Backbone Colorings for Graphs: Tree and Path Backbones*

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Abstract: We introduce and study backbone colorings, a variation on classical vertex colorings: Given a graph $G = (V, E)$ and a spanning subgraph H of G (the backbone of G), a backbone coloring for G and H is a proper vertex coloring $V \to \{1, 2, \ldots\}$ of G in which the colors assigned to adjacent vertices in H differ by at least two. We study the cases where the backbone is either a spanning tree or a spanning path. We show that for tree backbones of G the number of colors needed for a backbone coloring of G can roughly differ by a multiplicative factor of at most 2 from the chromatic number $\chi(G)$; for path backbones this factor is roughly $\frac{3}{2}$. We show that the computational complexity of the problem "Given a graph G, a spanning tree T of G, and an integer ℓ, is there a backbone coloring for G and T with at most ℓ colors?" jumps from polynomial to NP-complete between $\ell = 4$ (easy for all spanning trees) and $\ell = 5$ (difficult even for spanning paths). We finish the paper by discussing some open problems. J Graph Theory 55: 137–152, 2007

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1. INTRODUCTION AND RELATED RESEARCH

The work presented here is a full version of an extended abstract that appeared in the Proceedings of WG 2003 [5]. It is motivated by the general framework for coloring problems related to frequency assignment. In this application area graphs are used to model the topology and mutual interference between transmitters (receivers, base stations): the vertices of the graph represent the transmitters; two vertices are adjacent in the graph if the corresponding transmitters are so close (or so strong) that they are likely to interfere if they broadcast on the same or "similar" frequency channels. The problem in practice is to assign the frequency channels to the transmitters in such a way that interference is kept at an "acceptable level." This has led to various different types of coloring problems in graphs, depending on different ways to model the level of interference, the notion of similar frequency channels, and the definition of acceptable level of interference (see e.g., [15,20]). One way of putting these problems into a more general framework is the following:

> Given two graphs G_1 and G_2 with the property that G_1 is a spanning subgraph of G_2, one considers the following type of coloring problems: Determine a coloring of (G_1 and) G_2 that satisfies certain restrictions of type 1 in G_1, and restrictions of type 2 in G_2, using a limited number of colors.

Many known coloring problems related to frequency assignment fit into this general framework. We mention some of them here explicitly.

First of all suppose that $G_2 = G_1^2$, that is, G_2 is obtained from G_1 by adding edges between all pairs of vertices that are at distance 2 in G_1. If one just asks for a proper vertex coloring of G_2 (and G_1), this is known as the distance-2 coloring problem. Much of the research has been concentrated on the case that G_1 is a planar graph. We refer to [1,3,4,18,21,22] for more details. In some versions of this problem one puts the additional restriction on G_1 that the colors should be sufficiently separated, in order to model practical frequency assignment problems in which interference should be kept at an acceptable level. One way to model this is to use positive integers for the colors (modeling certain frequency channels) and to ask for a coloring of G_1 and G_2 such that the colors on adjacent vertices in G_2 are different, whereas they differ by at least 2 on adjacent vertices in G_1. This problem is known as the $L(2, 1)$-labeling problem and has been studied (under various names) in [2,7–11,19].

The so-called radio labeling problem (here also various names have been used) models a practical setting in which all assigned frequency channels should be distinct, with the additional restriction that adjacent transmitters should use sufficiently separated frequency channels. Within the above framework this can be modeled by considering the graph G_1 that models the adjacencies of n transmitters, and taking $G_2 = K_n$, the complete graph on n vertices. The restrictions are clear: one asks for a proper vertex coloring of G_2 such that adjacent vertices in G_1 receive colors that differ by at least 2. We refer to [14] and [17] for more particulars.

In this paper, we model the situation that the transmitters form a network in which a certain substructure of adjacent transmitters (called the backbone) is more crucial for the communication than the rest of the network. This means we should put more restrictions on the assignment of frequency channels along the backbone than on the assignment of frequency channels to other adjacent transmitters. The backbone could, for example, model so-called hot spots in the network where a very busy pattern of communications takes place, whereas the other adjacent transmitters supply a more moderate service. We consider the problem of coloring the graph G_2 (that models the whole network) with a proper vertex coloring such that the colors on adjacent vertices in G_1 (that model the backbone) differ by at least 2. Throughout the paper we consider two types of backbones: spanning trees and a special type of spanning trees also known as Hamiltonian paths. A recent paper [6] discusses the case where the backbone is a perfect matching or a collection of disjoint stars.

