Search problems on 1-skeletons of regular polyhedrons

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Abstract

For pursuit-evasion games played on 1-skeletons of regular polyhedrons some problems of finding the minimum number of pursuers needed for the capture (in various senses) of the mobile invisible evader are considered. The results obtained for these graphs are consequences of fundamental theorems in the theory of graph searching developed by authors in [1, 2, 3].

1 Introduction and statement of the problem

We study the following problem of guaranteed search. Let a pursuit-evasion game be played on a topological graph (embedded in an Euclidean space) where a team of pursuers tries to catch an evader who is "invisible" for them, all players making use of "simple motions" with fixed maximal speeds and the topological graph being a phase restraint for their trajectories. The problem is to find the minimum number of pursuers (searchers) needed to catch the evader (the fugitive) under conditions mentioned above.

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Initially, problems of guaranteed search on graphs were stated by Parsons in [4] and by Petrov in [5]. Since then such problems attracted the attention of many researchers because of their connections with different seemingly unrelated topics. Mention just some of them:

- linear graph layouts [6]
- problems of the fight against damage spread in complex systems, for instance, spread of the mobile computer virus in networks [7]
- "pebbling" games [8]
- the theory of graphs minors [9]
- problems of privacy in distributed systems [10].

See surveys [7] and [11] for references.

Further we use the word *graph* to denote a finite connected topological graph, embedded in $\mathbf{R}^3$. For simplicity, we shall assume that edges of a graph are polygonal lines.

n pursuers $P_1, \ldots, P_n$ and the evader E are on a graph Γ with the vertex set $V\Gamma$ and the edge set $E\Gamma$. We assume that in this space both pursuers and the evader possess simple motions:

$$\begin{array}{lll} (P_i): & \dot{x}_i = u_i, \quad \|u_i\| \leq 1, & i \in \overline{1,n}, \\ (E): & \dot{y} = u_0, \quad \|u_0\| \leq \mu. & \end{array}$$

($\| \cdots \|$ is the Euclidean norm.) Let us suppose that admissible controls u_i and u_0 are piecewise constant functions defined on arbitrary segments $[0, T]$. Also we presume that Γ is a phase restraint for all players. So admissible trajectories are piecewise affine vector functions with values in Γ.

Let ρ be the inner metric of Γ, *i.e.* $\rho(x, y)$ is the Euclidean length of the shortest path in Γ with ends in x and y. Denote by ε a nonnegative number characterizing the "radius of capture". The evader E is *caught* by a pursuer P_i at a moment $t \in [0, T]$, if $\rho(x_i(t), y(t)) \leq \varepsilon$. A family of trajectories $x_i\colon [0, T] \to \Gamma$, $i \in \overline{1,n}$ is called a pursuers' *program* on $[0, T]$. A program of n pursuers $(x_1, \ldots, x_n)$ on $[0, T]$ is said to be *winning*, iff for any trajectory of the evader $y\colon [0, T] \to \Gamma$, there exist $t \in [0, T]$ and $i \in \overline{1,n}$, such that $\rho(x_i(t), y(t)) \leq \varepsilon$. The problem is to determine the minimal number of pursuers that have a winning program. It is clear this number depends only on the graph Γ, μ and ε. Denote this number by $S^{\varepsilon}_{\mu}(\Gamma)$.

Note that this search problem can be interpreted as a problem of clearing the graph from a "diffused" evader. We say that in program $\Pi(x_1(t), \ldots, x_n(t))$, $t \in [0, T]$, point $x \in \Gamma$ is *contaminated* at a moment $t^* \in [0, T]$, if there exists a trajectory of the evader $y(t)$, $t \in [0, t^*]$, such that $y(t^*) = x$, and for any $i \in \overline{1, n}$ and $t \in [0, t^*]$, $\rho(y(t), x_i(t)) > \varepsilon$. Let us denote by $CONT(\Pi, \Gamma, t^*)$ the set of points of graph Γ contaminated at a moment t^*. $\Gamma \backslash CONT(\Pi, \Gamma, t^*)$ is called by the *cleared* set at the moment t^*. Thus we can state that program $\Pi(x_1(t), \ldots, x_n(t))$, $t \in [0, T]$ is winning if and only if for some $t^* \in [0, T]$ the whole graph Γ is cleared at the moment t^*.

