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On distance constrained labeling of disk graphs☆

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Abstract

A disk graph is the intersection graph of a set of disks in the plane. For a k-tuple $(p_1, \ldots, p_k)$ of positive integers, a distance constrained labeling of a graph G is an assignment of labels to the vertices of G such that the labels of any pair of vertices at graph distance i in G differ by at least p_i, for $i = 1, \ldots, k$. In the case when $k = 1$ and $p_1 = 1$, this gives a traditional coloring of G. We propose and analyze several online and offline labeling algorithms for the class of disk graphs.

Keywords: Disk graph; Labeling; Online; Approximation

1. Introduction

In traditional coloring of a graph, any pair of vertices in the graph gets distinct colors whenever they are adjacent by an edge, i.e. at graph distance one. For a long time coloring of simple graph classes, e.g. paths, cycles, grids, interval graphs, planar graphs, and etc., has been considered as a general model for the frequency assignment problem

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in radio networks [18,19,21,25,28], assuming that only frequencies used in "near" regions should be well separated. However, due to the rapid development of mobile networks, new theoretical approaches have emerged to model the problem assuming that frequencies used in both "near" and "distant" regions should be properly separated. One of these is distance constrained labeling, see e.g. [1,2,5,11–13,20,29,24].

1.1. Clique, independent set, coloring, and labeling

Let $G = (V, E)$ be a simple graph. A subset $V' \subseteq V$ is a *clique* if every two vertices in V' are joined by an edge in E. A *maximum clique* is, naturally, a clique whose number of vertices is at least as large as that for any other clique in the graph, and its size, $\omega(G)$, is called the *clique number* of G. A subset $V' \subseteq V$ is an *independent set* if no its vertices are adjacent. Similarly, a *maximum independent set* is an independent set whose number of vertices is at least as large as that for any other clique in the graph, and its size, $\alpha(G)$, is called the *independence number* of G. A (vertex) k-coloring of G is a function $c : V \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ whenever vertices u and v are joined by an edge in E, i.e. at graph distance 1. If a k-coloring of G exists, then G is called k-colorable. The *chromatic number* of G is defined as

$$\chi(G) = \min\{k\colon\ G \text{ is } k\text{-colorable}\}.$$

There are two simple facts. Let V' be a subset of V. If V' is an independent set, then the vertices of V' can be colored in one color. If V' is a clique, the vertices of V' must be colored in $|V'|$ distinct colors. There is a trivial bound as

$$\max\{\omega(G), |V|/\alpha(G)\} \leqslant \chi(G). \tag{1}$$

Let $k \geqslant 1$ be some integer. Let $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_k$ be a non-increasing sequence of positive integers, called *distance constraints*. An $L_{(p_1,\ldots,p_k)}$*-labeling*, or a *distance constrained labeling*, of a graph $G = (V, E)$ is a function $c : V \to \{1, \ldots, L\}$ such that $|c(u) - c(v)| \geqslant p_i$ whenever the graph distance between u and v is at least i, for $i = 1, \ldots, k$. If a $L_{(p_1,\ldots,p_k)}$-labeling of G exists, then G is called $L_{(p_1,\ldots,p_k)}$-labeled. The $(p_1, \ldots, p_k)$*-labeling number* of G is defined as

$$\chi_{(p_1,\ldots,p_k)}(G) = \min\{L\colon\ G \text{ is } L_{(p_1,\ldots,p_k)}\text{-labeled}\}.$$

First, we can observe the following simple facts. If $k = 1$ and $p_1 = 1$, then

$$\chi_{(1)}(G) = \chi(G), \tag{2}$$

where $\chi(G)$ is the chromatic number of G. If $p_1 = p_2 = \cdots = p_k = 1$, then

$$\chi_{(1,\ldots,1)}(G) = \chi(G^k), \tag{3}$$

where G^k is the kth power of G, i.e. a graph which arises from G by adding the edges which connect all the vertices at the graph distance at most k. Furthermore, as it was shown in [9,13], for any integer t it holds

$$\chi_{(tp_1,\ldots,tp_k)}(G) = t \cdot (\chi_{(p_1,\ldots,p_k)}(G) - 1) + 1. \tag{4}$$

Hence, we can assume w.l.o.g. that all integers $p_1, \ldots, p_k$ have no common divisor. Combining (3) and (4), we can bound

$$\begin{aligned} \chi_{(p_1,\ldots,p_k)}(G) &\leqslant \chi_{(p_1,\ldots,p_1)}(G) \\ &= 1 + p_1(\cdot\chi_{(1,\ldots,1)}(G) - 1) \\ &= 1 + p_1(\chi_{(1)}(G^k) - 1). \end{aligned} \tag{5}$$

Accordingly, for $k = 2$ and $(p_1, p_2) = (2, 1)$ we have

$$\begin{aligned} \chi_{(2,1)}(G) &\leqslant \chi_{(2,2)}(G) \\ &= 2(\chi_{(1,1)}(G) - 1) + 1 \\ &\leqslant 2\chi_{(1)}(G^2) \\ &= 2\chi(G^2). \end{aligned} \tag{6}$$

In [23] it was shown that for any fixed $k \geqslant 2$ finding the value of $\chi(G^k)$ is an NP-hard problem. Furthermore, even if one restricts to a planar graph G, computing $\chi(G^2)$ is still an NP-hard problem. There is the long-standing Wegner's conjecture [30]: For any planar graph G with the maximum degree $\Delta(G) \geqslant 8$, the chromatic number of the second power graph G^2 is at least $\lceil \frac{3}{2}\Delta \rceil + 1$. There are a number of recent results coming closer and closer to the conjectured bound. The current best result $\chi(G^2) \leqslant \frac{5}{3}\Delta + 78$ is due to [24].

The most intensively studied case of distance-constrained labeling is $k = 2$ and $(p_1, p_2) = (2, 1)$. The existence of an $L_{(2,1)}$-labeling was explored for different graph classes in [2,5,12,13,29]. The exact value of $\chi_{(2,1)}$ can be derived for *cycles*, and there are polynomial-time algorithms which compute the value of $\chi_{(2,1)}$ for *trees* and *co-graphs* [5]. For any fixed $L \geqslant 4$, the problem of recognizing graphs G such that $\chi_{(2,1)}(G) \leqslant L$ is NP-complete [10]. For a planar graph G, the problem of deciding whether $\chi_{(2,1)}(G) \leqslant 9$ was shown to be NP-complete in [2]. In [24] it was presented an approximation algorithm which produces an $L_{(p_1,p_2)}$-labeling of a planar graph G with the largest label at most $\frac{5}{3}(2p_2 - 1)\Delta(G) + 12p_1 + 144p_2 - 78$.

It is expected that for every k-tuple of distance constraints $(p_1, \ldots, p_k)$ and a graph G, there exists a bound L_0 such that for every $L \geqslant L_0$ the decision problem $\chi_{(p_1,\ldots,p_k)}(G) \leqslant L$ is NP-complete. So far, this conjecture has been proven for $k = 2$ and (p_1, p_2), where $p_1 \geqslant 2p_2$ [8].

1.2. Disk graphs

Let D be a set of disks in the Euclidian plane. Any disk in D is defined by its center and the value of its diameter. Then, the intersection graph G of the disks in D is called a *disk graph*, and D is called its *disk representation*. Let $d_{\min}$ and $d_{\max}$ be the minimum and maximum diameter values of the disks in D. Then, the value of $d_{\max}/d_{\min}$ is called the *diameter ratio* of D, denoted also by $\sigma(D)$. Let σ be some constant. A disk graph G is called a *σ-disk graph* if there exists its representation D whose diameter ratio $\sigma(D) \in (1, \sigma]$. If $\sigma(D) = 1$, then G is called a *unit disk graph*. In the latter case, we assume w.l.o.g. that all the disks in D have unit diameter.

Interestingly, every planar graph is a *coin graph*, that is, the intersection graph of interior-disjoint disks [17]. Hence, the class of disk graphs is more general than the class of planar

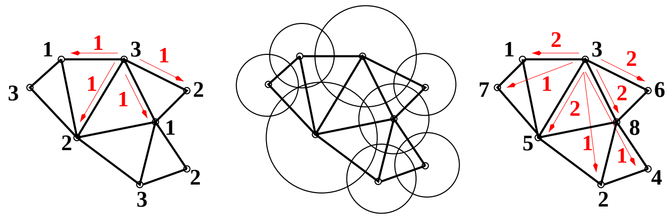

Fig. 1. Coloring-disk graph-$L_{(2,1)}$-labeling.

graphs. The recognition problem of a (unit, σ-) disk graph is NP-hard [3,4,15]. Hence, an algorithm that works on the set of graph's disks as the input is substantially weaker than one which works only on the sets of graph's vertices and edges. From this point of view, the requirement of a set of disks as the input is very strong. From another side, when dealing with real-world applications, e.g. in constructing interference graphs for radio and mobile telephony networks, some disk representation can be derived in a natural way.

There are a number of results on coloring of disk graphs. For a unit disk graph, the 3-coloring is NP-complete even when its disk representation is given [6]. There are a 3-approximation algorithm [3,26] and a 5-competitive algorithm [21,26]. These algorithms are given a set of unit disks as the input, but they can be also easily adjusted to the general case [7]. Regarding disk graphs, there is a 5-approximation algorithm which also works with a set of disks as the input [21]. On the other hand, there is no online coloring algorithm with a constant competitive ratio for planar graphs [14]. Hence, there is no such online algorithm for general disk graphs as well.

1.3. Our results

Here we consider the problem of distance-constrained labeling of σ-disk graphs, both given the disk representation and not. We present several offline and online algorithms for the case of general distance constraints $(p_1, \ldots, p_k)$ and for the case when $k = 2$ and $(p_1, p_2) = (2, 1)$. (For an illustration see Fig. 1.) We also derive several lower bounds. These provide the first step in the study of the distance-constrained labeling problem for disk graphs.

First, we deal with a fixed k-tuple of distance constraints $(p_1, \ldots, p_k)$. We give a simple online $L_{(p_1,\ldots,p_k)}$-labeling algorithm which is given a sequence of disks as the input. The algorithm is based on the so-called *hexagonal tiling*, *circular labeling*, and *first-fit* techniques. We derive an upper bound on its competitive ratio. We show for any fixed k-tuple $(p_1, \ldots, p_k)$ and any fixed diameter ratio σ the algorithm is constant competitive. As an example, we demonstrate the algorithm in the case when $k = 2$ and $(p_1, p_2) = (2, 1)$. We show that for σ-disk graphs with at least one edge and $\sigma \leqslant \sqrt{7}/2$ the competitive ratio of the algorithm is bounded by 16.67. The ratio also tens to 12.5 as the clique number of an input graph tens to infinity.

