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Interval degree and bandwidth of a graph

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Abstract

The interval degree id(G) of a graph G is defined to be the smallest max-degree of any interval supergraphs of G. One of the reasons for our interest in this parameter is that the bandwidth of a graph is always between id(G)/2 and id(G). We prove also that for any graph G the interval degree of G is at least the pathwidth of G^2. We show that if G is an AT-free claw-free graph, then the interval degree of G is equal to the clique number of G^2 minus one. Finally, we show that there is a polynomial time algorithm for computing the interval degree of AT-free claw-free graphs.

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Keywords: Interval degree; Interval graph; Bandwidth; Pathwidth; AT-free claw-free graph

1. Introduction and statement of the problem

Many interesting nontrivial graph parameters can be defined as the minimum of a more fundamental graph parameter (clique number, number of edges, max-degree, etc.) taken over all supergraphs of the given graph in a certain graph class. The last 20 years were the years of the intensive study of the parameters defined by using chordal and interval graphs. For example, the profile problem (fill-in problem) is for a given graph G to find an interval (chordal) supergraph of G with the minimum

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number of edges. Another famous example is the pathwidth (the treewidth) problem: To find an interval (chordal) supergraph of G with the smallest clique number. We refer to the surveys of Bodlaender [4] and Möhring [20] for further references. Chung and Mumford [7] studied the relevant problem of finding a chordal supergraph with the smallest maximum vertex degree.

Here we introduce the related problem of finding an interval supergraph with the smallest max-degree. Because of the great practical and theoretical importance of pathwidth, treewidth, fill-in and profile, we find the statement of the interval degree problem very natural and the study of this problem interesting. Another reason for our interest to this problem is (we prove it in this paper) that the interval degree is also closely associated with the bandwidth minimization problem. More precisely, we prove that for any graph G the interval degree of G is at least the bandwidth of G and at most twice the bandwidth of G. Finally, we observe the relation of the interval degree of a graph G and the pathwidth of the square G^2 of G.

The bandwidth problem has a long history and a number of practical applications. (See the classical survey of Chinn et al. [6].) The original motivation for the problem is the bandwidth minimization problem for matrices. This problem attracts the attention of numerous researchers over the last 30 years which makes it of the great theoretical importance. The problem is very hard from the computational point of view. It remains NP-complete even for very restricted graph classes. Monien [22] proved that the problem is NP-complete for a special class of trees, called caterpillars with hair length three and Parra and Scheffler [24] obtained NP completeness proof for cobipartite graphs. Recently many approximability results and nice techniques to obtain these results have appeared. Unger [27] showed that for any integer k there is no efficient approximation algorithm with performance ratio k (unless P=NP) of bandwidth for caterpillars with hair length three. Bandwidth approximation algorithms for restricted classes of graphs have been presented by Kloks et al. [17] and Kratsch and Stewart [18] among others. Feige [10] introduced volume respecting embedding technique to obtain a polylogarithmic factor approximation algorithm on bandwidth. Similar approach is discussed by Blum et al. [3]. (See also the paper of Dunagan and Vempala [8].) An overview of recent algorithmical results on bandwidth is given by Feige [11].

This paper is organized as follows. In Section 2 we give necessary definitions. In Section 3 we restate the problem of interval completion with the smallest max-degree in terms of linear layouts and obtain some bounds of interval degree in terms of bandwidth and pathwidth. In Section 4 we prove that the interval degree of AT-free claw-free graph is equal to the clique number of its square minus one. In Section 5 we introduce some complexity results. In Section 6 we give concluding remarks and leave some open questions.

2. Statement of the problem

We use the standard graph-theoretic terminology compatible with [5], to which we refer the reader for basic definitions. G is an undirected, simple (without loops and multiple edges) and finite graph with the vertex set $V(G)$ and the edge set $E(G)$.

Unless otherwise specified, n denotes the number of vertices of G. Let $\omega(G)$ denotes the clique number (maximum clique-size) of G. The degree of a vertex v in a graph G is denoted by $\deg_G(v)$ and the maximum degree of the vertices of a graph G by $\Delta(G)$. The closed neighborhood of a vertex v in a graph G (the set of neighbors of v in G with v) is denoted by $N_G[v]$. The *distance* $d_G(u,v)$ between two vertices u and v of G is the length of the shortest path between u and v in the graph G. Let G^k be the graph with vertex set $V(G)$ and such that two vertices u and v adjacent in G^k if and only if $d_G(u,v) \leqslant k$.

A graph G is called an *interval graph* provided we can assign to each $v \in V(G)$ an interval I_v so that $(u,v) \in E(G)$ if and only if $I_v \cap I_u \neq \emptyset$. Some interesting graph parameters can be defined in terms of interval supergraph. For example, the *pathwidth* $\mathrm{pw}(G)$ of a graph which was introduced by Robertson and Seymour [25] can be defined (see, e.g. [20]) as

$$\mathrm{pw}(G) := \min\{\omega(G') - 1\colon G' \text{ is an interval supergraph of } G\}.$$

The problem of finding the profile of a graph has applications in sparse matrix computations (see [6]). In terms of interval supergraphs the profile of a graph G can be defined (see, e.g. [2]) as

$$\mathrm{p}(G) := \min\{|E(G')|\colon G' \text{ is an interval supergraph of } G\}.$$

In the same manner we define a new graph parameter, namely the interval degree of a graph. The *interval degree* of a graph G is

$$\mathrm{id}(G) := \min\{\Delta(G')\colon G' \text{ is an interval supergraph of } G\}.$$

The problem of *interval completion with the smallest max-degree* is for a given graph G to find an interval supergraph I of G such that $(I) = \mathrm{id}(G)$.

