivalence may be seen by realizing). So since X is fibrant we get liftings for all injections $K\to L$ that are weak equivalences. If in addition $X\to *$ is a weak equivalence we get liftings for all inclusions by 1.4.6.

Proposition 1.4.11 *(the Whitehead theorem) If $X,Y\in\mathcal{S}$ are fibrant and $f\colon X\to Y$ is a weak equivalence, then f is a homotopy equivalence.*

Sketch proof: By factorization, we may assume that f is either a trivial fibration or both an injection and a weak equivalence (that this works needs a slight checking). Assume first f is a trivial fibration (in which case it is not needed that X and Y are fibrant). Then there is a lifting $s\colon X\to Y$ in the diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "\simeq"', "f"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

and we must show that s is a homotopy inverse. By construction fs is the identity, but what about sf? Consider the diagram

$$\begin{tikzcd} X\coprod X \arrow[r, "sf+id_X"] \arrow[d] & X \arrow[d, "\simeq"', "f"] \\ X\times\Delta[1] \arrow[r, "f\,pr_X"] & Y \end{tikzcd}$$

The left vertical map is the inclusion induced by $\partial\Delta[1]\subseteq\Delta[1]$, and since f is a trivial fibration we get a lifting of the homotopy, giving us the desired homotopy $X\times\Delta[1]\to X$ from sf to the identity.

On the other hand, assume that f is an inclusion and a weak equivalence. Then the diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[d] \\ Y \arrow[r] & * \end{tikzcd}$$

has a lifting $p\colon Y \to X$ since X is fibrant. We must show that fp is homotopic to the identity. Consider the diagram

$$\begin{tikzcd} Y\coprod_X (X\times\Delta[1])\coprod_X Y \arrow[r, "fp+f\,pr_X+id_Y"] \arrow[d] & Y \arrow[d] \\ Y\times\Delta[1] \arrow[r] & * \end{tikzcd}.$$

Again we get a lifting (since Y is fibrant and the left vertical map is an injection and a weak equivalence), and the desired homotopy is established.

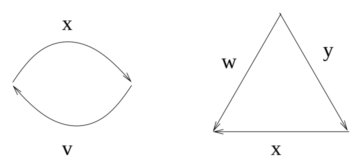
Example 1.4.12 Check that the above proof using factorizations give a proof in general.

2 Combinatorial homotopy groups

We are now in a position to define homotopy groups for simplicial sets entirely within $\mathcal{S}$ without reference to topological spaces. This can be done in one of many ways, the first we present is not very constructive since it involves the small object argument. The second is leaner and involves Kan's particular "fibrant replacement functor Ex^∞.

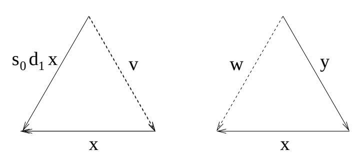
2.1 Homotopies and fibrant objects

Consider the path relation between vertices discussed in 3.0.2. There we mentioned that generally symmetry and transitivity was was a problem. This is not so for fibrant X.



The symmetry and reflexivity of the path relation in fibrant objects.

Exercise 2.1.1 Prove that in fibrant simplicial sets the path relation is an equivalence relation.



The idea of the proof: give 1-simplices x and y with $d_0x = d_1y$, find 2-simplices in X with the desired boundary.

This opens for the definition:

Definition 2.1.2 If $X \in \mathcal{S}_*$ is fibrant, then

$$\pi_q X = \pi_0 \underline{\mathcal{S}_*}(S^q, X).$$

Exercise 2.1.3 Prove that $\pi_1 X$ is a group. If you are really industrious you may want to show directly that $\pi_q X$ is an abelian group for $q > 1$.

Lemma 2.1.4 *If $X \in \mathcal{S}_*$ is fibrant, then there is a natural isomorphism $\pi_q X \cong \pi_*|X|$.*

Sketch proof: Since X is fibrant we have by 1.4.11 that the weak equivalence $X \to \operatorname{sing}|X|$ is a homotopy equivalence, and so that $\mathcal{S}_*(S^q, X) \to \mathcal{S}_*(S^q, \operatorname{sing}|X|) \cong \operatorname{sing}\left(|X|^{|S^q|}\right)$ is a homotopy equivalence. Hence they have the same π_0, giving the desired result.

So if $X \in \mathcal{S}_*$ we can choose any (pointed) weak equivalence $X \xrightarrow{\sim} Y$ with Y fibrant and define the homotopy groups of X to be those of Y. Of course, we want the choice of Y to be functorial, and one way is to use the functorial factorization of $X \to *$ into a weak equivalence that is an inclusion followed by a fibration. Another way is to simply let $Y = \operatorname{sing}|X|$.

Exercise 2.1.5 Let $f\colon X \to Y$ be a fibration and $y \in Y_0$, and define the *fiber* over y to be the simplicial set F gotten by the pullback

$$\begin{tikzcd} F \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ * \arrow[r, "y"] & Y \end{tikzcd}$$

(i.e., $F \cong f^{-1}(y)$). Prove that F is fibrant. In addition, if X and Y also are fibrant and X and F are pointed in an $x \in f^{-1}(y)$, prove the long exact sequence

$$\cdots \to \pi_{n+1}Y \to \pi_n F \to \pi_n X \to \pi_n Y \to \cdots \to \pi_0 Y.$$

Exercise 2.1.6 Use the fact that the result of the last exercise is true for arbitrary fibrations to prove that if $f\colon X \to Y$ is a trivial fibration, then f is a weak equivalence.

Exercise 2.1.7 Prove that if G is a simplicial group, then the path relation is an equivalence relation [in fact: G is fibrant]. Prove that $\pi_0 G$ is a group. Prove that $\mathcal{S}_*(S^q, G)$ is a simplicial group.

2.1.8 Subdivisions and Kan's Ex^∞

For completeness we present Kan's Ex^∞ which is a particularly compact "fibrant replacement functor" related to the small object argument. We refer to [6, III4] for proofs of the claims presented in this section.

Consider the category Inj with objects the finite subsets of $\mathbf{N}$, and with morphisms the inclusions (this is equivalent to the subcategory of Δ of all injective maps, but the combinatorics becomes easier this way).

The *over category* Inj_n is the category whose objects are subsets $S \subseteq \{0, 1, \ldots, n\}$ and where there is a single morphism from $T \subseteq \{0, 1, \ldots, n\}$ to $S \subseteq \{0, 1, \ldots, n\}$ if $T \subseteq S \subseteq \{0, 1, \ldots, n\}$

For every $\phi\colon [n] \to [m] \in \Delta$ we get a functor $\phi_*\colon \mathrm{Inj}_n \to \mathrm{Inj}_m$ by sending $S \subseteq \{0, 1, \ldots, n\}$ to $\phi(S) \subseteq \{0, 1, \ldots, m\}$ making

$$[n] \to \mathrm{Inj}_n$$

a cosimplicial category. The functor $\mathrm{Inj}_n \to [n]$ sending $\{i_0, \ldots, i_q\} \subseteq \{0, 1, \ldots, n\}$ to $i_q \in [n]$ becomes a map of cosimplicial categories when $[n]$ varies.