Next, we derive lower bounds for online coloring and labeling. We start with simple lower bounds for unit disk graphs. We consider the case when the input is given as a sequence of disks. We show that no online coloring algorithm can be better than 2-competitive, and no online $L_{(2,1)}$-labeling algorithm can be better than 5-competitive. Then, we consider σ-disk graphs. We prove that in the case when an algorithm is given a σ-graph in an online manner but neither its disk representation nor a bound on σ is given, the algorithm cannot achieve a constant competitive ratio. In addition, we give a lower bound on any general $L_{(p_1,\ldots,p_k)}$-labeling algorithm for σ-disk graphs. By using this result we show that our online labeling algorithm is asymptotically optimal for the class of disk graphs with at least one edge.

Finally, we deal with the offline setting. We explore the case $k = 2$ and $(p_1, p_2) = (2, 1)$. We present two approximation algorithms for unit disk graphs. The first algorithm is given a set of unit disks as the input, and it is based on the so-called *cutting* technique. The second algorithms is *robust*, what is, the algorithm is given a set of graph's vertices and a set of graph's edges as the input, and it either outputs a feasible labeling or shows that the input is not a unit disk graph. The approximation ratio of the *cutting* algorithm is bounded by 12, whereas the approximation ratio of the *robust* algorithm is bounded by 10, 67. The bounds also tend to 9 and to 10 as the clique number of an input graph tens to infinity, respectively. Finally, we present a simple general offline $L_{(p_1,\ldots,p_k)}$-labeling algorithm for σ-disk graphs. For any fixed σ and k the algorithm approximation ratio is constant $O(k^2\sigma^2)$.

The following table summarizes known and new results for (online, offline) coloring and labeling of unit disk graphs (UDG), σ-disk graphs (σ-DG), and general disk graphs (DG).

	Offline		Online	
	+	−	+	−
Coloring				
UDG	3 [26]	3 [26]	5 [21,26]	5 [21,26]
σ-DG	5 [21]	5 [21]	YES [∗]	YES [7]
DG	5 [21]	5 [21]	NO [7]	NO [14]
$L_{(2,1)}$-labeling				
UDG	12 [∗]	10.6 [∗]	16.67 [∗]	NO [∗]
$L_{(p_1,\ldots,p_k)}$-labeling				
UDG	YES [∗]	YES [∗]	YES [∗]	NO [∗]
σ-DG	YES [∗]	YES [∗]	YES [∗]	NO [∗]
DG	?	?	NO [∗]	NO [∗]

Here, "+/−" shows either the disk representation of graphs is given or not; "YES" means a constant competitive algorithm; "NO" means that no constant competitive algorithms can exist; "?" shows an open problem; "[∗]" means a result presented in this paper; "number" corresponds to the approximation ratio or the competitive ratio of the respective algorithm.

1.4. Last remarks

We say that an algorithm A is an *offline* $L_{(p_1,\ldots,p_k)}$*-labeling algorithm* if for any given graph G it runs in polynomial time and outputs a proper $L_{(p_1,\ldots,p_k)}$-labeling of G. If the maximum label used is at most $\rho \cdot \chi_{(p_1,\ldots,p_k)}(G)$, then A is called an *ρ-approximation* algorithm. The value ρ is called the *approximation ratio* of A. We say that an algorithm A is an *online $L_{(p_1,\ldots,p_k)}$-labeling algorithm* if for any graph G it properly labels the vertices of G one by one in an externally determined order $\prec$. If the maximum label used is at most $\rho \cdot \chi_{(p_1,\ldots,p_k)}(G)$, then A is called an *ρ-competitive* algorithm. The value ρ is called the *competitive ratio* of A. With respect to disk graphs, we always say whether disks are given the input or not.

The rest of this paper is organized as follows. In Section 2 we give some preliminary results. In Section 3 we introduce a *circular labeling*. In Section 4 we present a general online algorithm and derive an upper bound on its competitive ratio. In Section 5 we present lower bounds for online coloring and labeling. In Section 6 we present two offline $L_{(2,1)}$-labeling algorithms. In Section 7 we derive a general offline labeling algorithm. In the last section we give some concluding remarks.

2. Preliminaries

In this section we give some preliminary results which will be used throughout the paper. First, we introduce hexagonal cells on the plane and cell cliques in a disk graph. Then, we introduce the plane-mesh distance, and derive some simple results.

Let $\mathcal{E}$ be the Euclidean plane. Let x, y be coordinates in $\mathcal{E}$. For a graph G we will write $V(G)$ and $E(G)$ to denote the sets of G's vertices and edges. For a σ-disk graph G, we will use $D = \{D_1, \ldots, D_n\}$ to denote a disk representation of G. Then, for each D_i ($i = 1, \ldots, n$) we will use $d_i \in \mathbb{R}_+$ and (x_i, y_i) to denote the diameter and center of D_i, respectively. For each vertex $v \in V(G)$, we will use D_v to denote the disk of v. Thus, an edge $e = \{u, v\} \in E(G)$ iff $D_v \cap D_u \neq \emptyset$. We will also write $\sigma(D)$ to denote the value of $\max d_i / \min d_i$, that is, the diameter ratio of D. We always assume $\sigma(D)$ is at most σ. For simplicity, we associate a class of σ-disk graphs with its ratio bound σ. In many cases we assume that σ is given in the input.

2.1. Cells

We will use the following partition of the plane $\mathcal{E}$ into hexagons. For $i, j \in \mathbb{Z}$ we define a unit hexagon $C_{i,j}$ as the set of all points $(x, y) \in \mathcal{E}$ such that:

$$2i - j - 1 < \tfrac{4}{3}\sqrt{3}x \leqslant 2i - j + 1,$$

$$i + j - 1 < \tfrac{2}{3}(\sqrt{3}x + 3y) \leqslant i + j + 1,$$

$$-i + 2j - 1 < \tfrac{2}{3}(-\sqrt{3}x + 3y) \leqslant -i + 2j + 1.$$

Here, $C_{i,j}$ contains exactly two adjacent corners of the bounding simplex, see Fig. 2. The cell side is equal to $\frac{1}{2}$. The largest diameter of $C_{i,j}$ is equal to 1. So, the plane distance between every two points inside $C_{i,j}$ is at most 1. The smallest diameter of $C_{i,j}$ is equal to $\sqrt{3}/2$. This value is called the size of $C_{i,j}$. Furthermore, each point of plane $\mathcal{E}$ belongs to exactly one hexagon $C_{i,j}$, see Fig. 3. For simplicity, any $C_{i,j}$ will be called a *cell*, and $\mathcal{C}$

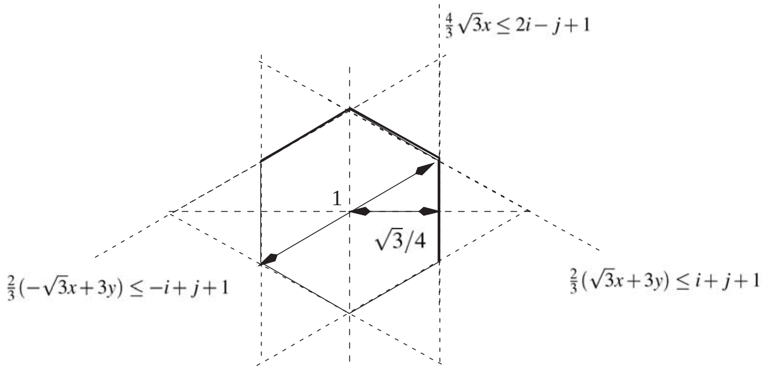

Fig. 2. A simplex C_{ij}.

will denote the set of all cells $C_{i,j}$, for $i, j \in \mathbb{Z}$. We will say that a disk D_i belongs to a cell $C_{i,j}$ iff the center (x_i, y_i) of D_i belongs to $C_{i,j}$.

2.2. Cell cliques

For a disk graph G given by a set D of disks, and a cell $C_{i,j}$ let

$$D(i, j) := \{D_k \mid D_k \in D \text{ and } (x_k, y_k) \in C_{i,j}\}$$

be the set of all disks which belong to $C_{i,j}$, and let

$$V(i, j) := \{v \in V(G) \mid D_v \in D(i, j)\}$$

be the set of all vertices whose disks are in $C_{i,j}$. Then, we can prove the following simple result.

Lemma 2.1. *For any disk graph G, any set $V(i, j)$ induces a clique. Hence, $|D(i, j)| = |V(i, j)|$ is at most the clique number $\omega(G)$.*

Proof. The distance between every two points inside cell C_{ij} is at most one. Hence, the disks of any pair in $D(i, j)$ intersect. This means that $\{u, v\} \in E(G)$ for any two $u, v \in V(i, j)$. Hence, $V(i, j)$ induces a clique in G. □

2.3. Plane and mesh distance

Let $\text{dist}_{\mathcal{E}}(p, p')$ be the standard plane distance between two points $p, p' \in \mathcal{E}$. Then, the *plane distance* between two cells C and C' is defined as

$$\text{dist}_{\mathcal{E}}(C, C') = \inf\{\text{dist}_{\mathcal{E}}(p, p') : p \in C, p' \in C'\}.$$

We define an infinite triangular mesh M. With every cell $C_{i,j} \in \mathcal{C}$ we simply associate a vertex (i, j), and connect any two vertices by an edge if the corresponding cells are

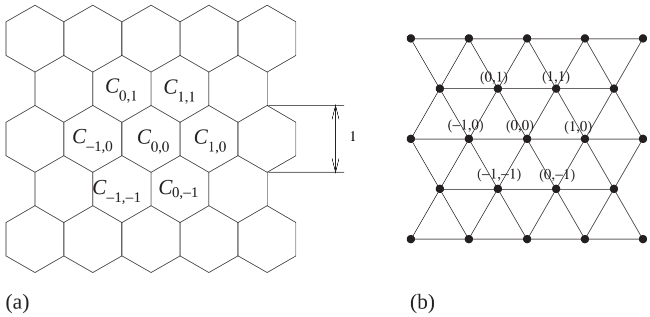

Fig. 3. Cells-Mesh: (a) cells in $\mathcal{C}$ and (b) mesh M.

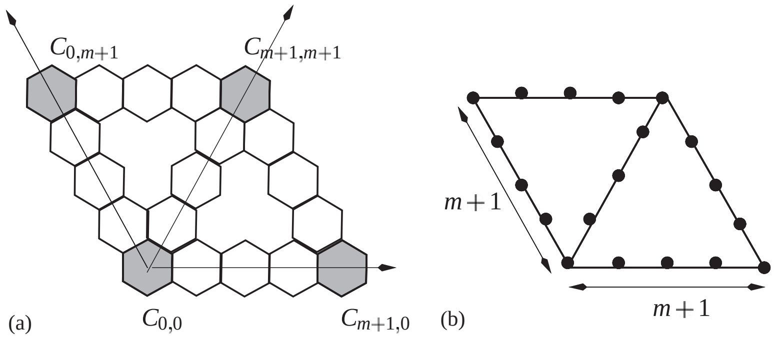

Fig. 4. Cells for $i = 0$ and $j = 0$ and $t = m + 1$.

neighbors. For an illustration see Fig. 3. Accordingly, we will write $\text{dist}_M(C_{i,j}, C_{s,t})$ to denote the *mesh distance* between two cells $C_{i,j}$ and $C_{s,t}$. This is measured as the number of edges in some shortest path connecting (i, j) and (s, t) in the mesh M.