3. Linear layouts

An interval representation of a graph naturally induces an ordering of its vertices and it is not surprising that sometimes interval completion problems can be 'rewritten' in terms of vertex orderings or linear layouts.

A *linear layout* of a graph G is a one-to-one mapping $f\colon V(G) \to \{1,\ldots,n\}$. For linear layout f of a graph G and $i \in \{1,\ldots,n\}$, we define

$$S_i(G,f) = |\{v \in V(G)\colon f(v) \leqslant i \text{ and } \exists\ (u,v) \in E(G), \text{ such that } f(u) > i\}|$$

and

$$\mathrm{vs}(G,f) = \max\{S_i(G,f)\colon i \in \{1,\ldots,n\}\}.$$

The *vertex separation number* of G [9] is

$$\mathrm{vs}(G) := \min\{\mathrm{vs}(G,f)\colon f \text{ is a linear layout of } G\}.$$

As noted by a number of researchers for any graph G, $\mathrm{vs}(G) = \mathrm{pw}(G)$ (see, e.g. [15] or [14]).

Profile also may be 'redefined' [2] as a graph invariant $\mathrm{p}(G)$ by finding a layout f of G which minimizes the sum

$$\sum_{u\in V(G)} (f(u) - \min\{f(v)\colon\ v\in V(G), v\in N_G[u]\}).$$

Another very important graph parameter, namely bandwidth, is defined by making use of linear layouts. For a linear layout f of G setting

$$W_i(G,f) = \max\{j-i\colon\ j\in\{i,\dots,n\}, f^{-1}(j)\in N_G[f^{-1}(i)]\}$$

and

$$\mathrm{bw}(G,f) = \max\{W_i(G,f)\colon\ i\in\{1,\dots,n\}\}$$

one can define the *bandwidth* of G as

$$\mathrm{bw}(G) := \min\{\mathrm{bw}(G,f)\colon\ f \text{ is a linear layout of } G\}.$$

The bandwidth problem has a long history and a number of practical applications.

In order to define the interval degree in terms of linear layouts, we introduce a 'hybrid' of the bandwidth and the vertex separation number. Let f be a linear layout of a graph G. We define

$$\mathrm{sw}(G,f) = \max\{S_{i-1}(G,f) + W_i(G,f)\colon\ i\in\{1,\dots,n\}\}$$

(putting $S_0(G,f)=0$) and

$$\mathrm{sw}(G) := \min\{\mathrm{sw}(G,f)\colon\ f \text{ is a linear layout of } G\}.$$

Notice that S_i is from the definition of the vertex separation number and W_i is from the definition of the bandwidth.

Theorem 1. *For any graph* G, $\mathrm{id}(G) = \mathrm{sw}(G)$.

Proof. First we prove that $\mathrm{id}(G) \geqslant \mathrm{sw}(G)$. Let G_I be an interval supergraph of G such that $\Delta(G_I) = \mathrm{id}(G)$. Without loss of generality we can assume that the left endpoints of the intervals that represent G_I are distinct integers $1,2,\dots,n$. Such a representation leads to a linear layout f of G: for $v\in V(G)$ $f(v)=i$ if and only if i is the left point of the corresponding interval of v. For a vertex v with $f(v)=i$ let us define

$$d_1(v) = |\{u\in V(G)\colon\ f(u) < i \text{ and } (u,v)\in E(G_I)\}|$$

and

$$d_2(v) = |\{u\in V(G)\colon\ f(u) > i \text{ and } (u,v)\in E(G_I)\}|.$$

Obviously, $S_{i-1}(G,f) \leqslant d_1(v)$, $W_i(G,f) \leqslant d_2(v)$ and $\deg_{G_I}(v) = d_1(v)+d_2(v)$. Choose a vertex v with a number i such that

$$S_{i-1}(G,f) + W_i(G,f) \text{ is maximum.}$$

Then $\Delta(G_I) \geqslant \deg_{G_I}(v) = d_1(v) + d_2(v) \geqslant S_{i-1}(G,f) + W_i(G,f) = \mathrm{sw}(G,f) \geqslant \mathrm{sw}(G)$.

We now turn to $\mathrm{id}(G) \leqslant \mathrm{sw}(G)$. Let f be a linear layout of G such that $\mathrm{sw}(G) = \mathrm{sw}(G, f)$. We assign to each vertex $v \in V(G)$ the interval $(f(v), r(v))$, where $r(v) = \max\{i \in \{f(v), \dots, n\}: f^{-1}(i) \in N_G[v]\} + \frac{1}{2}$. Denote the corresponding interval graph by G_I. If $(u, v) \in E(G)$ and $f(u) < f(v)$ then $f(v) < r(u)$; hence $(f(v), r(v)) \cap (f(u), r(u)) \neq \emptyset$. Consequently, $(u, v) \in E(G_I)$ and G_I is an interval supergraph of G. The further proof is straightforward once the following observations are made: for a vertex v with a number i, $S_{i-1}(G, f) \geqslant d_1(v)$ and $W_i(G, f) \geqslant d_2(v)$. □

The following Corollary is one of the reasons for our interest to the interval completion problem with the smallest max-degree.