For any simplicial set X Kan then defines

$$\mathrm{Ex}(X) = \{[q] \mapsto \mathcal{S}(N(\mathrm{Inj}_q), X)\}$$

This is a simplicial set, and $N(\mathrm{Inj}_q) \to N[q] = \Delta[q]$, defines an inclusion $X \subseteq \mathrm{Ex}(X)$. Set

$$\mathrm{Ex}^\infty X = \varinjlim_k \mathrm{Ex}^{(k)}(X).$$

Fact 2.1.9 Kan's subdivision functor Ex^∞ is a fibrant replacement functor, i.e.,

1. The inclusion $X \subseteq \mathrm{Ex}^\infty X$ is a weak equivalence (pointed if X is pointed).

2. $\mathrm{Ex}^{\infty}X$ is fibrant.

Kan then defines the homotopy groups of X without reference to topological spaces via

$$\pi_q X = \pi_0 \underline{\mathcal{S}_*}(S^q, Ex^{\infty}X).$$

Chapter III

Model categories

We are interested in homotopy theory in a variety of categories, and we want to compare these. There is an efficient machinery due to Quillen, which encodes this structure. We have used this language in our discussion of simplicial sets. In addition to weak equivalences (which is all that is needed to form the homotopy category) we have fibrations and cofibrations satisfying certain axioms. This structure ensures that the homotopy category actually exists, but more importantly it encodes the deeper homotopical structures, making a large class of arguments formal. It also makes comparison between different homotopical structures more transparent.

In particular, we are interested in functor categories. If I is a small category and $\mathcal{M}$ is a category "in which we know how to do homotopy theory", how can we do homotopy theory in the category of functors from I to $\mathcal{M}$? This question does not have a unique answer (which is a good thing since that different answers are serviceable in different situations), but there is still much we can say.

0.1 Liftings

As discussed at the beginning of the previous chapter, the formal side of homotopy theory is all about liftings.

Definition 0.1.1 Let $\mathcal{M}$ be a category, and $i\colon A \to B$ and $p\colon X \to Y$ be two maps in $\mathcal{M}$. We say that p has the *right lifting property* with respect to i (and that i has the *left lifting property* with respect to p) if for all commutative diagrams

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

in $\mathcal{M}$, there is a map $s\colon B \to X \in \mathcal{M}$ making the following diagram commutative

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, "s"] & Y \end{tikzcd}$$

1 The axioms

A model category is a category equipped with (more than) enough structure to do homotopy theory. The axioms are good in the sense that they apply to most of the situations we could imagine desirable, and still they are strong enough to carry important information.

Given the complexity of the axioms, it is perhaps surprising that they have not changed significantly during the 35 years that have passed since Quillen first proposed them. It is perhaps even more surprising that the modifications that have been suggested all have tended to make the axioms even more restrictive.

We give the version most common these days, following [4], see also [8]. Hovey [9] insists on a choice instead of merely existence in the factorization axiom $\mathcal{M}5$ below:

Definition 1.0.1 A *model category* is a category $\mathcal{M}$ together with three classes of maps $\mathrm{cof}\mathcal{M}$ (the *cofibrations*), $\mathrm{fib}\mathcal{M}$ (the *fibrations*) and $\mathrm{w}\mathcal{M}$ (the *weak equivalences*) satisfying the following axioms:

$\mathcal{M}1$: (Limit axiom) The category $\mathcal{M}$ is (co)complete (i.e., has all small (co)limits).

$\mathcal{M}2$: (Two out of three axiom) If

$$X \xrightarrow{g} Y \xrightarrow{f} Z \in \mathcal{M}$$

and two of f, g and fg are weak equivalences, then so is the third.

$\mathcal{M}3$: (Retract axiom) If the map $g \in \mathcal{M}$ is a retract of $h \in \mathcal{M}$ and h is in $\mathrm{cof}\mathcal{M}$, $\mathrm{fib}\mathcal{M}$ or in $\mathrm{w}\mathcal{M}$, then so is g.

$\mathcal{M}4$: (Lifting axiom) The cofibrations have the left lifting property with respect to the maps in $\mathrm{fib}\mathcal{M} \cap \mathrm{w}\mathcal{M}$ (the *trivial fibrations*) and the fibrations have the right lifting property with respect to the maps in $\mathrm{cof}\mathcal{M} \cap \mathrm{w}\mathcal{M}$ (the *trivial cofibrations*).

$\mathcal{M}5$: (Factorization axiom) If $g \colon X \to Y \in M$ there exist functorial factorizations

$$\begin{tikzcd}
 & Z_g \arrow[dr, two heads, "f_g"] & \\
X \arrow[ur, tail, "i_g", "\sim"'] \arrow[rr, "g"] \arrow[dr, tail, "j_g"'] & & Y \\
 & W_g \arrow[ur, two heads, "p_g"', "\sim"] &
\end{tikzcd}$$

where i_g is a trivial cofibration, f_g is a fibration, j_g is a cofibration and p_g is a trivial fibration.

Note 1.0.2 Note that a model category always have initial and final objects. An object for which the unique map from the initial object is a cofibration is said to be *cofibrant*, and – dually – an object for which the unique map to the final object is a fibration is said to be *fibrant*.

Theorem 1.0.3 *The categories $\mathcal{S}$, $\mathcal{S}_*$, $\mathcal{A}$, $\mathcal{T}op$, $\mathcal{T}op_*$ and $\mathcal{S}pt$ "are" model categories.*

By saying that they "are" model categories we mean that the categories have choices of subcategories which make them into model categories, and the weak equivalences are what you think they are.

To be a bit more precise:

1. ($\mathcal{S}$ and $\mathcal{S}_*$) The injections in $\mathcal{S}$ and $\mathcal{S}_*$ are the cofibrations (so all objects are cofibrant) and the weak equivalences and fibrations were discussed in the previous chapter.

2. ($\mathcal{T}op$ and $\mathcal{T}op_*$) The fibrations are the maps f such that $\operatorname{sing} f$ are fibrations (a.k.a. "Serre fibrations"), the weak equivalences are those inducing isomorphisms on π_* and the cofibrations are those that have the left lifting property with respect to the trivial fibrations. All objects are fibrant.

3. ($\mathcal{A}$) A map $f \in \mathcal{A} = s\mathcal{A}b$ is a weak equivalence (resp. fibration) if it is so when forgetting down to $\mathcal{S}$. The cofibrations are those with the left lifting property with respect to the trivial fibrations. All objects are fibrant (a map $X \to Y$ is a fibration if the map $X \to Y \times_{\pi_0 Y} \pi_0 X$ is a surjection, and the cofibration condition is very closely associated with free generation of non-degenerate elements).

4. ($\mathcal{S}pt$) A map in $\mathcal{S}pt$ is a weak equivalence if it is a stable equivalence. The fibrations and cofibrations will be discussed in section 5.