The case of $\mu = +\infty$ (the evader can move arbitrarily fast) and $\varepsilon = 0$ was initially considered by Parsons in [4]. Note that in this case the smallest number of searchers having a winning program on a graph Γ is independent of lengths of edges of Γ. The case of $\mu = +\infty$ and $\varepsilon \geq 0$ was studied by Golovach in [2] and the case of $\mu \geq 0$ and $\varepsilon = 0$ by Petrov in [5]. The problems of Golovach and Petrov can be regarded as natural generalizations of the "classical" Parsons' problem. Solutions of these problems can be found only in exceptional cases.

One of the basic tools of studying the "classical" Parsons' problem is the theorem of LaPaugh [12] which asserts that in the case of $\mu = +\infty$ and $\varepsilon = 0$ the "recontamination" does not help to search a graph, *i.e.,* if a team of pursuers can catch the evader then they can do it in a "monotone" fashion. In more general problems this principle does not work and we cannot apply standard technique. Hereby, in order to solve such problems we have to find other "tools".

Previously [1, 2, 3, 13] we proved a number of fundamental results on graph searching. The main purpose of this paper is to demonstrate how these results can be used in studying model problems.

In this paper we denote by T, C and O the graphs consisting of all edges and vertices of Tetrahedron, Cube and Octahedron respectively. We shall assume that edges of graphs T, C and O are one unit long.

The rest of the paper is organized as follows: In Section 2 we find numbers $S^{\varepsilon}_{\infty}(T)$, $S^{\varepsilon}_{\infty}(C)$ and $S^{\varepsilon}_{\infty}(O)$ for all $\varepsilon \geq 0$. In Section 3 we compute numbers $S^{0}_{\mu}(T)$, $S^{0}_{\mu}(C)$ and $S^{0}_{\mu}(O)$ for $\mu \geq 1$. Also we show that $S^{0}_{\mu}(T) \leq 2$ for $\mu \leq 1/3$, $S^{0}_{\mu}(C) \leq 3$ for $\mu < 0.5$ and $S^{0}_{\mu}(C) \leq 2$ for $\mu \leq 0.2$.

We conclude with section 4 which contains some open problems.

2 The case of $\mu = \infty$, $\varepsilon \geq 0$

Let us state without proofs some known facts which we will need in the sequel.

The following statement was proven in [13].

Theorem 2.1 *For any graph Γ with the minimal vertex degree $\delta \geq 3$, $S^0_\infty(\Gamma) \geq \delta + 1$.*

Intuition suggests that if ε is small then $S^\varepsilon_\infty(\Gamma)$ is close to $S^0_\infty(\Gamma)$. The following theorem (see [2] for the proof) confirms this suggestion and provides us with a useful tool.

Theorem 2.2 *Let l be the length of the shortest edge in a graph Γ. Then*

1. $S^\varepsilon_\infty(\Gamma) = S^0_\infty(\Gamma)$ *for any* $0 \leq \varepsilon < 0.25l$;
2. $S^\varepsilon_\infty(\Gamma) \geq S^0_\infty(\Gamma) - 1$ *for any* $0.25l \leq \varepsilon < 0.5l$.

Now we are ready to prove the following statement.

Theorem 2.3

$$S^\varepsilon_\infty(T) = \begin{cases} 4, & \textit{if } 0 \leq \varepsilon < 0.25, \\ 3, & \textit{if } 0.25 \leq \varepsilon < 0.5, \\ 2, & \textit{if } 0.5 \leq \varepsilon < 1.5, \\ 1, & \textit{if } 1.5 \leq \varepsilon. \end{cases}$$

Proof. Note that in all cases the construction of the corresponding winning program is easy: four pursuers can catch the evader for $\varepsilon \geq 0$, three for $\varepsilon \geq 0.25$, two for $\varepsilon \geq 0.5$ and one pursuer can win for $\varepsilon \geq 1.5$.

For $0 \leq \varepsilon < 0.5$ the answer is given by theorems 2.1 and 2.2. The case $\varepsilon \geq 0.5$ is trivial. □

In order to solve search problems for C and O we need stronger results.

A linear ordering of a graph Γ is a one-to-one mapping $f\colon V\Gamma \to \{1, \ldots, |V\Gamma|\}$.

