Available online at www.sciencedirect.com
SCIENCE DIRECT®
Discrete Applied Mathematics 141 (2004) 135–148
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

AT-free graphs: linear bounds for the oriented diameter☆

Fedor V. Fomin[a], Martín Matamala[a,b,*], Erich Prisner[c], Ivan Rapaport[a,b]

[a] *Centro de Modelamiento Matemático, Universidad de Chile and UMR 2071-CNRS, Casilla 170-3, Correo 3, Santiago, Chile*

[b] *Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Casilla 170-3, Correo 3, Santiago, Chile*

[c] *Department of Mathematics, University of Louisville, Louisville, KY 40292, USA*

Received 4 July 2001; received in revised form 9 April 2002; accepted 22 March 2003

Abstract

Let G be a bridgeless connected undirected (b.c.u.) graph. The oriented diameter of G, $OD(G)$, is given by $OD(G)=\min\{diam(H)\colon H \text{ is an orientation of } G\}$, where $diam(H)$ is the maximum length computed over the lengths of all the shortest directed paths in H. This work starts with a result stating that, for every b.c.u. graph G, its oriented diameter $OD(G)$ and its domination number $\gamma(G)$ are linearly related as follows: $OD(G) \leqslant 9\gamma(G) - 5$.

Since—as shown by Corneil et al. (SIAM J. Discrete Math. 10 (1997) 399)— $\gamma(G) \leqslant diam(G)$ for every AT-free graph G, it follows that $OD(G) \leqslant 9\,diam(G) - 5$ for every b.c.u. AT-free graph G. Our main result is the improvement of the previous linear upper bound. We show that $OD(G) \leqslant 2\,diam(G)+11$ for every b.c.u. AT-free graph G. For some subclasses we obtain better bounds: $OD(G) \leqslant \frac{3}{2}\,diam(G)+\frac{25}{2}$ for every interval b.c.u. graph G, and $OD(G) \leqslant \frac{5}{4}\,diam(G)+\frac{29}{2}$ for every 2-connected interval b.c.u. graph G. We prove that, for the class of b.c.u. AT-free graphs and its previously mentioned subclasses, all our bounds are optimal (up to additive constants).

MSC: 05C12; 05C20; 05C69; 05C62

Keywords: Diameter; Orientation; Domination; Asteroidal triple-free graph; Interval graph

☆ Partially supported by FONDAP on Applied Mathematics, Fondecyt 1990616 (I. Rapaport) and Fondecyt 1010442 (M. Matamala).

* Corresponding author. Department of Mathematical Engineering, University of Chile, Blanco Encalada 2120, Santiago, Chile.

E-mail address: mmatamal@dim.uchile.cl (M. Matamala).

doi:10.1016/S0166-218X(03)00376-7

1. Introduction

In this work we use standard notation as presented in [5]. An *orientation* of an undirected graph G is a directed graph whose arcs correspond to assignments of directions to the edges of G. An orientation H of G is *strongly connected* if every two vertices in H are mutually reachable in H by directed paths. The *oriented diameter* of a graph G is defined as

$$OD(G) = \min\{diam(H): H \text{ an orientation of } G\}.$$

In 1939 Robbins [15] proved that every undirected graph G admits a strongly connected orientation if and only if G is connected and bridgeless. Chung et al. [2] provided a linear-time algorithm for testing whether a graph has a strongly connected orientation and finding one if it does.

From now on G will always denote a bridgeless connected undirected graph. We will refer to it simply as *a graph*.

1If a graph G is thought as the plan of the system of two-way streets, then the orientations of G can be viewed as arrangements of one-way streets. Some variants of one-way street assignments were studied in [17–20]. Applications also appear in different network routing, broadcasting and gossip problems. (See [1,6,9] for surveys.)

Chvátal and Thomassen [3] initiated the study of $OD(G)$. They focused on the relation between $OD(G)$ and $diam(G)$ proving that $OD(G) \leqslant 2(diam(G))^2$.

Later this problem was studied for various classes of graphs including cartesian products of graphs, complete and complete bipartite graphs [8,10–13,21].

Our first result, presented in Section 2, proves a linear upper bound for the oriented diameter $OD(G)$ in terms of the domination number $\gamma(G)$ for an arbitrary graph G: $OD(G) \leqslant 9\gamma(G) - 5$.

This result motivates us to consider the class of bridgeless connected AT-free graphs (AT-free from now on). In fact, in this class the domination number and the diameter are linearly related [4]. Indeed, for every AT-free graph G it holds that $\gamma(G) \leqslant diam(G)$. This implies that $OD(G) \leqslant 9\,diam(G) - 5$.

In Section 3 we obtain better bounds for the class of AT-free graphs. More precisely, we prove that $OD(G) \leqslant 2\,diam(G) + 11$ for every AT-free graph G. We improve previous bound for some subclasses. For bridgeless connected interval graphs we obtain $OD(G) \leqslant \frac{3}{2}\,diam(G) + \frac{25}{2}$, and for 2-connected proper interval graphs we get $OD(G) \leqslant \frac{5}{4}\,diam(G) + \frac{29}{2}$. Finally, we show that each previous bound is optimal up to an additive constant.

2. Dominating sets

In this section we prove a linear relation between the oriented diameter of a graph G and its domination number.

Lemma 1. *Let G and G_D be graphs with G_D being a subgraph of G such that $V(G_D)$ is a dominating set in G. Then, for every strongly connected orientation H_D of G_D, there is an orientation H of G such that $diam(H) \leqslant diam(H_D) + 4$.*

Proof. For every connected component Q of $G \setminus V(G_D)$ we direct some edges having ends in Q as follows:

If Q consists of one vertex x then x is adjacent to at least two vertices of $V(G_D)$ (because G is bridgeless). We direct one edge from x and the second edge towards x. If there are more edges incident to x we direct them arbitrarily. Then, we have assured the existence of vertices $u, v \in V(G_D)$ such that $d_H(x, v) \leqslant 1$ and $d_H(u, x) \leqslant 1$.