A. Terminology and Notation

All graphs considered in this paper are assumed to be connected. Let $G = (V, E)$ be a connected finite undirected simple graph, and let $T = (V, E_T)$ be a spanning tree of G. A vertex coloring $f : V \to \{1, 2, 3, \ldots\}$ of V is *proper*, if $|f(u) - f(v)| \geq 1$ holds for all edges $uv \in E$. A vertex coloring is a *backbone* coloring for (G, T), if it is proper and if additionally $|f(u) - f(v)| \geq 2$ holds for all edges $uv \in E_T$ in the spanning tree T. The chromatic number $\chi(G)$ is the smallest integer k for which there exists a proper coloring $f : V \to \{1, \dots, k\}$. The backbone coloring number $\textsc{bbc}(G, T)$ of (G, T) is the smallest integer ℓ for which there exists a backbone coloring $f : V \to \{1, \dots, \ell\}$. When dealing with colorings, we say that two colors z_1 and z_2 are *adjacent* if and only if $|z_1 - z_2| = 1$.

A *Hamiltonian path* of the graph $G = (V, E)$ is a path containing all vertices of G, that is, a sequence $(v_1, v_2, \dots, v_n)$ such that $V = \{v_1, v_2, \dots, v_n\}$, all v_i are distinct, and $v_i v_{i+1} \in E$ for all $i = 1, 2, \dots, n - 1$.

B. Results

We start our investigations of the backbone coloring number by analyzing its relation to the classical chromatic number. How far away from $\chi(G)$ can $\textsc{bbc}(G, T)$ be in the worst case? For each integer $k \geq 1$ we define

$$\mathcal{T}(k) := \max\{\textsc{bbc}(G, T) : G \text{ a graph with spanning tree } T, \text{ and } \chi(G) = k\} \quad (1)$$

It turns out that $\mathcal{T}(k)$ behaves quite primitively:

Theorem 1. $\mathcal{T}(k) = 2k - 1$ *for all* $k \geq 1$.

The upper bound $\mathcal{T}(k) \leq 2k - 1$ in this theorem in fact is straightforward to see. Indeed, consider a proper coloring of G with colors $1, \dots, \chi(G)$, and replace every color i by a new color $2i - 1$. The resulting coloring uses only odd colors, and hence constitutes a "universal" backbone coloring for any spanning tree T of G. The proof of the matching lower bound $\mathcal{T}(k) \geq 2k - 1$ is more involved and will be presented in Section 2.

Next, let us discuss the situation where the backbone tree is a Hamiltonian path. Similarly as in (1), the values

$$\mathcal{P}(k) := \max\{\textsc{bbc}(G, P) : G \text{ a graph with Hamiltonian path } P, \text{ and } \chi(G) = k\} (2)$$

are considered. In Section 3, we will exactly determine all these values $\mathcal{P}(k)$ and observe that they roughly grow like $3k/2$. Their precise behavior is summarized in the following theorem.

Theorem 2. *For* $k \geq 1$ *the function* $\mathcal{P}(k)$ *takes the following values:*

(a) *For* $1 \leq k \leq 4$*:* $\mathcal{P}(k) = 2k - 1$*;*
(b) $\mathcal{P}(5) = 8$ *and* $\mathcal{P}(6) = 10$*;*
(c) *For* $k \geq 7$ *and* $k = 4t$*:* $\mathcal{P}(4t) = 6t$*;*
(d) *For* $k \geq 7$ *and* $k = 4t + 1$*:* $\mathcal{P}(4t + 1) = 6t + 1$*;*
(e) *For* $k \geq 7$ *and* $k = 4t + 2$*:* $\mathcal{P}(4t + 2) = 6t + 3$*;*
(f) *For* $k \geq 7$ *and* $k = 4t + 3$*:* $\mathcal{P}(4t + 3) = 6t + 5$*;*

In Section 4, we discuss the computational complexity of computing the backbone coloring number: *"Given a graph G, a spanning tree T, and an integer ℓ, is* $\text{BBC}(G, T) \leq \ell$*?"* Of course, this general problem is NP-complete. It turns out that for this problem the complexity jump occurs between $\ell = 4$ (easy for all spanning trees) and $\ell = 5$ (difficult even for Hamiltonian paths).

