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Planar Graph Coloring Avoiding Monochromatic Subgraphs: Trees and Paths Make It Difficult[1]

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Abstract. We consider the problem of coloring a planar graph with the minimum number of colors so that each color class avoids one or more forbidden graphs as subgraphs. We perform a detailed study of the computational complexity of this problem.

We present a complete picture for the case with a single forbidden connected (induced or noninduced) subgraph. The 2-coloring problem is NP-hard if the forbidden subgraph is a tree with at least two edges, and it is polynomially solvable in all other cases. The 3-coloring problem is NP-hard if the forbidden subgraph is a path with at least one edge, and it is polynomially solvable in all other cases. We also derive results for several forbidden sets of cycles. In particular, we prove that it is NP-complete to decide if a planar graph can be 2-colored so that no cycle of length at most 5 is monochromatic.

Key Words. Graph coloring, Graph partitioning, Forbidden subgraph, Planar graph, Computational complexity.

1. Introduction. We denote by $G = (V, E)$ a finite undirected and simple graph with $|V| = n$ vertices and $|E| = m$ edges. For any nonempty subset $W \subseteq V$, the subgraph of G induced by W is denoted by $G[W]$. A *clique* of G is a nonempty subset $C \subseteq V$ such that all the vertices of C are mutually adjacent. A nonempty subset $I \subseteq V$ is *independent* if no two of its elements are adjacent. An r-*coloring* of the vertices of G is a partition $V_1, V_2, \ldots, V_r$ of V; the r sets V_j are called the *color classes* of the r-coloring. An r-coloring is *proper* if every color class is an independent set. The *chromatic number* $\chi(G)$ is the minimum integer r for which a proper r-coloring of G exists.

Evidently, an r-coloring is proper if and only if for every color class V_j, the induced subgraph $G[V_j]$ does not contain a subgraph isomorphic to P_2. (We use P_k to denote the path on k vertices.) This observation leads to a number of interesting generalizations of the classical graph coloring concept. One such generalization was suggested by Harary

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[24]: Given a graph property π, a positive integer r, and a graph G, a π r-coloring of G is a (not necessarily proper) r-coloring in which each subgraph induced by a color class has property π. This generalization has been studied for the cases where the graph property π is acyclic, or planar, or perfect, or a path of length at most k, or a clique of size at most k. We refer the reader to the work of Brown and Corneil [7]–[9], Chartrand et al. [11]–[13], Farrguia [16], and Sachs [29] for more information on these variants.

In this paper we investigate graph colorings where the property π can be defined via some (maybe infinite) list of forbidden induced subgraphs. This naturally leads to the notion of $\mathcal{F}$-free colorings. Let $\mathcal{F} = \{F_1, F_2, \ldots\}$ be the set of so-called forbidden graphs. Throughout the paper we assume that the set $\mathcal{F}$ is nonempty, and that all graphs in $\mathcal{F}$ are connected and contain at least one edge. Moreover, to avoid technical difficulties in the proofs we assume that no graph of $\mathcal{F}$ is a proper subgraph of another graph of $\mathcal{F}$. For a graph G, a (not necessarily proper) r-coloring with color classes $V_1, V_2, \ldots, V_r$ is called *weakly $\mathcal{F}$-free*, if for all $1 \leq j \leq r$, the graph $G[V_j]$ does not contain any graph from $\mathcal{F}$ as an *induced* subgraph. Similarly, we say that an r-coloring is *strongly $\mathcal{F}$-free* if $G[V_j]$ does not contain any graph from $\mathcal{F}$ as an (induced or noninduced) subgraph. The smallest possible number of colors in a weakly (respectively, strongly) $\mathcal{F}$-free coloring of a graph G is called the *weakly* (respectively, *strongly*) *$\mathcal{F}$-free chromatic number*; it is denoted by $\chi^{\mathrm{W}}(\mathcal{F}, G)$ (respectively, by $\chi^{\mathrm{S}}(\mathcal{F}, G)$).

In the cases where $\mathcal{F} = \{F\}$ consists of a single graph F, we sometimes simplify the notation and omit the curly brackets: We write F-free short for $\{F\}$-free, $\chi^{\mathrm{W}}(F, G)$ short for $\chi^{\mathrm{W}}(\{F\}, G)$, and $\chi^{\mathrm{S}}(F, G)$ short for $\chi^{\mathrm{S}}(\{F\}, G)$. With this notation $\chi(G) = \chi^{\mathrm{S}}(P_2, G) = \chi^{\mathrm{W}}(P_2, G)$ holds for every graph G, and hence also

$$\chi^{\mathrm{W}}(\mathcal{F}, G) \leq \chi^{\mathrm{S}}(\mathcal{F}, G) \leq \chi(G).$$

It is easy to construct examples where both inequalities are strict. For instance, for $\mathcal{F} = \{P_3\}$ (the path on three vertices) and $G = C_3$ (the cycle on three vertices) we have $\chi(G) = 3$, $\chi^{\mathrm{S}}(P_3, G) = 2$, and $\chi^{\mathrm{W}}(P_3, G) = 1$.

Our main concern in the paper is planar graphs. Recall that a graph is *planar* if it can be drawn in the (Euclidean) plane without intersections of edges. Such a drawing is referred to as a *plane* graph. Hence a graph G is planar if and only if there exists a plane graph isomorphic to G. A planar graph is called *outerplanar* if it has a drawing such that all vertices lie on the boundary of the unbounded face (this face is usually referred to as the outer face).

1.1. *Previous Results*. The literature contains quite a number of papers on weakly and strongly $\mathcal{F}$-free colorings of graphs. One of the most general results is due to Achlioptas [1]: For any graph F with at least three vertices and for any $r \geq 2$, the problem of deciding whether a given input graph has a weakly F-free r-coloring is NP-hard. We often use weakly (strongly) $\mathcal{F}$-free r-coloring as shorthand for the corresponding decision problem.

The special case of weakly P_3-free coloring is known as the *subcoloring problem* in the literature. It has been studied by Broere and Mynhardt [4], by Albertson et al. [2], by Fiala et al. [18], or Gimbel and Hartman [21], and by Broersma et al. [6]. We further utilize especially the following result:

PROPOSITION 1.1 [18]. *Weakly P_3-free 2-coloring is NP-hard for triangle-free planar graphs.*

A $(1, 2)$-*subcoloring* of G is a partition of V into two sets S_1 and S_2 such that S_1 induces an independent set and S_2 induces a subgraph consisting of a matching and some (possibly none) isolated vertices. Le and Le [27] proved that recognizing if a graph is $(1, 2)$-subcolorable is NP-hard even for cubic triangle-free planar graphs.

The case of weakly P_4-free coloring has been investigated by Gimbel and Nešetřil [22] who study the problem of partitioning the vertex set of a graph into induced cographs. Since cographs are exactly the graphs without an induced P_4, the graph parameter studied in [22] equals the weakly P_4-free chromatic number of a graph. In [22] it is proved that the problems of deciding $\chi^{W}(P_4, G) \leq 2$, $\chi^{W}(P_4, G) = 3$, $\chi^{W}(P_4, G) \leq 3$, and $\chi^{W}(P_4, G) = 4$ are all NP-hard and/or coNP-hard for planar graphs. The work of Hoàng and Le [25] on weakly P_4-free 2-colorings was motivated by the Strong Perfect Graph Conjecture. Among other results, they show that weakly P_4-free 2-coloring is NP-hard for comparability graphs.

