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Pathwidth of Planar and Line Graphs*

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Abstract. We prove that for every 2-connected planar graph the pathwidth of its geometric dual is less than the pathwidth of its line graph. This implies that $\mathit{pathwidth}(H) \leq \mathit{pathwidth}(H^*) + 1$ for every planar triangulation H and leads us to a conjecture that $\mathit{pathwidth}(G) \leq \mathit{pathwidth}(G^*) + 1$ for every 2-connected graph G.

Key words. Pathwidth, Treewidth, Planar graphs, Line graphs

1. Definitions

We use the standard graph-theoretic terminology compatible with [4] where basic definitions may be found. We use the following notations: G is an undirected, simple (without loops and multiple edges) and finite graph with the vertex set $V(G)$ and the edge set $E(G)$; $\Delta(G)$ is the maximum degree of the vertices of G; $L(G)$ is the line graph of G. If G is a plane graph then G^* denotes its geometric dual.

A *tree decomposition* of a graph G is a pair $(T, \mathscr{X})$, where T is a tree and $\mathscr{X} = (X_i : i \in V(T))$ is a family of subsets of $V(G)$ indexed by $V(T)$ such that

(T1) $\bigcup_{i \in V(T)} X_i = V(G)$;
(T2) for every edge $\{u, v\} \in E(G)$ there is $i \in V(T)$ such that $u, v \in X_i$;
(T3) for every $i, j, k \in V(T)$ if j is on the path between i and k then $X_i \cap X_k \subseteq X_j$.

A *path decomposition* of G is a tree decomposition $(T, \mathscr{X})$ where T is a path.

The *width* of a decomposition $(T, \mathscr{X})$ is $\max_{i \in V(T)} |X_i| - 1$. Robertson and Seymour [18] define the *treewidth* tw(G) (the *pathwidth* pw(G)) of G is the minimum width over all tree decompositions (path decompositions) of G.

In this paper, we study the pathwidth of planar graphs. First, we prove that for any 2-connected plane graph G $\mathrm{pw}(G^*) < \mathrm{pw}(L(G))$. (We also demonstrate how

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our proof technique can be used to prove that $\mathrm{tw}(G^*) \leq \mathrm{tw}(L(G))$.) Then we show that for a graph G of $\Delta(G) \leq 3$ and line graph $L(G)$ 'width' parameters are 'close'. Our results imply that $\mathrm{pw}(G) \geq \mathrm{pw}(G^*) - 1$ for every 2-connected planar graph G of $\Delta(G) \leq 3$. Finally, we conjecture that every planar graph has a planar split of the same linear width and discuss how this conjecture implies that $\mathrm{pw}(G) \leq \mathrm{pw}(G^*) + 1$ for every 2-connected graph G.

2. Vertex Separators

There are different equivalent ways to define the treewidth and the pathwidth of a graph. Closer examination of these parameters may be found in survey papers of Bodlaender [3] and Reed [16] (see also the book of Diestel [7]). The following definitions are more convenient for our purposes.

For $S \subseteq V(G)$ we define

$$\partial S := \{u \in S \text{ and there exists } w \in V(G) \setminus S \text{ such that } \{u, w\} \in E(G)\}.$$

Let $\sigma = (v_1, v_2, \ldots, v_n)$ be an ordering of $V(G)$. For $j \in \{1, \ldots, n\}$ we put $V_j = \cup_{i=1}^{j} v_i$. Setting

$$\mathrm{vs}(G, \sigma) := \max_{i \in \{1, \ldots, n\}} |\partial V_i|,$$

we define the *vertex separation* of G (see [8] for more information on this parameter) as

$$\mathrm{vs}(G) := \min\{\mathrm{vs}(G, \sigma) \colon \sigma \text{is an ordering of } V(G)\}.$$

The following lemma is well known. (See the survey of Möhring [15] for an overview of the related results.) It follows directly form the results of Kirousis and Papadimitriou [11] on interval width (see also Kinnersley [10]).

Lemma 1. *For any graph G,* $\mathrm{vs}(G) = \mathrm{pw}(G)$.