Theorem 3.

(a) *The following problem is polynomially solvable for any* $\ell \leq 4$*: Given a graph G and a spanning tree T, decide whether* $\text{BBC}(G, T) \leq \ell$.
(b) *The following problem is NP-complete for all* $\ell \geq 5$*: Given a graph G and a Hamiltonian path P, decide whether* $\text{BBC}(G, P) \leq \ell$.

2. TREE BACKBONES AND THE CHROMATIC NUMBER

This section is devoted to a proof of the lower bound statement $\mathcal{T}(k) \geq 2k - 1$ in Theorem 1. Consider some arbitrary $k \geq 1$. We will construct a graph G with chromatic number $\chi(G) = k$, and a spanning tree T of G, such that $\text{BBC}(G, T) = 2k - 1$.

The graph G is a complete k-partite graph that consists of k independent sets $V_1, \ldots, V_k$ that are all of cardinality k^k. Clearly, $\chi(G) = k$. The spanning tree T is defined as the final tree in the following inductive construction: The tree T_0 is a star with root in V_1 and $k - 1$ leaves in the $k - 1$ sets $V_2, \ldots, V_k$, one in each set. For $j = 1, \ldots, k$ the tree T_j is constructed from the tree T_{j-1}, by creating $k - 1$ new vertices for every vertex v in T_{j-1} and by attaching them to v. If v is in the set V_q, then every independent set V_i with $i \neq q$ contains exactly one of these new vertices. Note that all newly created vertices are leaves in the tree T_j. It is easy to see that the tree T_j consists of k^{j+1} vertices that are equally distributed among the sets $V_1, \ldots, V_k$. We denote the vertex set of T_j by $V(T_j)$. Note that $V(T_j) \subset V(T_{j+1})$.

Consider a backbone coloring of (G, T) with ℓ colors where $T = T_k$ is the final tree in the above sequence of trees. Since G is complete k-partite, any color that is used in some set V_i cannot be used in any V_j with $j \neq i$. We denote by C_i the set of colors that are used on vertices in V_i. We now go through a number of steps; in every step, the colors in one of the color sets C_i are labeled with the labels A and B.

(Step s). If there exists some (yet unlabeled) color set C_i such that $|C_i| - 1$ of the colors in C_i are adjacent to a color with label A, then: Label these $|C_i| - 1$ colors with label B. Label the remaining color in C_i with label A.

Eventually, there will be no more color class that satisfies the condition in the if-part: Either, all colors have been labeled, or each of the remaining unlabeled color classes contains at least two colors that are not adjacent to any color with label A. If this is the case at the start, then $|C_i| \geq 2$ for all i, and we obtain $\ell \geq 2k$. We denote by $a \leq k$ the number of steps performed, and may assume $a \geq 1$. We denote by

$\pi(s)$ ($s = 1, \ldots, a$) the index of the color set that is labeled in step s. Moreover, we denote by $c_{\pi(s)}$ the unique color in $C_{\pi(s)}$ that is labeled A.

Lemma 4. *Let s be an integer with $1 \le s \le a$. Then the following statements hold.*

(L1) *In the backbone coloring, all vertices v in $V(T_{k-s}) \cap V_{\pi(s)}$ are colored by color $c_{\pi(s)}$.*
(L2) *The color $c_{\pi(s)}$ is not adjacent to any color $c_{\pi(q)}$ with $q < s$.*

Proof. The proofs of (L1) and (L2) are done simultaneously by induction on s. In step $s = 1$, only a color class $C_{\pi(1)}$ with $|C_{\pi(1)}| = 1$ can be labeled. Then the (unique) color in $C_{\pi(1)}$ is labeled by A, and thus becomes color $c_{\pi(1)}$. But by the definition of $C_{\pi(1)}$, in this case *all* vertices in $V_{\pi(1)}$ are colored by $c_{\pi(1)}$. Statement (b) is trivial for $s = 1$.