Let f be a linear ordering of a graph G. We put

$$cw_i(\Gamma, f) = |\{(u, v) \in E\Gamma\colon f(u) \leq i, f(v) > i\}|.$$

The cutwidth of Γ *with respect to an ordering* f is

$$cw(\Gamma, f) = \max_{i \in \overline{1, |V\Gamma|}} cw_i(\Gamma, f)$$

and *the cutwidth* of Γ (we denote it by $cw(\Gamma)$) is the minimum cutwidth over all linear orderings of Γ.

Makedon and Sudborough in [6] proved

Theorem 2.4 *For any graph Γ with the maximal degree ≤ 3, $S^0_\infty(\Gamma) = cw(\Gamma)$.*

Let Γ be a graph. Denote by $\Gamma'(v)$ the subgraph of Γ induced by the set of all vertices adjacent to v. Let $n_k(\Gamma, v)$ be the number of connected components of $\Gamma'(v)$ containing precisely k vertices.

Let us define

$$c(\Gamma, v) \triangleq \sum_{k=1}^{\infty} n_k(\Gamma, v) \left\lceil \frac{k}{2} \right\rceil,$$

and

$$c(\Gamma) \triangleq \min_{v \in V\Gamma} c(\Gamma, v).$$

The following theorem was proven in [3].

Theorem 2.5 *Let l be the length of the shortest edge in a graph Γ and let $0 \leq \varepsilon < l$. Then:*

1. *$S^\varepsilon_\infty(\Gamma) \geq c(\Gamma)$*
2. *if for any vertex $v \in V\Gamma$ all coefficients $n_k(\Gamma, v)$ with odd k are zeroes, then $S^\varepsilon_\infty(\Gamma) \geq c(\Gamma) + 1$.*

Now we consider 1-skeletons of Cube and Octahedron.

Theorem 2.6

$$S^\varepsilon_\infty(C) = \begin{cases} 5, & \text{if } 0 \leq \varepsilon < 0.25, \\ 4, & \text{if } 0.25 \leq \varepsilon < 0.5, \\ 3, & \text{if } 0.5 \leq \varepsilon < 1, \\ 2, & \text{if } 1 \leq \varepsilon < 3, \\ 1, & \text{if } 3 \leq \varepsilon. \end{cases}$$

Proof. Note that in all cases the construction of the corresponding winning program is easy.

Since any vertex of C has degree three we can conclude (theorem 2.4) that $cw(C) = S^0_\infty(C)$. An ordering f of C, $cw(C, f) = 5$ is shown on Figure 1 (we put here $f(u_i) = i$). Note that for any ordering f, $cw(C, f) \geq 5$. This inequality is true because any subgraph of C induced by five vertices contains at most five edges.

Thus lower bounds follow: for $0 \leq \varepsilon < 0.5$ from theorem 2.2, for $0.5 \leq \varepsilon < 1$ from theorem 2.5. The case $1 \leq \varepsilon < 3$ is trivial. □

Theorem 2.7

$$S_{\infty}^{\varepsilon}(O) = \begin{cases} 5, & \text{if } 0 \leq \varepsilon < 0.25, \\ 4, & \text{if } 0.25 \leq \varepsilon < 0.5, \\ 3, & \text{if } 0.5 \leq \varepsilon < 1, \\ 2, & \text{if } 1 \leq \varepsilon < 2, \\ 1, & \text{if } 2 \leq \varepsilon. \end{cases}$$

Proof. Theorem 2.1 implies that $S_{\infty}^{0}(O) \geq 5$. The rest of the proof is as in theorem 2.6.□

3 The case of $\mu \geq 0$, $\varepsilon = 0$

Let Γ be a graph with edges one unit long. We say that a program $\Pi(x_1(t), \ldots, x_n(t))$, $t \in [0, T]$, $T \in \mathbf{N}$, on Γ is *discrete*, if for any $k \in \overline{1, T}$ during the time $[k-1, k]$ every pursuer either stays in a vertex, or moves with the unit speed from vertex to vertex, *i.e.,* for any $i \in \overline{1, n}$ either $x_i(t) = v \in V\Gamma$, $t \in [k-1, k]$, or $x_i(k-1)$, $x_i(k) \in V\Gamma$, $x_i(k-1) \neq x_i(k)$ and $x_i(t) \notin V\Gamma$, $t \in (k-1, k)$.

The following theorem was proven in [1].