Theorem 2. *For any 2-connected plane graph G*

$$\mathrm{vs}(G^*) < \mathrm{vs}(L(G)).$$

Therefore,

$$\mathrm{pw}(G^*) < \mathrm{pw}(L(G)).$$

Proof. For a face v^* of G (or a vertex $v^* \in V(G^*)$) let $E(v^*)$ be the edges of the boundary of v^*.

Let $\sigma_l = (e_1, e_2, \ldots, e_m)$ be an ordering of $E(G)$ (or vertices of $L(G)$). To prove the theorem we construct an ordering $\sigma^* = (v_1^*, v_2^*, \ldots, v_k^*)$ of $V(G^*)$ such that

$$\mathrm{vs}(G, \sigma^*) < \mathrm{vs}(L(G), \sigma_l). \tag{1}$$

For $j \in \{1, \ldots, m\}$ we define $E_j = \cup_{i=1}^{j} e_i$. For $v^* \in V(G^*)$ let $l(v^*)$ be the smallest number $i \in \{1, \ldots, m\}$ such that

$$|E(v^*) \cap E_i| = 2.$$

Notice that at most two vertices in $V(G^*)$ have the same number $l(v^*)$. (In a planar graph only two faces can share a set of edges.) The vertex numbering l induces an ordering $\sigma^* = (v_1^*, v_2^*, \ldots, v_k^*)$ of the vertices of G^* where $i < j$ only if $l(v_i^*) \le l(v_j^*)$. Loosely speaking, we scan the list σ_l and add a vertex to the list σ^* after passing two edges of its boundary.

We put $V_i^* = \cup_{j=1}^{i} v_j^*$. To prove (1) we show that for any index $i \in \{1, \ldots, k-1\}$ one can choose

$$j = \begin{cases} l(v_i^*) & \text{if } l(v_i^*) = l(v_{i+1}^*), \\ l(v_{i+1}^*) - 1 & \text{if } l(v_i^*) \ne l(v_{i+1}^*) \end{cases}$$

such that

$$|\partial V_i^*| < |\partial E_j|.$$

First we prove that for any $v^* \in \partial V_i^*$

$$|E(v^*) \cap \partial E_j| \ge 2. \tag{2}$$

From $v^* \in \partial V_i^* \subseteq V_i^*$ it follows that $|E(v^*) \cap E_j| \ge 2$. If $|E(v^*) \setminus E_j| > 0$ then at least two edges of $E(v^*)$ are in ∂E_j. (The boundary $E(v^*)$ is a circuit of length ≥ 3 because G is 2-connected and has no multiply edges.) If $E(v^*) \subseteq E_j$ and there exists $e \in E(v^*) \cap \partial E_j$, then e is adjacent to an edge $e' \notin E_j$. Hence there is an edge $e'' \in E(v^*)$ adjacent to e and e'. Then $e', e'' \in E(v^*) \cap \partial E_j$. In summary, if $E(v^*) \nsubseteq E_j \setminus \partial E_j$ we obtain (2).

To conclude the proof of (2) we show that $E(v^*) \subseteq E_j \setminus \partial E_j$ cannot happen. Assume the converse. From $v^* \in \partial V_i^*$ it follows that there is a vertex $u^* \notin V_i^*$ that is adjacent to v^*. Let $e \in E(G)$ be the dual of $\{v^*, u^*\}$. Then $e \in E(v^*) \cap E(u^*)$. Let $e', e'' \in E(u^*)$ be adjacent to e (graphs are simple and every boundary has at least three edges). By assumption $e \notin \partial E_j$; then $e', e'' \in E_j$. This implies that $E(u^*)$ has at least three edges of E_j (the edges e, e' and e''). Hence $u^* \in V_i^*$. This contradiction proves (2).