A notion that is closely related to strongly F-free r-coloring is the so-called *defective* graph coloring. A defective (k, d)-coloring of a graph is a k-coloring in which each color class induces a subgraph with maximum degree at most d. Defective colorings have been studied for example by Archdeacon [3], by Cowen et al. [14], and by Frick and Henning [19]. Cowen et al. [15] have shown that the defective (3, 1)-coloring problem and the defective $(2, d)$-coloring problem for any $d \geq 1$ are NP-hard even for planar graphs. We observe that for any k, defective $(k, 1)$-coloring is equivalent to strongly P_3-free k-coloring, and hence we derive the following proposition.

PROPOSITION 1.2 [15].

(i) *Strongly P_3-free 2-coloring is NP-hard for planar graphs.*
(ii) *Strongly P_3-free 3-coloring is NP-hard for planar graphs.*

1.2. *Our Results*. We perform a complexity study of weakly and strongly $\mathcal{F}$-free coloring problems for *planar* graphs. By the Four Color Theorem, every planar graph G satisfies $\chi(G) \leq 4$. Consequently, every planar graph also satisfies $\chi^{W}(\mathcal{F}, G) \leq 4$ and $\chi^{S}(\mathcal{F}, G) \leq 4$, and so we may concentrate on 2-colorings and on 3-colorings. For the case of a single forbidden subgraph, we obtain the following results for 2-colorings:

- If the forbidden (connected) subgraph F is not a tree, then *every* planar graph is strongly and hence also weakly F-free 2-colorable. Therefore, the corresponding decision problems are trivially solvable.
- If the forbidden subgraph $F = P_2$, then F-free 2-coloring is equivalent to proper 2-coloring. It is well known that this problem is polynomially solvable.
- If the forbidden subgraph is a tree T with at least two edges, then both weakly and strongly T-free 2-colorings are NP-hard for planar graphs. Hence, these problems are intractable.

For 3-colorings with a single forbidden subgraph, we obtain the following results:

- If the forbidden (connected) subgraph F is not a path, then *every* planar graph is strongly and hence also weakly F-free 3-colorable. Hence, the corresponding decision problems are trivially solvable.
- For every path P with at least one edge, both weakly and strongly P-free 3-colorings are NP-hard for planar graphs. Hence, these problems are intractable.

Moreover, we derive several results for 2-colorings with certain forbidden sets of cycles:

- For the forbidden set $\mathcal{F}_{345} = \{C_3, C_4, C_5\}$, both weakly and strongly $\mathcal{F}_{345}$-free 2-colorings are NP-hard for planar graphs. In fact for any finite set $\mathcal{F}_{\geq 345} \supseteq \{C_3, C_4, C_5\}$ of cycles, both weakly and strongly $\mathcal{F}_{\geq 345}$-free 2-colorings are NP-hard for planar input graphs.
- Also for the forbidden set $\mathcal{F}_{\text{cycle}}$ of all cycles, both weakly and strongly $\mathcal{F}_{\text{cycle}}$-free 2-colorings are NP-hard for planar graphs.
- For the forbidden set $\mathcal{F}_{\text{odd}}$ of all cycles of odd lengths, *every* planar graph is strongly and hence also weakly $\mathcal{F}_{\text{odd}}$-free 2-colorable. This follows from (in fact, it is equivalent to) the Four Color Theorem.

2. The Machinery for Establishing NP-Hardness. Throughout this section let $\mathcal{F}$ denote some fixed set of forbidden planar subgraphs. We assume that all graphs in $\mathcal{F}$ are connected and contain at least two edges. We also assume that no graph of $\mathcal{F}$ is a (not necessarily induced) proper subgraph of another graph from $\mathcal{F}$. We develop a generic NP-hardness proof for certain types of weakly and strongly $\mathcal{F}$-free 2-coloring problems. The crucial concept is the so-called *equalizer* gadget. Before we define this gadget, we introduce the following technical concept of crossing graphs. We note that we distinguish between planar graphs and plane graphs (the latter being particular nonintersecting drawings of abstract planar graphs), but we use the same notation for a plane graph and its underlying abstract (planar) graph. When talking about more than one graph, we use subscripts to distinguish their vertex and edge sets (i.e., V_G and E_G denote the vertex and edge sets of a graph G).

DEFINITION 2.1. Given a plane graph G with outer face C and a set $S \subseteq V_G$ of vertices on the boundary of C (referred to as *contact points*), we say that another plane graph H is *crossing* G if the following assertions hold:

1. $G \cup H$ is a plane graph (i.e., no edge of G crosses any edge of H in the simultaneous drawing of G and H),
2. all edges of $E_H \backslash E_G$ are drawn in C,
3. no edge of $E_H \backslash E_G$ is incident with a vertex of $V_G \backslash S$,
4. $V_H \cap (V_G \backslash S) \neq \emptyset$.

If G is a plane graph with a set S of contact points, we say that a planar graph H *may cross* G if some nonintersecting planar drawing of a graph isomorphic to H is crossing G.

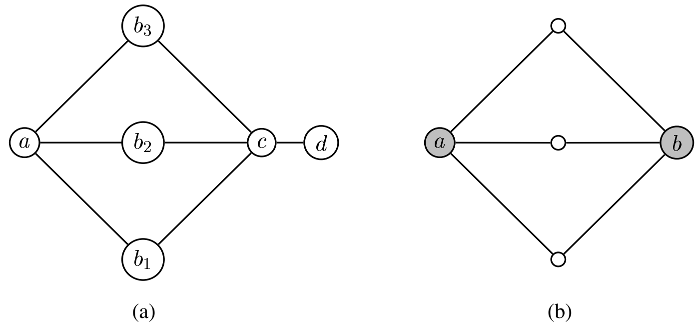

Fig. 1. Examples to illustrate Definitions 2.1 and 2.2.

In Figure 1(a) the graph H induced by the vertices b_2, c and d is crossing the graph G induced by the vertices the a, b_1, b_2, b_3, and c with $S = \{c\}$.

DEFINITION 2.2 (Equalizer). An (a, b)-*equalizer* for $\mathcal{F}$ is a plane graph $\mathcal{E}$ with two nonadjacent contact points a and b on the boundary of the outer face, which satisfies the following properties:

(i) In every weakly $\mathcal{F}$-free 2-coloring of $\mathcal{E}$, a and b receive the same color.
(ii) There exists a strongly $\mathcal{F}$-free 2-coloring of $\mathcal{E}$ such that a and b receive the same color, whereas no monochromatic copy of a graph in $\mathcal{F}$ may cross $\mathcal{E}$. Such a coloring is called a *good* 2-coloring of $\mathcal{E}$.

The graph $\mathcal{E}$ in Figure 1(b) is an (a, b)-equalizer for P_3. In every weakly P_3-free 2-coloring of $\mathcal{E}$ the vertices a and b should receive the same color; otherwise a monochromatic P_3 is unavoidable if we extend the 2-coloring. A good coloring of $\mathcal{E}$ can be obtained by assigning a and b the same color and all remaining vertices of $\mathcal{E}$ the other color.