Lemma 2.2. *For $m \geqslant 2$ and $i, j \in \mathbb{Z}$, each of cells $C_{i+t,j}$, $C_{i,j+t}$, $C_{i+t,j+t}$, where $t \in \{m + 1, -m - 1\}$, have mesh distance $m + 1$ and plane distance $(m\sqrt{3}/2)$ from $C_{i,j}$. Furthermore, any cell at mesh distance $m + 1$ from $C_{i,j}$ has plane distance at least $\lfloor \frac{m}{2} \rfloor + \frac{1}{2} \lceil \frac{m}{2} \rceil$.*

Proof. Recall that every cell has size $\sqrt{3}/2$, see Fig. 2. For simplicity, we consider the case when $i = 0$ and $j = 0$ and $t = m + 1$, see Fig. 4. Clearly, $C_{m+1,0}$, $C_{0,m+1}$ and $C_{m+1,m+1}$ are at mesh distance $m + 1$, see Fig. 4(b). Furthermore, there are m cells on the shortest line from $C_{0,0}$, see Fig. 4(a). Hence, the plane distance is $m \cdot (\sqrt{3}/2)$.

Now consider all the cells which are mesh distance $m + 1$ from $C_{0,0}$. From one side, the "corner" cells $C_{m+1,0}$ and $C_{m+1,m+1}$ are at the maximum plane distance from $C_{0,0}$. So, we need to consider some "middle" cells. One can see that, the "middle" cells, $C_{m+1,m/2}$ if

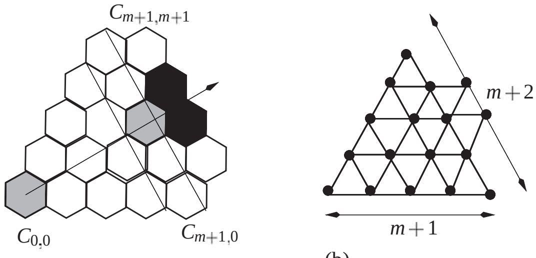

Fig. 5. Middle cells.

m is even and $C_{\lfloor(m+1)/2\rfloor,m+1}$, $C_{m+1-\lfloor(m+1)/2\rfloor,m+1}$ if m is odd, are at the minimum plane distance from $C_{0,0}$. For an illustration see Fig. 5. Then, the minimum plane distance can be bounded as $\lfloor\frac{m}{2}\rfloor$ times cell's diameter 1 and $\lceil\frac{m}{2}\rceil$ times the cell's side $\frac{1}{2}$. This is equal to $\lfloor\frac{m}{2}\rfloor + \frac{1}{2}\lceil\frac{m}{2}\rceil$. □

Corollary 2.3. *For $m \geqslant 2$ and $i, j \in \mathbb{Z}$, cells $C_{i,j}$, $C_{i+m+1,j}$, $C_{i,j+m+1}$, $C_{i+m+1,j+m+1}$ have pairwise mesh distance $m+1$ and plane distance $m\sqrt{3}/2$.*

Corollary 2.4. *Let $a = \lceil\frac{2k\sigma}{\sqrt{3}}\rceil$, where $k \geqslant 2$ and $\sigma \geqslant 1$. Then, cells $C_{i,j}$, $C_{i+t,j}$, $C_{i,j+t}$, $C_{i+t,j+t}$, where $t \in \{a+1, -a-1\}$, have pairwise mesh distance $a+1$ and pairwise plane distance greater than $k \cdot \sigma$.*

2.4. Patterns

Let $k \geqslant 2$ and $\sigma \geqslant 1$. As in Corollary 2.4, we define $a = \lceil\frac{2k\sigma}{\sqrt{3}}\rceil$. Then, the set of a^2 cells $C_{s,t}$ with $s, t \in \{0, \ldots, a\}$ is called a pattern. We say that a cell $C_{i,j} \in \mathcal{C}$ belongs to the (s, t)*th class* if

$$i - 1 = s \bmod a$$

and

$$j - 1 = t \bmod a.$$

In total, there are a^2 classes. Informally, by shifting the pattern around the plane, we "copy" its cells, see Fig. 6. Then, a cell $C_{i,j}$ belongs to the (s, t)th class if it is a "copy" of the (s, t)th cell in the pattern. Now we can prove the following simple result.

Lemma 2.5. *Any two cells in the same class have plane distance greater than $k \cdot \sigma$.*

Proof. The proof follows the definition of classes and Corollary 2.4. □

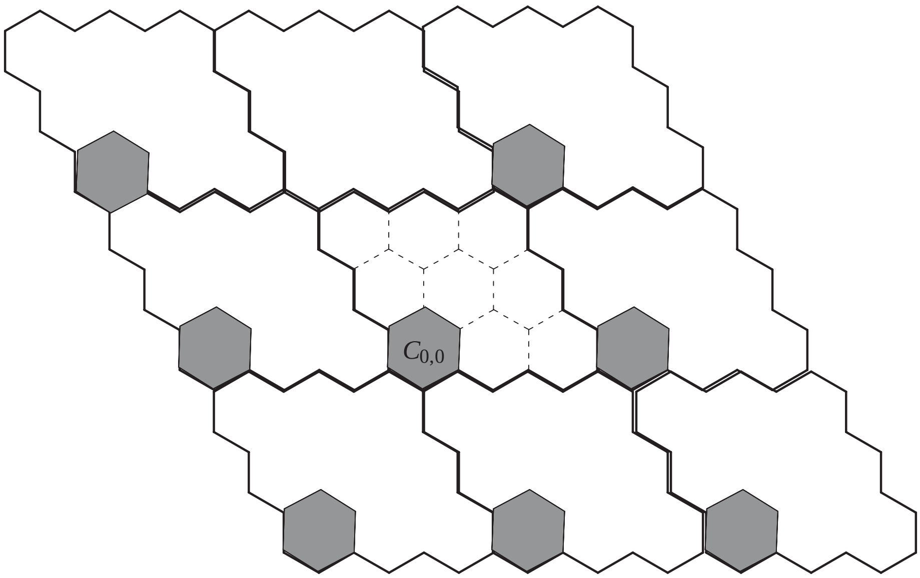

Fig. 6. Shifting the pattern and copies of $C_{0,0}$.

3. Circular labeling

Here we introduce and prove the existence of a special *circular labeling* for the cells in $\mathcal{C}$. This will be used later in Section 4.

Let $\sigma \geqslant 1$ be some constant. Let $(p_1, \ldots, p_k)$ be a k-tuple of distance constraints, where $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_k$. Let $\mathcal{C}$ be the set of cells C_{ij}, where $i, j \in \mathbb{Z}$. We say that a mapping $\varphi : \mathcal{C} \to \{1, 2, \ldots, \ell\}$ is an *ℓ-circular labeling* of $\mathcal{C}$ with respect to $(p_1, \ldots, p_k)$ and σ if for any two cells C' and C'' in $\mathcal{C}$ at plane distance $\mathrm{dist}_{\mathcal{E}}(C, C') \leqslant i \cdot \sigma$ it holds

$$\min\{|\varphi(C) - \varphi(C')|, \ell - |\varphi(C) - \varphi(C')|\} \geqslant p_i,$$

for all $i \in \{1, \ldots, k\}$.

For an illustration see Fig. 7. Informally, we take a circle with vertices $1, 2, \ldots, \ell$. Then, every cell C is assigned to a vertex $\varphi(C) \in \{1, 2, \ldots, \ell\}$. The "circular distance" between any two cells C and C' is equal to the number edges between vertices $\varphi(C)$ and $\varphi(C')$. This can be defined as

$$\min\{|\varphi(C) - \varphi(C')|, \ell - |\varphi(C) - \varphi(C')|\}.$$

Then, we require any two cells C and C' at plane distance at most $i \cdot \sigma$ to be at "circular distance" at least p_i, for all $i \in \{1, \ldots, k\}$.

The existence of such a circular labeling is guaranteed by the following result.

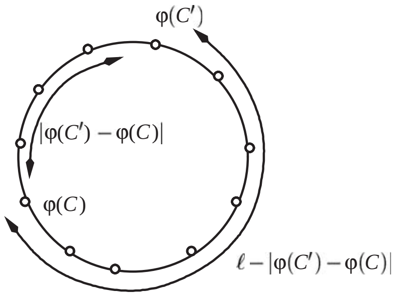

Fig. 7. A circle with ℓ vertices, and two cells C, C'.

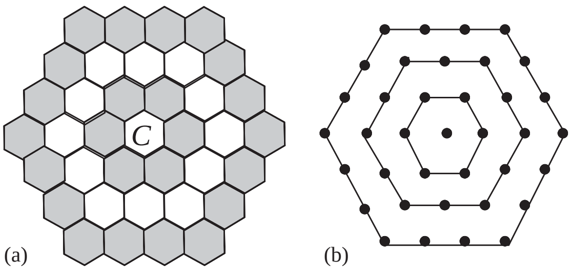

Fig. 8. Labeling of C.

Theorem 3.1. *For every k-tuple* $(p_1, \ldots, p_k)$ *and* $\sigma \geqslant 1$, *an* ℓ^**-circular labeling of* $\mathcal{C}$ *can be found in* $O(\ell^* \sigma^4 k^4)$ *time, where*

$$\ell^* := 1 + 6\left(3(2p_1 - 1) + \sum_{m=2}^{a} (m+1) \cdot (2p_{\lceil (3m-4)/4\sigma \rceil} - 1)\right).$$

Proof. Given k and $\sigma \geqslant 1$, we define $a = \lceil \frac{2k\sigma}{\sqrt{3}} \rceil$, and define a pattern with all cells $C_{s,t}$, where $s, t \in \{0, \ldots, a\}$.

We select the cells in the pattern one by one while labeling with an initial sequence of labels 1, 2, 3, ... in a *first-fit* manner. For a selected cell $C_{s,t}$ from the pattern we first find the least feasible label $\varphi_{s,t}$, and then we define $\varphi(C) = \varphi_{s,t}$ for any cell C in the (s, t)th class. By Lemma 2.5, any two cells in the same class have plane distance greater than $k \cdot \sigma$. Hence, at the end of the procedure we find a feasible circular labeling of $\mathcal{C}$.

In the following we show that ℓ^* is a upper bound on the largest $\varphi_{s,t}$ label used in the pattern, and the labeling procedure takes at most $O(\ell^* \sigma^4 k^4)$ steps. This will complete the proof of the theorem.