Suppose that there are at least two vertices in the connected component Q. Choose a spanning tree T in this component rooted in a vertex v. We orient the edges of this tree as follows. If a vertex x of the tree has an odd distance from v, then we orient all the tree edges adjacent to x towards x. If x has an even distance from v then we orient all the tree edges adjacent to x from x outwards. Also, for every such vertex x we orient the edges between x and G_D towards x if the distance from v in the tree is even, and towards G_D otherwise. The rest of the edges in the connected component Q are oriented arbitrarily.

In such orientation H, for every $x \in V(Q)$ there are vertices $u, v \in V(G_D)$ such that $d_H(x, v) \leqslant 2$ and $d_H(u, x) \leqslant 2$. Therefore, for every $x, y \in V(G)$ the distance between x and y in H is at most $diam(H_D) + 4$. □

Theorem 2. *For every graph G there exists a graph G_D being a subgraph of G such that $V(G_D)$ is a dominating set of G and $|G_D| \leqslant 9\gamma(G) - 8$.*

Proof. The case $\gamma(G) = 1$ is direct. Let D be a dominating set with $|D| = \gamma(G) \geqslant 2$. Iteratively, we construct a tree T_k for $k = 1, \ldots, |D|$. The tree T_1 is composed by one vertex x_1 in D. Among all the paths S connecting $D \setminus V(T_k)$ with $V(T_k)$ in $G \setminus V(T_k)$ and with minimum $|S \cap (V(T_k) \setminus D)|$, let P_k be the shortest one. Since D is a dominating set, the length of P_k is less than or equal to 3. We define $T_{k+1} := T_k \cup P_k$. Then $T := T_{|D|}$ is a tree which contains the set D and with $|T| \leqslant 3|D| - 2$.

Let $xy \in E(T)$ and let P^{xy} be a shortest path in $G - xy$ connecting the two subtrees of $T - xy$. Since G is bridgeless such path exists. Since D is a dominating set the path P^{xy} has length of at most 3. Clearly, in $T + P^{xy}$, the edge xy and the edges in P^{xy} are not bridges.

For each edge xy in T we add the path P^{xy} to T in order to obtain a graph G_D being a subgraph of G. Since $|P^{xy}| \leqslant 4$ and two vertices of P^{xy} belong to $V(T)$ we deduce that the new vertices in G_D are at most $2|E(T)| \leqslant 2(3|D| - 3)$. Finally, $|G_D| \leqslant 3|D| - 2 + 6|D| - 6 = 9|D| - 8$. □

Corollary 3. *For every graph G, $OD(G) \leqslant 9\gamma(G) - 5$.*

Proof. Since the graph G_D of Theorem 2 has an orientation H_D with diameter at most $|G_D| - 1 \leqslant 9\gamma(G) - 9$, the result follows from Lemma 1. □

3. AT-free graphs

The goal of this section is to prove that for every AT-free graph G, $OD(G) \leqslant 2\,diam(G) + 11$.

An independent set of three vertices is called an *asteroidal triple* if every two of them are connected by a path avoiding the neighborhood of the third. A graph is *AT-free* if it does not contain an asteroidal triple. Asteroidal triples were introduced to characterize interval graphs [14].

A pair of vertices x, y of a graph G form a *dominating pair* if the vertex set of every path connecting x with y is a dominating set of G. In their fundamental paper, Corneil et al. [4] showed that every AT-free graph G has a dominating pair. A dominating pair of an AT-free graph can be detected by a simple linear time algorithm, called `2LexBFS` [4].

Since a shortest path P between a dominating pair is a dominating set in an AT-free graph G, it follows that $\gamma(G) \leqslant |P| \leqslant diam(G)$. From Corollary 3 we conclude that $OD(G) \leqslant 9\,diam(G) - 5$. In this section our concern is to improve this linear upper bound.

A *chord* of a cycle C is an edge joining two non-consecutive vertices of C. A *chordless cycle* in G is a cycle of length more than three that has no chords. It is easy to see that an AT-free graph does not contain chordless cycles of length more than 5. We use this property several times in the following proofs.

Let u and v be two vertices of a graph G. For a (u,v)-path R and vertices $x, y \in V(R)$ we denote by xRy the subpath of R between x and y. We also denote uRy by Ry and xRv by xR.

Lemma 4. *Let G be an AT-free graph. Let u, v be two vertices of G. Then there are two edge-disjoint (u,v)-paths with one of them being a shortest path.*

Proof. We call *good* a vertex $x \in V(G)$ if the following two conditions hold.

(1) The vertex x belongs to some shortest (u,v)-path P.
(2) There exists a (u,x)-path $S_{u,x}$ which is edge-disjoint with the path P.

Notice that $d_G(u,x) < d_G(u,v)$ for every good vertex $x \neq v$. We have to show that v is good.

Let x be a good vertex having, among all the good vertices, maximum distance to u. Let P be a shortest (u,v)-path to which x belongs. Let $S_{u,x}$ denote some (u,x)-path edge-disjoint with P. We claim that $x = v$.

Let us assume $x \neq v$. We split P into Px and xP. Let $s \in V(S_{u,x})$ be the neighbor of x in the path $S_{u,x}$ and let $y \in V(P)$ be the neighbor of x in the path xP. Since y is not a good vertex it is not adjacent to any vertex in the path $S_{u,x}$. Let us denote by Q a shortest (u,y)-path avoiding the edge xy. Let z be the last vertex in Q which belongs to Px. Since y is not a good vertex, $z \neq x$. Let w denote the neighbor of z in zQ. The path P is a shortest path, therefore $w \neq y$. Since $V(wQ) \cap V(yP) \neq \emptyset$ and no vertex in yP is good, no vertex in zQ is adjacent to a vertex in $S_{u,x}$.

It follows that w is adjacent to y. Otherwise, the set $\{w, y, s\}$ would be an asteroidal triple. We conclude that z is the neighbor of x in the path Px because the path P is a shortest (u,v)-path.

The path P' obtained from P by replacing the edges zx and xy by the path zQ is also a shortest (u,v)-path. Moreover, y belongs to P' and there exists an edge-disjoint (u,y)-path S' obtained from S by adding the vertex y and the edge xy. Therefore, y is a good vertex with $d_G(u,y) > d_G(u,x)$. □

Let G be an AT-free graph and let u,v be a dominating pair. By Lemma 4 there exist two edge-disjoint (u,v)-paths, P and S, such that P is a shortest (u,v)-path. Without loss of generality we can assume that S is a shortest path among all the (u,v)-paths edge-disjoint with P. Notice that the set of vertices of both paths P and S are dominating sets. This property will be used in the rest of the paper without explicit mention.

