

# Complexity of Approximating the Oriented Diameter of Chordal Graphs

— Fedor V. Fomin,<sup>1</sup> Martín Matamala,<sup>2</sup> and Ivan Rapaport<sup>2</sup>

<sup>1</sup>HEINZ NIXDORF INSTITUTE AND UNIVERSITY OF PADERBORN  
FÜRSTENALLEE 11, D-33102 PADERBORN, GERMANY  
E-mail: fomin@upb.de

<sup>2</sup>DEPARTAMENTO DE INGENIERÍA MATEMÁTICA AND  
CENTRO DE MODELAMIENTO MATEMÁTICO UMR 2071-CNRS  
UNIVERSIDAD DE CHILE, CASILLA 170-3  
CORREO 3, SANTIAGO, CHILE  
E-mail: mmatamal, irapapor@dim.uchile.cl

Received August 29, 2002; Revised August 19, 2003

**Abstract:** The oriented diameter of a bridgeless connected undirected (*bcu*) graph  $G$  is the smallest diameter among all the diameters of strongly connected orientations of  $G$ . We study algorithmic aspects of determining the oriented diameter of a chordal graph. We (a) construct a linear-time approximation algorithm that, for a given chordal *bcu* graph  $G$ , finds a strongly connected orientation of  $G$  with diameter at most one plus twice the oriented diameter of  $G$ ; (b) prove that, for every  $k \geq 2$  and  $k \neq 3$ , to

---

Contract grant sponsor: FONDAP on Applied Mathematics; Contract grant sponsor: Fondecyt; Contract grant numbers: 1020611 (to I.R.), 1010442 (to M.M.); Contract grant sponsor: IST Programme of the EU; Contract grant number: IST-1999-14186 (ALCOM-FT).

decide whether a chordal (split for  $k = 2$ ) *bcu* graph  $G$  admits an orientation of diameter  $k$  is *NP*-complete; (c) show that, unless  $P = NP$ , there is neither a polynomial-time absolute approximation algorithm nor an  $\alpha$ -approximation algorithm that computes the oriented diameter of a *bcu* chordal graph for  $\alpha < \frac{3}{2}$ . © 2004 Wiley Periodicals, Inc. *J Graph Theory* 45: 255–269, 2004

Keywords: *algorithms; chordal graphs; oriented diameter*

## 1. INTRODUCTION

When the linkage structure of a communication networks is modeled by a graph, its diameter corresponds to the maximum number of links over which a message between two nodes must travel. In cases where the number of links in a path is roughly proportional to the time delay or signal degradation encountered by messages sent along the path, the diameter is then involved in the complexity analysis for the performance of the networks.

A variety of interrelated diameter problems are discussed in the literature (see [4] for a survey and Chapter 2 in [2]), including the problem of finding orientations for undirected or mixed graphs to minimize diameters. This problem has a long history. In 1939 Robbins proved that an undirected graph  $G$  admits a strongly connected orientation (one where every two vertices are mutually reachable by directed paths) if and only if  $G$  is connected and bridgeless [21]. More recently, Chung, Garey, and Tarjan provided a linear-time algorithm for testing whether a graph has a strongly connected orientation and finding one if it does [5]. From a graph theoretical point of view, this problem has been studied for particular graphs: the torus, the Cartesian product of complete graphs, complete bipartite graphs, and others [12,14–17,27]. This problem also appears in the study of network routing, broadcasting, gossip and one-way street arrangements [4,7,13,22–25].

Chvátal and Thomassen studied how the diameter of a bridgeless connected undirected graph  $G$  and the diameter of a strongly connected orientation of  $G$  are related [6]. They considered the following problems.

**Oriented Diameter Problem (ODP).** Given a bridgeless connected undirected graph  $G$ , find a strongly connected orientation  $H$  with the smallest diameter.

**Oriented Diameter Decision Problem (ODP<sub>D</sub>( $k$ )).** Given a bridgeless connected undirected graph  $G$ , decide whether there exists a strongly connected orientation  $H$  with diameter at most  $k$ .

For every  $k \geq 2$  and  $k \neq 3$ , Chvátal and Thomassen showed that ODP<sub>D</sub>( $k$ ) is *NP*-complete for general graphs. Therefore, there is a natural interest to study complexity issues of ODP<sub>D</sub>( $k$ ) for different graph classes.

Chordal graphs form a very well-investigated class of graphs (see [9] for classical results and [3] for more recent ones). They have well-understood properties

and many *NP*-hard problems, such as Coloring, Clique, and Independent Set, can be solved in polynomial time when the input is restricted to chordal graphs. However, some other problems remain *NP*-complete even when restricted to chordal graphs (for instance Pathwidth [11] and Bandwidth [20]). For some of these problems, the nice properties of chordal graphs has been useful for constructing approximation algorithms (for instance Bandwidth [10,18]).

To the best of our knowledge, no work on the algorithmic aspects of  $\text{ODP}$  and  $\text{ODP}_D$  for chordal graphs has been done. Even though, the work of Chvátal and Thomassen suggests the existence of a 3-approximation algorithm for  $\text{ODP}$  when the input is restricted to graphs where every edge belongs to a triangle (which holds for chordal graphs). Indeed, they proved that for every bridgeless connected graph  $G$ , there is a strongly connected orientation  $H$  such that, if an edge  $\{u, v\}$  belongs to a cycle in  $G$  of length  $k$ , then  $(u, v)$  or  $(v, u)$  belongs to a directed cycle in  $H$  of length at most  $(k - 2)2^{\lfloor (k-1)/2 \rfloor} + 2$ .

This paper is motivated by the searching of both approximation algorithms and hardness results for  $\text{ODP}$  when restricted to chordal graphs.

## A. Definitions

Let  $G$  be either an undirected graph or a directed graph with vertex set  $V(G)$  and edge set  $E(G)$ . By  $\{u, v\}$  we denote the undirected edge with ends in  $u$  and  $v$  and by  $(u, v)$  we denote the directed arc, directed from  $u$  towards  $v$ . The distance  $d_G(u, v)$  between two vertices  $u$  and  $v$  of  $G$  is the length of the shortest path (the shortest directed path if  $G$  is directed) between  $u$  and  $v$  in the graph  $G$  (from  $u$  to  $v$  if  $G$  is directed). If there is no path from  $u$  to  $v$  then we put  $d_G(u, v) = +\infty$ . The diameter of a graph  $G$ , denoted by  $\text{diam}(G)$ , is defined to be the maximum distance between two vertices of  $G$ . Thus  $\text{diam}(G) = \max\{d_G(u, v) : u, v \in V(G)\}$ . When  $G$  is a directed graph, we denote by  $\vec{d}_G(u, v) = \max\{d_G(u, v), d_G(v, u)\}$ .

