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On the monotonicity of games generated by symmetric submodular functions

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Abstract

Submodular functions have appeared to be a key tool for proving the monotonicity of several graph searching games. In this paper, we provide a general game theoretic framework able to unify old and new monotonicity results in a unique min–max theorem. Our theorem provides a game theoretic analogue to a wide number of graph theoretic parameters such as linear-width and cutwidth.

1. Introduction

A considerable part of graph theory is oriented to the following general problem: Given a non-trivial graph property P, find a complete characterization of the graphs that do not satisfy P. In general, such a characterization is achieved by describing some "forbidding" structure whose existence in G obstructs P from being satisfied. As a first example, we mention the Kuratowski theorem asserting that a graph is non-planar iff it topologically contains an obstructing structure of two forbidden graphs. Many

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graph theoretic parameters have been characterized by their obstructing analogue and this characterization is typically achieved by a so-called min–max theorem (for a good source of min–max theorems, see [5]). Several examples of such characterizations emerge from the work of Robertson and Seymour on their Graph Minors series. As a sample, we mention the characterization of treewidth via screens [20], of branchwidth via tangles [19], of pathwidth via blockages [3], and of carving-width via antipodalities [21] (see also [10,14]).

In many cases, it appears that min–max theorems are useful in the study of graph searching games. Roughly, a graph searching game has as objective to capture an omniscient fugitive residing on the vertices or/and the edges of a graph by systematically moving a specific number on searchers on it. Such a search strategy can, in general, allow "recontamination" in the sense that it can let the fugitive occupy parts of G that have already been searched. If this is not the case the strategy is called *monotone*. The "monotonicity question" for a graph searching game asks whether "allowing recontamination" makes the capturing of the fugitive easier than in the case where the player restricts his/her attention *only* to monotone strategies.

It frequently appears that the monotonicity question can be answered with the use of a suitable min–max theorem. The reason is that, in the majority of the cases, the minimum number of searchers required to capture the fugitive is equivalent to some known graph theoretic parameter. The existence of a min–max theorem for this parameter implies the existence of a forbidden structure that, in turn, indicates an escape strategy for the fugitive no matter whether the searching strategy is monotone or not. That way, the min–max theorems for treewidth and pathwidth, developed in [3,20], were the cornerstones for proving the monotonicity of the corresponding graph searching variants (see also [7]).

Our paper is motivated and constitutes an extension of the ideas in the proofs of the monotonicity of the agile fugitive search games examined in [2–4,12,13,15,17,22] as well as the proofs of the min–max theorems in [19,21]. Our main observation is that the kernel argument of all these proofs is based on the fact that, in any game variant, the cost of the search can be expressed by a connectivity function that is a non-negative-valued function on the set of subsets of a set M that is invariant over complement and satisfies the submodular property.

In this paper, we show how a graph parameter can be generated by a connectivity function and we prove a general min–max theorem for it. Moreover, we define for any such parameter a one-player conquest game and we use our min–max theorem in order to prove its monotonicity. Our game framework is general in the sense that it provides a big variety of games depending on the choice of the connectivity function α that generates it. In particular, for suitable choices of α, it provides obstruction characterizations, game counterparts, and monotonicity proofs for the parameters of linear-width, cutwidth and their extensions. Finally, our general min–max theorem implies in a uniform way the monotonicity proofs of all the agile fugitive search games developed so far in [4,12,15,17,22].

To illustrate the main motivation of our research let us give a simple example of an "expansion" game. Suppose that we have a set of countries subject to join some organization. At every moment of time we can either add a bounded number of countries to the union or expel an arbitrary number of countries. Adding countries and keeping them in the union needs some resources, say for guarding its border from the countries outside the union. The question is how to add all the countries using the minimal amount of resources.

More concretely, let M be the set of faces of a planar map, every face of M representing a country. For a set of countries $X \subseteq M$, let $\alpha(X)$ be the number of edges adjacent to a face in X and a face in $\bar{X} = M - X$. The question is: for a given $k \geqslant 0$ is there a sequence

$$(\emptyset = X_0, X_1, \ldots, X_n = M),$$

such that for every $1 \leqslant i \leqslant n$, $|X_i - X_{i-1}| \leqslant m$ (at every step we add at most m countries) and $\alpha(X_i) \leqslant k$ (at every step of the expansion the border of the union is of length at most k)?

There are a number of interesting questions concerning this game. One of them is the "monotonicity question": can it be useful at some step to expel countries from the union? Another is the "min–max question": what kind of structure in a map provides necessary and sufficient conditions for obstructing the intended expansion? In this paper, we provide the answers to both questions for a more general version of expansion game.