To understand the definition of an equalizer $\mathcal{E}$ better, we remark right away that if $\mathcal{F}$ contains a graph with a leaf, then an $\mathcal{F}$-free 2-coloring of $\mathcal{E}$ is good if and only if all neighbors of the contact points in $\mathcal{E}$ have the color that is not assigned to the contact points.

The rest of this section is devoted to the proof of the following (technical) main theorem. This theorem generates a number of NP-hardness statements in subsequent sections of the paper.

THEOREM 2.3. *Let $\mathcal{F}$ be a set of connected planar graphs that all contain at least two edges, such that no graph of $\mathcal{F}$ is a proper subgraph of another graph of $\mathcal{F}$. Suppose that*

- *$\mathcal{F}$ contains a graph on at least four vertices with a cut vertex, or a 2-connected graph with a planar embedding with at least five vertices on the boundary of the outer face;*
- *there exists an (a, b)-equalizer for $\mathcal{F}$.*

Then deciding weakly $\mathcal{F}$-free 2-colorability and deciding strongly $\mathcal{F}$-free 2-colorability are NP-hard problems for planar input graphs.

We postpone the proof of Theorem 2.3 to Section 2.2, but first introduce some additional tools.

2.1. *Gadgets for the NP-Hardness Proof.* We will design a series of gadgets that all use the equalizer gadget as an atomic component. In all constructions the only connections between an equalizer and the rest of the constructed graph will always be via the contact points. The use of the equalizer gadget is justified (and motivated) by the following lemma.

LEMMA 2.4. *Consider a nontrivial planar graph H and an edge $xy \in E_H$. Let the graph H^+ result from H by adding a vertex-disjoint copy $\mathcal{E}$ of an (a, b)-equalizer to H and then identifying vertex x with contact point a, and vertex y with contact point b. Then H^+ is a planar graph, and H^+ has a strongly/weakly $\mathcal{F}$-free 2-coloring if and only if H has a strongly/weakly $\mathcal{F}$-free 2-coloring in which x and y both receive the same color.*

PROOF. Since $\mathcal{E}$ is a plane graph with a and b on the boundary of the outer face, H^+ is also planar and it has a nonintersecting drawing such that all edges of H are drawn in the outer face of $\mathcal{E}$. For the proof of the "only if" part, observe that every strongly/weakly $\mathcal{F}$-free 2-coloring of H^+ induces a strongly/weakly $\mathcal{F}$-free 2-coloring of H. By property (i) in Definition 2.2, this induced coloring assigns the same color to x and y. For the proof of the "if" part, we construct a strongly/weakly $\mathcal{F}$-free 2-coloring of H^+: We use the strongly/weakly $\mathcal{F}$-free 2-coloring for the subgraph H, and we color the (a, b)-equalizer $\mathcal{E}$ using a good coloring in the sense of property (ii) in Definition 2.2. □

The negator gadget. An (a, b)-*negator* for $\mathcal{F}$ is a plane graph $\mathcal{N}$ with two nonadjacent contact points a and b on the boundary of the outer face, which satisfies the following properties:

(i) In every weakly $\mathcal{F}$-free 2-coloring of $\mathcal{N}$, a and b receive different colors.
(ii) There exists a strongly $\mathcal{F}$-free 2-coloring of $\mathcal{N}$ such that a and b receive different colors, whereas no monochromatic copy of a graph in $\mathcal{F}$ may cross $\mathcal{N}$. Such a coloring is called a *good* 2-coloring of $\mathcal{N}$.

Next we show how to construct such an (a, b)-negator from (a, b)-equalizers. We choose an arbitrary graph $F \in \mathcal{F}$, and take some fixed planar embedding of F to form the so-called *skeleton* of the negator. Let a' and b' denote two vertices on the boundary of the outer face of F. We partition V_F into two disjoint sets V_1 and V_2 in such a way that both $F[V_1]$ and $F[V_2]$ (the subgraphs of F induced by V_1 and V_2) are connected, and so that $a' \in V_1$ and $b' \in V_2$. For every edge $xy \in E_{F[V_1]} \cup E_{F[V_2]}$, we add an equalizer between x and y exactly in the way we described in Lemma 2.4. We introduce a new vertex a and connect it by an equalizer to a'; we create a new vertex b and connect it by an equalizer to b'. This completes the construction of $\mathcal{N}$. To see that (i) is fulfilled, consider some weakly $\mathcal{F}$-free 2-coloring of $\mathcal{N}$. Suppose that a and b receive the same color. Then the equalizers enforce that this color propagates to all vertices in the skeleton, and this yields a monochromatic induced copy of F, a contradiction. To see that (ii) is fulfilled, we may color $\{a\} \cup V_1$ with one color, and $\{b\} \cup V_2$ with the other color. The vertices inside the equalizer gadgets may be colored using a good coloring in the sense of Definition 2.2(ii). Any monochromatic copy of a graph $F' \in \mathcal{F}$ would either contain some edges of some equalizer gadget (which is impossible by the goodness of the equalizer coloring) or be a subgraph of $F[V_1]$ or $F[V_2]$ (which is impossible by the assumption we made on $\mathcal{F}$).

In our constructions, the negator gadget will be used similarly as the equalizer gadget described in Lemma 2.4. While the equalizer gadget can be used to enforce that a pair of vertices receives the same color, with the help of the negator gadget we can enforce that a pair of adjacent vertices in some planar graph must receive different colors in any weakly $\mathcal{F}$-free 2-coloring. We omit the details since the counterpart of Lemma 2.4 with respect to negators and its proof are straightforward variations on Lemma 2.4 and its proof.

For our NP-hardness proof (of Theorem 2.3) we need two additional gadgets.

The clause gadget with four contact points. The gadget $\mathcal{C}_4(a, b, c, d)$ is a plane graph $\mathcal{C}$ with pairwise nonadjacent contact points a, b, c, and d that lie in this (cyclic) ordering on the boundary of the outer face of $\mathcal{C}$. It has the following properties:

(i) In every weakly $\mathcal{F}$-free 2-coloring of $\mathcal{C}$, not all four contact points receive the same color.
(ii) Any 2-coloring of the four contact points that uses both colors can be extended to a strongly $\mathcal{F}$-free 2-coloring of the gadget $\mathcal{C}$, in such a way that no monochromatic copy of a graph in $\mathcal{F}$ may cross $\mathcal{C}$. Such a coloring is called a *good* 2-coloring of $\mathcal{C}$.

Now we construct such a clause gadget $\mathcal{C}_4(a, b, c, d)$. Suppose we are in the case assumed in Theorem 2.3. Hence the set $\mathcal{F}$ contains some graph F that can be planarly embedded such that there are four vertices a', b', c', d' on the boundary of the outer face. We choose this plane graph F to form the skeleton of the clause gadget. We create four new vertices a, b, c, and d. Each of these new vertices is connected by an equalizer to its corresponding primed vertex on the outer face of the skeleton. The vertices in the skeleton are partitioned into four components (with connecting edges between them) such that a', b', c', d' end up in different components. Within each component, we introduce equalizers along every edge in the way we described in Lemma 2.4. This completes the construction.

By now it is routine to verify that the construction indeed fulfills properties (i) and (ii). We leave the details to the reader.