Now assume that we have proved the statements up to step $s - 1 < a$, and consider step s. Every color in $C_{\pi(s)} - \{c_{\pi(s)}\}$ (if any) is labeled by B, and is adjacent to some color that has been labeled by A in an earlier step. Let D be the set of these adjacent colors. By the inductive assumption, the colors in D are the only possible colors (from their corresponding color sets) that can be used on the vertices in $V(T_{k-s+1})$. Every vertex v in $V(T_{k-s}) \cap V_{\pi(s)}$ is adjacent to $k - 1$ leaves in T_{k-s+1}, and therefore all the colors in D show up on these leaves. Consequently, they block all colors from $C_{\pi(s)}$ for vertex v except color $c_{\pi(s)}$. This proves statement (L1). In case color $c_{\pi(s)}$ was adjacent to some color x labeled by A in an earlier step, the above argument with $D \cup \{x\}$ instead of D yields that there is no possible color for vertex v. This proves statement (L2). ■

Let L^A denote the set of colors that are labeled by A. Since every step labels exactly one color by A, $|L^A| = a$. Let L^+ denote the set of colors z for which $z - 1$ is in L^A; clearly, $|L^+| \ge |L^A| - 1 = a - 1$. By statement (L2) in Lemma 4, the sets L^+ and L^A are disjoint. Moreover, there are $k - a$ color sets with unlabeled colors. Since they do not meet the condition in the if-part of the labeling step, each of them contains at least two colors that are not adjacent to any color with label A. These $2(k - a)$ colors are not contained in $L^A \cup L^+$. To summarize, we have found $|L^A| + |L^+| + 2(k - a)$ pairwise distinct colors in the range $1, \ldots, \ell$. Therefore,

$$\ell \ge |L^A| + |L^+| + 2(k - a) \ge a + (a - 1) + 2(k - a) = 2k - 1$$

Note that these arguments also go through in the extremal case $a = k$. This completes the proof of the lower bound statement in Theorem 1.

3. PATH BACKBONES AND THE CHROMATIC NUMBER

This section is devoted to a proof of Theorem 2. The upper bound is proved in Subsection 3A by case analysis. The lower bound is proved in Subsection 3B; this proof uses a similar idea as the proof in Section 2, but the actual arguments are quite different.

A. Proof of the Upper Bounds

We start with statement (c) in Theorem 2. Hence, consider a graph $G = (V, E)$ with $\chi(G) = 4t$ for some $t \geq 2$, and let $V_1, \ldots, V_{4t}$ denote the corresponding independent sets in the $4t$-coloring. Furthermore, let $P = (V, E_P)$ be a Hamiltonian path in G. Consider the following color sets:

- For $i = 1, \ldots, 3t$, we define the color set $C_i = \{2i - 1\}$.
- For $i = 1, \ldots, t$, we define the color set $C'_i = \{2i, 2t + 2i, 4t + 2i\}$.

Note that these $4t$ color sets are pairwise disjoint, and that all the used colors are from the range $1, \ldots, 6t$. Also note that all the colors of the sets C_i are odd, so these colors are pairwise at distance at least two.

We construct a backbone coloring for (G, P) that for $i = 1, \ldots, 3t$ colors the vertices in the independent set V_i with the color in color set C_i, and that for $i = 1, \ldots, t$ colors the vertices in the independent set V_{3t+i} with one of the three colors in color set C'_i. Clearly, with this assignment of colors all edges between the vertices from the sets V_i with $i = 1, \ldots, 3t$ satisfy the conditions of a backbone coloring (for any backbone of G). The vertices in $V_{3t+1}, \ldots, V_{4t}$ are colored greedily and in arbitrary order: Consider some vertex v in V_{3t+i} that is to be colored with one of the colors $2i, 2t + 2i, 4t + 2i$. In the worst case, the neighbors of v along the Hamiltonian path P have already been colored by colors x and y, and thus forbid the six colors $x - 1, x, x + 1, y - 1, y, y + 1$ for vertex v. Since $t \geq 2$, the three colors in $C'_i = \{2i, 2t + 2i, 4t + 2i\}$ are pairwise at distance at least four, whereas $x - 1, x + 1$ and $y - 1, y + 1$ are at distance two. Therefore, the intersection $C'_i \cap \{x - 1, x, x + 1, y - 1, y, y + 1\}$ contains at most two elements, and C'_i contains at least one feasible color for vertex v. This completes the proof of $\mathcal{P}(4t) \leq 6t$ for all $t \geq 2$.