1.1 Simple consequences

The axioms are stated in a form intended to make them easy to check. In practice, you often know that the category in question is a model category, and then it is better to know some of the implications.

Exercise 1.1.1 (see e.g., [8, 7.2]).

1. Let $g\colon X \to Y \in \mathcal{M}$ be factored as $g = pi$ where p has the right lifting property with respect to g. Show that g is a retract of i.

2. A map is in $\mathrm{cof}\mathcal{M}$ (resp. $\mathrm{cof}\mathcal{M} \cap \mathrm{w}\mathcal{M}$) if **and only if** it has the left lifting property with respect to the maps in $\mathrm{fib}\mathcal{M} \cap \mathrm{w}\mathcal{M}$ (resp. $\mathrm{fib}\mathcal{M}$). A map is in $\mathrm{fib}\mathcal{M}$ (resp. $\mathrm{fib}\mathcal{M} \cap \mathrm{w}\mathcal{M}$) if **and only if** it has the right lifting property with respect to the maps in $\mathrm{cof}\mathcal{M} \cap \mathrm{w}\mathcal{M}$ (resp. $\mathrm{cof}\mathcal{M}$).

3. Two of $\mathrm{cof}\mathcal{M}$, $\mathrm{fib}\mathcal{M}$ and $\mathrm{w}\mathcal{M}$ determine the third.

4. The classes of maps $\mathrm{cof}\mathcal{M}$, $\mathrm{fib}\mathcal{M}$ and $\mathrm{w}\mathcal{M}$ form subcategories of $\mathcal{M}$ containing all objects (and all isomorphisms).

5. If

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "g"'] & C \arrow[d, "f"] \\ B \arrow[r, "j"] & D \end{tikzcd}$$

is a commutative square in $\mathcal{M}$ we have that

(a) if the square is a pushout square and i is a (trivial) cofibration, then so is j.

(b) if the square is a pullback square and f is a (trivial) fibration, then so is g.

Note 1.1.2 On terminology: In [21] Quillen only required finite (co)limits, used a weaker version of the lifting axiom 1.0.14 (reserving the term *closed model category* for the current version) and did not require that the factorizations in the factorization axiom 1.0.15 had to be functorial.

He used the terms trivial (co)fibration in [21] and [19], but switched to *acyclic* (co)fibration in [20] (where he states that '"acyclic" is much preferable to the term "trivial"'). You will see both forms in the literature, and there are good arguments against both.

One of the most useful technical tools in model categories has become known as *Ken Brown's lemma* (earlier each new generation of students reading Quillen's works had to rediscover this lemma for themselves, but nowadays it is on the standard repertoire). We include the proof since it is a good example of the axioms at play.

Lemma 1.1.3 *Let $\mathcal{M}$ be a model category. If $f\colon A \to B$ is a map between cofibrant objects, then there exists a (functorial) diagram*

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, tail, "i_f"'] & & B \\
& C_f \arrow[ur, two heads, "p_f", "\sim"'] & B \arrow[l, tail, "j_f"', "\sim"] \arrow[u, "="']
\end{tikzcd}
$$

where i_f is a cofibration, p_f is a trivial fibration and j_f a trivial cofibration.

Dually, if $f\colon A \leftarrow B$ is a map between fibrant objects, then there exists a (functorial) diagram

$$
\begin{tikzcd}
A & & B \arrow[ll, "f"'] \arrow[dl, tail, "I_f"', "\sim"] \arrow[d, "="] \\
& Z_f \arrow[ul, two heads, "P_f"] \arrow[r, two heads, "Q_f", "\sim"'] & B
\end{tikzcd}
$$

where I_f is a trivial cofibration, P_f is a fibration and Q_f a trivial fibration.

Sketch proof: We prove only the first part. Using the factorization axiom $\mathcal{M}5$, factor the map

$$f + id_B \colon A \coprod B \to B$$

into a cofibration followed by a trivial fibration

$$
\begin{tikzcd}
A \coprod B \arrow[r, tail] & C_f \arrow[r, two heads, "p_f", "\sim"'] & B
\end{tikzcd}
$$

Since A and B are cofibrant, and cofibrations are closed under pushouts by 1.1.1.5, both the canonical maps $A \to A \coprod B \leftarrow B$ are cofibrations. Noting that the diagram

$$
\begin{tikzcd}
A \arrow[d, tail] \arrow[drr, "f"] & & \\
A \coprod B \arrow[r, tail] & C_f \arrow[r, two heads, "p_f", "\sim"'] & B \\
B \arrow[u, tail] \arrow[urr, "id_B"'] & &
\end{tikzcd}
$$

commutes, we are done (use that compositions of cofibrations are cofibrations (1.1.1.4) and the two out of three axiom $\mathcal{M}2$).

As an immediate consequence we have:

Corollary 1.1.4 *A functor $F\colon \mathcal{M} \to \mathcal{N}$ between model categories that preserve trivial cofibrations sends weak equivalences between cofibrant objects to weak equivalences.*

1.2 Proper model categories

Definition 1.2.1 Let $\mathcal{M}$ be a model category. We say that $\mathcal{M}$ is *left proper* if for all pushout diagrams

$$
\begin{tikzcd}
A \arrow[r, "f", "\sim"'] \arrow[d, tail, "i"] & B \arrow[d, tail] \\
C \arrow[r, "g"] & D
\end{tikzcd}
$$

with i a cofibration and f a weak equivalence, g is a weak equivalence.

Dually, $\mathcal{M}$ is *right proper* if for all pullback diagrams

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d] & B \arrow[d, two heads, "p"] \\ C \arrow[r, "f", "\sim"'] & D \end{tikzcd}$$

with p a fibration and f a weak equivalence, g is a weak equivalence.

If $\mathcal{M}$ is both left and right proper, we say that $\mathcal{M}$ is *proper*.

Example 1.2.2 The model category structures on $\mathcal{S}$, $\mathcal{S}_*$, $\mathcal{A}$, $\mathcal{T}op$, $\mathcal{T}op_*$ and $\mathcal{S}pt$ are all proper (see e.g., [8, 11.1], which also implies the case $\mathcal{A} = s\mathcal{A}b$. For spectra, see [1]).

For future reference we include the following definition

Definition 1.2.3 In a model category a (commuting) square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is a *homotopy pullback square* if for the functorial factorizations $B \overset{\sim}{\rightarrowtail} X \twoheadrightarrow D$ and $C \overset{\sim}{\rightarrowtail} Y \twoheadrightarrow D$ (into trivial cofibrations followed by fibrations) the canonical map $A \to X \times_D Y$ is a weak equivalence. The *homotopy pullback* is the object $X \times_D Y$.

The good thing about homotopy pullbacks in (right) proper model categories is that they are homotopy invariant, and if the square in the definition were already a (categorical) pullback diagram and $B \to D$ a fibration, then it would be homotopy cartesian, see [8].

1.3 Quillen functors

The morphisms in "the category of model categories" is not what you first come to think about (namely a functor preserving all the structure: this is much too rigid).