Theorem 3.1 *Let Γ be a graph with edges one unit long. The following statements are equivalent:*

1. *n pursuers have a winning program on Γ for $\mu = 1$, $\varepsilon = 0$.*

2. *n pursuers have a discrete winning program on Γ for $\mu = 1$, $\varepsilon = 0$.*

The following notions will be used in proofs of the next theorems.

Let Γ be a graph from the set $\{C, T, O\}$ and let Π be a discrete program on Γ, $\mu = 1$. It is easy to prove that if at a moment $k \in \mathbf{N}$ a point of an edge $e \in E\Gamma$ is contaminated then all points of e are contaminated at the moment k. So at any moment $k \in \mathbf{N}$ every edge of Γ is either cleared or contaminated. Denote the minimal number of pursuers which stay at i in a vertex v through $p_i(v)$. We say that v is a *special* vertex at the moment i if $p_i(v)$ is more than or equal to the number of contaminated edges incident to v. Denote the set of all vertices special at a moment i by A_i and the cardinality of A_i by a_i. We say that vertex v is k-*special* at the moment i if $v \in A_i$ and $p_i(v) = k$. We say that vertex $u \in A_k$ is an *old* one if $u \in A_{k-1}$ and that it is a *new* one otherwise. Vertex v is *bare* at i if all edges incident to v are contaminated. Denote by B_i the set of all bare vertices at the moment i and by b_i its cardinality.

Let v be a vertex of T, C or O. It is easy to verify that for any discrete program Π on T, C or O for $\mu = 1$ the following lemmas hold.

Lemma 3.1 *If $v \notin A_i$ and $p_{i+1}(v) = 0$ then $v \in B_{i+1}$.*

Lemma 3.2 *If v is a new special vertex at the moment $i+1$ then $p_{i+1}(v) \geq 2$.*

Theorem 3.2

(T1) $S^0_\mu(T) = 4 \Leftrightarrow \mu \geq 1$.

(T2) $S^0_\mu(T) \leq 2$ *for* $\mu \leq 1/3$.

Proof.
(T1). $\Leftarrow$: Let Π be an arbitrary program of three pursuers for $\mu = 1$ on T. Referring to theorem 3.1 we suppose that Π is a discrete one. For convenience, we think that at the initial moment all pursuers are in one vertex which is the only special vertex at this moment. We shall show that for any i, $a_i \leq 1$, which implies that Π is not a winning program.

Suppose that at some moment the number of special vertices is more than one and let $j \geq 1$ be the first such a moment. Thus $a_j \geq 2$. Lemma 3.2 implies that at moment j only one new special vertex appears, hence $a_{j-1} = 1$ and $a_j = 2$. It means that one vertex $u \in A_j$ is old and another vertex $v \in A_j$ is new.

By lemma 3.2, $p_j(v) \geq 2$, hence $p_j(u) \leq 1$. If $p_j(u) = 1$ then by lemma 3.1 $b_j = 2$ and u is not special. If $p_j(u) = 0$ then by the same lemma $b_j \geq 1$ and again u is not special.

Thus $S^0_\mu(T) > 3$ for $\mu \geq 1$ an by theorem 2.3 $S^0_\mu(T) = 4$ for $\mu \geq 1$.

⇒: Let us describe a winning program of three pursuers for $\mu < 1$. Let A, B, C, D be vertices of T. Denote by L the cycle (A, B, D, C, A). The pursuer P_1 is moving in one direction with the maximal speed on cycle L. Pursuer P_2 (P_3) moves on edge $[A, D]$ ($[B, C]$). Pursuers' actions are synchronized as follows: P_1 meets P_2 in A and D; P_1 meets P_3 in B and C.

If the evader does not leave L then in time $4(1-\mu)^{-1}$ he will be caught. On the other hand, taking into account actions of P_2 and P_3, it is easy to see that for the evader there is "no use" to leave L.

(T2) Denote by A, B, C, D the vertices of T. P_1 and P_2 are moving with the unit speed as follows:

$$\begin{array}{ll} P_1: & A \to C \to D \to A \to A \to C \to D \to B \to D \to C \to A \to D \\ P_2: & B \underset{1}{\to} C \underset{2}{\to} D \underset{3}{\to} B \underset{4}{\to} A \underset{5}{\to} B \underset{6}{\to} B \underset{7}{\to} B \underset{8}{\to} C \underset{9}{\to} C \underset{10}{\to} A \underset{11}{\to} B. \end{array}$$

Here $\xi \underset{i}{\to} \eta$ means that a pursuer during the time $[i-1, i]$ moves from ξ to η (or stays if $\xi = \eta$).