The cases $k = 4t + 1$, $k = 4t + 2$, $k = 4t + 3$ with $t \geq 2$ follow by simple modifications of the above argument: For $k = 4t + 1$, we add the color set $C_{3t+1} = \{6t + 1\}$. For $k = 4t + 2$, we furthermore add the color set $C_{3t+2} = \{6t + 3\}$. And for $k = 4t + 3$, we furthermore add the color set $C_{3t+3} = \{6t + 5\}$. This proves $\mathcal{P}(4t + 1) \leq 6t + 1$, $\mathcal{P}(4t + 2) \leq 6t + 3$, and $\mathcal{P}(4t + 3) \leq 6t + 5$ for all $t \geq 2$, and settles the upper bounds in Theorem 2 for all $k \geq 8$.

The upper bounds in Theorem 2 for all $k \leq 4$ follow trivially from Theorem 1. For $k = 5$, we use the above argument with five color sets

$$D_1 = \{1\}, \quad D_2 = \{3\}, \quad D_3 = \{5\}, \quad D_4 = \{8\}, \quad D_5 = \{2, 6, 7\}$$

For $k = 6$, we add a sixth color set $D_6 = \{10\}$. Finally, for $k = 7$ we use the seven color sets

$$D'_1 = \{1\}, \quad D'_2 = \{3\}, \quad D'_3 = \{5\}, \quad D'_4 = \{7\}, \quad D'_5 = \{9\}, \quad D'_6 = \{11\}, \quad D'_7 = \{2, 6, 10\}$$

These three constructions prove $\mathcal{P}(5) \leq 8$, $\mathcal{P}(6) \leq 10$, and $\mathcal{P}(7) \leq 11$. The proof of the upper bounds in Theorem 2 is complete.

B. Proof of the Lower Bounds

We consider a complete k-partite graph G with $k \geq 2$ that consists of k independent sets $V_1, \ldots, V_k$ that are all of cardinality $2\Pi_k$. Here Π_k denotes the number of different permutations of $1, 1, 2, 2, 3, 3, \ldots, k, k$ in which no two consecutive symbols are the same. Each such permutation is represented by different (pairs of) vertices in the sets $V_1, \ldots, V_k$, and corresponds to a segment, that is, a path on $2k$ vertices in G. Although we do not need this for our considerations, it is routine to deduce by inclusion-exclusion that $\Pi_k = \sum_{j=0}^{k}(-1)^j \binom{k}{j} \frac{(2k-j)!}{2^{k-j}}$. It is obvious that $\chi(G) = k$. The Hamiltonian path P consists of Π_k segments with $2k$ vertices each. Every such segment contains exactly two vertices of every independent set, since we let each segment correspond to one permutation π of the $2k$ indices $1, 1, 2, 2, 3, 3, \ldots, k, k$ that contributes to the total number of Π_k defined before, and we let the segment visit the independent sets exactly in the order $V_{\pi(1)}, V_{\pi(2)}, \ldots, V_{\pi(2k)}$.

The existence of all possible edges between the partite classes and the properties of the permutations (segments) enable us to join up all the paths on $2k$ vertices corresponding to the segments into one Hamiltonian path, in the following way. For every two different integers $i, j \in \{1, 2, \ldots, k\}$ there are the same positive number of segments with one endpoint in V_i and the other in V_j. Moreover, since segments induced by symmetric permutations have the same endpoints, this number is even. Consider an auxiliary multigraph M with vertices $V_1, \ldots, V_k$, in which edges represent segments with endpoints in different sets, and these edges join the vertices that represent these sets. Clearly, M is Eulerian. We construct a path in G containing all the vertices of the above segments (with endpoints in different sets) by following an Euler tour in M starting in V_1. The first, third, etc. edges in this tour are called forward edges, and the second, fourth, etc. edges are called backward edges. We start with a path in G consisting of one vertex of V_1 and extend the path step by step as follows. If the edge V_iV_j is a forward edge of the tour in M, then we extend the current path in G by joining the current end vertex in V_i of the path under construction with the endpoint in V_j of the segment corresponding to the edge V_iV_j by an edge (with exception of the first segment, which is simply the path on $2k$ vertices with the first vertex in V_1). If the edge V_iV_j is a backward edge, then we extend the current path in G by joining the end vertex that is in some V_k with $k \neq i$ with the endpoint in V_i of the segment corresponding to V_iV_j. To complete a Hamiltonian path we now consider segments with both endpoints in the same set. Note that for every $i \in \{1, 2, \ldots, k\}$ there are the same number of segments with both endpoints in V_i (for $k = 2$ this number is 0). For $k \neq 2$, we extend the path we had before by consecutively adding segments with endpoints in $V_1, V_2, \ldots, V_k$, one for each $i \in \{1, 2, \ldots, k\}$, using the edges between the partite classes; if there are still such segments left after this (if we do not have a Hamiltonian path in G), we repeat this procedure a number of times, until all segments are included. We end up with a Hamiltonian path P of G.