Definition 1.3.1 Let $\mathcal{M}$ and $\mathcal{N}$ be model categories. A *(left) Quillen functor* from $\mathcal{M}$ to $\mathcal{N}$ is a functor $F\colon \mathcal{M} \to \mathcal{N}$ such that

- F has a right adjoint
- F preserves cofibrations and trivial cofibrations.

A Quillen functor F with right adjoint G is called a *Quillen equivalence* if for all cofibrant $A \in \mathcal{M}$ and fibrant $X \in \mathcal{N}$ the bijection

$$\mathcal{M}(A, GX) \cong \mathcal{N}(FA, X)$$

restricts to a bijection

$$w\mathcal{M}(A, GX) \cong w\mathcal{N}(FA, X)$$

Example 1.3.2 1. The realization $\mathcal{S} \to \mathcal{T}op$ is a Quillen equivalence (see below)

2. The adding base point gives a Quillen functor $(-)_+\colon \mathcal{S} \to \mathcal{S}_*$.
3. The free functor $\tilde{\mathbf{Z}}\colon \mathcal{S}_* \to \mathcal{A}$ is a Quillen functor.
4. The suspension spectrum gives a Quillen functor $\Sigma^\infty\colon \mathcal{S}_* \to \mathcal{S}pt$.

Fact 1.3.3 If both a Quillen functor F and its right adjoint G preserve weak equivalences, F is a Quillen equivalence if and only if for all cofibrant and fibrant objects $A \in \mathcal{M}$ and cofibrant and fibrant objects $X \in \mathcal{N}$ the adjunction maps

$$A \to GFA \text{ and } FGX \to X$$

are weak equivalences.

From this we see

Theorem 1.3.4 *The realization $\mathcal{S} \to \mathcal{T}op$ is a Quillen equivalence.*

Exercise 1.3.5 Prove that the right adjoint of a (left) Quillen functor preserves fibrations and trivial fibrations.

2 Functor categories: the projective structure

Definition 2.0.1 Let $\mathcal{C}$ be a small category, and consider the category $\mathcal{S}^{\mathcal{C}}$ of functors $\mathcal{C} \to \mathcal{S}$. Call a natural transformation $X \to Y \in \mathcal{S}^{\mathcal{C}}$ a *pointwise weak equivalence* (resp. *pointwise fibration*) if for every $c \in \mathcal{C}$ the map $X(c) \to Y(c) \in \mathcal{S}$ is a weak equivalence (resp. fibration).

A *cellular inclusion* is a composite

$$X(0) \to X(1) \to \cdots \to \varinjlim_{i \in \mathbf{N}} X(i)$$

where each $X(i) \to X(i+1)$ is the pushout along a disjoint union of maps

$$\partial\Delta[n] \times \mathcal{C}(c, -) \subseteq \Delta[n] \times \mathcal{C}(c, -)$$

for $c \in \mathcal{C}$ and $n \geq 0$.

Theorem 2.0.2 *(see e.g., [2, p. 314]) The above gives a model category structure, called the projective structure, on $\mathcal{S}^{\mathcal{C}}$ in which a map is a cofibration iff it is a retract of a cellular inclusion.*

Sketch proof: One proof follows the line of the same claim for $\mathcal{S}$ (which relies on facts about the realization functor as discussed in the previous chapter). Alternatively one can use a lifting lemma approach as in [4].

Note 2.0.3 For the application to motivic homotopy theory (see chapter IV), this is not the model structure on the simplicial presheaves $Sm/S^{\mathrm{op}} \to \mathcal{S}_*$ usually considered. However, it is quite useful e.g., when considering spectra through motivic functors, partially because it is much "smaller" than the one considered by Jardine, Morel and Voevodsky (in which the cofibrations are the injections).

3 Cofibrantly generated model categories

There was one particularly cool thing about simplicial sets which we employed many times: that the "building blocks" for (trivial) cofibrations were easy to cope with.

Definition 3.0.1 Let I be a set of the morphisms in a cocomplete category $\mathcal{M}$. We let

1. $I - \text{inj}$ be the set of morphisms in $\mathcal{M}$ having the right lifting property with respect to the maps in I,

2. $I - \text{cof}$ be the set of morphisms in $\mathcal{M}$ having the left lifting property with respect to the maps in $I - \text{inj}$. The elements of $I - \text{cof}$ are referred to as *I-cofibrations*.

Previously when we discussed the small object argument, it referred to colimits indexed over the natural numbers. In what follows we will have to allow colimits over larger ordinals (satisfying the property that for all limit ordinals γ the map $\lim_{\overrightarrow{\beta<\gamma}} X(\beta) \to X(\gamma)$ is an isomorphism), but we refer to the literature for making this transfinite mumbojumbo precise e.g., [9, 2.1].

That a set I of maps permits the small object means that the domains of the maps are small with respect to (transfinite) compositions of pushouts of coproducts of maps in I (a.k.a. I-cell).

Definition 3.0.2 A *cofibrantly generated model category* is a model category $\mathcal{M}$ in which there exists

1. a set I of cofibrations which permits the small object argument and for which the cofibrations and I-cofibrations coincide.

2. a set J of trivial cofibrations which permits the small object argument and for which the trivial cofibrations and J-cofibrations coincide.

Example 3.0.3 All the model categories we have discussed so far are cofibrantly generated.

Lemma 3.0.4 *(Recognition principle). Let $\mathcal{M}$ be a (co)complete category with a subcategory $w\mathcal{M} \subseteq \mathcal{M}$ which is closed under retracts and satisfies the two-out-of-three axiom. Call the morphisms in $w\mathcal{M}$ weak equivalences.*

Let I and J be sets of maps in $\mathcal{M}$. Call the maps in $I - \mathrm{cof}$ (resp. $J - \mathrm{inj}$) cofibrations (resp. fibrations) and the maps in $J - \mathrm{cof}$ (resp. $I - \mathrm{inj}$) trivial cofibrations (resp. fibrations).

Assume that

1. *I and J permit the small object argument;*

2. *the trivial cofibrations are both cofibrations and weak equivalences and the trivial fibrations are both fibrations and weak equivalences;*

3. *and either all maps that are both cofibrations and weak equivalences are trivial cofibrations, or all maps that are both fibrations and weak equivalences are trivial fibrations.*

Then this structure defines a cofibrantly generated model structure on $\mathcal{M}$.

Sketch proof: The first axioms 1-3 follow automatically. The small object argument then gives 5. The third condition then implies one of the halves of 4 while the other follow from 5 and a retraction argument as in 1.1.1.1 (showing e.g. that all maps that are both weak equivalences and cofibrations necessarily are retracts of trivial cofibrations, and hence trivial cofibrations)

Let $\mathcal{C}$ be a small category, and consider the category $\mathcal{S}^{\mathcal{C}}$ of functors $\mathcal{C} \to \mathcal{S}$. Call a natural transformation $X \to Y \in \mathcal{S}^{\mathcal{C}}$ a *pointwise weak equivalence* (resp. *pointwise cofibration*) if for every $c \in \mathcal{C}$ the map $X(c) \to Y(c) \in \mathcal{S}$ is a weak equivalence (resp. cofibration).