Let  $G$  be a connected undirected graph. An *orientation* of  $G$  is a directed graph whose arcs correspond to assignments of directions to the edges of  $G$ . An orientation  $H$  of  $G$  is *strongly connected* if every two vertices in  $H$  are mutually reachable in  $H$  ( $\text{diam}(H) < +\infty$ ). An edge  $e$  of  $G$  is called a *bridge* if  $G - e$  is not connected. The graph  $G$  is *bridgeless* if  $G - e$  is connected for every edge  $e$ , that is, there is no bridge in  $G$ . The oriented diameter of  $G$  is defined as follows.

$$OD(G) = \min\{\text{diam}(H) : H \text{ is an orientation of } G\}.$$

It was proved by Robbins in 1939 that  $G$  is not connected or has a bridge if and only if there is no strongly connected orientation of  $G$ . In that case,  $OD(G) = +\infty$ . Further we consider only bridgeless connected graphs.

An algorithm  $\mathcal{A}$  is an  $(\alpha, k)$ -*approximation algorithm* for  $\text{ODP}$  if for every graph  $G$  it runs in polynomial time and outputs an orientation  $H$  of  $G$  such that  $\text{diam}(H) \leq \alpha OD(G) + k$ .

An  $(\alpha, 0)$ -approximation algorithm for ODP is called an  $\alpha$ -approximation algorithm for ODP and a  $(1, k)$ -approximation algorithm for ODP is called an absolute approximation algorithm for ODP (see [1]).

A chord of a cycle  $C$  in  $G$  is an edge not in  $C$  that has both ends in  $C$ . A chordless or induced cycle in  $G$  is a cycle of length more than three that has no chord. A graph  $G$  is chordal if it contains no chordless cycles.

## B. Our Contribution

In Section 2, we show that for every chordal graph  $G$  there exists a linear-time computable orientation  $H$  such that, for every pair of vertices  $u$  and  $v$ ,  $\vec{d}_H(u, v) \leq 2d_G(u, v) + 1$ . Therefore, for every chordal graph  $G$  there is an orientation  $H$  such that  $\text{diam}(H) \leq 2 \text{diam}(G) + 1$ . On one hand, this result implies that for chordal graphs  $OD(G) \leq 2 \text{diam}(G) + 1$ . We prove that the bound is sharp by constructing an infinite sequence of chordal graphs such that, for every graph  $G$  in this sequence, any strongly connected orientation of  $G$  has diameter  $2 \text{diam}(G) + 1$ . On the other hand, since  $\text{diam}(G) \leq OD(G)$ , we deduce that ODP is  $(2, 1)$ -approximable.

In Section 3, we prove that  $ODP_D(2)$  remains  $NP$ -complete in the subclass of chordal graphs called split graphs. We also prove that, for  $k \geq 4$ ,  $ODP_D(k)$  remains  $NP$ -complete for chordal graphs. Moreover, we prove two non-approximability results: first, for every  $\alpha < 3/2$  (ODP) is not  $\alpha$ -approximable in the class of split graphs; second, there is no absolute approximation algorithm for ODP when restricted to chordal graphs.

## 2. POSITIVE RESULTS

Our algorithmical contribution is stated in the following theorem.

**Theorem 2.1.** *There is a linear-time  $(2, 1)$ -approximation algorithm for ODP in the class of chordal graphs.*

Since for every  $u$  and  $v$ , we have  $d_G(u, v) \leq \text{diam}(G) \leq OD(G)$  and  $\text{diam}(H) = \max_{u, v \in V} \vec{d}_H(u, v)$ , Theorem 2.1 follows from the next much stronger result.

**Theorem 2.2.** *There is a linear-time algorithm such that, given a chordal graph  $G$ , it computes an orientation  $H$  of  $G$  satisfying, for every pair of vertices  $u$  and  $v$ ,  $\vec{d}_H(u, v) \leq 2d_G(u, v) + 1$ .*

Notice that Theorem 2.2 is an improvement of the bound of Chvátal and Thomassen applied to chordal graphs. Moreover, as stated in Theorem 2.3, this bound is the best possible.

**Theorem 2.3.** *For every  $n \geq 1$  there exists a chordal graph  $G^n$  such that  $\text{diam}(G^n) = 2n + 1$  and  $\text{diam}(H) = 2 \text{diam}(G^n) + 1$  for every strongly connected orientation  $H$  of  $G^n$ .*

**Proof.** In Figure 1 we show a chordal graph  $G^2$  of diameter 5 for which there is no orientation with diameter smaller than  $2 \cdot 5 + 1$ . This construction can be easily generalized to larger graphs. To obtain  $G^{n+1}$  from  $G^n$  it suffices to add, for every vertex  $x$  in  $G^n$  of degree 2, two new vertices  $x'$ , and  $x''$  and connect them in such a way that  $x, x'$ , and  $x''$  form a copy of  $K_3$ . Then  $\text{diam}(G^{n+1}) = \text{diam}(G^n) + 2$ . Moreover, every strongly connected orientation  $H$  of  $G^n$  orients every copy of  $K_3$  in a cyclic way. Therefore, for every strongly connected orientation  $H^{n+1}$  of  $G^{n+1}$  and every strongly connected orientation  $H^n$  of  $G^n$ , we have that  $\text{diam}(H^{n+1}) = \text{diam}(H^n) + 4$ . Assuming inductively that  $\text{diam}(H^n) = 2 \text{diam}(G^n) + 1$ , we get that  $\text{diam}(H^{n+1}) = 2 \text{diam}(G^{n+1}) + 1$ . ■

The rest of this section is devoted to the proof of Theorem 2.2. The proof of this theorem is indirect. First, we prove that every 2-connected chordal graph has a special orientation that can be obtained in linear-time (Lemma 2.3). Then we use this orientation to prove the theorem for 2-connected chordal graphs (Lemma 2.4) and only then we prove Theorem 2.2.