In order to prove the main theorem of this section we show first that the paths P and S are, roughly speaking, "close enough", and that the length of the path S is proportional to that of P.

Let $p_1 = u, p_2, \ldots, p_{l_{P+1}} = v$ be the vertices of P and let $s_1 = u, s_2, \ldots, s_{l_S+1} = v$ be the vertices of S. We define, for all $i \in \{2, \ldots, l_P + 1\}$,

$$a(i) = \min\{j \in \{1,2,\ldots,l_S+1\}\colon s_j \in (N[p_{i-1}] \cup N[p_i]) \setminus V(P)\},$$

$$b(i) = \max\{j \in \{1,2,\ldots,l_S+1\}\colon s_j \in (N[p_{i-1}] \cup N[p_i]) \setminus V(P)\}.$$

Lemma 5. *The indices $a(i)$ and $b(i)$ are well defined, for all $i \in \{2,\ldots,l_P+1\}$. Moreover, $a(2)=2$ and $b(l_P+1)=l_S$.*

Proof. We only have to show that $\Omega_i := (V(S) \setminus V(P)) \cap (N[p_{i-1}] \cup N[p_i]) \neq \emptyset$, for every $i=2,\ldots,l_P+1$. Since the vertices s_2 and s_{l_S} are not in $V(P)$ neither Ω_2 nor Ω_{l_P+1} is empty. Clearly $a(2)=2$ and $b(l_P+1)=l_S$. For sake of contradiction, let $3 \leqslant i \leqslant l_P$ be such that $\Omega_i = \emptyset$. Since S and P are edge-disjoint (u,v)-paths there is a subpath $S' = s'_1, \ldots, s'_r$ of S connecting $\{p_1, \ldots, p_{i-2}\}$ with $\{p_{i+1}, \ldots, p_{l_P+1}\}$, which only meets P in its end vertices. Let $t \leqslant i-2$ be the largest integer such that s'_2 is adjacent with p_t and let $m \geqslant i+1$ be the smallest integer such that s'_{r-1} is adjacent to p_m. Then $S'' = p_t, s'_2, \ldots, s'_{r-1}, p_m$ is a (p_t, p_m)-path edge-disjoint with P. Since P is a shortest path the length of S'' is at least the length of the path $p_t P p_m$. Then $C := S'' \cup p_t P p_m$ is an induced cycle of length at least $2(m-t) \geqslant 2(i+1-(i-2)) = 6$. Therefore, C contains an asteroidal triple which is a contradiction. □

By the definition of $a(i)$ and since at most one of any two consecutive vertices in $V(S)$ belongs to $V(P)$, no vertex in $\{s_2, \ldots, s_{a(i)}\}$ is equal to p_{i-1} or p_i.

Lemma 6. *Let G be an AT-free graph and let u,v be a dominating pair. Let P, S, $a(i)$ and $b(i)$ be defined as above. Then,*

(1) $\{s_1, \ldots, s_{a(i)-1}\} \cap N[\{p_i, \ldots, p_{l_P+1}\}] = \emptyset$, *for every* $i \in \{3, \ldots, l_P\}$.
(2) $\{s_{b(i)+1}, \ldots, s_{l_S+1}\} \cap N[\{p_1, \ldots, p_{i-1}\}] = \emptyset$, *for every* $i \in \{2, \ldots, l_P - 1\}$.
(3) $a(i) \leqslant b(i) \leqslant a(i)+6$, *for every* $i \in \{2, \ldots, l_P+1\}$.

Proof. (1) For sake of contradiction let us assume that the intersection is not empty and let $m < a(i)$ be the smallest index for which s_m is in the intersection. Since $i \geqslant 3$ and P is a shortest path $s_m \neq p_1$, then $m \geqslant 2$. We claim that neither s_m nor s_{m-1} belong to $V(P)$.

Since P is a dominating path and s_m is the first vertex adjacent to the set $\{p_i, \dots, p_{l_P+1}\}$, the vertex s_{m-1} is adjacent or equal to some vertex in $\{p_1, \dots, p_{i-1}\}$. But since s_{m-1} is not adjacent to p_i, it cannot be equal to p_{i-1}.

First, let us assume that $s_{m-1} \in V(P)$. Then it belongs to $\{p_1, \dots, p_{i-2}\}$. In this case $s_m \notin V(P)$. By the definitions of $a(i)$ and m, it is adjacent to some vertex p_t in $\{p_{i+1}, \dots, p_{l_P+1}\}$. That contradicts the fact that P is a shortest path. Hence, $s_{m-1} \notin V(P)$.

Now let us assume that $s_m = p_k$ for some $k \in \{1, 2, \dots, l_P + 1\}$. Since s_m is adjacent to the set $\{p_i, \dots, p_{l_P+1}\}$, we conclude $k \geqslant i - 1$. Since s_{m-1} is not adjacent to this set, we get $k \leqslant i - 1$, thus $k = i - 1$. But since s_{m-1} is not in $V(P)$, we get a contradiction with $m < a(i)$.

Having proven the claim, let us define p' to be the first vertex in $\{p_i, \dots, p_{l_P+1}\}$ adjacent to s_m and p'' to be the last vertex in $\{p_1, \dots, p_{i-1}\}$ adjacent to s_{m-1}. Notice that from the definition of $a(i)$, $p' \neq p_i$ and $p'' \neq p_{i-1}$.

Since the path (p'', s_{m-1}, s_m, p') has length three we deduce that $p'' = p_{i-2}$ and that $p' = p_{i+1}$. Since P is a shortest path and from the choice of m we deduce that the cycle $p'', p_{i-1}, p_i, p', s_m, s_{m-1}$ of length 6 is chordless and it would contains an asteroidal triple.

(2) It follows by symmetry from Part (1).