The paper is organized as follows. The main min–max theorem is proved in Section 2. In Section 3, we present our general game, and in Section 4, we present the consequences of the min–max theorem on its variants. In Section 5, we end up with further examples, remarks, and open problems.

2. A min–max theorem for connectivity functions

Given a finite set M, a function α mapping the subsets of M to integers is called a *connectivity function* for M [19] if the following two conditions are satisfied:

$$\forall A \subseteq M, \quad \alpha(A) = \alpha(\bar{A}) \text{ (we denote } \bar{A} = M - A), \tag{1}$$

$$\forall A, B \subseteq M, \quad \alpha(A \cup B) + \alpha(A \cap B) \leqslant \alpha(A) + \alpha(B). \tag{2}$$

Note that for any $X \subseteq M$, $\alpha(X) \geqslant \alpha(\emptyset)$ because $2\alpha(X) = \alpha(X) + \alpha(M - X) \geqslant \alpha(\emptyset) + \alpha(M) = 2\alpha(\emptyset)$. It is more convenient for our purposes to think that $\alpha(\emptyset) = 0$ (replacing $\alpha'(X) = \alpha(X) - \alpha(\emptyset)$ if it is not the case).

To warm up, let us consider some examples of connectivity functions. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$.

- Take $M = E(G)$ and for any $A \subseteq M$, let $\alpha(A)$ be the number of vertices incident both with an edge in A and an edge in $M - A$.
- Take $M = V(G)$ and for any $A \subseteq M$, let $\alpha(A)$ be the number of edges incident both with a vertex in A and a vertex in $M - A$.
- Another example is a matroid M with rank function χ, where we define, for any $A \subseteq M$, $\alpha(A) = \chi(A) + \chi(M - A)$.

For a given finite set M, let $\mathcal{O}$ be a set of subsets of M and let α be a connectivity function for M. A sequence $\mathscr{A}=(A_1,\dots,A_r)$ is a (k,m)-*expansion in* $\mathcal{O}$ (*for* M *with respect to a connectivity function* α) if

$$\forall i, 1\leqslant i\leqslant r, \quad A_i\subseteq M \quad \text{and} \quad \alpha(A_i)\leqslant k, \tag{3}$$

$$A_1, \bar{A}_r\in\mathcal{O}, \tag{4}$$

$$\forall i, 1\leqslant i\leqslant r-1, \quad |A_{i+1}-A_i|\leqslant m. \tag{5}$$

If, additionally,

$$\forall i, 1\leqslant i\leqslant r, \quad A_i\subseteq A_{i+1}, \tag{6}$$

then we say that $\mathscr{A}$ is *monotone*. Finally, a (k,m)-expansion in $\{\varnothing\}$, i.e. expansion with $A_1=\varnothing$ and $A_r=M$, is referred to as *complete*.

For facilitating the notation, in this section we will assume that all the expansions are defined with respect to a fixed connectivity function α. The following lemma uses the ideas of Bienstock–Seymour's monotonicity proof for graph crusades [4]. We include the proof of the lemma for completeness.

Lemma 1. *If there is a* (k,m)-*expansion in* $\mathcal{O}$ *for* M *then there is a monotone* (k,m)-*expansion in* $\mathcal{O}$ *for* M.

Proof. Let us choose a (k,m)-expansion $(A_1,A_2,\dots,A_r)$ in $\mathcal{O}$ for M such that

$$\sum_{i=1}^{r}\alpha(A_i) \text{ is minimum} \tag{7}$$

and, subject to (7),

$$\sum_{i=1}^{r}(|A_i|+1) \text{ is minimum.} \tag{8}$$

To prove (6), we will show that $A_{i-1}\subseteq A_i$ for $i\in\{2,\dots,r\}$.

We first claim that

$$\alpha(A_{i-1}\cup A_i)\geqslant\alpha(A_i). \tag{9}$$

Suppose that $\alpha(A_{i-1}\cup A_i)<\alpha(A_i)$. Then because $(A_{i-1}\cup A_i)-A_{i-1}=A_i-A_{i-1}$ we have that

$$|(A_{i-1}\cup A_i)-A_{i-1}|\leqslant m. \tag{10}$$

Also,

$$|A_{i+1}-(A_{i-1}\cup A_i)|\leqslant|A_{i+1}-A_i|\leqslant m. \tag{11}$$

Combining (10) and (11) we obtain that

$$(A_1,A_2,\dots,A_{i-1},A_{i-1}\cup A_i,A_{i+1},\dots,A_r)$$

is a (k,m)-expansion for M contradicting (7). This contradiction proves (9).