The proof that this program is winning for $\mu = 1/3$ is easy and left to the reader. □

Theorem 3.3 *$S^0_\mu(O) = 5 \Leftrightarrow \mu \geq 1$.*

Proof.

⇐: Let Π be an arbitrary discrete program of four pursuers on O. We shall prove that for $\mu = 1$ this program is not winning.

Since every vertex of octahedron is adjacent to all the rest vertices but one it is easy to prove the following lemma.

Lemma 3.3 *If $b_i \geq l$ then $p_i(v) \geq l - 1$ for any vertex $v \in A_i$.*

Let us show that for program Π all numbers $a_i \leq 2$ which implies that this program is not winning.

Obviously $a_0 \leq 1$. We prove that $a_{i-1} \leq 2$ implies $a_i \leq 2$. Note that by lemma 3.2 the number of new special vertices at any moment i is at most two. Hence, if $a_{i-1} = 0$ then $a_i \leq 2$.

Suppose that $a_{i-1} = 1$ and $a_i = 3$, *i.e.,* one special vertex u_1 is old and two special vertices u_2, u_3 are new. According to lemma 3.2 vertices u_2, u_3 at the moment i are 2-special; hence u_1 is 0-special. The latter implies $b_i \leq 1$.

On the other hand, by lemma 3.1 $b_i = 3$. We proved that if $a_{i-1} = 1$ then $a_i \leq 2$.

Let us consider the last possible case $a_{i-1} = 2$. First, we prove the following

Lemma 3.4 *If for some k, $a_{k-1} < 2$ and $a_k = 2$ then $a_{k+1} < 2$.*

Proof of lemma 3.4. If $a_{k-1} = 0$ then at the moment k two vertices are new 2-special; whereas lemma 3.1 implies that other vertices are bare. It is not difficult to see that such a situation is impossible and for the proof of the lemma it is sufficient to consider the case $a_{k-1} = 1$. There are two possibilities in this case:

either 1) from two vertices special at the moment k one is old and one is new,

or 2) from two vertices special at the moment k both are new.

Let us show that the first possibility cannot be realized. Suppose the opposite: at the moment k one special vertex u_1 is old and one—u_2 is new. Since $p_k(u_2) \geq 2$ (lemma 3.2) then $b_k \geq 2$ (lemma 3.1). By lemma 3.3 $p_k(u_1) \geq 1$ and again by lemma 3.1 $b_k \geq 3$. Then from lemma 3.3 it follows that $p_k(u_1) \geq 2$; hence $p_k(u_1) = p_k(u_2) = 2$. Lemma 3.1 implies that in such a situation other vertices are bare, *i.e.,* $b_k = 4$ which is impossible by lemma 3.3.

So we proved that if $a_{k-1} = 1$ and $a_k = 2$ then there is a vertex $u_1 \in A_{k-1} \setminus A_k$ and two new special vertices u_2, u_3. By lemma 3.2 these vertices are 2-special and $b_k = 3$ (note that u_1 cannot be bare). Such a situation can happen only if u_2 and u_3 are adjacent and each of them is adjacent to two bare vertices. Since u_2 and u_3 are 2-special and $b_k = 3$ we deduce that edge (u_2, u_3) is cleared. Furthermore, it is easy to check that edges (u_2, u_1) and (u_3, u_1) are cleared too. Thus the cycle induced by vertices u_1, u_2, u_3 is cleared at the moment k and $p_k(u_1) = 0$, $p_k(u_2) = p_k(u_3) = 2$. Note that such a situation is possible. Now it is clear that any action of pursuers from such a position leads to the case $a_{k+1} < 2$. Lemma 3.4 is proven. □

Lemma 3.4 implies that if $a_{i-1} = 2$ then $a_{i-2} < 2$ and $a_i < 2$. Thus we conclude that for the program Π all $a_i \leq 2$; hence this program is not winning.

Thus $S^0_\mu(O) > 4$ for $\mu \geq 1$ and by theorem 2.7 $S^0_\mu(O) = 5$ for $\mu \geq 1$.