Theorem 3.0.5 *(Heller, see [4]) The structure above gives rise to a model category structure on $\mathcal{S}^{\mathcal{C}}$ which is Quillen equivalent to the projective structure 2.0.2 through the identity map (from the projective to the current).*

Lemma 3.0.6 *(Lifting lemma). Let $\mathcal{M}$ be a cofibrantly generated model category, and let I (and J) be sets of generating (trivial) cofibrations. Let $\mathcal{C}$ be a (co)complete category and*

$$\mathcal{C} \underset{R}{\overset{L}{\leftrightarrows}} \mathcal{M}$$

be a pair of adjoint functors such that LI and LJ admit the small object argument and such that Rf is a weak equivalence for every $f \in LJ - \mathrm{cof}$.

Then $\mathcal{C}$ has a cofibrantly generated model structure with where a map $f \in \mathcal{C}$ is a weak equivalence (resp. fibration) if $Rf \in \mathcal{M}$ is. Furthermore the sets LI and LJ are generating (trivial) cofibrations and L is a Quillen functor.

Sketch proof: Use the recognition principle 3.0.4.

Exercise 3.0.7 Use the lifting lemma to prove that the structure on $\mathcal{A}$ given in the explanation following theorem 1.0.3 actually is a model structure.

4 Simplicial model categories

It is often the case that the set of maps assemble to a function *space* (or simplicial set), and that the model structure conforms nicely with the homotpy theory of these function spaces. In these cases many things become somewhat more transparent.

Definition 4.0.1 An $\mathcal{S}$*-category* $\mathcal{C}$ is a class of "objects" $ob\mathcal{C}$ together with a simplicial set

$$\underline{\mathcal{C}}(A, X) \in \mathcal{S}$$

for each pair of objects $A, X \in ob\mathcal{C}$ with a unital and associative composition (the usual axioms for a category, just allowing morphism sets to be simplicial sets).

Note 4.0.2 An $\mathcal{S}$-category $\mathcal{C}$ has an underlying category $\mathcal{C}$ by letting

$$\mathcal{C}(A, X) = \underline{\mathcal{C}}(A, X)_0.$$

Note that it takes no effort to replace $\mathcal{S}$ by similar animals like $\mathcal{S}_*$ or $\mathcal{A}$, giving rise to parallel theories.

Exercise 4.0.3 Give a definition of an $\mathcal{S}$-functor, which you think will be useful.

Definition 4.0.4 A $\mathcal{S}$-category is *tensored and cotensored* if for all $X \in \mathcal{C}$ the functor

$$Y \mapsto \underline{\mathcal{C}}(X, Y)$$

has a left adjoint $K \mapsto X \otimes X$, and if $K \mapsto X \otimes K$ has a right adjoint (everything natural).

Definition 4.0.5 A *simplicial model category* is a tensored and cotensored $\mathcal{S}$-category $\mathcal{M}$ with a model category structure on the underlying category, such that for all cofibrations $i\colon A \rightarrowtail B$ and fibrations $p\colon X \twoheadrightarrow Y$, the canonical map

$$(i, p)_*\colon \underline{\mathcal{M}}(B, X) \to \underline{\mathcal{M}}(B, Y) \times_{\underline{\mathcal{M}}(A,Y)} \underline{\mathcal{M}}(A, X) \in \mathcal{S}$$

is a fibration, and that furthermore, if in addition either i or p are weak equivalences, then so is $(i, p)_*$.

Note 4.0.6 Quillen referred to the "tensored and cotensored" part as axiom SM0 and to the condition on the map $(i, p)_*$ as axiom SM7. The terminology "simplicial model category" is obviously unfortunate, as one would think that it referred to a functor from Δ^{op} to some category of model categories, but the terminology is well established.

Note 4.0.7 In simplicial model categories we have the notion of (simplicial) homotopy at our disposal. This means that we have means of detecting weak equivalences at the function space level.

Proposition 4.0.8 *(see e.g., [8, 10.5.1]) Let $\mathcal{M}$ be a simplicial model category. A map $f\colon Y \to Z$ is a weak equivalence if either of the following conditions are satisfied:*

1. *For every fibrant object $X \in \mathcal{M}$ the induced map*

$$\underline{\mathcal{M}}(Z, X) \to \underline{\mathcal{M}}(Y, X) \in \mathcal{S}$$

is a weak equivalence.

2. *For every cofibrant object $A \in \mathcal{M}$ the induced map*

$$\underline{\mathcal{M}}(A, Y) \to \underline{\mathcal{M}}(A, Z) \in \mathcal{S}$$

is a weak equivalence.

In the case where both Y and Z are cofibrant the first condition is necessary and sufficient. Likewise with fibrant vs. the latter condition.

5 Spectra

As we discussed in I.6, in topology spectra are the necessary device for applying homotopy theory on the collection of cohomology theories. This is also true in motivic homotopy theory; in order to study cohomology theories like motivic cohomology and algebraic K-theory we introduce spectra. An interesting thing is what replaces the circle: spectra study "stable" phenomena, i.e., smashing with the circle shall be an equivalence of homotopy categories. In order to study this effectively we present a model structure on the category of spectra making this true.

5.1 Pointwise structure

This structure is just a technical device, preparing the ground for the stable model structure.

Definition 5.1.1 A pointwise weak equivalence (resp. fibration) of spectra is a map $X \to Y \in \mathcal{S}pt$ such that for every $n \geq 0$ the map $X^n \to Y^n$ is a weak equivalence (resp. fibration) in $\mathcal{S}_*$. A cofibration is a map with the left lifting property with respect to to the maps that are both pointwise equivalences and pointwise fibrations.

Exercise 5.1.2 The pointwise structure is a model category.

Note 5.1.3 As a matter of fact, $\mathcal{S}pt$ is isomorphic to a functor category $[\mathcal{S}_{S^1}, \mathcal{S}_*]$ if one allows the enrichment in $\mathcal{S}_*$ to play a rôle (see e.g., [8, 9.6.2]). The model structure follows from the simplicial analog of the projective structure discussion above. The cofibrations are of the form $R(-) \wedge \partial\Delta[n]_+ \to R(-) \wedge \Delta[n]_+$ where R runs over the representable functors, and likewise for the trivial cofibrations.

To be explicit, the objects of $\mathcal{S}_{S^1}$ are the S^n's for $n \geq 0$, and

$$\underline{\mathcal{S}_{S^1}}(S^n, S^{n+k}) = \begin{cases} S^k & \text{if } k \geq 0 \\ * & \text{otherwise} \end{cases}$$

(the composition is obtained from the natural isomorphisms $S^k \wedge S^l \cong S^{k+l}$).

Both symmetric spectra [11] and simplicial functors [15] can be treated in the same manner, making comparison easier.

5.2 Stable structure

Definition 5.2.1 A *stable fibration* is a map in $\mathcal{S}pt$ with the right lifting property with respect to all cofibrations that are stable equivalences I.6.3.3. The *stable structure* on $\mathcal{S}pt$ consists of the stable equivalences, the stable fibrations and the cofibrations.