(3) Let us assume that $b(i) \geqslant a(i) + 7$. Then no vertex in $\{p_{i-1}, p_i\}$ is adjacent to both $x := s_{a(i)}$ and $y := s_{b(i)}$. Hence, x and y have different neighbors in $\{p_{i-1}, p_i\}$. Let us call s_t one of the vertices $s_{a(i)+3}$ or $s_{a(i)+4}$ which is not in $V(P)$. Then $t - a(i) \geqslant 3$ and $b(i) - t \geqslant 3$ and then s_t has no neighbor in $\{p_{i-1}, p_i\}$. We get a contradiction by proving that $\Omega := \{x, s_t, y\}$ is an asteroidal triple. In fact, $x, s_t, y \notin V(P)$. Since S is a shortest path Ω is an independent set. From the definition of $a(i)$ and $b(i)$ there is a path of length at most three from x to y passing though the set $\{p_{i-1}, p_i\}$. Since s_t has no neighbor in $\{p_{i-1}, p_i\}$ we conclude that Ω is an asteroidal triple. □

Lemma 7. *Let G be an AT-free graph. Let u, v be a dominating pair. Let P, S, $a(i)$ and $b(i)$ defined as before. For all $3 \leqslant i \leqslant l_P$ the following properties hold*:

(1) $d_S(u, s_{a(i)}) \leqslant 3d_P(u, p_i) - 4$.
(2) $d_S(v, s_{b(i)}) \leqslant 3d_P(v, p_{i-1}) - 4$.

Proof. (1) We just need to prove that $a(i) - 1 \leqslant 3(i-1) - 4$. From Lemma 6 we deduce that the set $\{s_1, \dots, s_{a(i)-1}\}$ is dominated by the set $\{p_1, \dots, p_{i-1}\}$. Since S is a shortest (u, v)-path edge disjoint with P, the vertex p_1 dominates exactly the vertices $s_1 = p_1$ and s_2. By the same reason, every vertex p_j, for $j = 2, \dots, i-2$, dominates at most three vertices in $\{s_1, \dots, s_{a(i)-1}\}$. Finally, the only vertex in $\{s_1, \dots, s_{a(i)-1}\}$ which could be dominated by p_{i-1} is p_{i-2}. But this vertex has already been considered. Therefore, $a(i) - 1 \leqslant 2 + 3(i - 3) = 3(i - 1) - 4$.

(2) It follows by symmetry from the part (1). □

In the next lemma we prove that if we have a linear upper bound like those proved in Lemma 7 then it is possible to obtain a linear upper bound for $OD(G)$ in terms of $diam(G)$. Since in Section 4 we shall improve the bound of Lemma 7 for some subclasses of AT-free graphs (Lemmas 11 and 13) we prove the lemma in a general form.

Lemma 8. *Let G be an AT-free graph. Let u, v be a dominating pair of G and let P, S, $a(i)$ and $b(i)$ be defined as before. If there exist constants α and β satisfying for every $3 \leqslant i \leqslant l_P$*

(1) $d_S(u, s_{a(i)}) \leqslant \alpha d_P(u, p_i) + \beta$,
(2) $d_S(v, s_{b(i)}) \leqslant \alpha d_P(v, p_{i-1}) + \beta$,

then G has an orientation H such that
$diam(H) \leqslant \frac{\alpha+1}{2} diam(G) + \beta + \frac{1-\phi(\alpha)}{2}\alpha + \frac{29+\phi(\alpha)}{2}$, *where* $\phi(\alpha) = \lfloor 6/\alpha - 1 \rfloor$.

Proof. Let i be an integer with $3 \leqslant i \leqslant l_P$. Let p_t be a vertex in $\{p_{i-1}, p_i\}$ adjacent to $s_{a(i)}$. If $p_t \notin V(S)$ we denote by e the edge $s_{a(i)}p_t$. Otherwise $e = \emptyset$. We define an orientation H in the subgraph $G' := G[P \cup S]$ as follows. We orient Pp_t from u to p_t and p_tP from v to p_t. We orient $Ss_{a(i)}$ from $s_{a(i)}$ to u and $s_{a(i)}S$ from $s_{a(i)}$ to v. If $s_{a(i)}p_t$ has not been already oriented ($e \neq \emptyset$), we orient it from p_t to $s_{a(i)}$. All the remaining edges in G' are oriented from $V(S)$ to $V(P)$. Let us denote $L := d_P(u, v)$.

The distance in H from $x \neq s_{a(i)}$ to p_t is equal to $d_H(x, p_j) + d_H(p_j, p_t)$ where p_j dominates x or it is equal to x. If $e = s_{a(i)}p_t$ then $d_H(s_{a(i)}, p_t) = 1 + d_H(s_{a(i)-1}, p_t)$. Then for all $x \in V(G')$ we have that $d_H(x, p_t) \leqslant \max\{1 + d_P(u, p_t), 1 + d_P(p_t, v)\}$. Since $d_P(u, p_t) = t - 1$, $d_P(v, p_t) = L - (t-1)$ and $t \in \{i-1, i\}$ we get

$$d_H(x, p_t) \leqslant \max\{i, L - i + 3\}. \tag{1}$$

The distance in H from p_t to x, $d_H(p_t, x)$, satisfies:

$$d_H(p_t, x) \leqslant 1 + d_S(s_{a(i)}, x) \quad \text{when } x \in V(S) \setminus V(P),$$

$$d_H(p_t, x) \leqslant 1 + d_S(s_{a(i)}, s_{a(j)}) + 2 \quad \text{when } x = p_j \text{ with } j < t,$$

$$d_H(p_t, x) \leqslant 1 + d_S(s_{a(i)}, s_{a(j+1)}) + 2 \quad \text{when } x = p_j \text{ with } j > t.$$

Then $d_H(p_t, x) \leqslant \max\{3 + d_S(s_{a(i)}, u), 3 + d_S(s_{a(i)}, v)\}$. From the hypothesis and Lemma 6(3) we have

- $d_S(u, s_{a(i)}) \leqslant \alpha d_P(u, p_i) + \beta = \alpha(i-1) + \beta$.
- $d_S(s_{a(i)}, v) \leqslant d_S(s_{b(i)-6}, v) \leqslant 6 + \alpha d_P(p_{i-1}, v) + \beta = 6 + \alpha(L - (i-2)) + \beta$.