Combining (9) and (2), we obtain

$$\alpha(A_{i-1} \cap A_i) \leqslant \alpha(A_{i-1}). \tag{12}$$

Since

$$|(A_{i-1} \cap A_i) - A_{i-2}| \leqslant |A_{i-1} - A_{i-2}| \leqslant m$$

and

$$|A_i - (A_{i-1} \cap A_i)| = |A_i - A_{i-1}| \leqslant m,$$

it follows that

$$(A_0, A_1, \ldots, A_{i-2}, A_{i-1} \cap A_i, A_i, A_{i+1}, \ldots, A_r)$$

is a (k,m)-expansion for M. Taking into account (12), (7) and (8), we get $|A_{i-1} \cap A_i| \geqslant |A_{i-1}|$. Thus $A_{i-1} \subseteq A_i$ and $(A_1, A_2, \ldots, A_r)$ is a monotone (k,m)-expansion for M. □

Lemma 2. *Let $\mathscr{A} = (A_1, A_2, \ldots, A_r)$ be a monotone (k,m)-expansion for M. Then $\bar{\mathscr{A}} = (\bar{A}_r, \bar{A}_{r-1}, \ldots, \bar{A}_1)$ is also a monotone (k,m)-expansion for M.*

Proof. Since $\bar{\bar{A}}_1 = A_1$ and $\alpha(\bar{A}_i) = \alpha(A_i)$, we have that $\bar{\mathscr{A}}$ satisfy (3) and (4). For $i \in \{1, \ldots, r-1\}$, $A_{i+1} \supseteq A_i$ implies that $\bar{A}_i - \bar{A}_{i+1} = A_{i+1} - A_i$ and $\bar{A}_{i+1} \subseteq \bar{A}_i$. Therefore, (5) and (6) are also satisfied. □

For any integer k, we define a (k,m)-*obstacle* for M as the set $\mathcal{O}$ such that

(o1) Each $A \in \mathcal{O}$ is a subset of M with $\alpha(A) \leqslant k$.
(o2) If $A \in \mathcal{O}$, $B \subseteq A$ and $\alpha(B) \leqslant k$ then $B \in \mathcal{O}$.
(o3) If $A, B, C \subseteq M$, $A \cap B = \emptyset$, $\alpha(A) \leqslant k$, $\alpha(B) \leqslant k$, $|C| \leqslant m$, and $A \cup B \cup C = M$ then $(A \in \mathcal{O} \wedge B \notin \mathcal{O})$ or $(A \notin \mathcal{O} \wedge B \in \mathcal{O})$.

The aim of this section is to prove that the existence of a (k,m)-obstacle for M, obstructs the existence of a complete and monotone (k,m)-expansion for M and vice versa. In particular, we will prove the following min–max theorem.

Theorem 3. *Let M be a finite set and α a connectivity function on M. Then the following are equivalent*:

(i) *there exists no (k,m)-obstacle for M*;
(ii) *there exists a complete (k,m)-expansion for M*;
(iii) *there exists a complete and monotone (k,m)-expansion for M.*

For the proof of Theorem 3, we need first to prove Lemma 4 below. A set $\mathcal{O}$ is *partial (k,m)-obstacle* for M if it satisfies (o1) and (o2) and if there is no (k,m)-expansion for it. The next lemma is an analogue of the blockage Theorem (2.5) in [3].

Lemma 4. *Every partial (k,m)-obstacle for M is a subset of a (k,m)-obstacle for M.*

Proof. Let $\mathcal{O}$ be a partial (k,m)-obstacle for M. We assume that every $\mathcal{O}'$, where $|\mathcal{O}'| > |\mathcal{O}|$ and satisfying (o1) and (o2), either is not a partial (k,m)-obstacle for M, or is a subset of a (k,m)-obstacle for M. As base of the induction we can consider a set $\mathcal{O}_0$ containing any subset A of M, where $\alpha(A) \leqslant k$. Indeed the set $\mathcal{O}_0$ is not a partial (k,m)-obstacle. In fact, $\emptyset$, $M \in \mathcal{O}_0$ because $\alpha(\emptyset) = \alpha(M) = 0 \leqslant k$ and $(\emptyset)$ is a (k,m)-expansion in $\mathcal{O}_0$.

If $\mathcal{O}$ satisfies (o1)–(o3) then $\mathcal{O}$ is a (k,m)-obstacle for M and the lemma holds. Suppose that $\mathcal{O}$ does not satisfy (o3). Then there exist $A_1, A_2, C \subseteq M$ such that

- $\alpha(A_1) \leqslant k$, $\alpha(A_2) \leqslant k$, $|C| \leqslant m$;
- $A_1 \cup A_2 \cup C = M$ and $A_1 \cap A_2 = \emptyset$;
- $A_1, A_2 \in \mathcal{O}$ or $A_1, A_2 \notin \mathcal{O}$.