Using (2), we get

$$|\partial V_i^*| \le \frac{\sum_{v^* \in \partial V_i^*} |E(v^*) \cap \partial E_j|}{2}.$$

Every edge of G is adjacent to two faces of G and so the sum

$$\sum_{v^* \in \partial V_i^*} |E(v^*) \cap \partial E_j| \tag{3}$$

counts every edge of ∂E_j at most twice. Furthermore, $v_{i+1}^* \notin \partial V_i^*$ and by the definition of j, at least one edge $e \in E(v_{i+1}^*)$ is in E_j. We conclude that $e \in \partial E_j$ because otherwise $E(v_{i+1}^*)$ has at least three edges of E_j. This yields that at least one edge of $E(v_{i+1}^*)$ contributes in (3) at most once. Thus

$$\frac{\sum_{v^* \in \partial V_i^*} |E(v^*) \cap \partial E_j|}{2} < |\partial E_j|.$$

Finally, we have proved that for any index $i \in \{1, \dots, k\}$ there is $j \in \{1, \dots, m\}$ such that

$$|\partial V_i^*| \le \frac{\sum_{v^* \in \partial V_i^*} |E(v^*) \cap \partial E_j|}{2} < |\partial E_j|.$$

This concludes the proof of (1) and completes the proof of the theorem. □

3. Treewidth

The main purpose of this section is to show how the technique developed for the proof of Theorem 2 can be applied for treewidth of planar graphs.

For $S \subseteq V(G)$ and $v \in V(G) \setminus S$ we define

$$\partial_{|v} S := \{u \in S \text{ and there exists a } (u, v)\text{-path } P \text{ such that } V(P) \cap S = \{u\}\}.$$

Let $\sigma = (v_1, v_2, \dots, v_n)$ be an ordering of $V(G)$. For $j \in \{1, \dots, n\}$ we put $V_j = \cup_{i=1}^{j} v_i$ and

$$\mathrm{vs}(G, \sigma)_{|(\cdot)} := \max_{i \in \{2, \dots, n\}} |\partial_{|v_i} V_{i-1}|.$$

We define the *partial vertex separation* of G as

$$\mathrm{vs}(G)_{|(\cdot)} := \min\{\mathrm{vs}(G, \sigma)_{|(\cdot)} \colon \sigma \text{ is an ordering of } V(G)\}.$$

Lemma 3. *For any graph G,* $\mathrm{vs}(G)_{|(\cdot)} = \mathrm{tw}(G)$.

Proof. Let us give only a sketch of the proof. (The proof of the similar result in terms of graph searching is given by Dendris. Kirousis and Thilikos [6].) Let G be a graph of treewidth k. It is well known that $\mathrm{tw}(G) = k$ if and only if there is a chordal supergraph H of G with clique number $k + 1$.

For every chordal graph H there is an ordering $\sigma = (v_1, v_2, \ldots, v_n)$ of $V(H)$ such that for every $i \in \{1, \ldots, n\}$ the set of neighbours N_i of v_i in $H \setminus \{v_1, v_2, \ldots, v_{i-1}\}$ is a clique in H. (Such an ordering is often called a perfect elimination ordering.) Define $\sigma^{-1} = (v_n, v_{n-1}, \ldots, v_1)$. Then

$$\mathrm{vs}(G)_{|(\cdot)} \leq \mathrm{vs}(G, \sigma^{-1})_{|(\cdot)} = \max_{i \in \{1,\ldots,n\}} |N_i| = k.$$

To prove that $\mathrm{vs}(G)_{|(\cdot)} \geq \mathrm{tw}(G)$ we choose $\sigma = (v_1, v_2, \ldots, v_n)$ such that $\mathrm{vs}(G, \sigma)_{|(\cdot)} = k + 1$. Let H be a chordal supergraph of G with the minimum number of edges such that σ^{-1} is the perfect elimination ordering of $V(H)$. Then the clique number of H is at most $k + 1$. □

The proof of the next theorem is similar to the proof of Theorem

Theorem 4. *For any 2-connected plane graph G*

$$\mathrm{vs}(G^*)_{|(\cdot)} \leq \mathrm{vs}(L(G))_{|(\cdot)}.$$

Therefore,

$$\mathrm{tw}(G^*) \leq \mathrm{tw}(L(G)).$$

Proof. Let $\sigma_l = (e_1, e_2, \ldots, e_m)$ be an ordering of $E(G)$. We define E_i, V_i^*, $E(v^*)$, $l(v^*)$ and $\sigma^* = (v_1^*, v_2^*, \ldots, v_k^*)$ as in Theorem 2.