Proposition 5.2.2 *(see [1]) The stable structure defines a model category structure on $\mathcal{S}pt$.*

Note 5.2.3 [15] This model structure is cofibrantly generated: the generating cofibrations are the same as for the pointwise structure. The generating trivial cofibrations are given as follows using the notation of 5.1.3: consider the evaluation map

$$ev_n\colon \mathcal{S}_{S^1}(S^1 \wedge S^n, -) \wedge S^1 \cong \mathcal{S}_{S^1}(S^1, \mathcal{S}_{S^1}(S^n, -)) \wedge S^1 \to \mathcal{S}_{S^1}(S^n, -)$$

and let s_n be the inclusion of $\mathcal{S}_{S^1}(S^1 \wedge S^n, -) \wedge S^1$ into the mapping cylinder of ev_n. Let the generating trivial cofibrations be the generating trivial cofibrations from the pointwise structure plus all the s_ns.

Let

$$(QX)^n = \varinjlim_{i} \operatorname{sing} \Omega^i |X^{i+n}|.$$

Note that this defines a spectrum, and there is a canonical map $X \to QX$.

Exercise 5.2.4 Check the details about QX.

This construction is occasionally referred to as "spectrification". The important property of QX is that it is an Ω-spectrum, i.e., the adjoint $(QX)^n \to \Omega(QX)^{n+1}$ is a weak equivalence.

Fact 5.2.5 (see [1]) A map $X \to Y \in \mathcal{S}pt$ is a stable fibration if and only if it is a pointwise fibration such that for all $n \geq 0$ the square

$$\begin{tikzcd} X^n \arrow[r] \arrow[d] & (QX)^n \arrow[d] \\ Y^n \arrow[r] & (QY)^n \end{tikzcd}$$

is a homotopy pullback 1.2.3 (which means in this situation that the canonical map from X^n to the pullback of the rest of the diagram is supposed to be a weak equivalence).

Chapter IV

Motivic spaces and spectra

As our last application of the machinery discussed in these talks, we come to an approach to the main topic of this summer school: motivic spaces and spectra.

We first discuss the spaces and then the spectra (both very briefly). I emphasize that this is just one apprach, and that there are many others giving roughly the same result. See [22] for a more thorough discussion and [13] for background from algebraic geometry.

1 Motivic spaces

Let S be a Noetherian scheme and let Sm/S be the category of smooth scheme of finite type over S. Let $\mathcal{M}_S$ be the category of pointed simplicial presheaves on Sm/S, i.e., functors $(Sm/S)^{\mathrm{op}} \to \mathcal{S}_*$. This category is referred to as *the (motivic) category of spaces*, and is the basic object of study in this section.

We will provide $\mathcal{M}_S = \mathcal{S}_*^{(Sm/S)^{\mathrm{op}}}$ with several model category structures. Eventually we will have one whose homotopy category is worthy of the name *the unstable motivic homotopy category.* This is but one of many categories that can bear this name, but they are all Quillen equivalent, and the one we are going to present requires the least machinery.

We will study this category and also its functors briefly, before we go on to stabilize and so have motivic spectra in the next section.

Theorem III.2.0.2 provides a model structure on $\mathcal{M}_S = \mathcal{S}_*^{(Sm/S)^{\mathrm{op}}}$, called the projective structure. Furthermore (like any other functor category into $\mathcal{S}_*$), $\mathcal{M}_S$ has a smash product: if $X, Y \in \mathcal{M}_S$, then

$$(X \wedge Y)(U) = X(U) \wedge Y(U)$$

and an internal morphism object

$$\underline{\mathcal{M}}_S(X, Y) \in \mathcal{M}_S$$

right adjoint to the smash, and this satisfies the property corresponding to III.4.0.5, just that the morphism objects stays within $\mathcal{M}_S$ (they are *motivic* spaces, not just spaces). We say that $\mathcal{M}_S$ is a *monoidal model category.*

However, the projective structure is definitely not the structure we are interested in on $\mathcal{M}_S$. Firstly we have to take into account some topology, and secondly we will want the affine line to be contractible. This can be fixed as follows. The ideology is that we specify the fibrant objects as those objects having some desired property - at least up to homotopy - and model weak equivalences and cofibrations on them.

The topology we will consider here is the so-called *Nisnevich topology* (see [13] for this and other notions pertaining to algebraic geometry such as *smooth, étale and reduced*).

Definition 1.0.1 A space $X \in \mathcal{M}_S$ is *locally fibrant* if

1. $X(\emptyset) = *$,
2. for all $U \in Sm/S$ the pointed simplicial set $X(U)$ is fibrant, and
3. for all pullback squares
$$\begin{tikzcd} P \arrow[r] \arrow[d] & Y \arrow[d, "\phi"] \\ U \arrow[r, "i"] & Z \end{tikzcd} \quad \in Sm/S$$
where ϕ is étale, i is an open imbedding and $\phi(Z-U) \to Z-U$ is an isomorphism in the reduced structure, the induced square
$$\begin{tikzcd} X(P) & X(Y) \arrow[l] \\ X(U) \arrow[u] & X(Z) \arrow[l] \arrow[u] \end{tikzcd} \quad \in \mathcal{S}_*$$
is a homotopy pullback square (see III.1.2.3).

So, X is locally fibrant if it is a "Nisnevich sheaf up to homotopy".

For any $X \in \mathcal{M}_S$ let $* \rightarrowtail KX \overset{\sim}{\twoheadrightarrow} X$ be a functorial factorization (in the projective structure) of the canonical map $* \to X \in \mathcal{M}_S$.

Definition 1.0.2 A map $X \to Y \in \mathcal{M}_S$ is a *local equivalence* if for all locally fibrant Z the map
$$\underline{\mathcal{M}}_S(KY, Z) \to \underline{\mathcal{M}}_S(KX, Z)$$
is a pointwise equivalence. A map is a *local cofibration* if it is a projective cofibration, and a *local fibration* if it has the right lifting property with respect to to the local trivial cofibrations.

Note 1.0.3 The local structure is a model category, and on locally fibrant objects local weak equivalences are detected pointwise.

1.1 The $\mathbf{A}^1$-structure

In algebraic geometry the affine line $\mathbf{A}^1$ plays the rôle of the unit interval. We want the unit interval to be contractible, and again we let the fibrant objects be the "good ones".

Definition 1.1.1 An $\mathbf{A}^1$*-fibrant* object is a locally fibrant object $X \in \mathcal{M}_S$ such that
$$X(\mathbf{A}^1 \times_S -) \to X$$
is a pointwise weak equivalence.

An $\mathbf{A}^1$*-equivalence* is a map $X \to Y \in \mathcal{M}_S$ such that for all $\mathbf{A}^1$-fibrant Z the map
$$\underline{\mathcal{M}}_S(KY, Z) \to \underline{\mathcal{M}}_S(KX, Z)$$
is a pointwise weak equivalence (where K is the cofibrant replacement functor in the pointwise structure).