The clause gadget with five contact points. The gadget $\mathcal{C}_5(a, b, c, d_1, d_2)$ is a plane graph $\mathcal{C}$ with pairwise nonadjacent contact points a, b, c, d_1, and d_2 that lie in the (cyclic) ordering a–b–d_1–c–d_2 on the boundary of the outer face of $\mathcal{C}$. It has the following properties:

(i) In every weakly $\mathcal{F}$-free 2-coloring of $\mathcal{C}$, the vertices d_1 and d_2 receive the same color, and at least one of the vertices a, b, c receives the opposite color.
(ii) Any 2-coloring of the five contact points that assigns the same color to d_1 and d_2, and the opposite color to at least one of a, b, c, can be extended to a strongly $\mathcal{F}$-free 2-coloring of the gadget $\mathcal{C}$, in such a way that no monochromatic copy of a graph in $\mathcal{F}$ may cross $\mathcal{C}$. Such a coloring is called a *good* 2-coloring of $\mathcal{C}$.

Suppose we are in the case assumed in Theorem 2.3, hence the set $\mathcal{F}$ contains a graph on at least four vertices with a cut vertex, or a 2-connected graph with a planar embedding with at least five vertices on the boundary of the outer face.

We first discuss the case of a graph $F \in \mathcal{F}$ with a cut vertex d'. The skeleton of $\mathcal{C}_5(a, b, c, d_1, d_2)$ is formed by a planar embedding of F where d' is on the boundary of the outer face. Choose three vertices a', b', c' that all lie on the boundary of the outer face, and that do not belong to the same component of $F - d'$, such that we can move around the boundary of the outer face starting at a', then moving to b', then to d', then to c', then to d' again, and then returning to a' (maybe meeting other vertices, including d' and b', in between). For example, if $F = K_{1,k}$ is a star with $k \geq 3$ leaves, we choose d' as the center, and a', b' and c' as three successive end vertices in a cyclic ordering in a planar embedding of F. We can move around the boundary of the outer face from a' (via d') to b', then to d' and to c', and back to a' via d' (and alternating between d' and the possible other end vertices if $k \geq 3$). The other cases are similar. We create five new vertices a, b, c, d_1, and d_2, and we connect them by equalizers to a', b', c', d', and d', respectively, at the place where we hit the primed vertices in the above ordering a'–b'–d'–c'–d' while moving around the boundary of the outer face in the way we described. The vertices in the skeleton are partitioned into four components such that a', b', c', d' end up in different components. Within each component, we introduce equalizers along every edge in the way we described in Lemma 2.4. This completes the construction for the first case.

Next we discuss the case of a 2-connected planar graph $F \in \mathcal{F}$ that has a planar embedding with at least five vertices on the boundary of the outer face. We use such an embedding as the skeleton of $\mathcal{C}_5(a, b, c, d_1, d_2)$. Consider the cycle C that forms the boundary of the outer face. Choose five vertices v_0–v_1–v_2–v_3–v_4 in this order along C. Because all these v-vertices are on the outer face, only two subcases are possible:

- (Subcase 1) *There is a face D inside C that touches all these v-vertices*. Then we choose two nonadjacent vertices d_1' and d_2' from these five and three additional appropriate vertices a', b', c' from C such that the cyclic ordering along the cycle C is a'–b'–d_1'–c'–d_2'. Then we connect d_1' and d_2' by an equalizer that is put inside D. Notice that in the graph $F - \{d_1', d_2'\}$ the vertex c' is in a component different from the component containing a' or b'.
- (Subcase 2) *There is an i and a path P (possibly just one edge) internally disjoint from C that connects two vertices v_i and v_{i+3} (where the indices are taken modulo* 5). We put $d_1' = v_i$, $d_2' = v_{i+3}$ and call the remaining three v-vertices a', b', c' in such a way that the cyclic ordering along C is a'–b'–d_1'–c'–d_2'. For every edge of P we connect its incident vertices by an equalizer. Again notice that in the graph $F - V_P$ the vertex c' is in a component different from the component containing a' or b'.

In either subcase, we create five new vertices, a, b, c, d_1, d_2, and connect them by equalizers to their corresponding primed vertices on the outer face of the skeleton. Finally, we partition the vertices of the skeleton into five connected subgraphs, each containing one of the vertices a', b', c', d_1', d_2', and we introduce equalizers along the edges of these subgraphs as in Lemma 2.4. This completes the construction.

It can be verified that this construction in both cases and subcases indeed fulfills properties (i) and (ii).

2.2. *The NP-Hardness Argument*. Now we prove Theorem 2.3. The proof is done by a reduction from an NP-hard variant of the 3-satisfiability problem: Let Φ be a Boolean formula in conjunctive normal form over a set X of logical variables; every clause in Φ contains exactly three variables. With Φ we associate a graph Q_Φ. The vertices of Q_Φ are the clauses and the variables in Φ. There are two types of edges in Q_Φ. The first type belongs to a cycle that spans all the clauses in some ordering. The second type connects a variable $x \in X$ to a clause $\varphi \in \Phi$ if and only if x or $\bar{x}$ occurs as a literal in φ. We call a formula Φ *planar* if for some choice of the cycle spanning all the clauses of Φ the associated graph Q_Φ is planar. Fellows et al. [17] proved that the restriction of the 3-satisfiability problem to planar formulas is NP-hard. (To be precise, they only show the NP-hardness for formulas with *at most* three literals per clause. One may achieve exactly three literals per clause by dropping the requirement of distinctness of literals per clause. Since variable-clause incidences will later be replaced by gadgets with nonadjacent contact points, our final graph will have no multiple edges anyway.)

Consider an arbitrary planar formula Φ as described above, and let Q_Φ be an associated planar graph. We construct in polynomial time a planar graph G_Φ which has the following two properties: If formula Φ is satisfiable, then G_Φ has a strongly $\mathcal{F}$-free 2-coloring. If G_Φ has a weakly $\mathcal{F}$-free 2-coloring, then formula Φ is satisfiable. This clearly will prove Theorem 2.3.

Fix a planar embedding of Q_Φ. The cycle through the clause vertices divides the plane into a bounded and an unbounded region. Variables in X that are embedded in the unbounded region are called *outer* variables, and variables in the bounded region are called *inner* variables. As is usual in reductions from planar SAT, we construct a graph from the planar drawing of Q_Φ by a series of local replacements. Slightly informally described, we thicken the edges and the vertices in the planar embedding of Q_Φ such that they become streets and squares; this yields a map into which we will put our gadgets. For every variable $x \in X$, we put a vertex $v(x)$ into the square corresponding to x. For every clause $\varphi \in \Phi$, we put a corresponding clause gadget into the square corresponding to φ in the following way:

- If all three literals in clause φ belong to inner variables, then the clause gadget for φ is a clause gadget $\mathcal{C}_4(a, b, c, d)$ with four contact points. The contact point d lies in the center of the square of φ, and the contact points a, b, c lie at the beginning of the streets leading to these three inner variables.
- If two literals in clause φ belong to inner variables and one literal belongs to an outer variable, then the clause gadget for φ is a clause gadget $\mathcal{C}_5(a, b, c, d_1, d_2)$ with five contact points. The contact points d_1 and d_2 lie at the beginning of the streets that lead to the left and right neighbors of the clause φ on the clause cycle. The contact points a and b lie at the beginning of the streets that lead to the two inner variables. The contact point c lies at the beginning of the street that leads to the outer variable.
- The case of three outer variables, and the case of one inner and two outer variables are handled symmetrically to the above two cases.