Corollary 2. *For any graph G*

$$\mathrm{bw}(G) \leqslant \mathrm{id}(G) \leqslant 2\,\mathrm{bw}(G).$$

Therefore,

$$\frac{\mathrm{id}(G)}{2} \leqslant \mathrm{bw}(G) \leqslant \mathrm{id}(G).$$

Proof. It is easy to check that for any graph G and linear layout f, $\mathrm{vs}(G, f) \leqslant \mathrm{bw}(G, f)$. Then Corollary follows immediately from Theorem 1. □

Notice that $\Delta(G) \leqslant \mathrm{id}(G)$ for any graph G and Corollary 2 is the generalization of the well-known lower bound on the bandwidth of G (see, e.g. [6]) $\Delta(G)/2 \leqslant \mathrm{bw}(G)$. The bounds for the interval degree in terms of the bandwidth are tight. For example, for any star $K_{1,n}$, where $n = 2k$, $\mathrm{id}(K_{1,n}) = 2\,\mathrm{bw}(K_{1,n}) = n$ and for any path P_n with $n > 2$ vertices, $\mathrm{id}(P_n) = 2\,\mathrm{bw}(P_n) = 2$. For any complete graph K_n of n vertices $\mathrm{id}(K_n) = \mathrm{bw}(K_n) = n - 1$.

Corollary 3. *For any graph G*, $\mathrm{pw}(G^2) \leqslant \mathrm{id}(G)$.

Proof. We show that for every linear layout f, $\mathrm{vs}(G^2, f) \leqslant \mathrm{sw}(G, f)$. Because $\mathrm{pw}(G^2) = \mathrm{vs}(G^2)$ it will prove the corollary. Let f be a linear layout and let v be a vertex of G, $i = f(v)$. Let j be the smallest integer for which $f^{-1}(j)$ has a neighbor w in G with $f(w) \geqslant i$. Define $D^{j\leqslant}$ ($D^{<j}$) to be the set of all vertices x of G such that x is adjacent in G^2 to a vertex y having $f(y) > i$ and $j \leqslant f(x) \leqslant i$ ($f(x) < j$). It should be noted that $S_i(G^2, f) = |D^{j\leqslant}| + |D^{<j}|$. Using almost obvious inequalities

$$|D^{j\leqslant}| \leqslant W_j(G, f)$$

and

$$|D^{<j}| \leqslant S_{j-1}(G, f),$$

we arrive at

$$S_i(G^2, f) \leqslant W_j(G, f) + S_{j-1}(G, f).$$

The bound in Corollary 3 is sharp. Let us give some simple examples.

The disjoint union of graphs G and H is the graph $G\dot{\cup}H$ with the vertex set $V(G)\dot{\cup}V(H)$ and the edge set $E(G)\dot{\cup}E(H)$ (where $\dot{\cup}$ is the disjoint union on graphs and sets, respectively). We use $G \times H$ to denote the following type of 'product' of G and H: $G \times H$ is the graph with the vertex set $V(G)\dot{\cup}V(H)$ and the edge set $E(G)\dot{\cup}E(H) \cup \{(u,w):\ v \in V(G), w \in V(H)\}$). □

Since $(G \times H)^2$ is the complete graph, we obtain the following example to Corollary 3.

Example 4. Let G and H be graphs with $V(G) \cap V(H) = \emptyset$. Then $\mathrm{pw}((G \times H)^2) = \mathrm{id}(G \times H) = |V(G)| + |V(H)| - 1$.

A graph G is a *cograph* if it does not contain P_4 (a path with four vertices) as an induced subgraph.

It is well-known that a graph G is a cograph if and only if one of the following conditions is fulfilled:

(1) $|V(G)| = 1$;
(2) There are cographs $G_1, \ldots, G_k$ and $G = G_1\dot{\cup}G_2\dot{\cup}\cdots\dot{\cup}G_k$;
(3) There are cographs $G_1, \ldots, G_k$ and $G = G_1 \times G_2 \times \cdots \times G_k$.

Combining Example 4 with Corollary 3, we obtain the next example.

Example 5. Let G be a connected cograph. Then $\mathrm{pw}(G^2) = \mathrm{id}(G) = n - 1$.

A graph G is said to be *cobipartite* if it is the complement of a bipartite graph. Let a cobipartite graph G be the complement of a bipartite graph with bipartition (X, Y). We define $n_1 = |X|$ and $n_2 = |Y|$. The number of vertices of X (Y) that are adjacent in G to some vertices of Y (X) is denoted by m_1 (m_2). The proof of our last example is easy and we omit it here. Also this example is the direct consequence of Theorem 15 of the next section.

Example 6. Let G be a cobipartite graph. Then $\mathrm{pw}(G^2) = \mathrm{id}(G) = \max\{n_1 + m_2, n_2 + m_1\} - 1$.