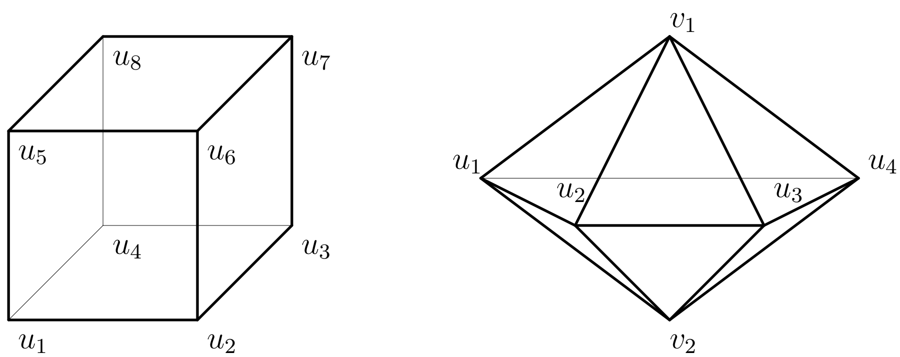

Figure 1: Cube and Octahedron

$\Rightarrow$: Let us describe a winning program of four pursuers for $\mu < 1$. Pursuer P_i, $i \in \overline{1,4}$, moves along the cycle L_i of the length three as follows (see Figure 1):

$$\begin{aligned} P_1: &\quad u_2 \to v_2 \to u_1 \to u_2 \to \dots \\ P_2: &\quad u_2 \to v_1 \to u_3 \to u_2 \to \dots \\ P_3: &\quad u_4 \to v_2 \to u_3 \to u_4 \to \dots \\ P_4: &\quad u_4 \to v_1 \to u_1 \to u_4 \to \dots \quad . \end{aligned}$$

If the evader does not leave a cycle L_i then in time $3\lceil(1-\mu)^{-1}\rceil$ he will be caught ($\mu < 1$) by pursuer P_i. On the other hand, changing cycles gives nothing to the evader. In fact, if the evader passes from L_i to L_j at a moment t (such transition can be realized only in a vertex of Octahedron) then at t he is proved to be "equally positioned" relatively pursuers P_i and P_j. $\square$

Theorem 3.4

(C1) $S^0_\mu(C) = 5 \Leftrightarrow \mu \geq 1$.

(C2) $S^0_\mu(C) \leq 3$ *for* $\mu < 0.5$.

(C3) $S^0_\mu(C) \leq 2$ *for* $\mu \leq 0.2$.

Proof.
(C1) $\Leftarrow$: Let us show that $S^0_1(C) = 5$. Let Π be an arbitrary discrete program of four pursuers. We shall show that the program Π is not winning for $\mu = 1$.

It is easy to prove the following lemma.

Lemma 3.5 *a) If there is one 0-special vertex at a moment i then $b_i \leq 4$.*

b) If there are two 0-special vertex at a moment i then $b_i \leq 2$.

Let us show that for Π all numbers $a_i \leq 3$ which implies that this program is not winning.

Let us suppose that at the initial moment all pursuers are in one vertex which is the only special vertex at this moment. Now we prove that $a_{i-1} \leq 3$ implies $a_i \leq 3$. Note that by lemma 3.2 there are at most two new special vertices at i. Thus if $a_{i-1} \leq 1$ then $a_i \leq 3$.

Suppose now that $a_{i-1} = 2$ but $a_i > 3$. Then $a_i = 4$, furthermore at the moment i there are two old special vertices u_1, u_2 and two new ones u_3, u_4. Since u_3, u_4 are 2-special at i then vertices u_1, u_2 are 0-special at this moment. Lemma 3.1 implies $b_i \leq 4$; on the other hand, $b_i \leq 2$ by lemma 3.5. Hence if $a_{i-1} = 2$ then $a_i \leq 3$.

In order to consider the case of $a_{i-1} = 3$ we prove the following lemma.

Lemma 3.6 *If for some $k \geq 1$, $a_{k-1} < 3$ and $a_k = 3$ then $a_{k+1} < 3$.*

The proof of lemma 3.6. If $a_{k-1} = 1$ then at the moment k the set of special vertices consists of one old vertex u_1 and two new vertices u_2, u_3. It follows from lemma 3.2 that u_2 and u_3 are 2-special at k; hence u_1 is 0-special at this moment. By virtue of lemma 3.1 $b_i \geq 5$ while lemma 3.5 implies $b_i \leq 4$. Thus $a_{k-1} = 2$.