Let us begin with some definitions. For a given chordal graph  $G$  and an orientation  $H$  of its edges, we say that an arc in  $H$  is *good* if it belongs to a directed triangle and it is *bad* otherwise. A good orientation is an orientation leaving every arc good. Let  $K_n$  be the complete graph with  $n$  vertices.

In order to orient chordal graphs, we first need to construct good orientations of complete graphs  $K_n$  for  $n \geq 5$ .

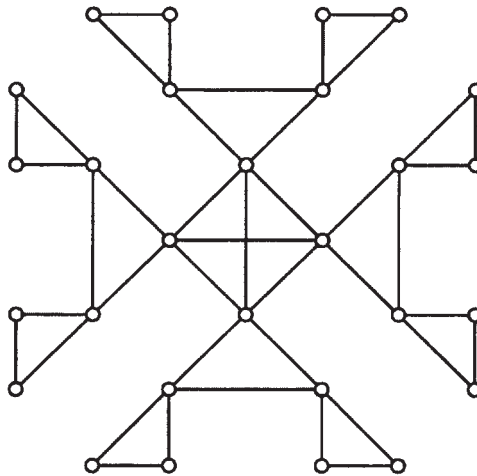


FIGURE 1. Connected bridgeless chordal graph of diameter 5.

**Lemma 2.1.** *For every  $n \geq 3$ ,  $K_n$  has a good orientation if and only if  $n \neq 4$ . Moreover, for every  $n \geq 5$ , every good orientation of  $K_n$  can be extended to a good orientation of  $K_{n+1}$  and this extension can be constructed in linear-time.*

**Proof.** We first consider the cases  $n = 3, 4$ , and  $5$ . It is easy to see that  $K_3$  has a good orientation and that  $K_4$  has no good orientations. Nevertheless,  $K_4$  has an orientation with exactly one bad arc. An orientation of  $K_5$  is obtained from an orientation of  $K_4$  with exactly one bad arc  $(v_1, v_2)$  by orienting the edges  $\{v_2, v_5\}$  and  $\{v_1, v_5\}$  as  $(v_2, v_5)$  and  $(v_5, v_1)$  and the edges  $\{v_3, v_5\}$  and  $\{v_4, v_5\}$  in such a way that they form a directed triangle with the arc associated with the edge  $\{v_3, v_4\}$ . This orientation of  $K_5$  is good.

Let us assume that  $n \geq 5$ . In order to construct a good orientation  $H_{n+1}$  for  $K_{n+1}$ , we use a good orientation  $H_n$  of  $K_n$ . Let us think that  $K_{n+1}$  is obtained from  $K_n$  by adding a new vertex  $v$ .

If  $n$  is even then a good orientation of  $K_{n+1}$  can be obtained from  $H_n$  by forming  $n/2$  directed triangles using all the edges adjacent to  $v$ . The orientation of every triangle is induced by the arc of  $H_n$ . Clearly, this orientation can be done in  $O(n)$  steps.

Suppose that  $n$  is odd. Since  $K_4$  has no good orientations, for any  $n \geq 5$  and every orientation  $H$  of  $K_n$  there are three vertices in  $K_n$  inducing a triangle that is not strongly connected. Let  $a, b$ , and  $c$  be such vertices for the orientation  $H_n$ . W.l.o.g. we may think that the arcs in  $H$  are of the form  $(a, b)$ ,  $(a, c)$ , and  $(b, c)$ . We orient the edges  $\{v, a\}$ ,  $\{v, b\}$ , and  $\{v, c\}$  as  $(v, a)$ ,  $(c, v)$ , and  $(b, v)$  in  $H_{n+1}$ . The remaining  $n - 3$  edges adjacent to  $v$  are in  $(n - 3)/2$  triangles in  $K_{n+1}$ , each of the triangles having one arc in  $H_n$ . Since  $n - 3$  is even, we orient these edges as in the previous case.

So to obtain the orientation of  $K_{n+1}$ , one should choose four arbitrary vertices in  $K_n$  and find three vertices that do not induce a strongly connected triangle. This can be done in a constant number of steps. And the orientation of the remaining  $n - 3$  edges can be done in  $O(n)$  steps. ■

In terms of diameter, we have the following corollary which will be used in Section 3.

**Corollary 2.1.** *For every  $n \geq 3$  there exists an orientation of  $K_n$  with diameter 2 if  $n \neq 4$ , and with diameter 3 if  $n = 4$ .*

A vertex  $v$  in a graph  $G$  is called *simplicial* if the graph induced by its neighborhood  $N_G(v)$  is a clique. Chordal graphs have been characterized as those having a *perfect elimination ordering* (*peo*) [8]. This is a vertex ordering  $\{v_1, \dots, v_n\}$  such that for every  $i \in \{1, \dots, n\}$ , the vertex  $v_i$  is simplicial in  $G[v_i, \dots, v_n]$  (where  $G[S]$  denotes the graph induced by the vertex set  $S$ ). We say that  $\{v_1, \dots, v_n\}$  is a *perfect construction ordering* (*pco*) if  $\{v_n, \dots, v_1\}$  is a *peo*. A *pco* of a chordal graph can be found in linear-time using the LexBFS algorithm [26]. Moreover, the first vertex can be chosen arbitrarily. Since LexBFS is a

special version of BFS, it follows that every graph  $G_i := G[v_1, \dots, v_i]$  is connected.

For a *pco*  $\{v_1, \dots, v_n\}$ , the idea of our construction is to orient all the edges incident to  $v_i$  in  $G[v_1, \dots, v_i]$  sequentially (following the *pco*).

Let  $\delta = \{v_1, \dots, v_n\}$  be a *pco* of a chordal graph  $G$ . We say that a vertex  $v_i$  is  $\delta$ -*super-simplicial* if  $N(v_i) \cap \{v_{i+1}, \dots, v_n\} = \emptyset$ . Notice that every  $\delta$ -super-simplicial vertex is simplicial in  $G$  but not vice versa.

A connected graph  $G$  is said to be *2-connected* if for every vertex  $v$ , the graph  $G - \{v\}$  is connected.

We need the following technical lemma about super-simplicial vertices in 2-connected graphs.