To prove the theorem, we show that for any $i \in \{1, \ldots, k-1\}$ and $j = l(v_{i+1}^*) - 1$

$$|\partial_{|v_{i+1}^*} V_i^*| \leq |\partial_{|e_{j+1}} E_j|.$$

As in Theorem 2, we claim that for every $v^* \in (\partial_{|v_{i+1}^*} V_i^* - v_i^*)$

$$|E(v^*) \cap \partial_{|e_{j+1}} E_j| \geq 2. \tag{4}$$

By definition $v^* \in \partial V_i^*$, and there exists a path

$$(v_{i+1}^*, u_1^*, u_2^*, \ldots, u_p^*, v^*)$$

such that

$$\cup_{k=1}^{p} u_k^* \cap V_i^* = \emptyset.$$

Since every facial boundary in G contains at least three edges, we have that for any adjacent vertices $x^*, y^* \in V(G^*)$ the set $E(x^*)$ contains at least three edges having neighbours (as vertices of $L(G)$) in $E(y^*)$. In addition, each of these three edges is adjacent to at least two edges in $E(y^*)$. For any $k \in \{1, \ldots, p\}$ the boundary $E(u_k^*)$ has at most one edge in E_j. Therefore there is $e \in E(u_p^*)$ (the case $e = e_{j+1}$ is possible) such that

1. $e \notin E_j$;
2. there is an (e_{j+1}, e)-path P in $L(G)$ such that $V(P) \cap E_j = \emptyset$;
3. e is adjacent to at least two edges of $E(v^*)$.

By definition of σ^*, $|E(v^*) \cap E_j| \geq 2$ and so there are at least two edges $e_1, e_2 \in E(v^*) \cap E_j$ such that (e, e_1) and (e, e_2)-paths do not internally intersect E_j. This concludes the proof of (4).

Notice that $|E(v_i^*) \cap E_j| \geq 1$. Combining the latter with (4), we obtain

$$|\partial_{|v^*_{i+1}} V_i^*| \leq \frac{\sum_{v^* \in \partial_{|v^*_{i+1}} V_i^*} |E(v^*) \cap \partial_{|e_{j+1}} E_j|}{2} + 1.$$

Since exactly two edges of $E(v^*_{i+1})$ are in E_{j+1}, we have that one of them is in $\partial_{|e_{j+1}} E_j$. Thus

$$\frac{\sum_{v^* \in \partial_{|v^*_{i+1}} V_i^*} |E(v^*) \cap \partial_{|e_{j+1}} E_j|}{2} < |\partial_{|e_{j+1}} E_j|.$$

(Each edge of ∂E_j is counted at most twice in the sum and $e = (E(v^*_{i+1}) \cap E_j)$ is counted once.)

Finally,

$$|\partial_{|v^*_{i+1}} V_i^*| \leq |\partial_{|e_{j+1}} E_j|,$$

which completes the proof. □

4. Line Graphs of Small Degree Graphs

Golovach in [9] obtained the following result about the vertex separation of line graphs and cutwidth (see Makedon and Sudborough [14] for definitions and further results on cutwidth).

Theorem 5 (Golovach, [9]). *For any graph G,*

$$\mathrm{cw}(G) \leq \mathrm{vs}(L(G)) \leq \mathrm{cw}(G) + \lfloor \Delta(G)/2 \rfloor - 1,$$

where cw(G) is the cutwidth of G.

The well known result of Makedon and Sudborough [14] is that for any graph G of $\Delta(G) \leq 3$ the cutwidth of G is equal to the edge search number. Since the edge search number of G is at most $\mathrm{vs}(G) + 2$, we obtain the following corollary of Golovach's theorem. (We refer the reader to the survey of Bienstock [1] for further more detailed information on graph searching.)

Lemma 6. *For any graph G of* $\Delta(G) \leq 3$, $\mathrm{vs}(L(G)) \leq \mathrm{vs}(G) + 2$.

Corollary 7. *For any 2-connected planar graph G of* $\Delta(G) \le 3$

$$\mathrm{pw}(G) \ge \mathrm{pw}(G^*) - 1.$$

Proof. By Lemma 6 $\mathrm{vs}(G) + 2 \ge \mathrm{vs}(L(G))$ and by Theorem 2 $\mathrm{vs}(L(G)) \ge \mathrm{vs}(G^*)+ 1$. Finally,

$$\mathrm{pw}(G) + 1 = \mathrm{vs}(G) + 1 \ge \mathrm{vs}(L(G)) - 1 \ge \mathrm{vs}(G^*) = \mathrm{pw}(G^*). \qquad \square$$

Corollary 7 can be restated in a weak form.