In this case one of the following possibilities occurs:
either 1) from three vertices special at the moment k two are old and one is new,
or 2) from three vertices special at the moment k one is old and two are new.

Let us show that the first possibility cannot be realized. Suppose that at the moment k vertices u_1, u_2 are old special ones and vertex u_3 is new. By lemma 3.2 $p_k(u_1) + p_k(u_2) \leq 2$. In this situation only three cases can take place.

a) $p_k(u_1) = p_k(u_2) = 0$. By lemma 3.5 $b_k \leq 2$ while lemma 3.1 implies $b_k \geq 3$.

b) $p_k(u_1) = 0$, $p_k(u_2) = 2$. By lemma 3.1 $b_k \geq 5$ while from lemma 3.5 it follows that $b_k \leq 4$.

c) $p_k(u_1) = 0$, $p_k(u_2) = 1$. By virtue of lemmas 3.1 and 3.5 we have $b_k = 4$. Three vertices adjacent to u_1 and vertex u_1 itself cannot be bare.

Consequently, u_2 is adjacent to u_1. Then u_2 is incident to two contaminated (at the moment k) edges and thus is not special.

d) $p_k(u_1) = p_k(u_2) = 1$, $p_k(u_3) = 2$. In this case $b_k = 5$. At the moment k vertex u_1 is adjacent to at least two cleared edges whose ends are not in B_k. Hence u_1 is adjacent to u_2 and u_3. By the same reason u_2 is adjacent to u_1 and u_3. The latter implies that vertices u_1, u_2, u_3 induce a cycle of the length three in C which is impossible.

We proved that if $a_{k-1} = 2$ and $a_k = 3$ then the second possibility takes place, *i.e.*, there exist vertex $u_0 \in A_{k-1} \setminus A_k$, vertex $u_1 \in A_{k-1} \cap A_k$ and two new vertices u_2, $u_3 \in A_k$. From lemma 3.1 it follows that all vertices excluding u_0,u_1,u_2,u_3 are bare at the moment k and thus $b_k = 4$. Also vertices u_0,u_2,u_3 are adjacent to vertex u_1. It is easy to see that such a situation is possible.

Now it is clear that any actions of pursuers in such a position lead to the case $a_{k+1} < 3$. Lemma 3.6 is proven. □

Lemma 3.6 implies that if $a_{i-1} = 3$ then $a_{i-2} < 3$ and $a_i < 3$. We conclude that for program Π all numbers $a_i \leq 3$ and thus Π is not winning.

Thus $S^0_\mu(C) > 4$ for $\mu \geq 1$ and by theorem 2.6 $S^0_\mu(C) = 5$ for $\mu \geq 1$.

$\Rightarrow$: Let us demonstrate how four pursuers can catch the evader for $\mu < 1$. Pursuer P_i, $i \in \overline{1,4}$, moves along the cycle L_i of the length four as follows (see Figure 1):

$$\begin{aligned}
P_1 :&\quad u_1 \to u_5 \to u_6 \to u_2 \to u_1 \to \dots \\
P_2 :&\quad u_1 \to u_5 \to u_8 \to u_4 \to u_1 \to \dots \\
P_3 :&\quad u_3 \to u_7 \to u_6 \to u_2 \to u_3 \to \dots \\
P_4 :&\quad u_3 \to u_7 \to u_8 \to u_4 \to u_3 \to \dots
\end{aligned}$$

As in theorem 3.3 it is easy to prove that in time $4\lceil(1-\mu)^{-1}\rceil$ the evader will be caught.

(C2) Let u_i, $i \in \overline{1,8}$, be vertices of C as in Figure 1. Define the following pursuers' trajectories:

$$\begin{aligned}
P_1 :&\quad u_1 \to u_2 \to u_2 \to u_3 \to u_3 \to u_4 \to u_4 \to u_1 \to u_1, \\
P_2 :&\quad u_5 \to u_6 \to u_6 \to u_7 \to u_7 \to u_8 \to u_8 \to u_5 \to u_5, \\
P_3 :&\quad u_1 \to u_2 \to u_6 \to u_7 \to u_3 \to u_4 \to u_8 \to u_5 \to u_1.
\end{aligned}$$