**Lemma 2.2.** *Let  $\delta = \{v_1, \dots, v_n\}$  be a *pco* of a 2-connected chordal graph  $G$ . If a vertex  $v_i$  is not  $\delta$ -super-simplicial, then there are  $k > i > l$  such that  $\{v_k, v_i, v_l\}$  is a clique in  $G$ .*

**Proof.** Let  $v_p$  and  $v_q, p > i > q$ , be vertices adjacent to  $v_i$  (they exist since  $G_i$  is connected and  $v_i$  is not a  $\delta$ -super-simplicial vertex). If  $\{v_p, v_q\} \in E(G)$  then  $\{v_p, v_q, v_i\}$  induce a clique and the lemma is proved. If  $\{v_p, v_q\} \notin E(G)$  then the vertices  $v_p, v_q, v_i$  belong to a cycle  $C$  in  $G$  ( $G$  is 2-connected). We choose  $C$  to have the shortest length among all cycles containing  $v_p, v_q, v_i$ . Notice that the length of  $C$  is at least 4. The cycle contains at least one vertex which is before  $v_i$  (in  $\delta$ ) and at least one vertex that is after  $v_i$ . Therefore, there are two adjacent vertices  $v_{p'}, v_{q'}$  with  $p' > i > q'$ . Since  $C$  is the shortest cycle, the only chords in this cycle are the edges adjacent to  $v_i$ . The chordality of  $G$  implies that  $v_i$  is adjacent to both  $v_{p'}$  and  $v_{q'}$  and the lemma is proved. ■

**Lemma 2.3.** *There exists a linear-time algorithm such that, given a 2-connected chordal graph  $G$  and a *pco*  $\delta = \{v_1, \dots, v_n\}$  of  $V(G)$ , it computes an orientation  $H$  with the following properties.*

- (a) Every maximal clique in  $G$  has at most one bad arc in  $H$ .
- (b) If  $(u, v)$  is a bad arc in  $H$  then  $u$  is a  $\delta$ -super-simplicial vertex of  $G$ .
- (c) For every  $v \in V(G)$ ,  $\vec{d}_H(v, v_1) \leq 2d_G(v, v_1)$ .
- (d) Every clique in  $H$  has diameter at most 3.

**Proof.** Iteratively, for  $k = 3, \dots, n$ , we construct an orientation  $H_k$  of  $G_k = G[v_1, \dots, v_k]$  with the following properties:

- (P1) Every bad arc belongs to a maximal clique (in  $H_k$ ) of size four.
- (P2) At most one arc is bad in each maximal clique.
- (P3) If  $(u, v)$  is a bad arc in  $H_k$  then  $u$  is either a  $\delta$ -super-simplicial vertex in  $G$  or the vertices  $u, v$  are used in some step  $j > k$  to form a new clique, that is,  $u, v \in N_{G_j}[v_j]$  for some  $j > k$ .



Let  $k \geq 2$  and let us denote  $N_{G_{k+1}}(v_{k+1}) = \{u_1, \dots, u_r\}$ . Since  $G$  is 2-connected, it follows that  $r \geq 2$ . Therefore  $G_3$  is a triangle, and obviously its cyclic orientation  $H_3$  satisfies properties (P1), (P2), and (P3).

Now we show how to extend, for every  $k = 3, \dots, n - 1$  the orientation  $H_k$  to an orientation  $H_{k+1}$  of  $G_{k+1}$  satisfying properties (P1), (P2), and (P3).

- (1) If  $r > 4$  then by (P1) we have that every arc in  $H_k[u_1, \dots, u_r]$  is good. We use Lemma 2.1 in order to get a good orientation of  $G[u_1, \dots, u_r, v_{k+1}]$ .
- (2) For  $r = 4$  we use the construction of a good orientation of  $K_5$  given in Lemma 2.1 to get a good orientation of  $G[u_1, \dots, u_4, v_{k+1}]$ . For  $r = 2$  we orient new edges in a directed triangle following the orientation given to  $\{u_1, u_2\}$ . In both cases the bad arc in  $H_k$  (if any) belongs to one of the directed triangles in  $H[u_1, \dots, u_r, v_{k+1}]$ . In other words, this arc becomes good in  $H_{k+1}$ .
- (3) For  $r = 3$  we consider three cases. Let us define  $H' := H_k[u_1, u_2, u_3]$ . By property (P2) we know that in  $H'$  there is at most one bad arc.
  - (i) If in  $H'$  there is one bad arc, let us say  $(u_1, u_2)$ , then we direct the new edges obtaining the following arcs:  $(u_2, v_{k+1})$  and  $(v_{k+1}, u_1)$ . Moreover, if  $(u_1, u_3) \in H'$  then we add  $(u_3, v_{k+1})$  to  $H_{k+1}$ . Otherwise  $(u_3, u_2)$  belongs to  $H'$  and we add  $(v_{k+1}, u_3)$  to  $H_{k+1}$ . Then the arcs  $(u_2, v_{k+1})$ ,  $(v_{k+1}, u_1)$ , and  $(u_1, u_2)$  are in a directed triangle and the arc between  $v_{k+1}$  and  $u_3$  is also in a directed triangle. Therefore all arcs in  $H_{k+1}[u_1, u_2, u_3, v_{k+1}]$  are good.
  - (ii) If  $H'$  has no bad arcs and  $v_{k+1}$  is not  $\delta$ -super-simplicial then by Lemma 2.2 at least one edge, say, with ends in  $\{v_{k+1}, u_1\}$ , is used in a step  $j > k$ . Then we direct edges  $\{v_{k+1}, u_2\}$  and  $\{v_{k+1}, u_3\}$  to form a directed triangle with the arc  $(u_2, u_3)$  (or  $(u_3, u_2)$ ) and we add the bad arc  $(v_{k+1}, u_1)$ .
  - (iii) The last case is when  $H'$  has no bad arcs and  $v_{k+1}$  is  $\delta$ -super-simplicial. To have property (c), we direct the edges  $\{v_{k+1}, u_2\}$  and  $\{v_{k+1}, u_3\}$  in order to form a directed triangle with the arc  $(u_2, u_3)$  (or  $(u_3, u_2)$ ), where the vertex  $u_2$  has, among all the  $u_i$ 's, the minimum distance in  $G$  to  $v_1$ . Finally, we add the bad arc  $(v_{k+1}, u_1)$ .