For our purpose it is more convenient to work with complements of A_1 and A_2. Let $T_1 = \bar{A}_1$ and $T_2 = \bar{A}_2$. Note that

$$\alpha(T_1) \leqslant k \quad \text{and} \quad \alpha(T_2) \leqslant k, \tag{13}$$

$$T_1 \cup T_2 = M, \tag{14}$$

$$|T_1 \cap T_2| \leqslant m. \tag{15}$$

We claim that

$$T_1, T_2 \notin \mathcal{O}. \tag{16}$$

Indeed, note first that, from (14), $A_1 \cap A_2 = \emptyset$ and thus $T_1 \supseteq A_2$. By (o2), if $T_1 \in \mathcal{O}$ then $A_2 \in \mathcal{O}$. Since $A_2 \in \mathcal{O}$, we have that $A_1 \in \mathcal{O}$. Then (A_1) is a (k,m)-expansion in $\mathcal{O}$, which is impossible. The proof of $T_2 \notin \mathcal{O}$ is similar.

We now choose T_1, T_2 satisfying (13)–(16) such that $|T_1|$ is minimal. We claim that

$$\text{If } X \subseteq T_1 \text{ and } \alpha(X) \leqslant k \text{ then either } X \in \mathcal{O}, \text{ or } X = T_1, \text{or } \bar{T}_1 \in \mathcal{O}. \tag{17}$$

We assume that $X \notin \mathcal{O}$ and $X \neq T_1$ and we will first prove that $\bar{X} \in \mathcal{O}$. For this, we will show that if $\bar{X} \notin \mathcal{O}$, then $X, \bar{X}$ satisfy (13)–(16), contradicting the choice of T_1, T_2. Indeed, (13) follows from $\alpha(X) \leqslant k$ and (1); (14) is obvious and (15) follows as $X \cap \bar{X} = \emptyset$. We now conclude the proof of (17) noting that $\bar{T}_1 \subset \bar{X}$ and (o2) gives $\bar{T}_1 \in \mathcal{O}$ as required.

For $i = 1,2$ we define $\mathcal{O}_i$ as the set of all $X \subseteq T_i$ with $\alpha(X) \leqslant k$. Note that, for $i = 1,2$, $T_i \notin \mathcal{O}$ implies $|\mathcal{O} \cup \mathcal{O}_i| > |\mathcal{O}|$. Therefore, the result follows from the induction hypothesis if we show that for some $i = 1,2$, $\mathcal{O} \cup \mathcal{O}_i$ is a partial (k,m)-obstacle for M. Clearly, (o1) and (o2) are satisfied for both $\mathcal{O} \cup \mathcal{O}_i$, $i = 1,2$ and therefore, it remains to prove that, for some $i = 1,2$, there is no (k,m)-expansion in $\mathcal{O} \cup \mathcal{O}_i$. Assume, towards a contradiction, that, for $i = 1,2$ there are (k,m)-expansions in $\mathcal{O} \cup \mathcal{O}_1$ and $\mathcal{O} \cup \mathcal{O}_2$ and, by Lemma 1, we can also assume that they are monotone.

Let us prove that

There is a monotone (k,m)-expansion

$$\mathscr{X} = (X_1, X_2, \dots, X_r) \text{ in } \mathscr{O}_1 \text{ with } X_1 = T_1 \text{ and } \bar{X}_r \in \mathscr{O}. \tag{18}$$

If $\bar{T}_1 \in \mathscr{O}$ then $\mathscr{X} = (T_1)$ is a (k,m)-expansion in $\mathscr{O}_1$ satisfying (18).

Suppose that $\bar{T}_1 \notin \mathscr{O}$. Let $\mathscr{X} = (X_1, X_2, \dots, X_r)$ be a monotone (k,m)-expansion in $\mathscr{O} \cup \mathscr{O}_1$. Then $X_1, \bar{X}_r \in \mathscr{O}_1$. By Lemma 2, $\bar{\mathscr{X}} = (\bar{X}_r, \bar{X}_{r-1}, \dots, \bar{X}_1)$ is also monotone (k,m)-expansion in $\mathscr{O} \cup \mathscr{O}_1$.

One of the sets X_1 and $\bar{X}_r$ is not in $\mathscr{O}$ because there is no (k,m)-expansion in $\mathscr{O}$. W.l.o.g. we can assume that $X_1 \notin \mathscr{O}$. (Otherwise we can consider $\bar{\mathscr{X}}$.) As $X_1 \in \mathscr{O} \cup \mathscr{O}_1$, we have that $X_1 \in \mathscr{O}_1$ and, from the definition of $\mathscr{O}_1$, we conclude that $X_1 \subseteq T_1$. By (17), (because $X_1 \notin \mathscr{O}$ and $\bar{T}_1 \notin \mathscr{O}$) $X_1 = T_1$.