Consider a cell C in the pattern, see Fig. 8. By Corollary 2.4, every cell which is at mesh distance at least $a+1$ is at plane distance greater than $k \cdot \sigma$. Hence, in order to find a feasible label for C we need to check all already labeled cells at mesh distance at most a.

There are six cells at mesh distance 1 from C, see Figs. 8(a) and (b). Each of these six cells has plane distance at most $1 \cdot \sigma$ from C. In the worst case, all six cells are labeled, and any two of the labels differ by $2p_1 - 1$. Hence, in order to select a feasible label for C we will "skip" at most $6(2p_1 - 1)$ "forbidden" numbers. Similarly, for 12 cells at mesh distance 2 from C, we will "skip" at most $12(2p_1 - 1)$ "forbidden" numbers.

For $m \geqslant 2$, there are $6(m+1)$ cells at mesh distance $m+1$ from C. By Lemma 2.2, the plane distance from C is at most $m\sqrt{3}/2$ but at least

$$\left\lfloor \frac{m}{2} \right\rfloor + \frac{1}{2}\left\lceil \frac{m}{2} \right\rceil.$$

By the definition of a circular labeling, we need to find the least integer $i \leqslant k$ such that

$$\left\lfloor \frac{m}{2} \right\rfloor + \frac{1}{2}\left\lceil \frac{m}{2} \right\rceil \leqslant i \cdot \sigma.$$

We can bound it as follows:

$$\begin{aligned} i &\geqslant \frac{1}{\sigma}\left(\left\lfloor \frac{m}{2} \right\rfloor + \frac{1}{2}\left\lceil \frac{m}{2} \right\rceil\right) \\ &\geqslant \frac{1}{\sigma}\left(\frac{m}{2} - 1 + \frac{m}{4}\right) \\ &= \frac{(3m-4)}{4\sigma}. \end{aligned}$$

Then, in the worst case, all $6(m+1)$ cells are labeled, and any two of the labels differ by

$$2p_{\lceil (3m-4)/4\sigma \rceil} - 1.$$

As before, in the worst case we will "skip" at most

$$6(m+1)(2p_{\lceil 3m-4/4\sigma \rceil} - 1)$$

"forbidden" numbers.

In total, summing up for mesh distance 1, 2 and over all $3 \leqslant m+1 \leqslant a$ at most

$$6\left(3(2p_1 - 1) + \sum_{m=2}^{a} (m+1) \cdot (2p_{\lceil (3m-4)/4\sigma \rceil} - 1)\right) = \ell^* - 1$$

numbers are "forbidden" be selected as a label for cell C in the pattern.

There are $a^2 = \mathrm{O}(k^2\sigma^2)$ cells in the pattern. For each cell C in the pattern we have to check all cells at mesh distance at most a, and each cell for at most ℓ^* numbers. Thus, the labeling procedure finds an ℓ^*-circular labeling of $\mathcal{C}$ in at most $\mathrm{O}(\ell^* k^4 \sigma^4)$ time steps. □

3.1. A circular 25-labeling for $(p_1, p_2) = (2, 1)$

Consider $k = 2$ and $(p_1, p_2) = (2, 1)$. We take a pattern with 25 cells, and label the cells of $\mathcal{C}$ as it is depicted in Fig. 9. One can see that any two cells with the same label are at the plane distance at least $2\sqrt{3}$. Furthermore, any two cells with ℓ and $\ell + 1$ labels ($\ell = 1, \ldots, 24$) are at the plane distance at least $\frac{\sqrt{7}}{2}$. If we define $\sigma = \frac{\sqrt{7}}{2}$, then $2\sigma < 2\sqrt{3}$. Hence, the depicted labeling is a 25-circular labeling with respect to $(p_1, p_2) = (2, 1)$ and $\sigma = \frac{\sqrt{7}}{2}$.

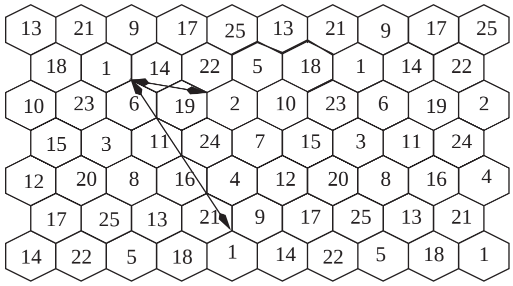

Fig. 9. A 25-circular labeling with $(p_1, p_2) = (2, 1)$, $\sigma = \frac{\sqrt{7}}{2}$.

4. General online labeling of σ-disk graphs

Let G be a σ-disk graphs given by a set $D = \{D_1, \ldots, D_n\}$ of n disks in $\mathcal{E}$. In the following we assume, w.l.o.g., that the coordinates of plane $\mathcal{E}$ are scaled such that minimum diameter is equal to 1 and the diameter ratio of D is at most σ. For a fixed k-tuple $(p_1, \ldots, p_k)$ of distance constraints, where $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_k$, and a fixed $\sigma \geqslant 1$, we describe the following online labeling algorithm:

ONLINE DISK LABELING (ODL):
Input: A k-tuple $(p_1, \ldots, p_k)$, $\sigma \geqslant 1$, and an ordered sequence of disks $D_1 \prec \cdots \prec D_n$.
Output: An $L_{(p_1,\ldots,p_k)}$-labeling c.
1. Find a circular ℓ^*-labeling $\varphi : \mathcal{C} \to \{1, \ldots, \ell^*\}$.
2. For all cells $C_{i,j} \in \mathcal{C}$ define $D(i, j) := \emptyset$.
3. Select the disks one by one in the given order.
4. For a disk D_v perform
 4a. Find $C_{i,j}$ such that $(x_v, y_v) \in C_{i,j}$.
 4b. Define $v \in V(G)$.
 4c. Define $c(v) := \varphi(C_{i,j}) + \ell^* \cdot |D(i, j)|$.
 4d. Put D_v into $D(i, j)$.

Informally, for every new disk the algorithm assigns a label which consists two parts: (1) the label of the cell which will contain this disk; (2) ℓ^* times the number of the disks which are already in the cell. The last part insures that all disk labels are properly separated. So, we can prove the following result.

Lemma 4.1. *The maximum label used by* ODL *is most* $\ell^* \cdot \max_{i,j} |D(i, j)|$.

Proof. The first disk in $D(i, j)$ will get a label equal to

$$\varphi(C_{i,j}) \leqslant \ell^*.$$

The last disk in $D(i, j)$ will get a label equal to

$$\varphi(C_{i,j}) + \ell^* \cdot (|D(i, j)|) \leqslant \ell^* \cdot \max_{i,j} |D(i, j)|.$$

Since, ODL handles all $D(i, j)$ separately, the maximum label used is bounded by

$$\ell^* \cdot \max_{i,j} |D(i, j)|. \qquad \square$$

Furthermore, we can prove the following result.

Lemma 4.2. *Let G be the disk graph given by a set D of disks. Then, for any k-tuple $(p_1, \dots, p_k)$ of distance constraints it holds that*

$$\chi_{(p_1,\dots,p_k)}(G) \geqslant 1 + p_1(\omega(G) - 1) \geqslant 1 + p_1 \left(\max_{i,j}\{|D(i, j)|\} - 1 \right).$$

Proof. Let K be a clique in G. Assume that one vertex in K has the least label 1, and other $|K| - 1$ vertices have larger labels. By the definition of a $L_{(p_1,\dots,p_k)}$-labeling, the labels of any two vertices in K should differ by at least p_1. Thus, the minimum label for K is at least

$$1 + p_1(|K| - 1).$$

By Lemma 2.1 for any set $D(i, j)$ of disks, the vertices of $V(i, j)$ form a clique in G and $|D(i, j)| = |V(i, j)|$ is at most the clique number $\omega(G)$. Thus, the $(p_1, \dots, p_k)$-labeling number of G is at least $1 + p_1(\omega(G) - 1)$. $\square$

Combining the above results, we can prove the following main theorem:

Theorem 4.3. *For every $(p_1, \dots, p_k)$ and $\sigma \geqslant 1$, the algorithm* ODL *is an online $L_{(p_1,\dots,p_k)}$-labeling algorithm for the class of σ-disk graphs, provided that it reserves a sequence of disks as the input. For any σ-disk graph G, the competitive ratio of* ODL *is bounded by*

$$\frac{\omega(G) \cdot \ell^*}{1 + (\omega(G) - 1) \cdot p_1} \leqslant \ell^*. \tag{7}$$

Proof. Let G be the σ-disk graph given by a disk set D. Notice that the value of $|D(i, j)|$ does not depend on an order in which the disks of D presented to ODL. Hence, ODL is an online $L_{(p_1,\dots,p_k)}$-labeling algorithm. Furthermore, by Lemmas 4.1 and 4.2, we can bound its competitive ratio as it is defined in (7). This completes the proof. $\square$

Corollary 4.4. *The algorithm* ODL *is $2\ell^*/(1 + p_1)$-competitive for the class of σ-disk graphs with at least one edge. Furthermore, the bound on its competitive ratio tens to ℓ^*/p_1 as the clique number of an input σ-disk graph grows to infinity.*

Proof. If a disk graph G has at least one edge, then $\omega(G) \geqslant 2$. From (7), for $w(G) = 2, 3, 4, \ldots$ we have

$$\frac{2\ell^*}{1+p_1} \geqslant \frac{3\ell^*}{1+2p_1} \geqslant \frac{4\ell^*}{1+3p_1} \geqslant \cdots \geqslant \frac{\ell^*}{p_1}.$$

This completes the proof. □

Corollary 4.5. *For* $(p_1, p_2) = (2, 1)$ *and* $\sigma = \frac{\sqrt{7}}{2}$, *there is an online* $L_{(2,1)}$*-labeling algorithm which competitive ratio is bounded by* 25 *for the class of* σ*-disk graphs, by* $\frac{50}{3} \approx 16.67$ *for the class of* σ*-disk graphs of with at least one edge, and the bound on its competitive ratio tens to* 12.5 *as the clique number of an input* σ*-disk graph grows to infinity*.

Proof. We use the algorithm ODL combined with a 25-circular labeling depicted in Fig. 9. □

5. Lower bounds: online coloring and labeling

Here we present some lower bounds for online coloring and labeling of disk graphs.

5.1. Coloring of unit disk graphs

We start with a simple lower bound for online coloring of unit disk graphs.

Lemma 5.1. *There is no* $(2-\varepsilon)$*-competitive coloring algorithm for the class of unit disk graphs, even if every unit disk graph occurs as a sequence of unit disks in the online input*.

Proof. Let A be an algorithm with competitive ratio $2-\varepsilon$, for some $\varepsilon > 0$. Consider a unit disk graph G_{bad} depicted in Fig. 10(a). Let the vertices of G_{bad} be ordered as shown in Fig. 10(b).