If the variable x occurs un-negated (respectively, negated) in the clause φ, then we put an equalizer (respectively, a negator) from $v(x)$ to the corresponding contact point in the clause gadget for φ. Finally, we put an equalizer gadget between the d-vertices into every street that connects a clause square to another clause square, and thus connect all clause gadgets into a ring. These equalizer gadgets connect contact points d from clause gadgets $\mathcal{C}_4(a, b, c, d)$, and contact points d_1 and d_2 from clause gadgets $\mathcal{C}_5(a, b, c, d_1, d_2)$ in an appropriate way. This completes the construction of the graph G_Φ which is easily seen to be planar.

Assume that formula Φ is satisfiable, and consider a satisfying truth assignment. Intuitively speaking, color 1 will correspond to TRUE and color 0 will correspond to FALSE. Color all contact points d, d_1, and d_2 of clause gadgets by color 0. If variable x is TRUE, then color the vertex $v(x)$ by color 1. If x is FALSE, then color $v(x)$ by 0. The equalizers and negators propagate the colors (respectively opposite colors) of the variables to the corresponding contact points a, b, c in the clause gadgets. Since the truth assignment is a satisfying truth assignment, in every clause gadget at least one of the contact points a, b, c is colored 1. Moreover, in every clause gadget the contact points d (respectively, d_1 and d_2) are colored 0. Therefore, we can use property (ii) of the clause gadgets to get a strongly $\mathcal{F}$-free 2-coloring of all used clause gadgets. Altogether, this yields a strongly $\mathcal{F}$-free 2-coloring for the graph G_Φ.

Now assume that G_Φ has a weakly $\mathcal{F}$-free 2-coloring. Because of the ring of equalizer gadgets that connect the clause gadgets to each other and property (i) of the $\mathcal{C}_5$-gadgets, all contact points d, d_1, d_2 of clause gadgets must receive the same color; without loss of generality we assume that this color is 0. We construct the following truth assignment for X: If $v(x)$ is colored 1, then x is set to TRUE. If $v(x)$ is colored 0, then x is set to FALSE. Suppose for the sake of contradiction that some clause φ in Φ is not satisfied by this truth setting. Then the three literals in φ are all FALSE, and hence all three contact points a, b, c in the corresponding clause gadget are colored 0. However, then *all* contact points of this clause gadget are colored 0, and by property (i) of the clause gadgets the coloring cannot be a weakly $\mathcal{F}$-free 2-coloring. This contradiction shows that Φ is satisfiable.

This completes the proof of Theorem 2.3.

3. Tree-Free 2-Colorings of Planar Graphs. The main result of this section is an NP-hardness result for weakly and strongly T-free 2-colorings of planar graphs for the case where T is a tree with at least two edges (see Theorem 3.3). The proof of this result is based on an inductive argument over the number of edges in T. The following two propositions are used as the base case of the induction.

Let $K_{1,k}$ denote the complete bipartite graph with one vertex in one color class and the other $k \geq 1$ vertices in the other color class. The leftmost drawing in Figure 2 shows a $K_{1,7}$ graph.

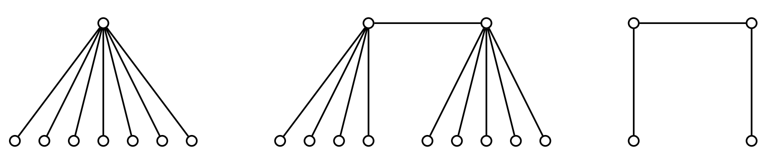

Fig. 2. The graph $K_{1,7}$ (left) and the double-stars $X_{4,5}$ and $X_{1,1}$.

PROPOSITION 3.1. *For every $k \geq 2$, it is NP-hard to decide whether a planar graph has a weakly (strongly) $K_{1,k}$-free 2-coloring.*

PROOF. For $k = 2$, the statement for weakly $K_{1,k}$-free 2-colorings follows from Proposition 1.1, and the statement for strongly $K_{1,k}$-free 2-colorings follows from Proposition 1.2(i).

For $k \geq 3$, we apply Theorem 2.3. The first condition in this theorem is fulfilled, since for $k \geq 3$, the star $K_{1,k}$ is a graph on at least four vertices with a cut vertex.

For the second condition, we note that the complete bipartite graph $K_{2,2k-1}$ with color classes I with $|I| = 2k-1$, and $\{a, b\}$, is an (a, b)-equalizer for $\mathcal{F} = \{K_{1,k}\}$. This graph satisfies Definition 2.2(i): In any 2-coloring, at least k of the vertices in I receive the same color, say color 0. If a and b are colored differently, then one of them is colored 0. This would yield an induced monochromatic $K_{1,k}$. A good coloring as required in Definition 2.2(ii) results from coloring a and b with the same color, and all vertices in I with the opposite color. This coloring has no monochromatic copy of $K_{1,k}$ itself, and since all neighbors of the contact points are colored with the other color than the contact points, no monochromatic copy of $K_{1,k}$ may cross the equalizer. □

As we mentioned in Section 1.1, Cowen et al. [15] have shown that the defective $(2, d)$-coloring problem for any $d \geq 1$ is NP-hard even for planar graphs. This implies that strongly $K_{1,k}$-free 2-coloring is NP-hard for planar graphs for any $k \geq 2$, so the above proof is needed for the weak case only.

For $1 \leq k \leq m$, a *double-star* $X_{k,m}$ is a tree of the following form: $X_{k,m}$ has $k+m+2$ vertices. There are two adjacent central vertices y_1 and y_2. Vertex y_1 is adjacent to k leaves, and y_2 is adjacent to m leaves. In other words, the double-star $X_{k,m}$ results from adding an edge between the centers (vertices of maximum degree) of $K_{1,k}$ and $K_{1,m}$. See Figure 2 for an illustration. Note that $X_{1,1}$ is isomorphic to the path P_4.

PROPOSITION 3.2. *For every $1 \leq k \leq m$, it is NP-hard to decide whether a planar graph has a weakly (strongly) $X_{k,m}$-free 2-coloring.*

PROOF. We apply Theorem 2.3. The first condition in this theorem is fulfilled, since $X_{k,m}$ is a graph on at least four vertices with a cut vertex. For the second condition, we construct an (a, b)-equalizer.