Therefore,

$$d_H(p_t, x) \leqslant 3 + \beta + \max\{\alpha(i-1), 6 + \alpha(L - (i-2))\}. \tag{2}$$

From inequalities (1) and (2) we deduce that for every x, y in G',

$$d_H(x, y) \leqslant 3 + \beta - \alpha + f(i),$$

where

$$f(i) := \max\{i(\alpha+1)+d_1, -i(\alpha-1)+d_2, i(\alpha-1)+d_3, -i(\alpha+1)+d_4\}$$

and $d_1=0$, $d_2=6+\alpha(L+3)$, $d_3=(L+3)$ and $d_4=L+9+\alpha(L+3)$. Then f is a piecewise linear function. We have that $d_1 < d_3 < d_2 < d_4$ and $-(\alpha+1) < -(\alpha-1) < 0 < \alpha-1 < \alpha+1$. Moreover, the solutions of the equations $i(\alpha+1)+d_1 = i(\alpha-1)+d_3$ and $-i(\alpha-1)+d_2 = -i(\alpha+1)+d_4$ are the same: $i_{13} = (L+3)/2$ and the solution of the equation $i(\alpha+1)+d_1 = -i(\alpha-1)+d_2$ is $(L+3)/2+3/\alpha > (L+3)/2$. Hence, f is given by:

$$f(i) = \begin{cases} -i(\alpha+1)+d_4, & i \in [0,(L+3)/2), \\ -i(\alpha-1)+d_2, & i \in [(L+3)/2,(L+3)/2+3/\alpha), \\ i(\alpha+1)+d_1, & i \in [(L+3)/2+3/\alpha, L]. \end{cases}$$

Let $\phi(\alpha)=\lfloor 6/\alpha-1 \rfloor$. Then either $i_1 := (L+3+\phi(\alpha))/2$ or $i_2 := (L+3+\phi(\alpha)+1)/2$ is an integer and both belong to the interval $[(L+3)/2,(L+3)/2+3/\alpha]$. Since $f(i_2) \leqslant f(i_1)$, we get $d_H(x,y) \leqslant 3+\beta-\alpha+f(i_1)$. We finally obtain $d_H(x,y) \leqslant [(\alpha+1)/2]L+\beta+(1-\phi(\alpha))/2\alpha+(21+\phi(\alpha))/2$. Since (u,v) is a dominating pair $V(P)$ is a dominating set in G. Moreover G' contains $V(P)$. Then from Lemma 1 we conclude that G has an orientation with diameter at most $(\alpha+1)/2L+\beta+(1-\phi(\alpha))/2\alpha+(29+\phi(\alpha))/2$. □

Theorem 9. *For every AT-free graph G, $OD(G) \leqslant 2\,diam(G)+11$.*

Proof. From Lemma 7 we know that $d_S(u,s_{a(i)}) \leqslant 3d_P(u,p_i)-4$ and that $d_S(v,s_{b(i)}) \leqslant 3d_P(v,p_{i-1})-4$. Then by taking $\alpha=3$ and $\beta=-4$ in Lemma 8 we get that $\phi(3)=1$. Then $OD(G) \leqslant 2\,diam(G)+\frac{30}{2}-4 = 2\,diam(G)+11$. □

4. Better upper bounds for classes of interval graphs

In this section we improve the upper bound of Theorem 9 for the following subclasses of AT-free graphs: interval, proper interval and 2-connected proper interval graphs.

A graph G is an interval graph if it is the intersection graph of a finite family $\{I_1,\ldots,I_n\}$ of intervals of the real line. An interval graph G is a proper interval graph if in the family $\{I_1,\ldots,I_n\}$, no two intervals I_i and I_j properly contain each other. An interval graph G is a unit interval graph if in the family $\{I_1,\ldots,I_n\}$ each interval I_i has unit length. A graph G is chordal if it has no induced cycles of length greater than three. A claw is a tree with three leaves and four vertices. A graph G is claw-free if it contains no induced claws.

Moreover, we have the following characterizations.

- A graph G is interval if and only if G is AT-free and chordal [14].
- A graph G is proper interval if and only if G is unit interval [16].
- A graph G is proper interval if and only if G is interval and claw-free [22].

In the following the paths P and S, as well as $a(i)$ and $b(i)$, are defined as in the previous section.

A k-balloon G' in G is a subgraph of G which consists of a cycle C of length k and a vertex $x \notin V(C)$ adjacent to a unique vertex v_x in C. If G has an induced k-balloon with $k \geqslant 4$ then the cycle C is an induced cycle of length greater than three and the closed neighborhood of v_x in G is a claw. Therefore, G is neither chordal nor claw-free.

When we defined $a(i)$ and $b(i)$ in Section 3 we already showed that in an AT-free graph G, every pair of consecutive vertices of P has some neighbor in $V(S) \setminus V(P)$. If we add the property of being chordal or claw-free, *each vertex* of P has such a neighbor. Moreover,

Lemma 10. *Let G be an AT-free graph. Let P, S, $a(i)$ and $b(i)$ defined as before and $l_P \geqslant 4$. If G is chordal or claw-free then for all $i = 3, \ldots, l_P$*

(1) *The vertex $s_{a(i)}$ is adjacent to p_{i-1}.*
(2) *The vertex $s_{b(i)}$ is adjacent to p_i.*

Proof. (1) Let us assume that $s_{a(i)}$ is not adjacent to p_{i-1} for some $i = 3, \ldots, l_P$. We show that G has an induced k-balloon with $k \geqslant 4$ or an asteroidal triple. From the definition of $a(i)$ the vertex $s_{a(i)}$ is adjacent to p_i.

Let us suppose that $s_{a(i)-1} \notin V(P)$. From the definition of $a(i)$ the vertex $s_{a(i)-1}$ is not adjacent to p_{i-1}. From Lemma 6 there exists $t \leqslant i-2$ such that $s_{a(i)-1}$ is adjacent to p_t. Let t be the largest index with this property. If $p_t, \ldots, p_{i-1}, p_i, s_{a(i)}, s_{a(i)-1}, p_{i+1}$ is not an induced k-balloon then $s_{a(i)}$ is adjacent to p_{i+1}. Since $p_t, s_{a(i)-1}, s_{a(i)}, p_{i+1}$ is a path of length three and P is a shortest path we deduce that $t = i-2$. But then $\{p_{i-1}, s_{a(i)-1}, p_{i+1}\}$ is an asteroidal triple.