From one side, vertices 1–6 form an independent set. The algorithm A has to color them by the same color. If it is not the case, then A is not $(2-\varepsilon)$-competitive. From another side, vertices 1–12 form a bipartite graph. To color them properly, the algorithm A needs exactly two more colors. Then, vertices 13–15 require three extra colors. These vertices form a triangle, so they cannot share the same color, and each of them is adjacent to three vertices among 1–12 that are colored by three distinct colors.

In other words, A is forced to use at least six colors for online coloring of G_{bad}. However, the graph is 3-colorable. Hence, A is not an $(2-\varepsilon)$-competitive algorithm. □

5.2. Labeling of unit disk graphs

Now we present a simple lower bound for online $L_{(p_1,p_2)}$-labeling of unit disk graphs.

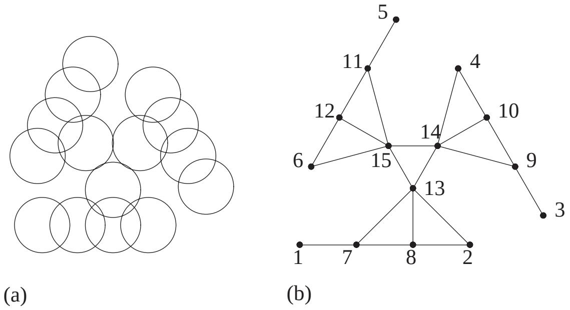

Fig. 10. Graph G_{bad} for coloring.

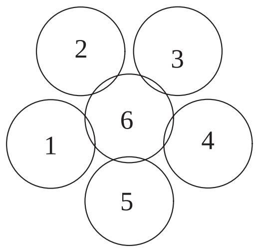

Fig. 11. Graph G_{bad} for $L_{(2,1)}$-labeling.

Lemma 5.2. *For any* 2*-tuple* (p_1, p_2) *of distance constraints and* $\varepsilon > 0$*, there is no* $(4p_2 + 1 - \varepsilon)$*-competitive* $L_{(p_1,p_2)}$*-labeling algorithm for the class of unit disk graphs, even if every unit disk graph occurs as a sequence of unit disks in the online input.*

Proof. Consider a unit disk graph G_{bad} given by five "outer" unit disks 1, 2, 3, 4, 5 depicted in Fig. 11. No two of these five disks intersect. Hence, in the offline case, one needs exactly one label for G_{bad}. Hence, we have that $\chi_{(2,1)}(G_{\mathrm{bad}}) = 1$.

Let A be an online $L_{(p_1,p_2)}$-labeling for the class of unit disk graphs. For any online input of a unit disk G, A always outputs a feasible $L_{(p_1,p_2)}$-labeling of G.

It is not a matter in which order we present the disks of G_{bad}, any two labels assigned by A must differ by at least p_2. If it is not the case, then adding the "central" unit disk 6 leads to a non-feasible labeling of the unit disk graph given by all disks 1, 2, 3, 4, 5, 6. This gives a contradiction.

Thus, the maximum label assigned by A to the disks of G_{bad} is at least

$$1 + p_2 + p_2 + p_2 + p_2 = 1 + 4p_2.$$

However, $\chi_{(2,1)}(G_{\mathrm{bad}}) = 1$. Hence, the competitive ratio of A is at least $4p_2 + 1$. $\square$

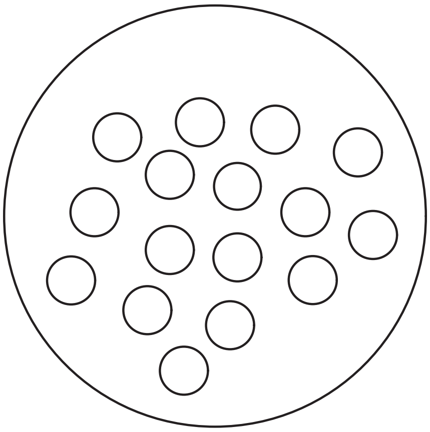

Fig. 12. A set D of disks.

5.3. General labeling of disk graphs

Let $k = 2$ and (p_1, p_2) be a 2-tuple of distance constraints. The following simple result demonstrates the importance of information received in the online input.

Lemma 5.3. *There is no constant competitive online $L_{(p_1,p_2)}$-labeling algorithm for the class of σ-disk graphs, unless there is an upper bound on σ and any σ-disk graph occurs as a sequence of disks in the online input.*

Proof. Let D be a set of n mutually disjoint disks. For an illustration see Fig. 12. Let G a disk graph given by D. Then, there are no edges in G, and $\chi_{(p_1,p_2)}(G) = 1$.

Let A be a general online $L_{(p_1,p_2)}$-labeling algorithm. We present the vertices v in $V(G)$ in an arbitrary order. Assume that there exists a pair of vertices in $V(G)$ which are assigned the same label by A. Then we simply add a new disk to D such that these two vertices get connected by a path of length 2. The new set of disks gives an “extended” disk graph. In this case, A outputs a non-feasible labeling for it. This gives a contradiction. Hence, A must use $|D|$ distinct labels for all the vertices in $V(G)$.

Thus, the maximum label used by A for G is at least $|D| = n$. However, $\chi_{(p_1,p_2)}(G) = 1$. Hence, the competitive ratio of A is bounded by n from below. □

Notice that this result can be generalized for any k-tuple $(p_1, p_2, \ldots, p_k)$ of distance constrains. Now we are ready to present a general lower bound.

Theorem 5.4. *Let $(p_1, \ldots, p_k)$ be a fixed k-tuple of distance constraints, $\sigma \geqslant 1$ be some constant, and let*

$$\bar{\rho} = 1 + \frac{\sigma^2}{9} \max_{i=2,\ldots,k} \{i^2 p_i\}.$$

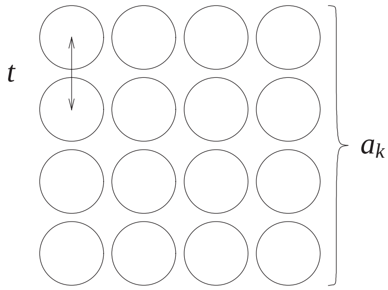

Fig. 13. The set D of a_k^2 unit disks.

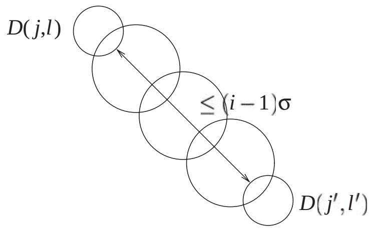

Fig. 14. Disks $D_{j,l}$ and $D_{j',l'}$.

Then, there is no $(\bar{\rho} - \varepsilon)$-competitive online $L_{(p_1,\ldots,p_k)}$-labeling algorithm for the class of σ-disk graphs, even if there is an upper bound on σ and any σ-disk graph occurs as a sequence of disks in the online input.

Proof. Take any $t \in (1, \sqrt{2})$ and define $a_k = \lfloor(((k-1)\sigma + 1)/t\sqrt{2}) + 1\rfloor$. Next, define a set $D = \{D_{1,1}, D_{1,2}, \ldots, D_{a_k,a_k}\}$ of a_k^2 unit disks, where each disk $D_{j,l}$ is defined by its center in $(j \cdot t, l \cdot t)$, and all j, l are integers from $\{1, 2, \ldots, a_k\}$. All disks are mutually disjoint and the centers of any two closest disks are at plane distance t. For an illustration see Fig. 13.

Consider the unit disk graph G given by D. Clearly, G consists of a_k^2 independent vertices (disks). In the offline case, we only need one label for G, i.e.,

$$\chi_{(p_1,\ldots,p_k)}(G) = 1.$$

Now consider two disks $D_{j,l}$ and $D_{j',l'}$ in D with coordinates j, l and j', l', respectively. Let $a_i = \lfloor(((i-1)\sigma + 1)/t\sqrt{2}) + 1\rfloor$ for $i = 2, \ldots, k$. Let i be the minimum such that $|j - j'| \leqslant a_i$ and $|l - l'| \leqslant a_i$. Then, $D_{j,l}$ and $D_{j',l'}$ are at plane distance at most $(i-1) \cdot \sigma$. We construct a set $D(j, l, j', l')$ of $(i-1)$ disks of diameter σ which will connect $D_{j,l}$ and $D_{j',l'}$ by a path of length at most i. For an illustration see Fig. 14. In other words, in the σ-disk graph $G(j, l, j', l')$ given by $D \cup D(j, l, j', l')$ the vertices of disks $D_{j,l}$ and $D_{j',l'}$ are at graph distance i.

Let A be a required online $L_{(p_1,\ldots,p_k)}$-labeling algorithm for the class of σ-disk graphs. We present the disks of D in an arbitrary order to A. For some i from $\{2, \ldots, k\}$, let $D_{j,l}$ and

$D_{j',l'}$ be any two disks in D such that $|j - j'| \leqslant a_i$ and $|l - l'| \leqslant a_i$. If A assigns the labels to $D_{j,l}$ and $D_{j',l'}$ which differ by at most $p_i - 1$, then we add the disks of $D(j, l, j', l')$ to D. In this case, A outputs a non-feasible labeling for a σ-disk graph $G(j, l, j', l')$ given by $D \cup D(j, l, j', l')$. This is a contradiction.

In total, for each $i = 2, \ldots, k$, and for any two disks from set $D_i = \{D_{j,l} | 1 \leqslant j, l \leqslant a_i\}$ of a_i^2 disks, A assigns the labels which differ by at least p_i. As in Lemma 5.2, for each $i = 2, \ldots, k$ the maximum label used by A is at least

$$1 + p_i \cdot (a^2 - 1) = 1 + \left(\left\lfloor \frac{(i-1)\sigma + 1}{t\sqrt{2}} + 1 \right\rfloor^2 - 1 \right).$$

In total, the maximum label used by A for a σ-disk graph G given by D is at least

$$1 + \max_{i=2,\ldots,k} \left\{ \left(\left\lfloor \frac{(i-1)\sigma + 1}{t\sqrt{2}} + 1 \right\rfloor^2 - 1 \right) p_i \right\}$$

and for $t = \frac{3}{2\sqrt{2}}$

$$\bar{\rho} = 1 + \frac{\sigma^2}{9} \max_{i=2,\ldots,k} \{i^2 \cdot p_i\}.$$

From another side, $\chi_{(p_1,\ldots,p_k)}(G) = 1$. Hence, A cannot be better than $(\bar{\rho} - \varepsilon)$-competitive, for any $\varepsilon > 0$. □

From Theorems 4.3 and 5.4 we have the following result.

Corollary 5.5. *For any fixed k-tuple* $(p_1, \ldots, p_k)$ *of distance constraints* $(k \geqslant 2)$, *the competitive ratio of the algorithm* ODL *is at most* $O(\log k)$ *times larger than the competitive ratio of any online* $L_{(p_1,\ldots,p_k)}$*-labeling algorithm for the class of σ-disk graphs with at least one edge. Therefore, the algorithm* ODL *is asymptotically optimal*.