Corollary 8. *For any planar triangulation H*

$$\mathrm{pw}(H) \le \mathrm{pw}(H^*) + 1.$$

5. Concluding Remarks

Let v be a vertex in a graph G and $N[v]$ be the set of all vertices adjacent to v. Consider a partition of the set $N[v]$ into any two sets M and N. (Note that M or N may be empty.) Let us transform G as follows: delete v with all incident edges, add new vertices u and w with edge $\{u, w\}$, and make u adjacent to all vertices of M and w to all vertices of N. We say that the result of this transformation is obtained from G by *vertex splitting* of v. A graph H is said to be a *split* of G if H is obtained from G by a sequence of vertex splittings.

To state Conjecture 5 we need the notion of linear width. This notion was introduced by Thomas [21] and is closely related to crusades of Bienstock and Seymour [2] (see also Bienstock's survey [1]). For $X \subseteq E(G)$ let $\delta(X)$ be the set of all vertices incident to edges in X and $E(G) \setminus X$. Let $\sigma = (e_1, e_2, \ldots, e_m)$ be an ordering of $E(G)$. For $i \in \{1, \ldots, m\}$ we put $E_i = \cup_{j=1}^{i} e_j$. We define

$$\mathrm{lw}(G, \sigma) := \max_{i \in \{1,\ldots,m\}} |\delta(E_i)|,$$

and the *linear width* of G as

$$\mathrm{lw}(G) := \min\{\mathrm{lw}(G, \sigma) \colon \sigma \text{ is an ordering of } E(G)\}.$$

The results of Bienstock and Seymour [2] imply that for graphs without vertices of degree 1, the linear width has a game theoretic interpretaton in terms of mixed search number. (See also the article by Takahashi, Ueno and Kajitani [20] on further discussions of mixed search number.)

Notice that for any graph G with minimum vertex degree at least 2

$$\mathrm{vs}(G) \le \mathrm{lw}(G) \le \mathrm{vs}(G) + 1.$$

This fact follows from the game-theoretical interpretation of these parameters. For any graph G, $\mathrm{vs}(G)$ is equal to the node search number of G minus one, *i.e.*, $ns(G) - 1 = \mathrm{vs}(G)$ (see the paper of Kirousis and Papadimitriou [12] for the proof). By the result of Bienstock and Seymour [2] for a graph G with minimum vertex degree at least 2 the mixed search number $ms(G)$ of G is equal to $\mathrm{lw}(G)$ and it is well known that

$$ns(G) - 1 \leq ms(G) \leq ns(G).$$

The reader is also referred to Bienstock's survey [1] on graph searching.

Let us remark that not every split of a planar graph is planar. But every planar graph G has a planar split H of $\Delta(H) \leq 3$. It is also easy to show that for every planar graph G there is split H such that $\Delta(H) \leq 3$ and $\mathrm{lw}(G) = \mathrm{lw}(H)$.

We conjecture that the following statement is true.

Conjecture. *For every planar graph G there is planar split H such that* $\Delta(H) \leq 3$ *and* $\mathrm{lw}(G) = \mathrm{lw}(H)$.

Our Conjecture is related to the following statement of Robertson and Seymour [17]:

It seems that the tree-width of a planar graph and the tree-width of its geometric dual are approximately equal – indeed, we have convinced ourselves that they differ by at most one.

Lapoire [13] proved this result using algebraic approach. Recently Bouchitté, Mazoit and Todinca [5] obtain nice combinatorial proof of this result by clever usage of minimal separators. It is also worth to mention the results of Seymour and Thomas [19] based on the heavy machinery developed in Graph Minors Theory which imply that the branchwidth of a planar graph is equal to the branchwidth of its dual. (The branchwidth of a graph is the graph parameter related to linear width.)