Notice that a bad arc appears in  $H_{k+1}[u_1, \dots, u_r, v_{k+1}]$  only when  $r = 3$  and  $H_{k+1}[u_1, u_2, u_3]$  has no bad arcs. In this case, it is  $(v_{k+1}, u_1)$ . It is easy to see that the orientation  $H_{k+1}$  satisfies properties (P1), (P2), and (P3).

Clearly the orientation  $H := H_n$  satisfies properties (a) and (b). We prove that  $H$  satisfies property (c) by induction in  $k$ . Since the pco is the ordering obtained for a LexBFS started at  $v_1$ , we know that  $d_G(v_1, v_k) = d_{G_k}(v_1, v_k)$ . As before, let  $N_{G_{k+1}}(v_{k+1}) = \{u_1, \dots, u_r\}$ . Let us assume that for all  $v \in G_k$

$$\vec{d}_H(v_1, v) \leq 2d_G(v_1, v). \tag{1}$$



If there are no bad arcs in  $H$  connecting  $v_{k+1}$  with  $u_1, \dots, u_r$  then we have that  $\vec{d}_H(u_i, v_{k+1}) \leq 2$  for all  $i = 1, \dots, r$  and (1) holds for  $G_{k+1}$ .

If  $v_{k+1}$  is connected to some  $u_i$  by a bad arc in  $H$  then  $r = 3, H[u_1, u_2, u_3]$  has no bad arcs and  $v_{k+1}$  is a  $\delta$ -super-simplicial vertex of  $G$ . Moreover  $d_G(v_1, v_{k+1}) = d_G(v_1, u_2) + 1$  where  $u_2$  is, among all the  $u_i$ 's, the vertex having minimum distance to  $v_1$  in  $G$ .

By the construction of  $H$ , there exists a directed triangle that contains  $v_{k+1}$  and  $u_2$ , which implies that  $\vec{d}_H(u_2, v_{k+1}) \leq 2$ . Therefore  $\vec{d}_H(v_1, v_{k+1}) \leq 2d_G(v_1, u_2) + 2 = 2d_G(v_1, v_{k+1})$ . Property (d) follows from (P2).

Finally, we claim that for every  $k$ , the orientation of the arcs adjacent to  $v_{k+1}$  during the extension of the orientation  $H_k$  to  $H_{k+1}$  can be performed in  $O(|N(v_{k+1})|)$ . We assume that the set of  $\delta$ -super-simplicial vertices is known. (Clearly this set can be computed in linear-time  $O(\sum_{v \in V(G)} |N(v)|) = O(|E(G)|)$ .)

If we are in the cases 1 or 2 then the orientation of arcs can be performed in  $O(|N(v_{k+1})|)$  steps by Lemma 2.1. If we are in case 3 then subcase (i) is performed in a constant number of steps. For subcases (ii) we should be able to find a vertex from  $\{u_1, u_2, u_3\}$  which has a common neighbor with  $v_{k+1}$  in  $\{v_{k+2}, \dots, v_n\}$ . Every neighbor of  $v_{k+1}$  in  $\{v_{k+2}, \dots, v_n\}$  has at most three neighbors in  $\{v_1, v_2, \dots, v_k\}$  and such a vertex can be found in  $O(|N(v_{k+1})|)$  steps. The subcase (iii) takes constant number of steps.

Therefore, the complexity of the algorithm is

$$O\left(\sum_{1 \leq k \leq n} |N(v_k)|\right) = O(|E(G)|). \quad \blacksquare$$

**Lemma 2.4.** *There exists a linear-time algorithm such that, given a 2-connected chordal graph  $G$ , it computes an orientation  $H$  satisfying, for every pair of vertices  $u$  and  $v$ ,  $\vec{d}_H(u, v) \leq 2d_G(u, v) + 1$ .*

**Proof.** Given  $G$  the algorithm first computes (in linear-time) a pco and then the orientation  $H$  (in linear time) given by Lemma 2.3. We prove that  $H$  has the desired property.

Take  $u, v \in V(G)$  and let  $P$  be a shortest  $(u, v)$ -path in  $G$ . If  $d_G(u, v) = 1$  then  $u, v$  are in some clique  $C, |C| \geq 3$ . From Property (d) we have  $\vec{d}_H(u, v) \leq 3$ .

Suppose that  $d_G(u, v) > 1$ . Let us assume that  $P = (u, x, \dots, y, v)$  (where  $x$  and  $y$  could be equal). Then  $d_H(u, v) \leq d_H(u, x) + d_H(x, y) + d_H(y, v)$ . Clearly, the inner vertices of  $P$  cannot be simplicial; therefore, each arc in  $H$  associated to some inner edge of  $P$  is contained in a directed triangle in  $H$ . Thus  $d_H(x, y) \leq 2d_G(x, y)$ . In order to finish the proof, we need to prove that  $d_H(u, x) + d_H(y, v) \leq 5$ . Since every clique in  $H$  has diameter at most three, it suffices to show that  $d_H(u, x) = 2$ . If  $(u, x) \in H$  then  $d_H(u, x) = 1$ . Otherwise,  $(x, u) \in H$ . Since  $x$  is not simplicial,  $(x, u)$  is a good arc and then  $d_H(u, x) = 2$ . A similar analysis shows that  $d_H(v, u) \leq d_H(v, y) + d_H(y, x) + d_H(x, u) \leq 2d_G(u, v) + 1$  (by proving that  $d_H(v, y) \leq 2$ ).

**Proof of Theorem 2.2.** The set of 2-connected components has a tree-like structure  $T$ . By the classical result of Tarjan [28], the tree-like structure of the 2-connected components can be computed in linear-time. We choose one 2-connected component  $C_0$  as a root of  $T$ . Notice that once a root has been defined the notion of father and sons of a 2-connected component is well defined. For every 2-connected component  $C$ , we define its *father-cut vertex* as the unique vertex in  $C$  which belongs to its father.

In each 2-connected component other than the root, we compute a pco starting in its father-cut vertex and orient it as in Lemma 2.3. For  $C_0$  we compute a pco starting at any vertex and orient it as in Lemma 2.3. Let  $H$  be the orientation of  $G$  so obtained. In each 2-connected component, the construction of  $H$  is done in linear-time by Lemma 2.3.