The (a, b)-equalizer $\mathcal{E} = (V', E')$ consists of $2m+k-1$ independent copies (V^i, E^i) of the double-star $X_{k,m}$ where $1 \leq i \leq 2m + k - 1$. Moreover, there are five special vertices a, b, v_1, v_2, and v_3. We define

$$V' = \{v_1, v_2, v_3, a, b\} \cup \bigcup_{1 \leq i \leq 2m+k-1} V^i \quad \text{and}$$

$$E' = \{v_1v_2, v_2v_3, v_1v_3, av_3, bv_3\} \cup \bigcup_{1 \leq i \leq 2m+k-1} E^i$$

$$\cup \bigcup_{1 \leq i \leq m} \{v_1v \colon v \in V^i\} \cup \bigcup_{m+1 \leq i \leq 2m} \{v_2v \colon v \in V^i\}$$

$$\cup \bigcup_{2m+1 \leq i \leq 2m+k-1} \{v_3v \colon v \in V^i\}.$$

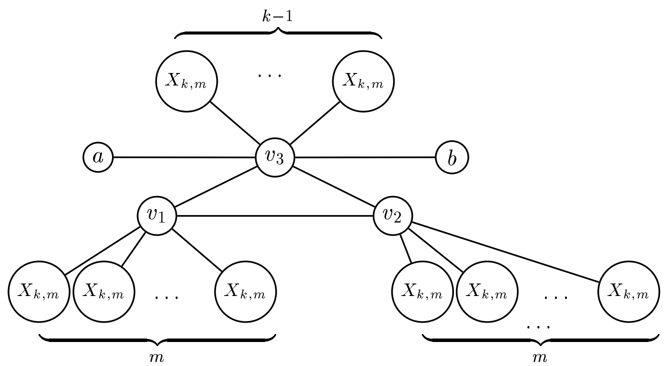

Fig. 3. An equalizer for the double-star $X_{k,m}$.

See Figure 3 for an illustration.

We claim that every 2-coloring of $\mathcal{E}$ with a and b colored differently contains a monochromatic induced copy of $X_{k,m}$: Consider some weakly $X_{k,m}$-free 2-coloring of $\mathcal{E}$. Then each copy (V^i, E^i) of $X_{k,m}$ must have at least one vertex that is colored 0 and at least one vertex that is colored 1. If v_1 and v_2 had the same color, then together with appropriate vertices in V^i, $1 \le i \le 2m$, they would form a monochromatic copy of $X_{k,m}$. Hence, we may assume by symmetry that v_1 is colored 1, and that v_2 and v_3 are colored 0. Suppose for the sake of contradiction that a and b are colored differently. Then one of them would be colored 0, and there would be a monochromatic copy of $X_{k,m}$ with center vertices v_3 and v_2. Thus $\mathcal{E}$ satisfies property (i) in Definition 2.2.

To show that property (ii) in Definition 2.2 is also satisfied, we construct a good 2-coloring: The vertices a, b, v_1 are colored 0, and v_2 and v_3 are colored 1. In every set V^i with $1 \le i \le m$, one vertex is colored 0 and all other vertices are colored 1. In every set V^i with $m + 1 \le i \le 2m + k - 1$, one vertex is colored 1 and all other vertices are colored 0. This coloring has no monochromatic copy of $X_{k,m}$, and since vertex v_3 as the only neighbor of the contact points a, b, is colored differently than a, b, no monochromatic copy of any tree may cross $\mathcal{E}$. □

Now we are ready to prove the main result of this section.

THEOREM 3.3. *Let T be a tree with at least two edges. Then it is NP-hard to decide whether a planar input graph G has a weakly* (*strongly*) *T-free* 2-*coloring*.

PROOF. By induction on the number ℓ of edges in T. If T has $\ell = 2$ edges, then $T = K_{1,2}$, and NP-hardness follows by Proposition 1.1. If T has $\ell \ge 3$ edges, then we consider the so-called *shaved* tree T^* of T that results from T by removing all the leaves. If the shaved tree T^* is a single vertex, then T is a star, and NP-hardness follows by Proposition 3.1. If the shaved tree T^* is a single edge, then T is a double-star, and NP-hardness follows by Proposition 3.2.

Hence, it remains to settle the case where the shaved tree T^* contains at least two edges. In this case we know from the induction hypothesis that weakly (strongly) T^*-free

2-coloring is NP-hard. Consider an arbitrary planar input graph G^* for weakly (strongly) T^*-free 2-coloring. To complete the NP-hardness proof, we construct in polynomial time a planar graph G that has a weakly (strongly) T-free 2-coloring if and only if G^* has a weakly (strongly) T^*-free 2-coloring: Let Δ be the maximum number of leaves pending on a vertex of T. For every vertex v in G^*, we create Δ independent copies $T_1(v), \ldots, T_\Delta(v)$ of T, and we join v to all vertices of all these copies.

Assume first that G^* is weakly (strongly) T^*-free 2-colorable. We extend this coloring to a weakly (strongly) T-free 2-coloring of G by taking a proper 2-coloring of every subgraph $T_i(v)$ in G, such that for every $v \in V_{G^*}$, exactly one vertex of each $T_i(v)$ receives the same color as v. It can be verified that this extended 2-coloring for G does not contain any monochromatic copy of T.

Now assume that G is weakly (strongly) T-free 2-colorable, and let c be such a 2-coloring. Every subgraph $T_i(v)$ in G must meet both colors. This implies that every vertex v in the subgraph G^* of G has at least Δ neighbors of color 0 and at least Δ neighbors of color 1 in the subgraphs $T_i(v)$. Any monochromatic (induced) copy of T^* in G^* would then extend to a monochromatic (induced) copy of T in G, and hence the restriction of the coloring c to the subgraph G^* is a weakly (strongly) T^*-free 2-coloring. This concludes the proof of the theorem. □

Using the same ideas as in the proofs of the previous theorem and propositions, one can prove the following more general statement about larger sets of forbidden graphs.

THEOREM 3.4. *Let $\mathcal{F}$ be a finite set of graphs containing at least one tree with at least two edges. Then both weakly and strongly $\mathcal{F}$-free 2-coloring are NP-hard.*

PROOF. Let $T \in \mathcal{F}$ be a tree in $\mathcal{F}$ with the minimum number of edges. If $T = P_3$, then the remaining graphs in $\mathcal{F}$ must be complete (every noncomplete connected graph contains P_3 as an induced subgraph), so they could be only K_3 or K_4. The NP-hardness of $\mathcal{F}$-free 2-coloring then follows from Proposition 1.1, since for coloring triangle-free graphs, graphs containing triangles are irrelevant as forbidden subgraphs.

For T being a star (with at least three leaves) or a double-star, the result follows directly from the construction of equalizers in the proofs of Propositions 3.1 and 3.2, since the good colorings presented there are such that the neighbors of the contact points receive a different color from the contact points, and the only connected monochromatic subgraphs of the equalizers are singletons (in case of $T = K_{1,k}$) or smaller double-stars ($X_{k-1,m}$ in case of $T = X_{k,m}$). Since T itself may be used as the skeleton of the negator and the clause gadgets, the good colorings of these gadgets also do not contain a monochromatic copy of any graph of $\mathcal{F}$ (neither induced nor noninduced).

If T is such that the shaved tree T^* has at least two edges, we proceed by induction similarly as in the proof of Theorem 3.3. For any graph $F \in \mathcal{F}$, we denote by F^* the shaved copy of F, i.e., the graph obtained from F by removing all leaves (vertices of degree 1). Let Δ be the maximum number of leaves pending on a vertex of a graph from $\mathcal{F}$. Construct the graph G from a graph G^* as in the proof of Theorem 3.3 and use the fact that G has a weakly (strongly) $\mathcal{F}$-free 2-coloring if and only if G^* has a weakly (strongly) $\mathcal{F}^*$-free 2-coloring, where $\mathcal{F}^* = \{F^*\colon\ F \in \mathcal{F}\}$. □

4. Cycle-Free 2-Colorings of Planar Graphs. In this section we turn to the case when the forbidden graph F is not a tree and hence contains a cycle (we assume F is connected).