We find Example 6 to be interesting for the following reason. Parra and Scheffler [24] proved that bandwidth equals pathwidth and treewidth for cobipartite graphs. Then by the celebrated result of Arnborg et al. [1] all these problems are NP-hard even for cobipartite graphs. Cobipartite graphs form a subclass of AT-free claw-free graphs and we generalize this example in the next section.

4. Minimal triangulations and AT-free claw-free graphs

A *chord* of a cycle C is an edge not in C that has endpoints in C. A *chordless cycle* in G is a cycle of length more than three in G that has no chord. A graph G

is *chordal* if it does not contain a chordless cycle. A set of three vertices x, y, z of a graph G is called an *asteroidal triple* (abbr. AT) if for any two of these vertices there exists a path joining them that avoids the (closed) neighborhood of the third. A graph G is called an *asteroidal triple-free* (abbr. AT-free) graph if G does not contain an asteroidal triple. This notion was introduced by Lekkerkerker an Boland for the following characterization of interval graphs.

Theorem 7 (Lekkerkerker and Boland [19]). *G is an interval graph if and only if it is chordal and AT-free.*

A graph isomorphic to $K_{1,3}$ is referred to as a *claw*, and a graph that does not contain an induced claw is said to be *claw-free*. Notice that cobipartite graphs form a subclass of AT-free claw-free graphs. Another subclass of AT-free claw-free graphs form proper interval graphs. An interval graph G is a *proper interval graph* if it is claw-free. Thus G is a proper interval graph if and only if it is chordal and AT-free claw-free. The following characterization was observed by As it was observed by Kloks et al. [17] every connected AT-free claw-free graph G is a claw-free cocomparability graph or is a complement of a triangle-free graph.

Because of Example 6, one can conjecture that $\mathrm{vs}(G^2) = \mathrm{id}(G)$ for any AT-free claw-free graph G. In order to prove this conjecture we restate the interval completion problem in terms of minimal triangulations. A *triangulation* of a graph G is a graph H on the same vertex set as G that contains all edges of G and is chordal. A *minimal* triangulation of G is a triangulation H such that no proper subgraph of H is a triangulation of G.

Möhring generalized Theorem 7 in the following way.

Theorem 8 (Möhring [21]). *Every minimal triangulation of an AT-free graph is an interval graph.*

Möhring's theorem implies

Corollary 9. *For any AT-free graph G,* $\mathrm{id}(G)$ *is equal to the smallest max-degree over all minimal triangulations of G.*

Lemma 12 provides us with some information on the structure of minimal triangulations. This information is strongly used in the proof of Theorem 15. In order to obtain Lemma 12 we need additional 'tools'. A subset S of vertices of a connected graph G is called an *a,b-separator* for non adjacent vertices a and b in $V(G) \setminus S$ if a and b are in different connected component of the subgraph of G induced by $V(G) \setminus S$. If no proper subset of an a,b-separator S separates a and b in this way, then S is called a *minimal a,b-separator*. A subset S is referred to as a *minimal separator*, if there exist non adjacent vertices a and b for which S is a minimal a,b-separator.

The following characterization of minimal separators is well-known (see, e.g. [12]).

Lemma 10. *Let S be an a,b-separator of G and let G_a, G_b be two components of $G \setminus S$ containing a and b, respectively. Then S is a minimal a,b-separator if and only if every vertex $s \in S$ is adjacent to a vertex in each of these components.*

The following lemma is an immediate consequence of characterizations of a minimal triangulation by 'completing' minimal separators, see [17].

Lemma 11. *Let H be a minimal triangulation of G. If an edge $e=(x,y) \in E(H) \setminus E(G)$ then there is a minimal separator of G containing x and y.*

The next lemma is related to a well-known theorem of Rose et al. [26] on minimal triangulations.

Lemma 12. *Let H be a minimal triangulation of G. If an edge $e=(x,y) \in E(H) \setminus E(G)$, then there is an induced cycle C of length $\geqslant 4$ in G such that $x, y \in V(C)$.*

Proof. By Lemma 11 there exists a minimal a,b-separator S in G containing x and y. Let G_a, G_b be components of $G \setminus S$ containing a and b, respectively. By Lemma 10 vertices x and y have neighbors in G_a and G_b. Hence there exist inclusion-minimal paths $(x, a_1, \ldots, a_k, y)$, $a_i \in V(G_a)$, and $(x, b_1, \ldots, b_l, y)$, $b_i \in V(G_b)$. Since for no pair of vertices a_i and b_j $(a_i, b_j) \in E(G)$, we have that vertices $(x, a_1, \ldots, a_k, y, b_l, \ldots, b_1)$ induce a cycle of length $\geqslant 4$ in G. □

Lemma 12 implies some interesting corollaries.

Corollary 13. *Let G be an AT-free graph. Then* $\mathrm{id}(G) \leqslant \omega(G^4) - 1$. *In particular*, $\mathrm{pw}(G^2) \leqslant \mathrm{id}(G) \leqslant \mathrm{pw}(G^4)$.

Proof. Let G_I be an interval supergraph of G such that $\Delta(G_I) = \mathrm{id}(G)$. By Corollary 9 G_I is a minimal triangulation of G. Since G is AT-free, we have that it does not contain a chordless cycle of length at least 6. Let O be a vertex of the maximal degree in G_I. By Lemma 12 $d_G(u,w) \leqslant 4$ for all $u, w \in N_{G_I}[v]$. Hence $\deg_{G_I}(O) \leqslant \omega(G^4) - 1 \leqslant \mathrm{pw}(G^4)$. □

Corollary 14. *Let G be an AT-free graph. Then* $\Delta(G^2)/4 \leqslant \mathrm{bw}(G) \leqslant \Delta(G^2)$.