Then $s_{a(i)-1} \in V(P)$. From Lemma 6 $s_{a(i)-1} = p_t$ with $t \leqslant i-2$. Since P is a shortest path the vertex $s_{a(i)}$ is not adjacent to $\{p_1, \ldots, p_{i-3}\}$. Then $t = i-2$ and $\{p_{i-2}, p_{i-1}, p_i, p_{i+1}, s_{a(i)}\}$ is an induced 4-balloon in G.

(2) It follows by symmetry from part (1). □

Lemma 11. *Let G be an AT-free graph. If G is chordal or claw-free then for all $3 \leqslant i \leqslant l_P$*

(1) $d_S(u, s_{a(i)}) \leqslant 2d_P(u, p_i) - 2$.
(2) $d_S(v, s_{b(i)}) \leqslant 2d_P(v, p_{i-1}) - 2$.

Proof. (1) We prove the property by induction. We have to prove that $a(i)-1 \leqslant 2(i-1)-2$, that is $a(i) \leqslant 2(i-1)-1$. We first prove that $a(3) \leqslant 3$. For sake of contradiction let us assume that $a(3) > 3$. Then the vertices s_2 and s_3 are adjacent neither to p_2 nor to p_3. Hence they do not belong to $V(P)$. From Lemma 6 we deduce that s_3 is adjacent to p_1. Then we obtain a (u, v)-path shorter than S edge-disjoint with P which is a contradiction.

Let us assume that $a(i) \leqslant 2(i-1)-1$ for $3 \leqslant i \leqslant l_P - 1$. We prove that $a(i+1) \leqslant 2i-1$ by showing that $a(i+1) \leqslant a(i)+2$. From Lemma 10 we know that $s_{a(i+1)}$ is adjacent to p_i and $s_{a(i)}$ is adjacent to p_{i-1}. We assume that $a(i+1) > a(i)$. From the definition of $a(i)$ the vertex $s_{a(i)}$ is not adjacent to p_i. Now, if $s_{a(i+1)}$ is adjacent to p_{i-1} or $s_{a(i+1)}$ is adjacent to $s_{a(i)}$ or $s_{a(i+1)-1}$ is adjacent to $s_{a(i)}$ then $a(i+1) \leqslant a(i)+2$. We will obtain a contradiction by assuming that $s_{a(i+1)}$ is not adjacent to p_{i-1}, $s_{a(i+1)}$ is not adjacent to $s_{a(i)}$ and $s_{a(i+1)-1}$ is not adjacent to $s_{a(i)}$. If $s_{a(i+1)-1}$ is adjacent to p_{i-1} then $p_{i-1}, p_i, s_{a(i+1)}, s_{a(i+1)-1}, s_{a(i)}$ is an induced 4-balloon. Then $s_{a(i+1)-1}$ is not adjacent to p_{i-1}. Thus $\{s_{a(i)}, s_{a(i+1)-1}, p_i\}$ is an asteroidal triple.

(2) It follows by symmetry from part (1). □

Theorem 12. *Let G be an AT-free graph. If G is chordal or claw-free then* $OD(G) \leqslant \frac{3}{2}\,diam(G) + \frac{25}{2}$.

Proof. From Lemma 11 we know that $d_S(u, s_{a(i)}) \leqslant 2d_P(u, p_i) - 2$ and that $d_S(v, s_{b(i)}) \leqslant 2d_P(v, p_{i-1}) - 2$ for all $3 \leqslant i \leqslant l_P$. Then by taking $\alpha = 2$ and $\beta = -2$ in Lemma 8 we get that $\phi(2) = 2$. Then $OD(G) \leqslant \frac{3}{2}\,diam(G) - 2 - 1 + \frac{31}{2} = \frac{3}{2}\,diam(G) + \frac{25}{2}$. □

Lemma 13. *Let G be an AT-free graph. If G is claw-free and 2-connected then for all* $i \in \{3, \ldots, l_P\}$

(1) $d_S(u, p_i) \leqslant \frac{3}{2}(d_P(u, p_i) + 1)$.
(2) $d_S(p_i, v) \leqslant \frac{3}{2}(d_P(v, p_{v-1}) + 1)$.

Proof. (1) Let us denote $\tilde{G} = G - E(P)$. For every $1 \leqslant i \leqslant l_P - 1$ we prove the following properties:

(a) $d_{\tilde{G}}(p_i, p_{i+2}) \leqslant 3$.
(b) $d_{\tilde{G}}(p_i, p_{i+3}) \leqslant 6$, $i \neq l_P - 1$.

Let i be with $1 \leqslant i \leqslant l_P - 1$. Among all the paths connecting $\{p_1, \ldots, p_i\}$ and $\{p_{i+2}, \ldots, p_{l_P+1}\}$ in $G \setminus p_{i+1}$ let $Q = \{q_1, \ldots, q_r\}$ be a shortest one. Then $q_2, \ldots, q_{r-1} \in V(G) \setminus V(P)$ and hence $E(Q) \cap E(P) = \emptyset$.

(a) Since P is a shortest path if $r = 3$ then $q_1 = p_i$ and $q_3 = p_{i+2}$ which proves the statement. Let us assume that $r \geqslant 4$. Then $q_3 \notin V(P)$. We shall prove that q_3 is adjacent to p_{i+2} and that q_2 is adjacent to p_i. Since Q is a shortest path the vertex q_3 is not dominated by $\{p_1, \ldots, p_i\}$. Let p_t be the first vertex in $\{p_{i+1}, \ldots, p_{l_P+1}\}$ which dominates q_3. Since q_3 is not adjacent to p_i we deduce that p_{t+1} also dominates q_3 otherwise we get an induced claw $\{p_{t-1}, p_t, p_{t+1}, q_3\}$. Since the path q_1, q_2, q_3, p_{t+1} has length 3 we deduce that $i+1 \leqslant t \leqslant i+2$ and then $i+2 \in \{t, t+1\}$ that is p_{i+2} is adjacent to q_3. We now prove that q_2 is adjacent to p_i. Since P is a shortest path the vertex q_2 can not be adjacent to any p_j with $j < i-1$. The set $\{p_1, \ldots, p_i\}$ dominates q_2 then q_2 is adjacent to p_i or to p_{i-1}. If q_2 is not adjacent to p_i then it is adjacent to p_{i-1} and we get that $\{p_{i-2}, p_{i-1}, p_i, q_2\}$ is an induced claw. Then (p_i, q_2, q_3, p_{i+2}) is a path of length three between p_i and p_{i+2} with no edges in $E(P)$.