From the monotonicity of $\mathscr{X}$ it follows that $T_1 = X_1 \subseteq X_r$. We claim that either $\bar{X}_r \notin \mathscr{O}_1$ or $\bar{X}_r = \emptyset$. Indeed, if $\bar{X}_r \in \mathscr{O}_1$, then the definition of $\mathscr{O}_1$ implies that $\bar{X}_r \subseteq T_1$. Moreover, $T_1 \subseteq X_r$ implies that $\bar{X}_r \subseteq \bar{T}_1$ and $\bar{X}_r = \emptyset$ follows.

Note now that (18) holds trivially if $\bar{X}_r = \emptyset \in \mathscr{O}$. Moreover, if $\bar{X}_r \notin \mathscr{O}_1$, then $\bar{X}_r \in \mathscr{O} \cup \mathscr{O}_1$ implies that $\bar{X}_r \in \mathscr{O}$ and this concludes the proof of (18).

Let $\mathscr{Y} = (Y_1, Y_2, \dots, Y_s)$ be a monotone (k,m)-expansion in $\mathscr{O}_2$. Since there is no (k,m)-expansion in $\mathscr{O}$, we have that either Y_1 or $\bar{Y}_s$ is not in $\mathscr{O}$. W.l.o.g. we can assume that $Y_1 \notin \mathscr{O}$ (otherwise we replace $\mathscr{Y}$ by $\bar{\mathscr{Y}}$). Then $Y_1 \subseteq T_2$, yielding $|Y_1 - \bar{X}_1| \leqslant |T_2 - \bar{T}_1| \leqslant m$.

We now claim that $\bar{Y}_r \notin \mathscr{O}$. Indeed, if this is not the case, then

$$(\bar{X}_r, \bar{X}_{r-1}, \dots, \bar{X}_1, Y_1, Y_2, \dots, Y_s)$$

is a (k,m)-expansion in $\mathscr{O}$, contradicting to the fact that there are no (k,m)-expansions in $\mathscr{O}$.

Recall that $\bar{Y}_s \in \mathscr{O} \cup \mathscr{O}_2$ and, as $\bar{Y}_s \notin \mathscr{O}$, we get that $\bar{Y}_s \in \mathscr{O}_2$, thus $\bar{Y}_s \subseteq T_2$. But $Y_s \supseteq \bar{T}_2$ combined with (15), implies $|X_1 - Y_s| = |T_1 - Y_s| \leqslant |T_1 - \bar{T}_2| = |T_1 \cap T_2| \leqslant m$. As a consequence,

$$(\bar{X}_r, \bar{X}_{r-1}, \dots, \bar{X}_1, Y_1, Y_2, \dots, Y_s, X_1, X_2, \dots, X_r)$$

is a (k,m)-expansion in $\mathscr{O}$. This is again a contradiction and the lemma is proved. □

We can now proceed with the proof of Theorem 3.

Proof of Theorem 3. The fact that (ii) ⇒ (iii) follows directly from Lemma 1.

Next, we will prove that (iii) ⇒ (i). Suppose, on contrary that $\mathscr{O}$ is a (k,m)-obstacle for M and $\mathscr{A} = (A_1, \dots, A_r)$ a complete and monotone (k,m)-expansion for M. Recall that from (3) we have that $\alpha(A_1) \leqslant k$ and from (o2), $\emptyset = A_1 \in \mathscr{O}$. Condition (o3) now implies $A_r = M \notin \mathscr{O}$. Let i be the smallest i, $0 \leqslant i < r$ such that $A_{i+1} \notin \mathscr{O}$. Let $A_{i+1} - A_i = C$ and (5) gives $|C| \leqslant m$. Note that $A_i \cup C \cup \bar{A}_{i+1} = M$. Moreover, $A_i \cap \bar{A}_{i+1} = \emptyset$ and therefore (o3) implies that $A_i \notin \mathscr{O}$, a contradiction.

It remains now to prove (i) ⇒ (ii). It is enough to show that there exists some (k,m)-expansion in $\{\emptyset\}$. Indeed, if this is not the case and since $\{\emptyset\}$ satisfies (o1) and (o2), we get that $\{\emptyset\}$ is a partial (k,m)-obstacle for M. From Lemma 4, $\{\emptyset\}$ is the subset of some (k,m)-obstacle for M, a contradiction. □

3. A general framework for conquest games

In this section, we will introduce a general framework of a game on graphs, hypergraphs and sets based on the notion of (k,m)-expansions.