Pursuers repeat such actions $(1-2\mu)^{-1}$ times. It is easy to prove that for $\mu < 0.5$ the program described above is winning. In fact, pursuers P_1 and P_2 force the evader to leave edges of cycles u_1, u_2, u_3, u_4, u_1 and u_5, u_6, u_7, u_7, u_5 and then the evader will be caught by P_3.
(C3) Let u_i, $i \in \overline{1,8}$, be vertices of C as in Figure 1. Define the following pursuers' trajectories:

$$\begin{array}{ll}
P_1: & u_6 \to u_7 \to u_8 \to u_5 \to u_6 \to u_6 \to u_6 \to u_2 \to u_3 \to u_7 \to u_3 \to u_2 \to u_6 \\
P_2: & u_3 \underset{1}{\to} u_7 \underset{2}{\to} u_8 \underset{3}{\to} u_4 \underset{4}{\to} u_3 \underset{5}{\to} u_7 \underset{6}{\to} u_6 \underset{7}{\to} u_5 \underset{8}{\to} u_8 \underset{9}{\to} u_4 \underset{10}{\to} u_3 \underset{11}{\to} u_4 \underset{12}{\to} u_8 \\
P_1: & u_6 \to u_5 \to u_1 \to u_2 \to u_6 \to u_2 \to u_3 \to u_7 \to u_8 \to u_4 \to u_1 \to u_2 \to u_6 \\
P_2: & u_8 \underset{13}{\to} u_5 \underset{14}{\to} u_1 \underset{15}{\to} u_4 \underset{16}{\to} u_8 \underset{17}{\to} u_4 \underset{18}{\to} u_3 \underset{19}{\to} u_2 \underset{20}{\to} u_1 \underset{21}{\to} u_4 \underset{22}{\to} u_3 \underset{23}{\to} u_2 \underset{24}{\to} u_6.
\end{array}$$

The proof that this program is winning for $\mu = 0.2$ is easy and left to the reader.

□

4 Open problems

In section 2 we found all numbers S^{ε}_{∞} for 1-skeletons of Tetrahedron, Octahedron and Cube. We do not know the complete solution of this problem for Icosahedron and Dodecahedron but suppose that the following assertions are true.

Conjecture 1 *Let $I(D)$ be 1-skeleton of Icosahedron (Dodecahedron). Then*

$$S^{\varepsilon}_{\infty}(I) = \begin{cases} 7, & \text{if } 0 \le \varepsilon < 0.25, \\ 6, & \text{if } 0.25 \le \varepsilon < 0.5, \\ 4, & \text{if } 0.5 \le \varepsilon < 1, \\ 3, & \text{if } 1 \le \varepsilon < 1.25, \\ 2, & \text{if } 1.25 \le \varepsilon < 3, \\ 1, & \text{if } 3 \le \varepsilon. \end{cases} \qquad S^{\varepsilon}_{\infty}(D) = \begin{cases} 7, & \text{if } 0 \le \varepsilon < 0.25, \\ 6, & \text{if } 0.25 \le \varepsilon < 0.5, \\ 4, & \text{if } 0.5 \le \varepsilon < 1.5, \\ 3, & \text{if } 1.5 \le \varepsilon < 2.25, \\ 2, & \text{if } 2.25 \le \varepsilon < 5, \\ 1, & \text{if } 5 \le \varepsilon. \end{cases}$$

In section 3 we proved that $S^0_{\mu}(T) = 4$ and $S^0_{\mu}(C) = 5$ if and only if $\mu \ge 1$. We constructed a winning program of two pursuers on T for $\mu \le 1/3$. Also we proved (by producing winning programs) that $S^0_{\mu}(C) = 2$ for $\mu \le 0.2$ and $S^0_{\mu}(C) \le 3$ for $\mu < 0.5$. We suppose that these bounds are sharp:

Conjecture 2

$$S^0_\mu(T) = \begin{cases} 4, & \textit{if } \mu \geq 1, \\ 3, & \textit{if } \ 1/3 < \mu < 1, \\ 2, & \textit{if } \ 0 < \mu \leq 1/3, \\ 1, & \textit{if } \ \mu = 0. \end{cases} \qquad S^0_\mu(C) = \begin{cases} 5, & \textit{if } \mu \geq 1, \\ 4, & \textit{if } \ 0.5 \leq \mu < 1, \\ 3, & \textit{if } \ 0.2 < \mu < 0.5, \\ 2, & \textit{if } \ 0 < \mu \leq 0.2, \\ 1, & \textit{if } \ \mu = 0. \end{cases}$$

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