Proof. Taking into account Corollary 2 and Lemma 12, we obtain that $\mathrm{bw}(G) \leqslant \mathrm{id}(G) \leqslant \Delta(G^2)$. It is well-known (see [6]) that for any graph G $\Delta(G) \leqslant 2\,\mathrm{bw}(G)$ and $\mathrm{bw}(G^2) \leqslant 2\,\mathrm{bw}(G)$. Hence, $\Delta(G^2) \leqslant 4\,\mathrm{bw}(G)$. □

We are now in a position to state the main result of this section.

Theorem 15. *Let G be an AT-free claw-free graph. Then* $\omega(G^2) - 1 = \mathrm{id}(G)$.

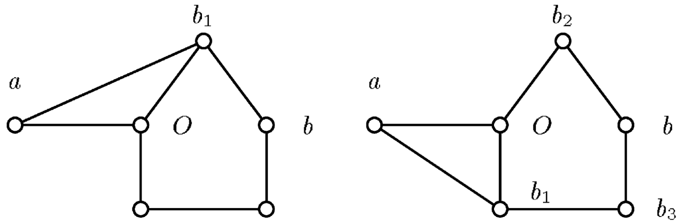

Fig. 1. Case 2.

Proof. By definition, $\omega(G) - 1 \leqslant \mathrm{pw}(G)$ for any graph G. Therefore, the application of Corollary 3 yields $\omega(G^2) - 1 \leqslant \mathrm{id}(G)$.

Now we prove $\omega(G^2) - 1 \geqslant \mathrm{id}(G)$. Let G_I be an interval supergraph of G such that $\Delta(G_I) = \mathrm{id}(G)$. Notice that owing to Corollary 9 G_I may be treated as a minimal triangulation of G. Let O be a vertex of the maximal degree in G_I. Then $|N_{G_I}[O]| - 1 = \Delta(G_I) = \mathrm{id}(G)$. Let a, b be vertices of $N_{G_I}[O]$. We show that $d_G(a,b) \leqslant 2$ which implies the existence of the clique in G^2 containing all vertices of $N_{G_I}[O]$.

Let us consider three cases.

Case 1: If $a, b \in N_G[O]$ then obviously $d_G(a,b) \leqslant 2$.

Case 2: Assume that $a \in N_G[O]$ and $b \notin N_G[O]$. Then $(O,b) \in E(G_I) \setminus E(G)$ and by Lemma 12 there is an induced cycle C_b in G of length at least four such that $O, b \in V(C_b)$. Because G is AT-free, the length of C_b is at most five. Hence if $a \in V(C_b)$ then $d_G(a,b) \leqslant 2$. Suppose that $a \notin V(C_b)$. C_b is a chordless cycle and G is claw-free; hence a is adjacent in G to at least one neighbor of O in C_b. Let b_1 be such a neighbor. If $d_G(b_1,b) = 1$ (see the left half of Fig. 1) then $d_G(a,b) = 2$. Let $d_G(b_1,b) = 2$. Denote the second neighbor of O in C_b by b_2 and the vertex that is placed between b_1 and b in C_b by b_3 (see the right graph in Fig. 1). Then a is adjacent to at least one of the vertices b_2, b_3 and b because a, b_2, b_3 cannot form an AT in G. Thus $d_G(a,b) \leqslant 2$.

Case 3: Suppose that $a, b \notin N_G[O]$. Let C_a and C_b be chordless cycles of length at least four in G that contain vertices O, a and O, b, respectively.

If $|E(C_a \cap C_b)| \geqslant 3$ then the proof is obvious (the diameter of the graph $C_a \cup C_b$ is at most two). Supposing $|E(C_a \cap C_b)| < 3$, we arrive at three possible cases.

Case 3.1: $|E(C_a \cap C_b)| = 2$. It is easy to check that if C_a and C_b have a common edge that is not incident to O then the diameter of $C_b \cup C_a$ is at most two. Because of this, we can assume that O has the same neighbors in C_a and C_b, i.e. $V(C_a) \cap N_G[O] = V(C_b) \cap N_G[O]$. We denote these neighbors by x and y (see the left graph in Fig. 2). Since G is claw-free and C_a, C_b are chordless cycles, we have that the neighbor of x in $C_a \setminus C_b$ is adjacent to the neighbor of x in $C_b \setminus C_a$ and the neighbor of y in $C_a \setminus C_b$ is adjacent to the neighbor of y in $C_b \setminus C_a$. This situation is illustrated in the right half of Fig. 2. Then the distance in G between any two vertices from $V(C_b) \cup V(C_a)$ is at most two.