Proof. Take a set D of unit disks as described in the proof of Theorem 5.4. Add a pair of new intersecting disks. These two disks intersect no disk in D.

Let G be a σ-disk graph given by D and the new disks. There is only one edge in G. We can use label 1 for all disks in D, and use labels 1 and $p_1 + 1$ for the new disks. Hence, we can show that

$$\chi_{(p_1,\ldots,p_k)} = p_1 + 1.$$

Then, following the proof of Theorem 5.4 we can show that a lower bound on the competitive ratio of any online algorithm is at least

$$\frac{1 + (\sigma^2/9) \max_{i=2,\ldots,k} \{i^2 p_i\}}{1 + p_1} \geqslant c \cdot \frac{\sigma^2 \max_{i=2,\ldots,k} \{i^2 p_i\}}{1 + p_1}, \tag{8}$$

where c is some suitable constant which neither depends on σ nor $(p_1, \ldots, p_k)$.

From another side, by using Theorems 3.1 and 4.3, we can show that an upper bound on the competitive ratio of our algorithm ODL is at most

$$\frac{2\ell^*}{1+p_1} = 2 \cdot \frac{1 + 6(4(2p_1-1) + \sum_{m=2}^{a}(m+1)\cdot(2p_{\lceil(3m-4)/4\sigma\rceil}-1))}{1+p_1}$$
$$\leqslant c' \cdot \frac{\sigma^2 \sum_{i=2}^{k} i p_i}{1+p_1} + \mathrm{O}(1), \tag{9}$$

where c' is some suitable constant which also neither depends on σ nor $(p_1, \ldots, p_k)$.

Let $s \geqslant 2$ be such that $p_i \leqslant (s^2/i^2) \cdot p_s$ for all $i = 2, \ldots, k$. Here $s \in \{2, \ldots, k\}$ delivers the maximum to $i^2 \cdot p_i$. Then,

$$\sum_{i=2}^{k} i \cdot p_i \leqslant \sum_{i=2}^{k} \left(\frac{s^2}{i}\right) \cdot p_s = s^2 \cdot p_s \left(\sum_{i=2}^{k} \frac{1}{i}\right) \leqslant \max_{i=2,\ldots,k}\{i^2 p_i\} \cdot \mathrm{O}(\log k). \tag{10}$$

Indeed, we can combine (8)–(10). This will show that the competitive ratio of our algorithm OLD is at most $\mathrm{O}(\log k)$ times the competitive ratio of any online $L_{(p_1,\ldots,p_k)}$-labeling algorithm. □

6. Offline labeling of unit disk graphs

Here we explore the offline version of the distance-constrained labeling problem in the case when $k = 2$ and distance constrains $(p_1, p_2) = (2, 1)$. We deal with unit disk graphs. First, we consider the case when the disk representation of unit disk graphs is given, and present a simple approximation algorithm which is based on the so-called *cutting* technique. Then, we present a robust algorithm, i.e., it does not require the disk representation and either outputs a feasible labeling, or shows that the input graph is not a unit disk graph.

6.1. *Cutting technique and strip graphs*

The main idea of our cutting technique is rather simple: We "cut" the plane into strips of small width. Then, we take a unit disk graph and split it into several "strip" unit disk graphs which are induced by the strips. Finally, we label each strip disk graph, and combine all these together into one labeling for the original unit disk graph.

A unit disk graph G is called a $\frac{1}{\sqrt{2}}$*-strip unit disk graph* if there is a mapping $f: V(G) \to \mathbb{R} \times [0, \frac{1}{\sqrt{2}}]$ such that $(u, v) \in E(G)$ iff $\mathrm{dist}_{\mathcal{E}}(f(u), f(v)) \leqslant 1$. Informally, G is given by a set D of unit disks such that each disk from D has its center in a *strip* of width $\frac{1}{\sqrt{2}}$. For an illustration see Fig. 15.

We will use the following simple properties which were mentioned in the introduction. Let G be a graph. Let G^2 be the second power of G, i.e. a graph which arises from G by adding the edges which connect all vertices at graph distance 2. Then, a coloring of G^2 is an $L_{(1,1)}$-labeling of G and vise versa, i.e.

$$\chi_{(1,1)}(G) = \chi(G^2).$$

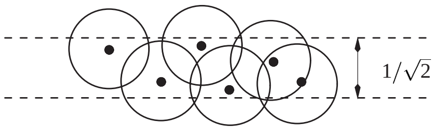

Fig. 15. A $\frac{1}{\sqrt{2}}$-strip unit disk graph.

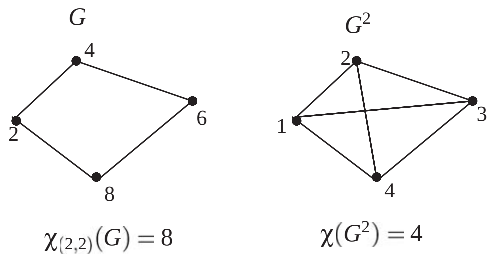

Fig. 16. An $L_{(2,2)}$-labeling of G and a coloring of G^2.

Furthermore, by multiplying all labels in an $L_{(1,1)}$-labeling for G by 2 we can obtain an $L_{(2,2)}$-labeling for G, i.e.

$$\chi_{(2,1)}(G) \leqslant \chi_{(2,2)}(G) \leqslant 2 \cdot \chi_{(1,1)}(G).$$

For an illustration see Fig. 16.

6.2. Coloring and labeling of strip graphs

We start with the following result.

Lemma 6.1. *Let G be a $\frac{1}{\sqrt{2}}$-strip unit disk graph and let v be a vertex such that the unit disk corresponding to v has the least x-coordinate. Then, for G^2, the cardinality of the vertex set*

$$N_{G^2}(v) = \{u \in V(G) - \{v\}: \mathrm{dist}_G(u, v) \leqslant 2\}$$

is at most $3\omega(G) - 1$.

Proof. There is a *strip* of width $\frac{1}{\sqrt{2}}$, and each vertex v in G corresponds to a unit disk D_v with the center in this strip. Let v be a vertex in G which unit disk D_v has the smallest x-coordinate. For an illustration see Figs. 17.

Consider all vertices u in $V(G)$ which are at graph distance at most 2 from v, i.e. $\mathrm{dist}_G(u, v) \leqslant 2$. Then, for each such u, the x-coordinate of disk D_u and disk D_v differ by at most 2, see Figs. 17(a) and (b).

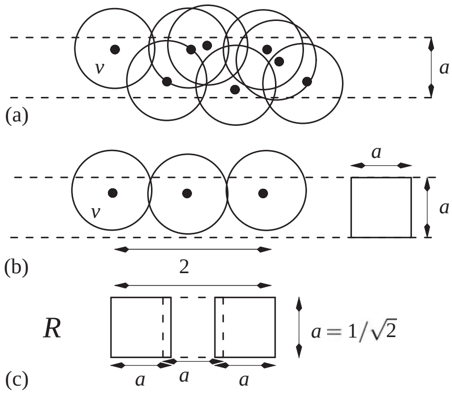

Fig. 17. A $\frac{1}{\sqrt{2}}$-strip unit disk graph.

Consider all disks in a square of side $\frac{1}{\sqrt{2}}$, see Fig. 17(b). Clearly, all of them intersect in pairs. This forms a clique in G. Hence, we can bound the maximum number of the disks in a square by $\omega(G)$.

Consider all disks D_u in a rectangle R of width $\frac{1}{\sqrt{2}}$ and length 2, see Fig. 17(c). It can be covered by three squares of width $\frac{1}{\sqrt{2}}$. Hence the maximum number of disks in R is at most $3\omega(G)$.

Consider vertices u from $N_{G^2}(v)$. Each u is at graph distance at most 2 from v in G. Hence, each disk D_u is in a rectangle R having the center of disk D_v on its left side. For an illustration see Fig. 18. Excepting disk D_v the number of such disks D_u in R is at most $3\omega(G) - 1$. Hence, we can bound $|N_{G^2}(v)|$ by $3\omega(G) - 1$. □

Let G be $\frac{1}{\sqrt{2}}$-strip unit disk graph. Let D_v be the disk of $v \in V(G)$. We order vertices v in $V(G)$ such that the x-coordinate of disks D_v does not increase. If $|V(G)| = n$, then such an *decreasing* order $\prec$ for the vertices of $V(G)$ can be found in $O(n \log n)$ time.

Informally, given a vertex v and all vertices u in $V(G)$ such that $v \prec u$, disk D_v has the least x-coordinate within all disks D_u. For an illustration see Fig. 18. Then, by using Lemma 6.1, for each vertex v we can bound the number of such vertices u in $N_{G^2}(v)$ by $3\omega(G) - 1$.

This helps in the following coloring algorithm:

FIRST FIT COLORING (FFC):
Input: A $\frac{1}{\sqrt{2}}$-strip unit disk graph G,
Output: A coloring of G^2.
Select vertices v from $G(V)$ in a *decreasing* order $\prec$ while coloring with an initial sequence of colors $1, 2, \ldots$. Assign the vertex v the least color that has not already been assigned to any vertex u adjacent to v in G^2.

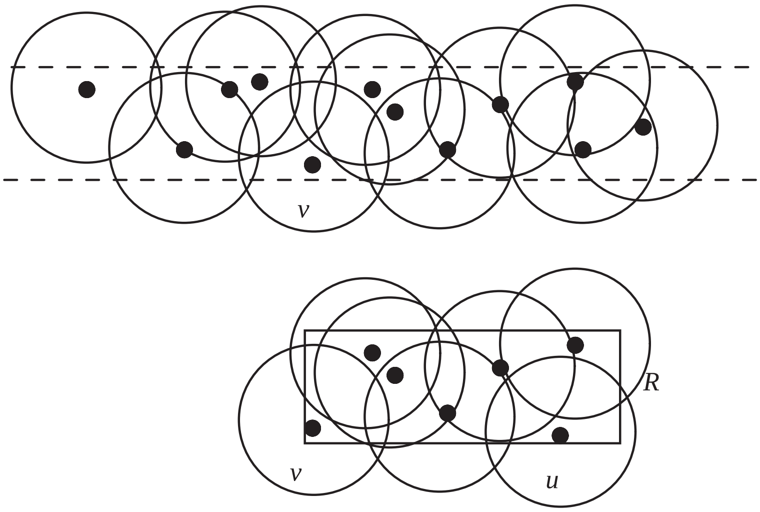

Fig. 18. A vertex $v \in V(G)$ and a vertex $u \in N_{G^2}(v)$.

Lemma 6.2. *The maximum color used by the algorithm* FFC *is bounded by* $3\omega(G)$.