If Conjecture 5 is true then $\mathrm{pw}(G) \geq \mathrm{pw}(G^*) - 1$ for any 2-connected planar graph G. Indeed, suppose that for a planar graph G there is a planar split H such that $\Delta(H) \leq 3$ and $\mathrm{lw}(G) = \mathrm{lw}(H)$. Then by Lemma 6, $\mathrm{lw}(H) \geq \mathrm{vs}(L(H)) - 1$ and by Theorem 2, $\mathrm{vs}(L(H)) > \mathrm{vs}(H^*)$. G^* is a subgraph of H^* and

$$\mathrm{pw}(G) + 1 \geq \mathrm{lw}(G) = \mathrm{lw}(H) \geq \mathrm{vs}(L(H)) - 1 \geq \mathrm{vs}(H^*) \geq \mathrm{vs}(G^*) = \mathrm{pw}(G^*).$$

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References

1. Bienstock, D.: Graph searching, path-width, tree-width and related problems (a survey). DIMACS Ser. Discrete Math. Theor. Comput. Sci. **5**, 33–49 (1991)
2. Bienstock, D., Seymour, P.: Monotonicity in graph searching. J. Algorithms **12**, 239–245 (1991)

3. Bodlaender, H.L.: A partial k-arboretum of graphs with bounded treewidth. Theor. Comput. Sci. **209**, 1–45 (1998)
4. Bondy, J.A.: Basic graph theory: Paths and circuits. In: R. L. Graham, M. Grötschel, and L. Lovász: Handbook of Combinatorics, vol. 1, pp. 3–110, Elsevier Science B.V. 1995
5. Bouchitté, V., Mazoit, F., Todinca, I.: Treewidth of planar graphs: connections with duality. 2001 (manuscript)
6. Dendris, N.D., Kirousis, L.M., Thilikos, D.M.: Fugitive-search games on graphs and related parameters. Theor. Comput. Sci. **172**, 233–254 (1997)
7. Diestel, R.: Graph theory. New York: Springer Verlag, 1997 (Translated from the 1996 German original)
8. Ellis, J.A., Sudborough, I.H., Turner, J.: The vertex separation and search number of a graph. Inf. Comput. **113**, 50–79 (1994)
9. Golovach, P.A.: The cutwidth of a graph and the vertex separation number of the line graph. Discrete Math. Appl. **3**, 517–521 (1993) Diskretn. Mat. **5**, 76–80 (1993) (translation)
10. Kinnersley, N.G.: The vertex separation number of a graph equals its path width. Inf. Proccess. Lett. **42**, 345–350 (1992)
11. Kirousis, L.M., Papadimitriou, C.H.: Interval graphs and searching. Discrete Math **55**, 181–184 (1985)
12. Kirousis, L.M., Papadimitriou, C.H.: Searching and pebbling. Theor. Comput. Sci. **47**, 205–218 (1986)
13. Lapoire, D.: Structuration des graphes planaires. Ph.D thesis, Universite de Bordeaux, France 1999
14. Makedon, F.S., Sudborough, I.H.: On minimizing width in linear layouts. Discrete Appl. Math. **23**, 243–265 (1989)
15. Möhring, R.H.: Graph problems related to gate matrix layout and PLA folding. In: E. Mayr, H. Noltemeier, and M. Sysło: Computational Graph Theory, Comuting Suppl. 7, pp. 17–51, Springer Verlag 1990
16. Reed, B.: Treewidth and tangles: a new connectivity measure and some applications. In: R.A. Bailey: Surveys in Combinatorics, pp. 87–162, Cambridge University Press 1997
17. Robertson, N., Seymour, P.D.: Graph minors. III. Planar tree-width. J. Comb. Theory, Ser. B **36**, 49–64 (1984)
18. Robertson, N., Seymour, P.D.: Graph minors. II. Algorithmic aspects of tree-width. J. Algorithms, **7**, 309–322 (1986)
19. Seymour, P.D., Thomas, R.: Call routing and the ratcatcher. Combinatorica, **14**, 217–241 (1994)
20. Takahashi, A., Ueno, S., Kajitani, Y.: Mixed-searching and proper-path-width. Theor. Comput. Sci. **137**, 253–268 (1995)
21. Thomas, R.: Tree decompositions of graphs. Lecture notes. Atlanta: Georgia Institut of Technology 1996

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