If F contains an odd cycle, then the Four Color Theorem (4CT) shows that any planar graph G has a strongly F-free 2-coloring: a proper 4-coloring of G partitions V_G into two sets S_1 and S_2, each inducing a bipartite graph. Coloring all the vertices of S_i by color i yields a strongly F-free 2-coloring of G. If we extend the set of forbidden graphs to all cycles of odd length, denoted by $\mathcal{F}_{\mathrm{odd}}$, then the converse is also true: In any $\mathcal{F}_{\mathrm{odd}}$-free 2-coloring of G both monochromatic subgraphs of G are bipartite, yielding a 4-coloring of G. To summarize we obtain the following.

PROPOSITION 4.1. *The statement* "$\chi^{\mathrm{S}}(\mathcal{F}_{\mathrm{odd}}, G) \leq 2$ for every planar graph G" *is equivalent to the* 4*CT.*

In case F is just the triangle C_3, one can avoid using the heavy 4CT machinery to prove that $\chi^{\mathrm{S}}(C_3, G) \leq 2$ for every planar graph G by applying a result due to Burstein [10]. A brief sketch of the argument follows. Prove by induction that in any plane triangulation, any nonmonochromatic precoloring of the outer face (triangle) can be extended to a coloring which avoids monochromatic triangles.

If F contains no triangles, a result of Thomassen [31] can be applied. He proved that the vertex set of any planar graph can be partitioned into two sets, each of which induces a subgraph with no cycles of length exceeding 3. Hence every planar graph is strongly $\mathcal{F}_{\geq 4}$-free 2-colorable, where $\mathcal{F}_{\geq 4}$ denotes the set of all cycles of length exceeding 3. The following theorem summarizes the above observations.

THEOREM 4.2. *If the forbidden connected subgraph F is not a tree*, *then every planar graph is strongly and hence also weakly F-free* 2*-colorable*.

The picture changes if one forbids all cycles, or a combination of cycles including the triangle. A result of Stein [30] states that the vertex set of a plane triangulation G can be partitioned into two sets, each inducing a forest if and only if the plane dual of G is hamiltonian. Since deciding hamiltonicity of planar cubic graphs is NP-hard (see [20]), this implies that deciding whether a (maximal) planar graph has a weakly (strongly) cycle-free 2-coloring is NP-hard. We are able to strengthen this statement slightly.

THEOREM 4.3. *Let $\mathcal{F}_{345} = \{C_3, C_4, C_5\}$ be the set of cycles of lengths* 3, 4, *and* 5. *Then the problem of deciding whether a given planar graph has a weakly* (*strongly*) *$\mathcal{F}_{345}$-free* 2*-coloring is NP-hard*.

We prove this theorem in a more general setting as a corollary of Theorem 4.4. However, first we note that $\{C_3, C_4, C_5\}$ is a minimal set of cycles which determines an NP-hard instance of the $\mathcal{F}$-free 2-coloring problem. Indeed, if $\mathcal{F} \subset \{C_3, C_4, C_5\}$ is a proper subset, then every planar graph is strongly $\mathcal{F}$-free 2-colorable. We have noted this already for $\mathcal{F} \subseteq \{C_3, C_5\}$ and $\mathcal{F} \subseteq \{C_4, C_5\}$, and the last case $\mathcal{F} = \{C_3, C_4\}$ is covered by the result of Kaiser and Škrekovski [26] who proved that every planar graph is strongly $\{C_3, C_4\}$-free 2-colorable.

THEOREM 4.4. *Let $\mathcal{F}$ be a finite set of planar 2-connected graphs. If there exists a planar graph which is not weakly (strongly) $\mathcal{F}$-free 2-colorable, then weakly (strongly) $\mathcal{F}$-free 2-coloring is NP-hard for planar input graphs.*

PROOF. Consider the graphs of $\mathcal{F}$ with some fixed plane embeddings. If every face of every graph $F \in \mathcal{F}$ is C_3 or C_4, then every planar graph is strongly $\mathcal{F}$-free 2-colorable by the main result in [26]. If not, then there is an $F \in \mathcal{F}$ with a face of size at least 5 and the first assumption of Theorem 2.3 is met.

Next we show how to construct an equalizer. Let H' be a smallest (by the number of edges) planar graph which is not weakly (strongly) $\mathcal{F}$-free 2-colorable. Take an edge $xy \in E_{H'}$ and denote by H the graph obtained from H' by deleting this edge. Then H is weakly (strongly) $\mathcal{F}$-free 2-colorable, and in every weakly (strongly) $\mathcal{F}$-free 2-coloring of H the vertices x and y receive the same color. We construct an equalizer $\mathcal{E}$ by concatenating sufficiently many copies of H. More formally, choose a number k to be larger than the order of any graph in $\mathcal{F}$. The copies of H will be $H_i = (V_i, E_i)$ with $V_i = \{v_i\colon v \in V_H\}$ and $E_i = \{u_i v_i\colon uv \in E_H\}$, for $i = 1, 2, \ldots, k$. For $i = 1, 2, \ldots, k-1$, we identify y_i with x_{i+1}, and we set $a = x_1$ and $b = y_k$ to be the contact points.

Clearly, $\mathcal{E}$ is planar and in every weakly (strongly) $\mathcal{F}$-free 2-coloring of $\mathcal{E}$ the vertices x_i, y_i for $i = 1, 2, \ldots, k$, and hence also a and b, receive the same color.

Let c be a weakly (strongly) $\mathcal{F}$-free 2-coloring of H. Color $\mathcal{E}$ using c on every H_i. Consider a graph $F \in \mathcal{F}$. No copy of F which lies entirely in $\mathcal{E}$ is monochromatic, since the 2-connectedness of F implies that such a copy of F lies entirely within one of the H_i's. Therefore this 2-coloring of $\mathcal{E}$ is weakly (strongly) F-free. It also follows from the 2-connectedness of F that every copy of F which crosses $\mathcal{E}$ contains a path from a to b through $\mathcal{E}$. However, every such path has more vertices than F. Hence the 2-coloring of $\mathcal{E}$ is good. □

To conclude the proof of Theorem 4.3 it would suffice to construct a planar graph which is not weakly $\mathcal{F}_{345}$-free 2-colorable. It is, however, equally simple to describe an equalizer for $\mathcal{F}_{345}$ (and exploit the fact that $C_5 \in \mathcal{F}_{345}$ is 2-connected and every plane drawing contains a face of size 5): Let $\Theta(x, y)$ be the graph depicted in Figure 4. This graph has the following important property: In any weakly $\mathcal{F}_{345}$-free 2-coloring of $\Theta(x, y)$, the vertices x and y have different colors (we leave the simple proof of this fact to the reader). The (a, b)-equalizer is constructed from a graph $\Theta(a, x)$ and a graph $\Theta(b, y)$ by identifying the two vertices x and y. A good 2-coloring of the (a, b)-equalizer is induced by the 2-coloring indicated in Figure 4.

The following statement is now a direct corollary of Theorems 4.4 and 4.3.

COROLLARY 4.5. *For any finite set $\mathcal{F}_{\geq 345} \supseteq \{C_3, C_4, C_5\}$ of cycles, both weakly and strongly $\mathcal{F}_{\geq 345}$-free 2-coloring are NP-hard for planar input graphs.*

5. 3-Colorings of Planar Graphs A *linear forest* is a disjoint union of paths (some of which may be trivial). The following result was proved independently in [23] and [28].