Case 3.2: C_a and C_b have only one common edge. If this edge is not incident to O then it is easy to check that the diameter of $C_a \cup C_b$ is at most two. Suppose that $V(C_b) = \{O, b_1, b_2, b_3, b_4\}$ and $V(C_a) = \{O, a_1, a_2, a_3, b_4\}$. This case is illustrated in the

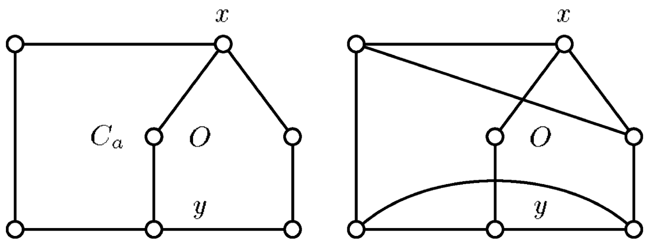

Fig. 2. Case 3.1.

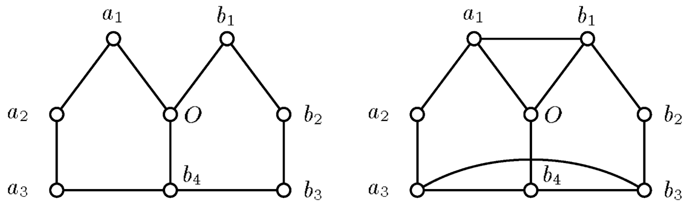

Fig. 3. Case 3.2.

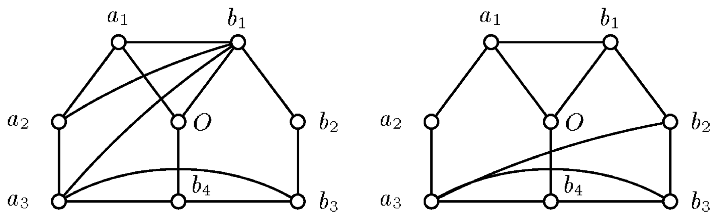

Fig. 4. Case 3.2: $(a_3, b_1) \in E(G)$ and $(a_3, b_1) \notin E(G)$.

left half of Fig. 2 (if C_a or C_b is a cycle of length four then the proof is the same). Since G is claw-free, we conclude that $(a_1, b_1) \in E(G)$ and $(a_3, b_3) \in E(G)$ (see the right graph in Fig. 3). If $(a_3, b_1) \in E(G)$ then a_2 is adjacent to b_1 in G (a_2, a_3, b_1, b_4 induces a claw otherwise) (see the left graph in Fig. 4). If $(a_3, b_1) \notin E(G)$ then a_3 is adjacent to b_2 in G because a_3, b_2, O cannot form an AT in G (see the right graph in Fig. 4). For both graphs in Fig. 4 for $i, j = \{2, 3\}$ the distance between a_i and b_j is at most two.

Case 3.3: $E(C_a \cap C_b) = \emptyset$. It is easy to see that if $|V(C_a \cap C_b)| \geqslant 2$ then the distance between any two vertices of the set $V(C_a \cup C_b) \setminus N_G[O]$ is at most two.

Suppose that $V(C_a \cap C_b) = O$ and $V(C_b) = \{O, b_1, b_2, b_3, b_4\}$, $V(C_a) = \{O, a_1, a_2, a_3, a_4\}$ (if C_a or C_b is the cycle of length four the proof is similar). Because G is claw-free and C_a, C_b are chordless in G then $(a_1, b_1), (a_4, b_4) \in E(G)$ (or $(a_1, b_4), (a_4, b_1) \in E(G)$ but this is the 'symmetric' case). (See the left graph in Fig. 5.)

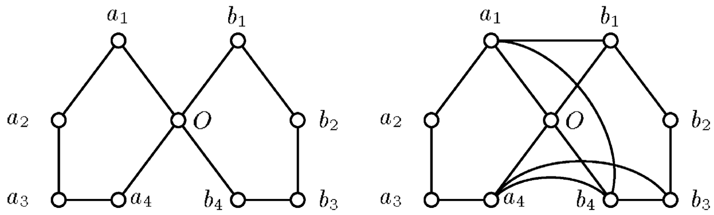

Fig. 5. Case 3.3: $V(C_a \cap C_b) = O$ and Case 3.3.1.

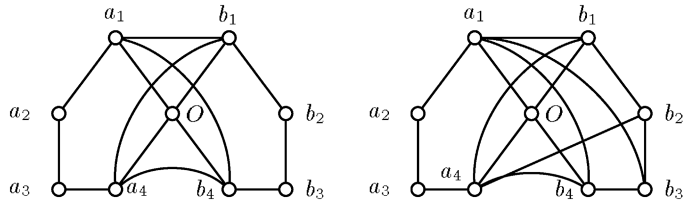

Fig. 6. Case 3.3.3.

Vertices a_1, b_2, b_4 and a_4, b_1, b_3 do not form ATs in G, so at least one pair from each triple $\{(a_1, b_2), (a_1, b_3), (a_1, b_4)\}$ and $\{(a_4, b_1), (a_4, b_2), (a_4, b_3)\}$ is an edge of G. If $(a_1, b_3) \in E(G)$ (or $(a_4, b_2) \in E(G)$ for the second triple) then (a_1, b_2) or (a_1, b_4) is in $E(G)$ ((a_4, b_1) or (a_4, b_3) is in $E(G)$) because vertices b_2, b_3, b_4 and a_1 (b_1, b_2, b_3 and a_4) otherwise induce a claw in G. Thus at least one edge of each pair $\{(a_1, b_2), (a_1, b_4)\}$, $\{(a_4, b_1), (a_4, b_3)\}$ is in $E(G)$.