In particular, we assume that α is a connectivity function for some set M. The elements of M represent countries to be conquested. In the beginning, all the elements of M are considered unoccupied. The conquest proceeds in steps. Each step can either be the occupation of at most m new elements of M or the retreat from some already occupied elements of M. We define a *move* as a pair $P=(S,\omega)$, where S is a set of elements of M and ω equals 0 when these elements are removed from the set of conquested elements (retreat move) and 1 when these elements are added to the set of conquested elements (attack move). A *conquest strategy* for M is a sequence $\mathscr{E}=(P_1,\dots,P_r)$ of moves. The *aggressivity* of a conquest strategy is $\max\{|S| \mid (S,1)\in\mathscr{E}\}$, i.e. the maximal number of elements occupied at some step. Given a conquest strategy $\mathscr{E}$, we recursively define the *occupation sequence* of $\mathscr{E}$ as the sequence $(T_0,\dots,T_r)$ where $T_0=\emptyset$, $T_i=T_{i-1}\cup S$ if $P_i=(S,1)$, and $T_i=T_{i-1}-S$ if $P_i=(S,0)$. The *α-cost* (or simply *cost* when there is no doubt about α) of a conquest strategy $\mathscr{E}$ is defined as $\max_{0\leqslant i\leqslant r}\alpha(T_i)$. We call a conquest strategy *monotone* if it does not contain any retreat move. We call a conquest strategy *successful* if T_r is the set of all the elements of M.

Lemma 5. *For any set M and any connectivity function α, there exists a successful (monotone) conquest strategy with α-cost k and aggressivity m if and only if there exists a complete (monotone) (k,m)-expansion for M with respect to α.*

Proof. Let $(A_1,\dots,A_r)$ be a complete (monotone) (k,m)-expansion for M with respect to α. We construct a sequence of moves $\mathscr{E}=(P_1',P_1,\dots,P_{r-1}',P_{r-1})$ such that for every $1\leqslant i\leqslant r-1$,

$$(P_i',P_i)=\begin{cases}((A_i-A_{i+1},0),(A_{i+1}-A_i,1)) & \text{if } \alpha(A_{i-1}\cap A_i)\leqslant k,\\ ((A_{i+1}-A_i,1),(A_i-A_{i+1},0)) & \text{if } \alpha(A_{i-1}\cup A_i)\leqslant k.\end{cases}$$

Note that, (2) and the fact that $\alpha(A_i)+\alpha(A_{i+1})\leqslant 2k$ imply that either $\alpha(A_{i-1}\cap A_i)\leqslant k$ or $\alpha(A_{i-1}\cup A_i)\leqslant k$. Therefore, $\mathscr{E}$ is well defined. It now follows directly from the definitions that $\mathscr{E}$ is a successful (monotone) conquest strategy with α-cost k and aggressivity m. Given now a successful (monotone) conquest strategy for G with α-cost k and aggressivity m, it is easy to see that its occupation sequence is a complete (monotone) (k,m)-expansion for $V(G)$ with respect to the function α.

Given a set M and a connectivity function α, we define a (k,m)-ordered partition of M as a linear ordering $(B_1,\dots,B_r)$ where $\{B_1,\dots,B_r\}$ is a partition of M, $\forall_{i,1\leqslant i\leqslant r}\,|B_i|\leqslant m$, and $\forall_{i,1\leqslant i\leqslant r}\,\alpha(B_1\cup\cdots\cup B_i)\leqslant k$. For any set M, we define the

(α, m)-*width* of M as the minimum k for which there exists a (k,m)-ordered partition of G. □

We can now conclude this section with the following.

Theorem 6. *Let M be a set, α be a connectivity function for M, and k, m two integers. The following assertions are equivalent*:

(1) *There exists a complete (k,m)-expansion for M with respect to the function α.*
(2) *There exists a complete monotone (k,m)-expansion for M.*
(3) *There exists no (k,m)-obstacle for M with respect to the function α.*
(4) *The (α, m)-width of M is at most k.*
(5) *There exists a successful conquest strategy for G with aggressivity m and α-cost k.*
(6) *There exists a successful monotone conquest strategy for G with aggressivity m and α-cost k.*

Proof. The equivalence of (1)–(3) follows directly from Theorem 3. The equivalence of (1) and (5) as well as the equivalence of (2) and (6) follows from Lemma 5. It remains to prove that (4) and (6) are equivalent. Let $(V_1,\dots,V_r)$ be a (k,m)-ordered partition of $V(G)$. We construct $\mathscr{E}=((S_1,1),\dots,(S_r,1))$ where for $i=1,\dots,r$, $S_i=V_1$ and observe that $\mathscr{E}$ is a complete monotone (k,m)-expansion for $V(G)$, as required. Finally, if $\mathscr{E}=((S_1,1),\dots,(S_r,1))$ is a successful monotone conquest strategy, then it is enough to note that $(S_1,\dots,S_r)$ is a (k,m)-ordered partition of $V(G)$. □

4. Applications

In this section, we will give some examples of games on graphs where the conquest game framework introduced in Section 2 can provide a min–max theorem.