Let  $u$  and  $v$  be two vertices of  $G$  and let  $P$  be a  $(u, v)$ -shortest path in  $G$ . If  $P$  has no cut vertices then  $u$  and  $v$  lie in the same 2-connected component. From Lemma 2.4 and the construction of  $H$  we have  $\vec{d}_H(u, v) \leq 2d_G(u, v) + 1$ . Otherwise, let  $u_1, \dots, u_r$  be the cut vertices in  $P$  and  $C_i$  the 2-connected component containing  $u_i$  and  $u_{i+1}$  for  $i = 1, \dots, r - 1$ . Notice that for at most one  $i_0$  neither  $u_{i_0}$  nor  $u_{i_0+1}$  are father-cut vertices of  $C_{i_0}$ . From the construction of  $H$ , we know that  $\vec{d}_H(u_i, u_{i+1}) \leq 2d_G(u_i, u_{i+1})$  for all  $i = 1, \dots, r$  not equal to  $i_0$  (in this case,  $\vec{d}_H(u_{i_0}, u_{i_0+1}) \leq 2d_G(u_{i_0}, u_{i_0+1}) + 1$ ). Therefore,  $\vec{d}_H(u, v) \leq 2d_G(u, v) + 1$ . ■

### 3. NEGATIVE RESULTS

Our first step is to prove the  $NP$ -completeness of  $ODP_D(k)$  for chordal graphs. In fact, we will prove two results: the  $NP$ -completeness of  $ODP_D(2)$  for split graphs and the  $NP$ -completeness of  $ODP_D(k)$  for chordal graphs for every  $k \geq 4$ .

A graph  $G$  is a split graph if its vertex set  $V(G)$  can be partitioned into sets  $C$  and  $I$  such that  $C$  is a clique and  $I$  is an independent set. Split graphs form a subclass of chordal graphs. Our proof, inspired by the one of Chvátal and Thomassen [6], relies on the  $NP$ -completeness of the 2-coloring problem for hypergraphs [19]. Let us recall that a hypergraph  $\mathcal{H}$  is called 2-colorable if its vertices can be colored red and blue in such a way that every edge includes at least one vertex of each color.

**Lemma 3.1.** *For every  $t \geq 0$  and for every hypergraph  $\mathcal{H}$ , there exists a polynomial time-computable chordal graph  $G_{\mathcal{H}}^t$  (split graph for  $t = 0$ ) such that if  $\mathcal{H}$  is 2-colorable then  $OD(G_{\mathcal{H}}^t) = 2(t + 1)$  and if  $\mathcal{H}$  is not 2-colorable then  $OD(G_{\mathcal{H}}^t) = 3(t + 1)$ .*

**Proof.** We first consider the case  $t = 0$ . For a given hypergraph  $\mathcal{H}$ , we will construct a split graph  $G_{\mathcal{H}}^0 = G_{\mathcal{H}}$  such that if  $\mathcal{H}$  is 2-colorable then  $OD(G_{\mathcal{H}}) = 2$  and if  $\mathcal{H}$  is not 2-colorable then  $OD(G_{\mathcal{H}}) = 3$ . Let  $\mathcal{H}$  be a hypergraph with vertex set  $V$  of size  $n$  and edge set  $E$  of size  $m$ .

The clique  $C$  of  $G_{\mathcal{H}}$  contains  $n + 2m + 5$  vertices. More precisely,  $C = W \cup Y$  with  $W = V \cup \{\gamma, \delta, \eta\}$  and  $Y = \{\alpha, \beta\} \cup E_1 \cup E_2$  where  $E_1$  and  $E_2$  are copies of the edge set  $E$  of  $\mathcal{H}$ . The independent set  $I$  of  $G_{\mathcal{H}}$  contains  $m + 1$  vertices. More precisely,  $I = \{x\} \cup E$ .

Now let us explain how to connect the vertices of  $I$  with those of  $C$ . The vertex  $x$  is connected to all the vertices of  $W$ . A vertex  $e \in E$  is connected to a vertex  $v \in V$  if and only if  $v \in e$  (in the hypergraph  $\mathcal{H}$ ). Finally, every vertex  $y \in Y$  is connected to every vertex  $e$  of  $E$ .

We construct an orientation  $H$  of  $G_{\mathcal{H}}$  with the required properties in two steps. In the first step, we orient all the edges not connecting  $V$  with  $I$ . The orientation of these edges will not depend on the 2-colorability of  $\mathcal{H}$ . In the second step, we orient the remaining edges: those connecting  $V$  with  $I$ .

**Step 1.** Notice first that the graph induced by  $E \cup E_1 \cup E_2$  contains a copy of  $K_{m,m,m}$ . Gutin has proven that for all  $m \geq 1$ , there is an orientation of  $K_{m,m,m}$  with diameter 2 [12]. We use this orientation for the edges between  $E, E_1$  and  $E_2$ . The orientation of the edges inside  $E_1$  and inside  $E_2$  is irrelevant. We assume that  $n \geq 5$  and we orient the edges inside  $V$  as in Lemma 2.1. Then the diameter of the graph  $G_{\mathcal{H}}[V]$  is 2.

The rest of the edges not connecting  $V$  and  $I = \{x\} \cup E$  are oriented as indicated in the following 0 – 1 matrix. A value 1 in the position  $(P, Q)$  means that all the edges connecting the vertices of  $P$  with those of  $Q$  are oriented from  $P$  towards  $Q$  and we denote it by  $P \rightarrow Q$ .

	$x$	$\gamma$	$\eta$	$\delta$	$V$	$\alpha$	$\beta$	$E$	$E_1$	$E_2$
$x$	–	1	1	–	?	–	–	–	–	–
$\gamma$	–	–	–	1	1	1	1	–	–	–
$\eta$	–	1	–	–	1	–	–	–	1	1
$\delta$	1	–	1	–	–	1	1	–	–	–
$V$	?	–	–	1	–	–	–	?	1	1
$\alpha$	–	–	1	–	1	–	–	1	1	–
$\beta$	–	–	1	–	1	1	–	–	–	1
$E$	–	–	–	–	?	–	1	–	–	–
$E_1$	–	1	–	1	–	–	1	–	–	–
$E_2$	–	1	–	1	–	1	–	–	–	–