There is a need to examine the following cases:

Case 3.3.1: (a_1, b_4), $(a_4, b_3) \in E(G)$. (See the right graph in Fig. 5.) $(a_4, b_3) \in E(G)$ implies (vertices a_3, a_4, O, b_3 cannot induce a claw) $(a_3, b_3) \in E(G)$. From $(a_1, b_4) \in E(G)$ it follows $(a_2, a_1, b_4, b_1$ do not induce a claw) that $(a_2, b_1) \in E(G)$ or $(a_2, b_4) \in E(G)$). If $(a_2, b_2) \in E(G)$ then the diameter of $C_a \cup C_b$ is at most two. If $(a_2, b_4) \in E(G)$ then $(a_2, b_4, O, b_3$ do not induce a claw) $(a_2, b_3) \in E(G)$. Thus the distance between any two vertices $a_i, b_j \in V(C_a \cup C_b) \setminus N_G[O]$ in G is at most two.

Case 3.3.2: (a_1, b_2), $(a_4, b_1) \in E(G)$. This case is 'symmetric' about the previous case.

Case 3.3.3: (a_1, b_4), $(a_4, b_1) \in E(G)$ (see the left graph in Fig. 6). If $(a_1, b_2) \in E(G)$ then we arrive at Case 3.3.2. If $(a_4, b_3) \in E(G)$ then this is Case 3.3.1. Assuming that $(a_1, b_2), (a_4, b_3) \notin E(G)$ we obtain $(a_1, b_3) \in E(G)$ (a_4, b_4, b_3, a_1 induce a claw otherwise) and $d_G(a_2, b_3) \leqslant 2$. Furthermore, vertices a_2, a_1, b_3, b_1 do not induce a claw in G; hence $(a_2, b_3) \in E(G)$ or $(a_2, b_2) \in E(G)$. Therefore, $d_G(a_2, b_2) \leqslant 2$. Vertices a_4 b_2 are adjacent in G because a_1, b_1, b_2, a_4 otherwise induce a claw (see the right graph in Fig. 6). Then $d_G(a_3, b_2) \leqslant 2$. The graph induced by a_3, a_4, b_2, b_4 is not a claw; hence $d_G(a_3, b_3) \leqslant 2$. Thus for $i, j = \{2, 3\}$, $d_G(a_i, b_j) \leqslant 2$.

Case 3.3.4: (a_1, b_2), $(a_4, b_3) \in E(G)$. The claw-free condition for a_1, a_2, O, b_2 implies $(a_2, b_2) \in E(G)$ and the claw-free condition for a_3, a_4, O, b_3 implies $(a_3, b_3) \in E(G)$. This implies that for any i,j except $i=3$ and $j=1$ the distance from a_i to b_j is at most 2. To fix the case a_3, b_1 we consider the triple a_2, a_4, b_1. The AT-free condition for these vertices combining with the fact that cycles C_a and C_b are chordless, implies that at least one edge from the set $\{(a_2,b_1),(a_3,b_1),(a_4,b_1)\}$ is in $E(G)$. Therefore, the distance between a_3 and b_1 is at most 2.

We proved that for any $a,b \in N_{G_I}[O]$ the distance between a and b in G is at most two. Therefore, there is a clique K in G^2 such that $N_{G_I}[O] \subseteq K$. To conclude the proof, it remains to note that

$$\mathrm{id}(G) = \deg_{G_I}(O) = |N_{G_I}[O]| - 1 \leqslant |K| - 1 \leqslant \omega(G^2) - 1.$$

The following lemma is the corollary of a more general result of Ho et al. [13] about powers of graphs with bounded asteroidal numbers.

Lemma 16. *If G is AT-free then G^2 is AT-free.*

Lemma 17. *Let G be an AT-free claw-free graph. Then G^2 is AT-free claw-free.*

Proof. By Lemma 16 G^2 is AT-free. Suppose that there exist vertices b,c,d and a inducing the claw K in G^2, where a is the vertex of degree three. Note that at least two edges of K are from $E(G^2)\setminus E(G)$. If all edges of K are from $E(G^2)\setminus E(G)$ then vertices b,c,d form the AT in G.

Assume that only two edges, say (b,a) and (c,a), are in $E(G^2)\setminus E(G)$. Then $d_G(b,a) = d_G(c,a) = 2$. Let x be a vertex adjacent to vertices a, b and y be a vertex adjacent to a and c in G. Vertices b and c are not adjacent to d in G^2; hence x and y are not adjacent to d in G. Because G is claw-free, $(x,y)\in E(G)$; whence it follows that b,c,d form the AT in G. This is a contradiction and concludes the proof of Lemma 17.

The following statement is due to Parra and Scheffler.

Theorem 18 (Parra and Scheffler [24]). *Let G be an AT-free claw-free graph. Then* $\mathrm{bw}(G) = \mathrm{pw}(G)$.

There are different ways to define the treewidth of a graph (see, e.g. [16]). For more information on this parameter the reader is referred to the recent survey paper of Bodlaender [4]. The following definition is more convenient for our purposes. The *treewidth* $\mathrm{tw}(G)$ of a graph G is the smallest clique number over all triangulations of G decreased by one.