4.1. Definitions

We give first general definitions of some well-known width-type parameters for graphs.

The cutwidth of graphs has been extensively considered and emerged as a tool for the study of VLSI layouts (see [1,16] for further references). We give below a natural generalization of its definition.

Let H be a graph with the vertex set $V(H)$ and the edge set $E(H)$. For $X \subseteq V(H)$ let $\alpha_0(X)$ be the number of edges incident to vertices in X and $V(H)-X$. Let $\phi=(B_1,B_2,\dots,B_n)$, $\max_{1\leqslant i\leqslant n}|B_i| \leqslant m$, be an ordered partition of $V(H)$. For $1\leqslant i\leqslant n$ we put $V_i=\bigcup_{j=1}^{i} B_j$ and

$$m\text{-cw}(H,\phi)=\max_{i\in\{1,\dots,n\}} \alpha_0(V_i).$$

The *m-cutwidth* of H is $\min\{m\text{-cw}(H,\phi)\colon \phi$ is an ordering of $V(H)\}$. So, in our terms, the m-cutwidth of a graph H is equivalent to its (α_0, m)-*width*. If in the definition above we set $m=1$, we have the definition of *cutwidth*.

The notion of linear width for graphs was introduced by Thomas [25]. Let G be an undirected and finite graph with vertex set $V(G)$ and edge set $E(G)$. For $X \subseteq E(G)$, let $\alpha_1(X)$ be the number of vertices incident to edges in X and $E(G)-X$. Let $\sigma=(e_1,e_2\ldots,e_m)$ be an ordering of $E(G)$. For $i\in\{1,\ldots,m\}$ we put $E_i=\bigcup_{j=1}^{i} e_j$. We define

$$\mathrm{lw}(G,\sigma) := \max_{i\in\{1,\ldots,m\}} \alpha_1(E_i)$$

and the *linear-width* of G is $\min\{\mathrm{lw}(G,\sigma)\colon \sigma$ is an ordering of $E(G)\}$. In a similar way, one can define the notion of linear-width for hypergraphs. In our terms the linear-width of G is the $(\alpha_1, 1)$-*width* of G. Certainly, it is possible to extend the linear-width to m-linear-width considering ordered partitions of edges in the fashion we did for m-cutwidth. This would define a parameter equivalent to (α_1, m)-*width*. Note that both definitions given in this subsection can be generalized to hypergraphs and can have various extensions based on a weight assignment function for the vertices or the (hyper)edges of G. Any of these versions are equivalent to some (α, m)-width for suitable choices of M, α and m.

4.2. A game for cutwidth

We consider first the case where M is the vertex set of a graph G and α maps any of its subsets S to the number of edges with endpoints in both S and $V(G)-S$. That way, (α, m)-width is the m-cutwidth of G and Theorem 3 provides an obstruction characterization and a min–max theorem for m-cutwidth (which in case $m=1$ is simply the cutwidth of G). For a more intuitive approach, we can consider the vertices of G as the countries of the world where existence of "common borders" between two countries implies the existence of an edge between the corresponding edges. The aggressivity of the game indicates how many countries are permitted to be occupied after each attack. The cost function α may indicate the resources the player can use in order to keep the occupied positions. For a more realistic scenario, we can assign a cost on each of the edges indicating the length of the corresponding border portion or simply the cost of guarding it. It is easy to check that, in any case, the cost function is a connectivity function and, therefore, Theorem 3 indicates that the player *can restrict* his/her attention only to monotone conquest strategies (those that do not contain retreats) as this does not imply any deterioration of his/her ability to take over the world.

4.3. Linear-width and search games on graphs

Recall that, using the terminology of the previous sections, the linear-width of a graph G is equivalent to its $(\alpha_1, 1)$-width. Theorem 3 provides an obstruction characterization and a conquest game for linear-width. In what follows, we will show how this conquest game can modelize the three variants of the graph searching game.

A *mixed searching game* is defined in terms of a graph representing a system of tunnels where an agile and omniscient fugitive with unbounded speed is hidden (alternatively, we can formulate the same problem considering that the tunnels are contaminated by some poisonous gas). The object of the game is to *clear* all edges, using one or more *searchers*. An edge of the graph is cleared if any of the following occurs:

A : *both of its endpoints are occupied by a searcher*,
B : *a searcher slides along it*, i.e., a searcher is moved from one endpoint of the edge to the other endpoint.

A search is a sequence containing some of the following moves: (i) placing a new searcher on v, (ii) deleting a searcher from v, (iii) sliding a searcher on v along $\{v,u\}$ and placing it on u.