**Step 2.** We first prove that, no matter the 2-colorability of the hypergraph  $\mathcal{H}$ , we can orient the remaining edges to obtain an orientation  $H$  of the whole graph with diameter less than or equal to 3. Using the orientation  $V \rightarrow \{x\}$  and  $V \rightarrow E$ , we obtain an orientation  $H$  satisfying  $d_H(u, v) \leq 3$ , for all  $u, v \in V(G_{\mathcal{H}})$ . To see this, it is enough to exhibit a  $(u, v)$ -directed path of length at most three, for all  $u, v \in V(G_{\mathcal{H}})$ . Here we only show the property for the case  $u = x$  and for the case  $u \in E$ . The remaining cases can be verified easily. Since  $x \rightarrow \{\gamma, \eta\}$ ,  $\gamma \rightarrow \{\delta, V, \alpha, \beta\}$ , and  $\eta \rightarrow \{E_1, E_2\}$ , we have that  $d_H(x, v) \leq 2$  for every  $v \notin E$ . For a vertex  $u \in E$ , we proceed analogously: since the orientation of the copy of

$K_{m,m,m}$  in  $E_1 \cup E_2 \cup E$  has diameter 2, there exists an arc  $(u, u')$  in  $H$  with  $u' \in E_1 \cup E_2$ . Since  $E_1 \cup E_2 \rightarrow \{\gamma, \delta\}, E \rightarrow \beta$ , and  $\beta \rightarrow \{\eta, V, \alpha\}$ , we have that  $d_H(u, v) = 2$ , for all  $u \in E$  and all  $v \neq x$ . Moreover, if  $d_H(u, v) = 3$  then  $u$  and  $v$  respectively belong to  $\{x\}$  and  $E$ .

In order to conclude that if  $\mathcal{H}$  is 2-colorable then  $OD(G_{\mathcal{H}}) = 2$  and if  $\mathcal{H}$  is not 2-colorable then  $OD(G_{\mathcal{H}}) = 3$ , it is enough to prove that

$$\mathcal{H} \text{ is 2-colorable} \Leftrightarrow OD(G_{\mathcal{H}}) = 2.$$

( $\Leftarrow$ ) Let  $H$  be an orientation of  $G_{\mathcal{H}}$  of diameter 2. The way we color every vertex  $v$  of the hypergraph  $\mathcal{H}$  is the following: if according to  $H$ , the edge connecting  $x$  with  $v$  is oriented towards  $v$ , then we color it red. Otherwise we color it blue. Since for every vertex  $e \in E$ , the distance  $d_H(x, e) = d_H(e, x) = 2$ , it follows that every edge  $e$  in  $\mathcal{H}$  contains a red and a blue vertex.

( $\Rightarrow$ ) Now let us suppose that  $\mathcal{H}$  is 2-colorable. Let us denote by  $R$  and  $B$  the set of red and blue vertices in  $V$ . We orient all the edges in  $G_{\mathcal{H}}$  except those with ends in  $V$  and  $E$  or with one of their ends being  $x$  as given by the partial orientation obtained in Step 1. In order to achieve  $\text{diam}(H) = 2$  we only have to reach  $x$  from  $E$  with paths of length two and vice versa. We orient all the edges between  $R$  and  $E$  towards  $E$  and all the edges between  $B$  and  $E$  towards  $B$ . Finally, we orient  $\{x, v\}$  as  $(x, v)$  if  $v \in R$  and as  $(v, x)$  otherwise. Since  $\mathcal{H}$  is 2-colorable there is a directed path of length 2 in  $H$  from  $x$  to every  $e \in E$  and from every  $e \in E$  to  $x$ .

For  $t > 0$  the chordal graph  $G_{\mathcal{H}}^t$  is constructed from  $t + 1$  copies of the split graph  $G_{\mathcal{H}}^0 = (C, \{e_1, \dots, e_m, x\})$  where the vertex  $x$  of the  $i$ th copy is identified with some element in  $E$  in the  $(i - 1)$ th copy, for  $i = 1, \dots, t$  (see Fig. 2). It is easy to see that  $G_{\mathcal{H}}^t$  is a chordal graph and that if  $\mathcal{H}$  is 2-colorable then  $OD(G_{\mathcal{H}}^t) = 2(t + 1)$  and if  $\mathcal{H}$  is not 2-colorable then  $OD(G_{\mathcal{H}}^t) = 3(t + 1)$ .

**Theorem 3.1.** *ODPD(2) is NP-complete for split graphs and, for every  $k \geq 4$ , ODPD( $k$ ) is NP-complete for chordal graphs.*

*Proof.* By using Lemma 3.1 we can reduce, in polynomial time, the 2-coloring hypergraph problem to ODPD( $k$ ). For  $k = 2$  we take  $t = 0$  and for  $k \geq 4$  we take  $t = \lfloor \frac{k}{2} \rfloor - 1$ . ■

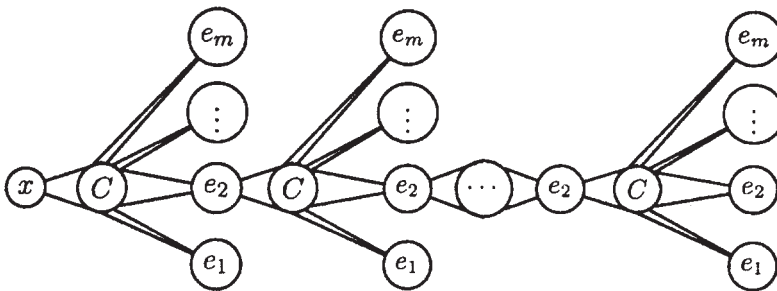


FIGURE 2. Path-like structure of  $G_{\mathcal{H}}^t$ .

Now we prove two results concerning the hardness of approximating the oriented diameter.