A map is an $\mathbf{A}^1$*-cofibration* if it is a local cofibration (i.e., a projective cofibration) and an $\mathbf{A}^1$*-fibration* if it has the right lifting property with respect to the trivial $\mathbf{A}^1$-cofibrations.

This is called the $\mathbf{A}^1$*-structure* on $\mathcal{M}_S$ and

Theorem 1.1.2 *The $\mathbf{A}^1$-structure on $\mathcal{M}_S$ is a model category whose homotopy category is equivalent to the "unstable homotopy category" of Voevodsky's lecture.*

We will refer to $\mathcal{M}_S$ with the $\mathbf{A}^1$-structure as "the category of spaces" where "pointed" and "motivic" may be inserted if clarity demands it.

2 Motivic functors

2.1 Two questions

1. How can one study (co)homology theories in $\mathcal{M}_S$ (with multiplicative structures) such as motivic cohomology?
2. More generally: How does one study $\mathcal{M}_S$-valued "continuous" and $\mathbf{A}^1$-homotopy invariant functors?

Remark 2.1.1 *1. The relevant multiplicative structure comes from the (pointwise) smash. Motivic homology theories must be stable under smashing with the Tate object*

$$T = S^1 \wedge \mathbf{G}_m.$$

where $\mathbf{G}_m$ is $A^1 - \{0\}$ pointed at 1.

2. Homology theories should commute with filtered colimits. We only define them on the category $f\mathcal{M}_S \subseteq \mathcal{M}_S$ of finite spaces.

A common answer to the above questions may be found by studying the category

$$\mathcal{F}_S$$

of "continuous" functors $X\colon f\mathcal{M}_S \to \mathcal{M}_S$. Continuous means that X induces a map of morphism spaces $\underline{\mathcal{M}}_S(v,w) \to \underline{\mathcal{M}}_S(X(v),X(w)) \in \mathcal{M}_S$ (not just on the underlying sets). Such functors are referred to by the category theorists as "enriched functors". You have seen before how this is useful when we discussed the Eilenberg-Mac Lane spectrum, and got a map $X \wedge \tilde{\mathbf{Z}}[Y] \to \tilde{\mathbf{Z}}[X \wedge Y]$. This is done by noting that $\tilde{\mathbf{Z}}$ - when considered as a functor on $\mathcal{S}_*$ - induces maps on function spaces, so that you get a chain

$$\begin{aligned}\mathcal{S}_*(X\wedge Y, X\wedge Y) &\cong \mathcal{S}_*(X, \underline{\mathcal{S}_*}(Y, X\wedge Y))\\ &\to \mathcal{S}_*(X, \underline{\mathcal{S}_*}(\tilde{\mathbf{Z}}[Y], \tilde{\mathbf{Z}}[X\wedge Y])) \cong \mathcal{S}_*(X\wedge \tilde{\mathbf{Z}}[Y], \tilde{\mathbf{Z}}[X\wedge Y]).\end{aligned}$$

So if you start with the identity $X \wedge Y = X \wedge Y$ you end up with the desired map. This is the underlying idea for the connection between continuous functors X and spectra: you have maps $S^1 \wedge X(S^n) \to X(S^{n+1})$, and so $n \mapsto X(S^n)$ defines a spectrum.

2.2 Algebraic structure

A successful study of homology theories and homotopy functors must take care of multiplicative properties. This is provided to us by

Theorem 2.2.1 *(Day) The category $\mathcal{F}_S$ has an internal morphism object and an adjoint associative, commutative and unital smash. The smash of X and Y is built as a filler (up to natural transformation) in*

$$\begin{tikzcd} f\mathcal{M}_S \times f\mathcal{M}_S \arrow[r, "(u,v)\mapsto X(u)\wedge Y(v)"] \arrow[d, "(u,v)\mapsto u\wedge v"'] & \mathcal{M}_S \\ f\mathcal{M}_S \arrow[ur, dashed, "v\mapsto (X\wedge Y)(v)"'] & \end{tikzcd}.$$

Day would say that $\mathcal{F}_S$ is a "closed symmetric monoidal category" (and he'd be quite right: look it up in your Mac Lane, where you can check up on Kan extensions while you're at it).

Although this definition is rather abstract, in the situations you actually need the smash product it is easy to interpret: the set of maps $X \wedge Y \to Z$ is in one-to-one correspondence with the set of maps $X(u) \wedge Y(v) \to Z(u \wedge v)$ that are natural in u, v. Compare this with the interpretation of the tensor product in terms of bilinear maps.

We will use this to build our algebra.

Example 2.2.2 Examples of algebras in $\mathcal{F}_S$.

1. **The motivic sphere spectrum.** This is simply the inclusion

$$\mathbf{S}\colon f\mathcal{M}_S \to \mathcal{M}_S.$$

The multiplication comes in form of an isomorphism $\mathbf{S} \wedge \mathbf{S} \cong \mathbf{S}$. As a matter of fact, $\mathbf{S}$ is the initial algebra (just like the integers is the initial ring).

2. **The motivic Eilenberg-Mac Lane spectrum.** Recall the category Cor_S of correspondences. Its objects are the same as Sm/S, but

$$Cor_S(U,V) = \mathbf{Z}\{\text{closed irreducible } C \subseteq U \times V \text{ finite and surjective over a component of } U\}.$$

Let Tr_S be the category of (linear) presheaves $Cor_S^{\mathrm{op}} \to \mathcal{A} = s\mathcal{A}b$. The graph induces a functor

$$Sm/S \to Cor_S,$$

and also a functor $u\colon Tr_S \to \mathcal{M}_S$. This last functor has a left adjoint

$$\gamma\colon \mathcal{M}_S \to Tr_S$$

and γ sends the smash in $\mathcal{M}_S$ to a slightly more complicated tensor which resides in Tr_S.

We define

$$M\mathbf{Z}\colon f\mathcal{M}_S \to \mathcal{M}_S$$

to be the composite

$$f\mathcal{M}_S \subseteq \mathcal{M}_S \xrightarrow{\gamma} Tr_S \xrightarrow{u} \mathcal{M}_S.$$

The unit map $\mathbf{S} \to M\mathbf{Z}$ is induced by the unit of adjunction $1 \to u\gamma$, and the multiplication $M\mathbf{Z} \wedge M\mathbf{Z} \to M\mathbf{Z}$ comes from the maps $u\gamma(v \wedge w) \cong u(\gamma v \otimes \gamma w) \to u\gamma(v)u\gamma(w)$ which the reader may think of as beefed-up versions of the corresponding maps for the free-forgetful pair between abelian groups and sets.

2.3 Wanted

Model structures on $\mathcal{F}_S = \{\text{continuous functors } f\mathcal{M}_S \to \mathcal{M}_S\}$ that lift to structures on the categories of algebras and their modules.

It should be noted that such a structure is at present needed just in order to state that there are corresponding algebraic objects on the homotopy categories.

Really, we want one structure to account for homotopy functors among motivic spaces, and one to model the stable situation.

Fact This exists and the stable situation has a homotopy category which is equivalent to the "stable category" in Voevodsky's talk. This is the second theme of the last section.