(b) Let $1 \leqslant i < l_P - 1$. Let Q be defined as above. We already know that the vertex q_3 in Q is adjacent to p_{i+2}. If q_3 is adjacent to p_{i+3} then we obtain the conclusion. Otherwise q_3 is adjacent to p_{i+1}. Therefore $d_{\tilde{G}}(p_i, p_{i+1}) \leqslant 3$. Using the first property for $i+1$ we obtain the conclusion.

From previous properties we deduce that $d_{\tilde{G}}(u, p_i) \leqslant 3(i-1)/2$, for all odd integer $i \in \{3, \ldots, l_P\}$ and that $d_{\tilde{G}}(u, p_i) \leqslant d_{\tilde{G}}(u, p_{i-3}) + d_{\tilde{G}}(p_{i-3}, p_i) \leqslant 3(i-4)/2 + 6 \leqslant 3i/2$, for all even integer $i \in \{4, \ldots, l_P\}$. Since S is a shortest (u, v)-path edge disjoint with P we conclude the result.

(2) It follows by symmetry from (1). □

Theorem 14. *Let G be an AT-free graph. If G is claw-free and 2-connected then* $OD(G) \leqslant \frac{5}{4}\, diam(G) + \frac{29}{2}$.

Proof. From Lemma 13 we know that $d_S(u, s_{a(i)}) \leqslant \frac{3}{2} d_P(u, p_i) + \frac{3}{2}$ and that $d_S(v, s_{b(i)}) \leqslant \frac{3}{2} d_P(v, p_{i-1}) + \frac{3}{2}$. Then by taking $\alpha = \frac{3}{2}$ and $\beta = \frac{3}{2}$ in Lemma 8 we get that $\phi(\frac{3}{2}) = 3$.

Therefore, $OD(G) \leqslant \frac{5}{4}\, diam(G) - 3 + 16 + \frac{3}{2} = \frac{5}{4}\, diam(G) + \frac{29}{2}$. □

Corollary 15. *Let G be a graph.*

- *If G is an interval graph then* $OD(G) \leqslant \frac{3}{2}\, diam(G) + \frac{25}{2}$.
- *If G is a proper interval graph then* $OD(G) \leqslant \frac{3}{2}\, diam(G) + \frac{25}{2}$.
- *If G is a 2-connected proper interval graph then* $OD(G) \leqslant \frac{5}{4}\, diam(G) + \frac{29}{2}$.

5. Tightness results

Here we show that all our upper bounds are tight up to additive constants. For this purpose we exhibit families of graphs reaching the upper bounds. Moreover, in the case of AT-free graphs the exhibited family belongs to the class of cocomparability graphs (known to be a subclass of AT-free graphs [7]).

Theorem 16. *For every $d \geqslant 3$ there is a*

- *Cocomparability graph G with $diam(G) = d$ and* $OD(G) \geqslant 2\, diam(G) - 1$.
- *Proper interval graph G with $diam(G) = d$ and* $OD(G) \geqslant \frac{3}{2}\, diam(G)$.
- *2-connected proper interval graph G with $diam(G) = d$ and* $OD(G) \geqslant \frac{5}{4}\, diam(G)$.

Proof. See the constructions of Figs. 1–3. □

The previous examples in Figs. 1 and 2 contain a lot of cut vertices, thus the question whether the bounds could possibly be improved if we require the graph to be k-connected for some $k \geqslant 2$ arises.

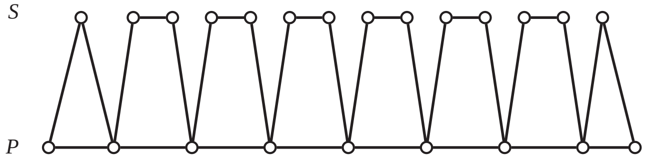

Fig. 1. A cocomparability graph with $OD(G) \geqslant 2\,diam(G) - 1$.

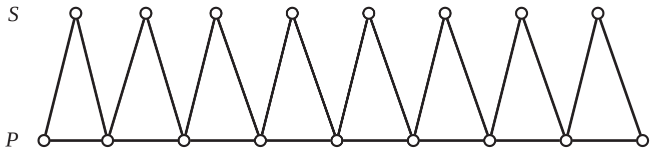

Fig. 2. A proper interval graph with $OD(G) \geqslant \frac{3}{2}\,diam(G)$.

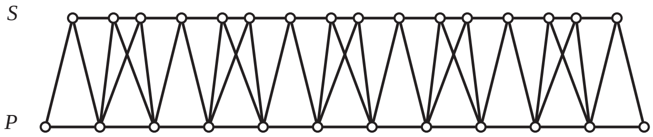

Fig. 3. A 2-connected proper interval graph with $OD(G) \geqslant \frac{5}{4}\,diam(G)$.

We need the operation of *replacing a vertex v by a complete graph K_k* in a graph G. This is done by deleting the vertex v, adding k new vertices $v_1, v_2, \ldots, v_k$, adding an edge between each pair $v_i, v_j, i \neq j$, and adding all edges $v_i y$ for all neighbors y of v in G and all $1 \leqslant i \leqslant k$. Note that if G is a proper interval graph, or interval graph, or cocomparability graph, or AT-free, then the graph obtained is also a proper interval graph, or an interval graph, or a cocomparability graph, or AT-free, respectively. Moreover, the diameter of the new graph equals that of G.

Let G be a graph and $F \subseteq E(G)$. A *partial F-orientation* of G is a graph obtained by orienting all the edges in F, and replacing every other edge by two antiparallel arcs.