Proof. For the first vertex in the order the algorithm FFC uses color 1. Then, for each next vertex v the algorithm FFC assigns the least color which is not used for vertices u in $N_{G^2}(v)$. As we know, the number of colored vertices u in $N_{G^2}(v)$ is bounded by $3\omega(G) - 1$. Hence, FFC only uses colors from $\{1, 2, \ldots, 3\omega(G)\}$. □

Now we can give the following simple labeling algorithm:

STRIP LABELING (SL):
Input: A $\frac{1}{\sqrt{2}}$-strip unit disk graph G,
Output: An $L_{(2,1)}$-labeling of G.
1. Find an $L_{(1,1)}$-labeling for G.
2. Multiply all labels by 2.

Lemma 6.3. *The maximum label used by the algorithm* SL *is bounded by* $6\omega(G)$. *Furthermore, all labels used are even.*

Proof. By Lemma 6.2 we can color G^2 with at most $3\omega(G)$ colors. This gives a feasible $L_{(1,1)}$-labeling for G. Then, we multiply all labels by 2. This gives a feasible $L_{(2,2)}$-labeling for G which is also a feasible $L_{(2,1)}$-labeling for G. Thus, all labels used are even, and the maximum label used is at most $2 \cdot (3\omega(G)) = 6\omega(G)$. □

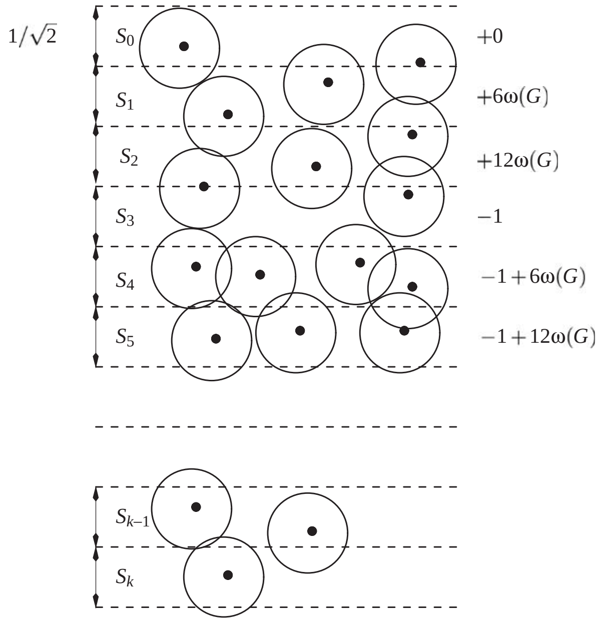

Fig. 19. Strips $S_0, S_1, \ldots, S_k$.

6.3. *Cutting of unit disk graphs*

Now we are ready to describe an approximation algorithm for labeling of unit disk graphs. W.l.o.g. we assume that a unit disk graph G is connected and has at least one edge, i.e. $\omega(G) \geqslant 2$.

Given a unit disk graph G, we partition the plane into $k = O(|V(G)|)$ strips S_0, S_1, ..., S_k of width $\frac{1}{\sqrt{2}}$. Strip S_0 contains a disk with the most y-coordinate and S_k contains a disk the least y-coordinate. All other strips are numbered from top to bottom, respectively. For an illustration see Fig. 19. This partition induces a partition of G into $\frac{1}{\sqrt{2}}$-strip unit disk graphs $G_0, \ldots, G_k$. In the case of disks with centers in two strips ties are broken arbitrarily.

Our main idea is as follows. Consider consecutive strips S_0, S_1, S_2 and S_3, S_4, S_5. The width of each strip is $\frac{1}{\sqrt{2}}$, and the width of two consecutive strips $\sqrt{2}$ is larger than the diameter of a unit disk. Thus, two disks in S_0, S_1, S_2 or S_3, S_4, S_5 can intersect. However, no disk in S_0 (S_1,S_2) can intersect with a disk in S_3 (S_4, S_5), see Fig. 19.

We are interested in an $L_{(2,1)}$-labeling. Hence, any two vertices in $\cup_{i=1}^{3} G_i$ or in $\cup_{i=3}^{5} G_i$ may require their labels be different by 2, and any vertex in G_0 (G_1,G_2) and any vertex in G_3 (G_4, G_5) may require their labels be different by 1. By using the algorithm SL we find an $L_{(2,1)}$-labeling for each G_i, $i = 0, \ldots, 5$. By Lemma 6.3, we can bound the maximum label used as $\max_i \omega(G_i) \leqslant \omega(G)$. Furthermore, all labels are even.

To obtain a feasible $L_{(2,1)}$-labeling for $\cup_{i=1}^{3} G_i$, we let the labels of G_0 be the same (increase by 0), and increase the labels of G_1 and G_2 by $6\omega(G)$ and $12\omega(G)$, respectively. This defines all labels be even, and any two labels be different by at least 2. To obtain a feasible $L_{(2,1)}$-labeling for $\cup_{i=3}^{5} G_i$, we decrease the labels of G_3 by 1 (increase by -1), and increase the labels of G_4 and G_5 by $6\omega(G) - 1$ and $12\omega(G) - 1$, respectively. (Remember $\omega(G) \geqslant 2$.) This defines all labels be odd, and any two labels of $\cup_{i=3}^{5} G_i$ be different by at least 2. Finally, we simply combine both parts. Since the labels of $\cup_{i=1}^{3} G_i$ are even and the labels of $\cup_{i=3}^{5} G_i$ are odd, it holds that any vertex in G_0 (G_1,G_2) and any vertex in G_3 (G_4, G_5) differ by 1. Hence, we have found a feasible $L_{(2,1)}$-labeling for $\cup_{i=0}^{5} G_i$.

By generalizing this idea we present the final algorithm:

CUTTING DISTANCE LABELING (CDL):
Input: A unit disk graph G,
Output: An $L_{(2,1)}$-labeling for G.
1. Partition the plane into $k = O(V(G))$ strips $S_0, \dots, S_k$ of width $\frac{1}{\sqrt{2}}$.
2. For each $i \in \{0, \dots, k\}$ find an $L_{(2,1)}$-labeling of G_i.
3. Change the labels of graph G_i by adding integer $\#_{(i \bmod 6)}$, where

$$(\#_0, \dots, \#_5) = (0,\ 6\omega(G),\ 12\omega(G),\ -1,\ 6\omega(G) - 1,\ 12\omega(G) - 1).$$

Theorem 6.4. *The maximum label used by the algorithm* CDL *is at most* $18\omega(G)$.

Proof. By Lemma 6.3, the maximum label used on every G_i ($i = 1, \dots, k$) is at most $6\omega(G)$. Hence, the maximal label assigned by the algorithm CDL is at most $12\omega(G) + 6\omega(G)$. □

Corollary 6.5. *The approximation ratio of the algorithm* CDL *is bounded by* 12, *and the bound tens to* 9 *as the clique number* $\omega(G)$ *of unit disk graphs grows to infinity*.

Proof. W.l.o.g. we can assume that $\omega(G) \geqslant 2$. Then, in order to label a clique of size $\omega(G)$ we must use the maximum label at least $1 + p_1(\omega(G) - 1)$, where $p_1 = 2$. Thus, by Theorem 6.4, the approximation ratio of CDL is bounded by

$$\frac{18\omega(G)}{2\omega(G) - 1}.$$

For $\omega(G) = 2$, the bound is equal to 12. If $\omega(G)$ grows to infinity, then the bound tens to 9. □

As the last note, it is not hard to observe that $\frac{1}{\sqrt{2}}$-strips were used in the description of the algorithm to simplify the explanation. To avoid irrational numbers, $\frac{1}{\sqrt{2}}$-strips in the algorithm can be replaced by c-strips, where c is any rational number between $\frac{2}{3}$ and $\frac{1}{\sqrt{2}}$.

6.4. Robust algorithms

Here we present an approximation labeling algorithm which does not need the disk representation of a unit disk graph as a part of the input. (Recall that it is NP-hard to recognize unit disk graphs.)

An algorithm which solves an optimization problem on a class $\mathcal{C}$ of inputs is called *robust* if it satisfies the following conditions [27]:

1. Whenever the input is in $\mathcal{C}$, the algorithm finds the correct solution.
2. If the input is not in $\mathcal{C}$, then the algorithm either finds the correct solution, or reports that the input is not in $\mathcal{C}$.

Based on the ideas of [6], a robust algorithm computing the maximal clique of a unit disk graph is given in [27]. Every unit disk graph has an edge ordering $e_1 \prec_e \cdots \prec_e e_m$ such that for every edge e_i the neighbors of its endpoints induce a cobipartite subgraph C_i (i.e., the complement of a bipartite graph) of a graph induced by $\{e_1, \ldots, e_i\}$. If such an ordering $\prec_e$ exists, then each clique is contained in the cobipartite graph C_i for some edge e_i. The robust algorithm first constructs (if any exists) an edge ordering $\prec_e$ in time $O(m^2 n)$, and then the algorithm finds a maximal clique in each graph C_i. This is equivalent to finding the maximum independent set in a bipartite graph which can be done in $O(m\sqrt{n})$ time by using the matching technique [16]. Therefore, the running time of the entire algorithm is $O(m^2 n)$.

Let G be a unit disk graph and let G^2 be the second power of G, i.e. a graph which arises from G by adding the edges which connect all vertices at graph distance 2. Then, we can prove the following simple result:

Lemma 6.6. *Every unit disk graph G has a vertex v such that the set*

$$N_G(v) = \{u \neq v : \{u, v\} \in E(G)\} \tag{11}$$

contains at most $3\omega(G) - 3$ *vertices and the set*

$$N_{G^2}(v) - N_G(v) \tag{12}$$

contains at most $11\omega(G)$ *vertices.*

Proof. Let G be a unit disk graph. Let D_v be the unit disk of $v \in V(G)$. Then, we can select a vertex v such that D_v has the least y coordinate. For an illustration see Fig. 20.

Now consider the sector partition around v depicted in Fig. 21. There are 14 sectors S_i, $i = 1, \ldots, 14$. Consider a vertex u in $V(G)$. We say D_u is in S_i ($i = 1, \ldots, 14$) if its center in S_i. To break ties, any disk on a border of two sectors is in the sector with smaller index.

Then, we have the following property. If $u \in N_G(v)$, i.e. D_u intersects D_v, then D_u in one of sectors S_i, $i = 1, 2, 3$. If $u \in N_{G^2}(v) - N_G(v)$, i.e. there is a disk which intersects D_v and D_u, then D_u in one of sectors S_i, $i = 4, \ldots, 14$.

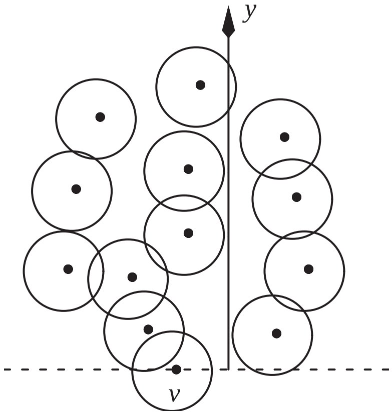

Fig. 20. A vertex v with the least y-coordinate.