The object of a mixed search is to clear all edges using a search. The search number of a search is the maximum number of searchers on the graph during any move. The mixed search number, $\mathrm{ms}(G)$, of a graph G is the minimum search number over all the possible searches of it. A move causes *recontamination* of an edge if it causes the appearance of a path from an uncleared edge to this edge not containing any searchers on its vertices or its edges. (Recontaminated edges must be cleared again.) A search without recontamination is called *monotone*.

The *node* (*edge*) *search number*, $\mathrm{ns}(G)$ ($\mathrm{es}(G)$) is defined similarly to the mixed search number with the difference that an edge can be cleared only if A (B) happens.

The following is a combination of results in [4,24].

Theorem 7. *For any graph G the following hold*:

- *If G^p is the graph occurring from G after subdividing its pendant edges, then* $\mathrm{ms}(G)=\mathrm{lw}(G^p)$. (*We call pendant any edge with an endpoint of degree* 1.)
- *If G^e is the graph occurring from G after subdividing each of its edges, then* $\mathrm{es}(G)=\mathrm{lw}(G^e)$.
- *If G^n is the graph occurring from G after replacing every edge in G with two edges in parallel, then* $\mathrm{ns}(G)=\mathrm{lw}(G^n)$.

We mention that the mixed search number is equivalent to the parameter of proper-pathwidth defined by Takahashi, Ueno, and Kajitani in [22,23]. It is also known that the node search number is equivalent to the pathwidth, the interval thickness, and the vertex separation number (see [8,11–13,18]).

Theorem 7 gives a way to transform any searching game to a conquest game for linear-width. Therefore, the obstruction characterization for linear-width provided by Theorem 3, can serve as an obstruction characterization for any variant of the search parameters. Applying Theorems 3 and 7, we can directly derive the monotonicity results of [2,13,15] for the three variants of the search game. In particular, we have the following.

Theorem 8. *Let G be a graph and k,m two positive integers. The following assertions are equivalent*:

(1) *There exists a k-expansion for $E(G^n)/E(G^e)/E(G^p)$ with respect to the function α_1.*
(2) *There exists a monotone k-expansion for $E(G^n)/E(G^e)/E(G^p)$ with respect to the function α_1.*
(3) *There exists no (k,m)-obstacle for $E(G^n)/E(G^e)/E(G^p)$ with respect to the function α_1.*
(4) *There exists a node/edge/mixed search for G using k searchers.*
(5) *There exists a monotone node/edge/mixed search for G using k searchers.*

5. Conclusions

It appears that Theorem 6 gives a framework to prove the monotonicity for any search/conquest game based on some connectivity function α. Several interesting versions of the conquest game can be generated like that. In particular, we can set α to be any connectivity function mapping vertex sets of $V(G)$ to non-negative integers. As an example, we can define $\alpha : V(G) \to \{0,\ldots,|V(G)|\}$ such that for $A \subseteq V(G)$, $\alpha(A)$ is the number of endpoints of the edges in $\{\{v,u\} \mid v \in A,\ u \in V(G)-A\}$, i.e. the number of countries lying on both sides of the frontier of the occupied area defined by A. By observing that this version of α is indeed a connectivity function, we can directly apply Theorem 3 and derive a min–max theorem as well as the monotonicity property of the corresponding game.

As a last application of Theorem 3, we define a conquest game on a plane graph where now the regions represent countries and the cost of the strategy is the maximum number of vertices incident to its frontier. This game seems similar to the "dual" version of the game of Section 4.3 when restricted to planar graphs. The only difference is that now the cost is determined by the vertices of the frontier and not by the edges. In both cases, the cost function is a connectivity function and the monotonicity property is a consequence of Theorem 3. We conjecture that the two games are equivalent, in the sense that when they have the same aggressivity, the existence of an optimal complete conquest strategy for one implies the existence of an optimal complete conquest strategy for the other.

Another way of obtaining game theoretical approaches to width-type parameters is the study of conquest games with an "average" criteria of optimality. For example, instead of the *α-cost* of a conquest strategy $\mathscr{E}$, one can define the *α-sum-cost* of a conquest strategy $\mathscr{E}$ as $\sum_{0 \leqslant i \leqslant r} \alpha(T_i)$. Similar to Lemma 1, it is possible to prove monotonicity results for expansion games with minimal sum-cost. This idea can be used to obtain the monotonicity result for the game related to the sum bandwidth problem [6] (or optimal linear arrangement). (See also [9] for similar approach to the interval completion problem.) Nevertheless, to find an analogue of Theorem 3, if any exists, appears to be an interesting and hard open problem.

In conclusion, we find interesting the question whether there are monotone search or conquest games where the cost is not expressed by a connectivity function.

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