3 Model structures of motivic functors and relation to spectra

In the last section we defined the category of motivic spaces as the category of simplicial presheaves $Sm/S^{\mathrm{op}} \to \mathcal{S}_*$ together with a suitable model structure for which the fibrant objects

1. are Nisnevich sheaves up to homotopy,
2. do not see the difference between $\mathbf{A}^1$ and a point.

In this section we will study two structures on the category $\mathcal{F}_S$ of continuous functors $f\mathcal{M}_S \to \mathcal{M}_S$ from finite spaces to spaces. The first one will have fibrant objects the functors that are blind to the difference between $\mathbf{A}^1$-equivalent spaces, and in the second one they are the functors that think smashing with the Tate object 2.1.1.1 should be an invertible operation. Detail can be found in [3].

3.1 The homotopy functor model structure.

In practice, one is interested in homotopy functors, i.e., motivic functors that preserve $\mathbf{A}^1$-equivalences. However, the category of homotopy functors is very badly behaved. The trick is to study all motivic functors, but with a model category focusing on the homotopy functors.

Similar to the case for other functor categories (like the projective structure for motivic spaces), there is a model structure - called the *pointwise structure* - on $\mathcal{F}_S$ in which a map $X \to Y$ is a weak equivalence (resp. fibration) if for every $v \in f\mathcal{M}_S$ the map $X(v) \to Y(v) \in \mathcal{M}_S$ is an $\mathbf{A}^1$-equivalence (resp. fibration), and where the cofibrations are the maps having the usual left lifting property.

Let $* \rightarrowtail KX \twoheadrightarrow X$ be the functorial factorization of $* \to X$ in the pointwise structure.

Definition 3.1.1 An *ht-fibrant* object is a pointwise fibrant object $X \in \mathcal{F}_S$ such that for all $\mathbf{A}^1$-equivalences

$$\phi\colon v \xrightarrow{\sim} w \in f\mathcal{M}_S$$

the induced map

$$X(v) \to X(w) \in \mathcal{M}_S$$

is an $\mathbf{A}^1$-equivalence.

An *ht-equivalence* is a map $X \to Y \in \mathbf{F}_S$ such that for all ht-fibrant $Z \in \mathbf{F}_S$ the induced map

$$\underline{\mathcal{F}}_S(KY, Z) \to \underline{\mathcal{F}}_S(KX, Z)$$

is a pointwise $\mathbf{A}^1$-equivalence.

Being an *ht-cofibration* is the same as being a cofibration in the pointwise structure, and an *ht-fibration* is defined by the right lifting property.

Theorem 3.1.2 *The ht-structure is a model category structure on $\mathcal{F}_S$. This model category lifts to model structures on algebras and modules in $\mathcal{F}_S$.*

Note 3.1.3 In this structure a map between homotopy functors is an ht-equivalence if and only if it is a pointwise equivalence, and so the homotopy category of the ht-structure is the same as that of the pointwise when restricting to the homotopy functors. Furthermore, any motivic functor is ht-equivalent to a homotopy functor, and the ht-structure gives a functorial replacement making a motivic functor into a homotopy functor.

We may alternatively characterize the ht-fibrations as the pointwise fibrations $X \to Y$ such that for all $\mathbf{A}^1$-equivalences $v \xrightarrow{\sim} w \in f\mathcal{M}_S$ the square

$$\begin{array}{ccc} X(v) & \longrightarrow & Y(v) \\ \downarrow & & \downarrow \\ X(w) & \longrightarrow & Y(w) \end{array}$$

is a homotopy pullback.

3.2 Motivic spectra

Recall the Tate object $T = S^1 \wedge \mathbf{G}_m$ 2.1.1.1. Motivic cohomology, algebraic K-theory and so on are stable with respect to both S^1 and $\mathbf{G}_m$, and are represented by T-spectra. Naïvely one can model T-spectra in the same way as we did for spectra in spaces in III.5.

Definition 3.2.1 A T-spectrum is a sequence

$$E^0, E^1, \cdots \in \mathcal{M}_S = \mathcal{S}_*^{Sm/S^{\mathrm{op}}}$$

together with structure maps

$$T \wedge E^n \to E^{n+1}.$$

T-spectra are also referred to as *motivic spectra.*

As in the simplicial set case, these form a category $\mathcal{S}pt_S$ that comes equipped with a model structure whose homotopy category is equivalent to the "motivic stable homotopy category" of Voevodsky's talk [22]. The nicest way (following Hovey [10]) to describe the stable equivalences is the following: construct a "stably fibrant replacement functor", along the lines of the Q in 5.2 (makes the spectrum into an "Ω-spectrum"), and declare that a map $X \to Y$ of spectra is a *stable equivalence* if $(QX)^n \to (QY)^n$ is an $\mathbf{A}^1$-equivalence for each n. Similarly, stable fibrations are defined, giving rise to the *stable structure.*

In search for a deeper structure on this category we may mimic one of the ways of of constructing such structures for ordinary spaces. Among approaches we may mention symmetric spectra [11], S-modules [5] and simplicial functors [15]. Since I personally like the last approach best, I will outline how this goes.

3.3 The connection $\mathcal{F}_S \to \mathcal{S}pt_S$.

Let $T^k = T \wedge T^{k-1}$ and $T^0 = S^0$ (the constant two-point functor: it is the unit of the $\wedge$-structure on $\mathcal{M}_S$). For any continuous $X \colon f\mathcal{M}_S \to \mathcal{M}_S$ let the *evaluation* $evX \in \mathcal{S}pt_S$ be the spectrum with n-th term $X(T^n)$. The before-mentioned map $T \wedge X(T^n) \to X(T^{n+1})$ coming from the continuity of X ensures that evX actually is a spectrum. Obviously $ev \colon \mathcal{F}_S \to \mathcal{S}pt_S$ preserves both limits and colimits.

The idea is to transport the stable equivalences from $\mathcal{S}pt_S$ to $\mathcal{F}_S$, and this almost forces the stable structure on $\mathcal{F}_S$. More precisely, to check whether $X \to Y$ is a *stable equivalence* one first uses the ht-structure to replace X and Y by homotopy functors, and then asks whether the induced map of T-spectra is a stable equivalence.

Call the left adjoint of ev

$$F \colon \mathcal{S}pt_S \to \mathcal{F}_S.$$

Theorem 3.3.1 *There is a model category structure, called the stable structure, on $\mathcal{F}_S$ such that*

$$F \colon \mathcal{S}pt_S \to \mathcal{F}_S$$

is a Quillen equivalence.

Furthermore, the smash in $\mathcal{F}_S$ gives a smash in the stable homotopy category, agreeing with that discussed in the other talks. The stable model structure induces model category structures on algebras and modules over $\mathcal{F}_S$.

In algebraic topology there is an interesting interpretation of the stable structure in terms of linear functors in the sense of Goodwillie calculus. This connection is not as straight-forward in the motivic world. In [3] what we here have called the stable structure is called the spherewise structure.

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Index