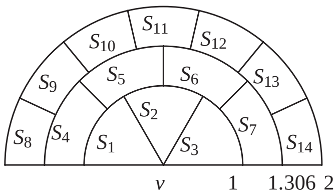

Fig. 21. The sector partition around a vertex v.

The sectors are constructed such that any two unit disks in one sector intersect. Thus, for each sector S_i, $i = 1, \dots, 14$, vertices u from $V(G)$ with disks D_u in S_i form a clique. Hence, for each sector S_i, $i = 1, 2, 3$, we can bound the number of the disks by $\omega(G) - 1$ (excepting our D_v), and for each sector S_i, $i = 4, \dots, 14$, we can bound the number of disks by $\omega(G)$. In total, we can bound $|N_G(v)|$ by $3(\omega(G) - 1)$, and $N_{G^2}(v) - N_G(v)$ by $(14 - 3)\omega(G)$. □

We say that a vertex ordering $v_1 \prec \cdots \prec v_n$ of G is *good* if for every $2 \leqslant i \leqslant n$: (i) $|N_G(v_i) \cap \{v_1, \dots, v_{i-1}\}| \leqslant 3\omega(G) - 3$; (ii) $|(N_{G^2}(v_i) - N_G(v_i)) \cap \{v_1, \dots, v_{i-1}\}| \leqslant 11\omega(G)$.

Notice, that by Lemma 6.6 every unit disk graph has a good vertex ordering. Also, for a graph G one can either find a good vertex ordering, or conclude that there is no good ordering for G. Furthermore, if G has n vertices, this can be done in $O(n^3)$ time.

Now we are ready to present a robust $L_{(2,1)}$-labeling approximation algorithm for unit disk graphs. The algorithm described below, called RDL, does not require the disk representation.

It either concludes that a graph G is not a unit disk graph, or it finds an $L_{(2,1)}$-labeling of G.

ROBUST DISTANCE LABELING (RDL):
Input: A graph G given as an adjacency list.
Output: An $L_{(2,1)}$-labeling c of $V(G)$, or the conclusion that G is not a unit disk graph.
1. Run the robust algorithm to compute $\omega(G)$. This algorithm either computes $\omega(G)$ or concludes that G is not a unit disk graph.
2. Find a good vertex ordering $v_1 \prec \ldots \prec v_n$. If there is no such ordering, then conclude that G is not a unit disk graph.
3. Label vertices sequentially in the order $\prec$ as follows:
 3a. Let vertices $v_1, \ldots, v_{i-1}$ be already labeled.
 3b. Let $\lambda \geqslant 1$ be the smallest integer which is neither a label of vertices in

$$N_{G^2}(v_i) \cap \{v_1, \ldots, v_{i-1}\}$$

nor a member of the set

$$\bigcup_{j<i:\, v_j \in N_G(v_i)} \{c(v_j) - 1, c(v_j), c(v_j) + 1\}.$$

 3c. Label v_i by $c(v_i) = \lambda$.

Theorem 6.7. *For any graph G, the algorithm* RDL *either produces an $L_{(2,1)}$-labeling for G with the maximum label at most* $20\omega(G) - 8$, *or concludes that G is not a unit disk graph*.

Proof. Suppose that the algorithm RDL outputs that G is not a unit disk graph. If it occurs after the first step, then G has no edge ordering $\prec_e$ and therefore is not a unit disk graph. If the algorithm halts at the second step, then its conclusion is verified by Lemma 6.6.

Suppose that RDL outputs a labeling. Let us first show that the maximum label used by the algorithm is not larger than $20\omega(G) - 8$. We proceed by induction. The vertex v_1 is labeled by 1, hence both sets declared in 3b are empty. Suppose that we have labeled vertices $v_1, \ldots, v_{i-1}$. We need to assign a label to v_i. If a neighbor of v_i has a label x then labels $x - 1$, x and $x + 1$ are “forbidden” for v_i. If a vertex at distance two from v_i has a label x then x is “forbidden” for v_i. By (11), v_i has at most $3\omega(G) - 3$ labeled vertices in $N_G(v_i)$. By (12), there are at most $11\omega(G)$ labeled vertices in $N_{G^2}(v_i) - N_G(v_i)$. Hence, the total number of “forbidden” labels for v_i is at most

$$3 \cdot (3\omega(G) - 3) + 11\omega(G) = 20\omega(G) - 9.$$

Since there are $20\omega(G) - 8$ labels, it holds $c(v_i) \leqslant 20\omega(G) - 8$. □

Corollary 6.8. *The approximation ratio of the algorithm* RDL *is bounded by* $\frac{32}{3} \approx 10.67$, *and the bound tens to 10 as the clique number of an input graph grows to infinity*.

Proof. W.l.o.g. we can assume that $\omega(G) \geqslant 2$. Then, in order to label a clique of size $\omega(G)$, the maximum label used is at least $1 + p_1(\omega(G) - 1)$, where $p_1 = 2$. Thus, by Theorem 6.7, the performance ratio of RDL is bounded by

$$\frac{20\omega(G) - 8}{2\omega(G) - 1}.$$

For $\omega(G) = 2$, the bound is equal to $\frac{32}{3} \approx 10.67$. If $\omega(G)$ grows to infinity, then the bound tens to 10. □

7. General offline labeling of σ-disk graphs

Here we discuss an offline labeling algorithm for σ-disk graphs. We assume that the disk representation of σ-disk graphs is not given. We will need the following simple result:

Lemma 7.1. *For each vertex v in a σ-disk graph G, the set*

$$N_G^{(k)}(v) = \{u \neq v : \mathrm{dist}_G(u, v) \leqslant k\}$$

consists of at most $(8k)^2\sigma^2\omega(G)$ *vertices*.

Proof. Let D_v be the disk for $v \in V(G)$. Assume w.l.o.g. that the smallest disk diameter is equal to 1, and the largest disk diameter is equal to σ.

Take a vertex $v \in V(G)$ and consider $u \in N_G^{(k)}(v)$. The centers of D_v and D_u are at plane distance at most $k\sigma$ from each other. For illustration see Fig. 22.

Consider a square S of width $4k\sigma$. We put the center of S at the center of D_v. Then, all disks D_u, $u \in N_G^{(k)}(v)$, fall into S. Next, we partition S into $(4)^2(2)^2k^2\sigma^2$ small squares of width 1/2. For an illustration see Fig. 23. Any two disks that fall into a small square intersect. Hence, the set of vertices $u \in N_G^{(k)}(v)$ which have disks D_u in one small square form a clique. Thus, the number of vertices in any such set is bounded by the maximum clique number $\omega(G)$. In total, we can bound $|N_G^{(k)}(v)|$ by $(8)^2k^2\sigma^2\omega(G)$. □

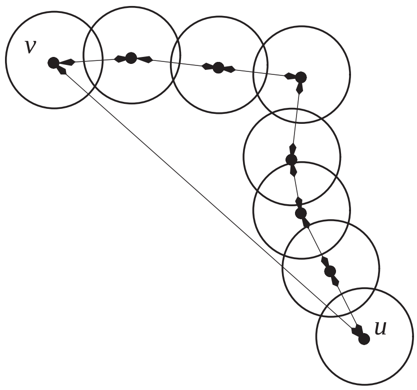

Fig. 22. Vertices v and v'.

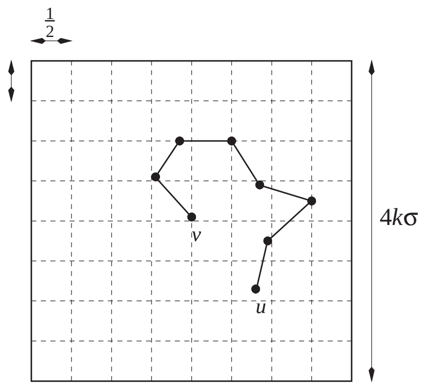

Fig. 23. A square S at a vertex v.

Consider the following algorithm:

FIRST FIT LABELING (FFL):
Input: A σ-graph G in an adjacency list, and a k-tuple $(p_1, p_2, \ldots, p_k)$ of distance constraints.
Output: An $L_{(p_1,\ldots,p_k)}$-labeling for G.
For each $v \in V(G)$ find $N_G^{(k)}(v)$. Select vertices v from $G(V)$ in an arbitrary order while labeling with an initial sequence of labels 1, 2, … . Assign the vertex v the least feasible label which respects $(p_1, \ldots, p_k)$.

Now we can prove the following main result.

Theorem 7.2. The algorithm FFL is an $O(k^2\sigma^2)$-approximate $L_{(p_1,\ldots,p_k)}$-labeling algorithm for the class of σ-disk graphs.

Proof. Let G be a σ-disk graph. Assume w.l.o.g. that the clique number $\omega(G) \geqslant 2$.

By Lemma 7.1, for any vertex v the number of vertices in $N_G^{(k)}(v)$ is bounded by $(8k)^2\sigma^2\omega(G)$. Even if any two labels for $u \in N_G^{(k)}(v)$ differ by $2p_1$, that is more than $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_k$, the label assigned to v by FFL is most

$$1 + 2p_1((8k)^2\sigma^2\omega(G)).$$

From another side, in labeling a clique of size $\omega(G)$ the maximum label is at least

$$1 + p_1(\omega(G) - 1).$$

Since $\omega(G) \geqslant 2$, the approximation ratio of FFL is bounded by

$$\frac{1 + 2p_1((8k)^2\sigma^2\omega(G))}{1 + p_1(\omega(G) - 1)} = \mathrm{O}(k^2\sigma^2). \qquad \square$$

8. Conclusions

The distance constrained labeling problem, which is a natural generalization of the coloring problem, has only recently received increasing attention. In this paper, we considered the distance constrained labeling problem for the class of disk graphs. We presented a number of approximation and online algorithms for different variants of disk graphs and distance constraints, obtaining the first results in this direction. The techniques used, e.g. hexagonal tiling, circular labeling, plane cutting and neighborhood sectoring, are quite general and can be used in the design of online and offline algorithms for many other variants of the labeling problem. Furthermore, these techniques are very simple and do not require larger computational resources, see a realization in [22].

Indeed, there are still many open questions. We name just a few of them. Concerning the complexity, there is a need to understand the status of the general labeling problem, previously studied in [8], and $L_{(p_1,p_2)}$-labeling for planar graphs. Regarding disk graphs, there is a need to clarify the importance of disk representation, robustness. Regarding distance constrains, one can consider $L_{(3,2,1)}$-labeling for simple graph classes. Notice also that even in the case of $L_{(2,1)}$-labeling of unit disk graphs, only very simple lower bounds have been found so far. It is highly interesting to see any improvement on their values.

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