**Theorem 3.2.** *Let  $\alpha < \frac{3}{2}$ . Unless  $P = NP$ , ODP has no  $\alpha$ -approximation algorithm for split graphs.*

*Proof.* Let  $A(G)$  be the orientation assigned to a graph  $G$  by an  $\alpha$ -approximation algorithm for ODP. If  $\mathcal{H}$  is 2-colorable then  $G_{\mathcal{H}}$  has diameter 2. Thus  $\text{diam}(A(G_{\mathcal{H}})) \leq 2\alpha < 3$ . On the other hand, if  $\mathcal{H}$  is not 2-colorable then every orientation of the graph  $G_{\mathcal{H}}$  has diameter at least 3. Whence  $\text{diam}(A(G_{\mathcal{H}})) \geq 3$ . Therefore, since  $G_{\mathcal{H}}$  can be constructed in polynomial time, by knowing  $\text{diam}(A(G_{\mathcal{H}}))$  we could decide the 2-colorability of  $\mathcal{H}$ . ■

**Theorem 3.3.** *Unless  $P = NP$ , there is no absolute approximation algorithms for ODP when restricted to chordal graphs.*

*Proof.* Let us assume that there exist  $K$  and an absolute approximation algorithm  $A$  for ODP such that  $\text{diam}(A(G)) \leq OD(G) + K$ . By using this algorithm, we could decide the 2-coloring problem for hypergraphs. Let  $\mathcal{H}$  be a hypergraph and  $t \geq K$ . From Lemma 3.1, there exists a chordal graph  $G_{\mathcal{H}}^t$  computable in polynomial time such that, if  $\mathcal{H}$  is 2-colorable, then  $G_{\mathcal{H}}^t$  has diameter  $2t + 2$ . Thus  $\text{diam}(A(G_{\mathcal{H}}^t)) \leq 2t + 2 + K \leq 3t + 2$ . And, if  $\mathcal{H}$  is not 2-colorable then  $G_{\mathcal{H}}^t$  has diameter  $3t + 3$ . Thus  $\text{diam}(A(G_{\mathcal{H}}^t)) \geq 3t + 3$ . ■

#### 4. CONCLUDING REMARKS

In this paper we have provided a linear-time  $(2, 1)$ -approximation algorithm for the problem of finding the oriented diameter of chordal graphs. On the other hand, we have proved that for every  $\alpha < 3/2$ , finding an orientation with diameter at most  $\alpha$  times the oriented diameter is  $NP$ -hard. The challenge is to decrease the gap between these lower and upper bounds. But even the existence of a 2-approximation algorithm is an interesting open problem.

#### REFERENCES

- [1] G. Ausiello, P. Crescenzi, G. Gambosi, V. Kann, A. Marchetti-Spaccamela, and M. Protasi, *Complexity and Approximation*, Springer, 1999.
- [2] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer-Verlag, London Springer Monographs in Mathematics, 2000.
- [3] A. Brandstädt, V. B. Le, and J. P. Spinrad, *Graph classes: A survey*, SIAM Monographs on Discrete Mathematics and Applications, Philadelphia, 1999.
- [4] F. R. K. Chung, *Diameters of communication networks*, Mathematics of Information Processing (Louisville, Ky., 1984), Amer Math Soc, Providence, R.I., 1986, pp. 1–18.

- [5] F. R. K. Chung, M. R. Garey, and R. E. Tarjan, Strongly connected orientations of mixed multigraphs, *Networks* 15 (1985), 477–484.
- [6] V. Chvátal and C. Thomassen, Distances in orientations of graphs, *J. Combin Theory Ser B* 24 (1978), 61–75.
- [7] P. Fraigniaud and E. Lazard, Methods and problems of communication in usual networks, *Discrete Appl Math* 53 (1994), 79–133.
- [8] D. Fulkerson and O. Gross, Incidence matrices and interval graphs, *Pacific J Math* 15 (1965), 835–855.
- [9] M. C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Academic Press, New York, 1980.
- [10] A. Gupta, Improved bandwidth approximation for trees and chordal graphs, *J Algorithms* 40(1) (2001), 24–36.
- [11] J. Gustedt, On the pathwidth of chordal graphs, *Discrete Appl Math* 45 (1993), 233–248.
- [12] G. Gutin, Minimizing and maximizing the diameter in orientations of graphs, *Graphs Combin* 10 (1994), 225–230.
- [13] S. M. Hedetniemi, S. T. Hedetniemi, and A. L. Liestman, A survey of gossiping and broadcasting in communication networks, *Networks* 18 (1988), 319–349.
- [14] K. M. Koh and E. G. Tay, On optimal orientations of Cartesian products of even cycles, *Networks* 32 (1998), 299–306.
- [15] K. M. Koh and E. G. Tay, On optimal orientations of Cartesian products with a bipartite graph, *Discrete Appl Math* 98 (1999), 103–120.
- [16] K. M. Koh and E. G. Tay, On optimal orientations of Cartesian products of graphs, II. Complete graphs and even cycles, *Discrete Math* 211 (2000), 75–102.
- [17] J. -C. König, D. W. Krumme, and E. Lazard, Diameter-preserving orientations of the torus, *Networks* 32 (1998), 1–11.
- [18] D. Kratsch and L. Stewart, Approximating bandwidth by mixing layouts of interval graphs, *STACS 99 (Trier)*, (1999) Springer, Berlin, pp. 248–258.
- [19] L. Lovász, Coverings and coloring of hypergraphs, *Proceedings of the Fourth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1973)*, Utilitas Math Winnipeg, Man 1973, pp. 3–12.
- [20] B. Monien, The bandwidth minimization problem for caterpillars with hair length 3 is NP-complete, *SIAM J Algebr Discrete Methods* 7(4) (1986), 505–512.
- [21] H. E. Robbins, A theorem on graphs with an application to a problem of traffic control, *Am Math Monthly* 46 (1939), 281–283.
- [22] F. S. Roberts and Y. Xu, On the optimal strongly connected orientations of city street graphs, I. Large grids, *SIAM J Discrete Math* 1 (1988), 199–222.

- [23] F. S. Roberts and Y. Xu, On the optimal strongly connected orientations of city street graphs, II. Two east-west avenues or north-south streets, *Networks* 19 (1989), 221–233.
- [24] F. S. Roberts and Y. Xu, On the optimal strongly connected orientations of city street graphs, III. Three east-west avenues or north-south streets, *Networks* 22 (1992), 109–143.
- [25] F. S. Roberts and Y. Xu, On the optimal strongly connected orientations of city street graphs, IV. Four east-west avenues or north-south streets, *Discrete Appl Math* 49 (1994), 331–356. *Viewpoints on optimization* (Grimentz, 1990; Boston, MA, 1991).
- [26] D. Rose, R. E. Tarjan, and G. Lueker, Algorithmic aspects on vertex elimination on graphs, *SIAM J Comput* 5 (1976), 566–579.
- [27] L. Šoltés, Orientations of graphs minimizing the radius or the diameter, *Math Slovaca* 36 (1986), 289–296.
- [28] R. Tarjan, Depth-first search and linear graph algorithms, *SIAM J Comput* 1 (1972), 146–160.