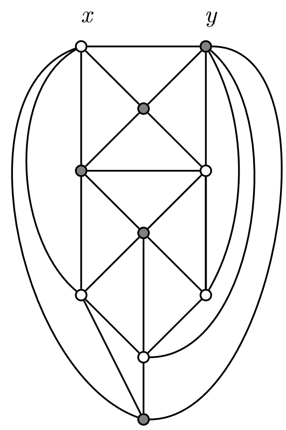

Fig. 4. A gadget for the forbidden set $\mathcal{F}_{345}$ in Theorem 4.3.

PROPOSITION 5.1 [23], [28]. *Every planar graph has a partition of its vertex set into three subsets such that every subset induces a linear forest.*

This result immediately implies that if a connected graph F is not a path, then $\chi^{\mathrm{W}}(F, G) \leq 3$ and $\chi^{\mathrm{S}}(F, G) \leq 3$ hold for *all* planar graphs G. Hence, these coloring problems are trivially solvable in polynomial time.

We now turn to the remaining cases of F-free 3-coloring for planar graphs where the forbidden graph F is a path. We start with a technical lemma that will yield a gadget for the NP-hardness argument.

LEMMA 5.2. *For every $k \geq 2$, there exists an outerplanar graph Y_k that satisfies the following properties*:

(i) *Y_k is not weakly P_k-free 2-colorable.*
(ii) *There exists a strongly P_k-free 3-coloring of Y_k, in which at least one color class being is an independent set.*

PROOF. The skeleton of the graph Y_k is formed by a regular tree, in which every inner vertex has exactly k children, and all paths from the root to a leaf have exactly k vertices. Additionally to the edges in this regular tree, the children of every inner vertex are connected by a path. See Figure 5 for an illustration.

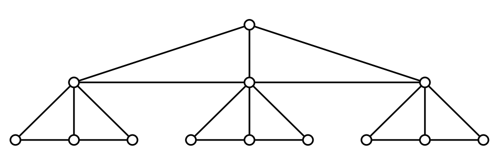

Fig. 5. Example for the graph Y_3.

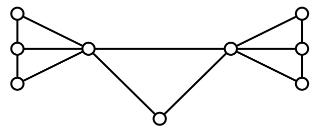

Fig. 6. The graph Z in the proof of Theorem 5.3.

Suppose there was a P_k-free 2-coloring of Y_k. Since the children of any vertex induce a path P_k, both colors must show up at the children. Consequently, for every inner vertex v at least one of its children must get the same color as v. However, this yields a monochromatic induced P_k running from the root to some leaf. This contradiction proves property (i). Property (ii) is straightforward to prove: The graph Y_k is outerplanar, and hence properly 3-colorable. □

THEOREM 5.3. *For any path P_k with $k \geq 2$, it is NP-hard to decide whether a planar input graph has a weakly (strongly) P_k-free 3-coloring.*

PROOF. We use induction on k. The basic cases are $k = 2$ and $k = 3$. For $k = 2$, both weakly and strongly P_2-free 3-colorings are equivalent to proper 3-coloring which is well known to be NP-hard for planar graphs.

Next, consider the case $k = 3$. Proposition 1.2(ii) yields NP-hardness of strongly P_3-free 3-coloring for planar graphs. For weakly P_3-free 3-coloring, we sketch a reduction from proper 3-coloring of planar graphs. As a gadget, we use the outerplanar graph Z depicted in Figure 6. The crucial property of Z is that it does not allow a weakly P_3-free 2-coloring, as is easily checked. Now consider an arbitrary planar graph G. From G we construct the planar graph G': For every vertex v in G, create a copy $Z(v)$ of Z, and add all possible edges between v and $Z(v)$. Since $Z(v)$ is outerplanar, the new graph G' is planar. It is easy to verify that $\chi(G) \leq 3$ if and only if $\chi^{\mathrm{W}}(P_3, G') \leq 3$.

For $k \geq 4$, we give a reduction from weakly (strongly) P_{k-2}-free 3-coloring to weakly (strongly) P_k-free 3-coloring. Consider an arbitrary planar graph G, and construct the following planar graph G': For every vertex v in G, create a copy $Y_k(v)$ of the graph Y_k from Lemma 5.2, and add all possible edges between v and $Y_k(v)$. Since Y_k is outerplanar, the new graph G' is planar. If G has a weakly (strongly) P_{k-2}-free 3-coloring, then this can be extended to a weakly (strongly) P_k-free 3-coloring of G' by coloring the subgraphs $Y_k(v)$ according to Lemma 5.2(ii). If G' has a weakly (strongly) P_k-free 3-coloring, then by Lemma 5.2(i) this induces a weakly (strongly) P_{k-2}-free 3-coloring for G. □

6. Concluding Remarks and Open Problems

6.1. *Triangle-Free Graphs*. By modifying the gadgets for the equalizers in such a way that the planar graph G_Φ constructed in the proof of Theorem 2.3 becomes triangle-free, one might be able to prove complexity results for weakly (strongly) F-free 2-coloring restricted to triangle-free planar graphs. In fact, it is not difficult to apply this method to prove that for $F = K_{1,k}$ with $k \geq 2$, weakly (strongly) F-free 2-coloring remains NP-hard for triangle-free planar graphs.

PROBLEM. Is it true that every triangle-free planar graph G is P_4-free 2-colorable? This would imply that for every connected graph F of diameter at least 3 there is a weakly F-free 2-coloring of G.

6.2. *Monotonicity.* All our NP-hardness techniques are such that hardness proofs for $\mathcal{F}$-free 2-colorability extend naturally to NP-hardness of $\mathcal{F}'$-free 2-colorability for any finite $\mathcal{F}' \supseteq \mathcal{F}$. This raises the following question.

PROBLEM. For finite sets of graphs $\mathcal{F}' \supseteq \mathcal{F}$, is it true that $\mathcal{F}'$-F-2-CPG $\propto$ $\mathcal{F}$-F-2-CPG? ($\mathcal{F}$-F-2-CPG standing for $\mathcal{F}$-Free-2-Coloring-Planar-Graphs.)

Note that this is not necessarily true for infinite sets of forbidden graphs. The infinite set $\mathcal{F}_{\text{cycle}}$ of all cycles has uncountably many subsets, and if each of these defines a different problem, infinitely many of them will have to be undecidable, whereas deciding the existence of an $\mathcal{F}_{\text{cycle}}$-free 2-coloring is surely in NP.

6.3. *Forbidden Sets of Cycles*. It would be interesting to characterize for which particular (finite) sets of forbidden cycles the $\mathcal{F}$-free 2-coloring problem on planar graphs is feasible and for which it is hard. In particular, for two cycles this question remains open if one of them is the triangle and the other one is an even cycle of length greater than 4.

PROBLEM. For which $k > 2$ does there exist a planar graph which is not $\{C_3, C_{2k}\}$-free 2-colorable?

6.4. *Equalizers*. Despite our inductive proof of NP-hardness for forbidden trees, it would be interesting to know whether one can use the equalizer gadget machinery directly.

PROBLEM. Does there exist an equalizer for any tree T?

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