The next Theorem summarize the results of this section.

Theorem 19. *For any AT-free claw-free graph G,*

$$\mathrm{id}(G) = \omega(G^2) - 1 = \mathrm{pw}(G^2) = \mathrm{tw}(G^2) = \mathrm{bw}(G^2).$$

Proof. Let G be an AT-free claw-free graph. By Lemma 17 G^2 is also AT-free claw-free. By Theorem 8 $\mathrm{pw}(G^2) = \mathrm{tw}(G^2)$ and by Theorem 18 $\mathrm{pw}(G^2) = \mathrm{bw}(G^2)$. Since for any graph G, $\mathrm{pw}(G) \geqslant \omega(G) - 1$ we have that Corollary 3 and Theorem 15 imply $\mathrm{pw}(G^2) = \omega(G^2) - 1 = \mathrm{id}(G)$.

5. Complexity results

In [27] Unger proved that for any integer k there is no efficient approximation algorithm with performance ratio k (unless P=NP) of bandwidth for caterpillars with hair length three. Combining Unger's results with Corollary 2 we obtain the following complexity result.

The problem of Interval Completion With the Smallest Max-degree:

Instance: A graph G and an integer k.
Question: Is there an interval supergraph I of G such that $\Delta(I) \leqslant k$?

is NP-complete even when G is stipulated to be a caterpillar with hair length three.

However, the interval degree of AT-free claw-free graphs can be computed efficiently. We need the following result of Müller.

Lemma 20 (Müller [23]). *Let G be an AT-free claw-free graph. Then G^2 is a chordal graph.*

As far as we know the proof of Lemma was not published and we give it here for completeness.

Proof. Let $C = (x_1, x_2, \ldots, x_l)$, be a chordless cycle in G^2 and let $X = \{x_1, x_2, \ldots, x_l\}$. Since C is chordless, we have that for every index $i \in \{1, \ldots, l\}$ at least one of the edges (x_{i-1}, x_i) and (x_i, x_{i+1}) (summation is taken modulo l) is not present in G. For every edge (x_i, x_{i+1}) not present in G there is a vertex y_i such that $N_G[y_i] \cap X = \{x_i, x_{i+1}\}$. Let F be the subgraph of G induced by $X \cup Y$. F has a hamiltonian cycle Z created from C by adding y_i between x_i and x_{i+1}. Since the number of edges of Z is at least six, we have that Z has at least one chord e. C is chordless in G^2 and therefore at least one end of e is not in X.

If one end of e, say y_i, is in Y and another end, say x_j, is in X (notice that $i \neq j,\ j+1$) then $(x_i, x_j), (x_j, x_{i+1}) \in E(G^2)$ and C is not chordless. Hence both ends of e, say y_i, y_j are in Y. The latter contradicts G being claw-free because x_i, x_{i+1}, y_i, y_j induce a claw in G.

Lemmas 17 and 20 together imply that if G is AT-free claw-free and chordal then G squared is AT-free claw-free and chordal, which in turn implies

Corollary 21. *Let G be an AT-free claw-free graph. Then G^2 is a proper interval graph.*

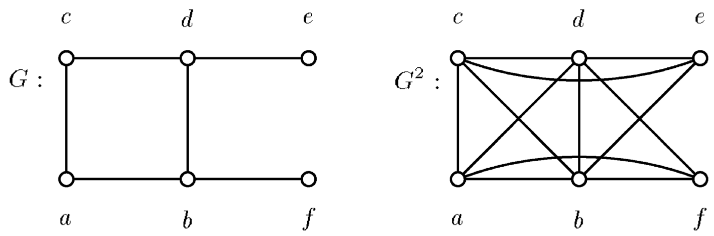

Fig. 7. Example of $\mathrm{id}(G) = \mathrm{pw}(G^2) + 1$. $\mathrm{id}(G) = 4$; every triangulation of G contains (a, d) or (b, c). $\mathrm{pw}(G^2) = 3$; G^2 is an interval graph and $\omega(G^2) = 4$.

This corollary suggest simple algorithm computing interval degree of AT-free claw-free graphs. (Let us remind that "related" pathwidth and bandwidth problems remain NP-complete on this graph class.)

In fact, it easy to check that G^2 can be constructed in $O(n|E(G)|)$ time. Since the clique number of an interval graph can be calculated in a linear time (see, e.g. [12]) we have that Theorem 19 and Corollary 20 imply.

Corollary 22. *For any AT-free claw-free graph* G, $\mathrm{id}(G)$ *can be calculated in* $O(n|E(G)| + |E(G^2)|)$ *time.*

6. Concluding remarks

We leave many questions unanswered, a few of them are:

(1) Fig. 7 shows that the claw-free condition in Theorem 19 is necessary. Is it true that there exists $k \geqslant 0$ such that for any AT-free graph G $\mathrm{id}(G) \leqslant \mathrm{pw}(G^2) + k$?
(2) Kloks et al. [17] obtained an $O(|E(G)| + n \log n)$ algorithm to approximate the bandwidth of an AT-free graph within a factor 4. Because of Corollary 14 it is interesting to know whether calculation of the max-degree of an AT-free graph squared can be done faster.
(3) Is it possible to improve the time bound in Corollary 22? Probably the construction of the square is not necessary for calculating $\omega(G^2)$.

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