Lemma 17. *Let G be a graph and $W \subseteq V(G)$. Let us call G' the graph resulting from replacing each vertex $w \in W$ by a complete graph with at least two vertices. Assume that each vertex $w \in W$ is replaced by a complete graph with at least two vertices, the resulting graph is called G'. Let F be the set of those edges of G between non members of W. Then the minimum diameter of a strongly connected orientation of G' is greater or equal to the minimum diameter of a strongly connected partial F-orientation of G.*

Proof. Let H' be an optimum orientation of G'. By orienting all the edges in F like in H', we get a partial F-orientation H of G. The "projection" of every directed path in H' is a directed path in H, whence $diam(H) \leqslant diam(H')$. □

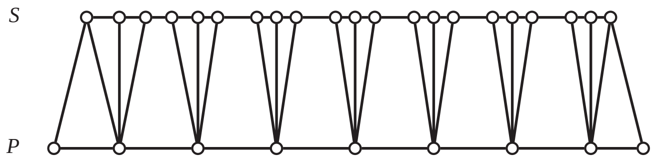

Fig. 4. A 2-connected cocomparability graph with $OD(G) \geqslant 2\,diam(G) - 1$.

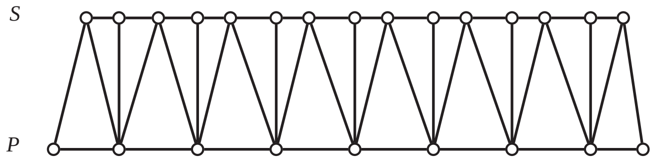

Fig. 5. A 2-connected interval graph with $OD(G) \geqslant \frac{3}{2}\,diam(G)$.

Theorem 18. *For every k and for every $d \geqslant 3$ there is a k-connected*

(a) *Cocomparability graph G with $diam(G) \geqslant d$ and $OD(G) \geqslant 2\,diam(G) - 1$.*
(b) *Interval graph G with $diam(G) \geqslant d$ and $OD(G) \geqslant \frac{3}{2}\,diam(G)$.*
(c) *Proper interval graph G with $diam(G) \geqslant d$ and $OD(G) \geqslant \frac{5}{4}\,diam(G)$.*

Proof. The case $k = 2$ is shown in Figs. 3–5. The set of vertices of these graphs can be divided in two sets: $V(P)$ and $V(S)$, where P is the shortest path between the ends u and v and S is a (u, v)-path vertex-disjoint with P. For $k \geqslant 3$ we replace every vertex in $V(S)$ by a k-complete graph obtaining a $k + 1$ connected graph. The proof is finished by taking $F = E(P)$ and $W = V(S)$ in Lemma 17. □

References

[1] F.R.K. Chung, Diameters of communication networks, in: Mathematics of Information Processing (Louisville, KY, 1984), Proc. Sympos. Appl. Math., 34, American Mathematical Society, Providence, RI, 1986, pp. 1–18.

[2] F.R.K. Chung, M.R. Garey, R.E. Tarjan, Strongly connected orientations of mixed multigraphs, Networks 15 (1985) 477–484.

[3] V. Chvátal, C. Thomassen, Distances in orientations of graphs, J. Combin. Theory Ser. B 24 (1978) 61–75.

[4] D.G. Corneil, S. Olariu, L.K. Stewart, Asteroidal triple-free graphs, SIAM J. Discrete Math. 10 (1997) 399–430.

[5] R. Diestel, Graph Theory, 2nd Edition, Springer, New York, 2000.

[6] P. Fraigniaud, E. Lazard, Methods and problems of communication in usual networks, Discrete Appl. Math. 53 (1994) 79–133.

[7] M.C. Golumbic, C.L. Monma, W.T. Trotter, Tolerance graphs, Discrete Appl. Math. 9 (1984) 157–170.

[8] G. Gutin, Minimizing and maximizing the diameter in orientations of graphs, Graphs Combin. 10 (1994) 225–230.

[9] S.M. Hedetniemi, S.T. Hedetniemi, A.L. Liestman, A survey of gossiping and broadcasting in communication networks, Networks 18 (1988) 319–349.

[10] K.M. Koh, E.G. Tay, On optimal orientations of Cartesian products of even cycles, Networks 32 (1998) 299–306.
[11] K.M. Koh, E.G. Tay, On optimal orientations of Cartesian products with a bipartite graph, Discrete Appl. Math. 98 (1999) 103–120.
[12] K.M. Koh, E.G. Tay, On optimal orientations of Cartesian products of graphs, II, Complete graphs and even cycles, Discrete Math. 211 (2000) 75–102.
[13] J.-C. Konig, D.W. Krumme, E. Lazard, Diameter-preserving orientations of the torus, Networks 32 (1998) 1–11.
[14] C.G. Lekkerkerker, J.C. Boland, Representation of a finite graph by a set of intervals on the real line, Fund. Math. 51 (1962) 45–64.
[15] H.E. Robbins, A theorem on graphs with an application to a problem of traffic control, Amer. Math. Monthly 46 (1939) 281–283.
[16] F.S. Roberts, Indifference graphs, in: F. Harary (Ed.), Proof Techniques in Graph Theory, Academic Press, New York, 1969, pp. 301–310.
[17] F.S. Roberts, Y. Xu, On the optimal strongly connected orientations of city street graphs, I, Large grids, SIAM J. Discrete Math. 1 (1988) 199–222.
[18] F.S. Roberts, Y. Xu, On the optimal strongly connected orientations of city street graphs, II, Two east–west avenues or north–south streets, Networks 19 (1989) 221–233.
[19] F.S. Roberts, Y. Xu, On the optimal strongly connected orientations of city street graphs, III, Three east–west avenues or north–south streets, Networks 22 (1992) 109–143.
[20] F.S. Roberts, Y.H. Xu, On the optimal strongly connected orientations of city street graphs, IV, Four east–west avenues or north–south streets, Discrete Appl. Math. 49 (1994) 331–356 (Viewpoints on optimization (Grimentz, 1990; Boston, MA, 1991)).
[21] L. Šoltés, Orientations of graphs minimizing the radius or the diameter, Math. Slovaca 36 (1986) 289–296.
[22] G. Wegner, Eigenschaften der Nerven homologisch-einfacher Familien im $\mathbb{R}^n$, Dissertation Thesis, University of Gttingen, 1967.