Consider some fixed backbone coloring of (G, P) with colors $1, \ldots, \ell$. Since G is complete k-partite, any color that shows up in some set V_i cannot show up in any V_j with $j \neq i$. We denote by C_i the set of colors that are used on vertices in V_i. If $|C_i| = 1$, then V_i is called *mono-chromatic*; if $|C_i| = 2$, then V_i is *bi-chromatic*; if $|C_i| \geq 3$, then V_i is *poly-chromatic*. We denote by s_1, s_2, and s_3 the number of mono-chromatic, bi-chromatic, and poly-chromatic sets, respectively. Then clearly

$$s_1 + s_2 + s_3 = k \tag{3}$$

and

$$s_1 + 2s_2 + 3s_3 \leq \ell \tag{4}$$

Colors that are used on mono-chromatic, bi-chromatic, poly-chromatic sets, are called mono-chromatic, bi-chromatic, poly-chromatic colors, respectively. We say that two bi-chromatic colors x, y with $1 \leq x < y \leq \ell$ are *partner* colors, if $C_i = \{x, y\}$ holds for some bi-chromatic set V_i.

Clearly, we may assume there are mono-chromatic colors. Now consider the following process that labels some of the colors in $\{1, 2, \ldots, \ell\}$ with the labels A and B, and that creates a number of arcs among the labeled colors.

(Phase 1). All mono-chromatic colors are labeled by label A.

(Phase 2). Repeat the following step over and over again, as long as the condition in the if-part is met:

If there exists an unlabeled bi-chromatic color y that is adjacent to another color z that has already been labeled A at an earlier point in time, then y is labeled B and its partner color x is labeled A. Moreover, we create an arc going from z to y, and another arc going from y to x.

This process eventually terminates, since the step in the second phase can be performed at most s_2 times. We denote by a and b the number of A-labels and B-labels in the final situation after termination.

Lemma 5. *After termination, the following properties are satisfied.*

(T1) $a = b + s_1$.

(T2) *For every labeled color z, there is a unique directed path from some mono-chromatic color to z.*

(T3) *For two adjacent colors z and $z + 1$, at least one of them is not labeled A.*

Proof. Proof of (T1). After the first phase, there are exactly s_1 colors with A-labels and no vertices with B-labels. Every time the step in the second phase is performed, exactly one new label A and one new label B are created.

Proof of (T2). This is straightforward from the definition of the second phase.

Proof of (T3). Suppose for the sake of contradiction that the adjacent colors z and $z+1$ are both labeled A. By (T2), there exists a directed path from some mono-chromatic color $x_{\phi(0)}$ to z (note that $x_{\phi(0)} = z$ might hold). This path goes through colors $x_{\phi(0)}, y_{\phi(1)}, x_{\phi(1)}, y_{\phi(2)}, x_{\phi(2)}, \ldots, y_{\phi(f)}, x_{\phi(f)}$, with $x_{\phi(f)} = z$. Every color $x_{\phi(i)}$ has an A-label, and every color $y_{\phi(i)}$ has a B-label. Every color $y_{\phi(i)}$ is adjacent to color $x_{\phi(i-1)}$. Moreover, the colors $x_{\phi(i)}$ and $y_{\phi(i)}$ are used on the independent set $V_{\phi(i)}$. By similar considerations, we find a directed path from some mono-chromatic color $x_{\psi(0)}$ to $z+1$ that goes through colors $x_{\psi(0)}, y_{\psi(1)}, x_{\psi(1)}, \ldots, y_{\psi(g)}, x_{\psi(g)}$, with $x_{\psi(g)} = z+1$. Every color $x_{\psi(i)}$ has an A-label, and every color $y_{\psi(i)}$ has a B-label. Colors $x_{\psi(i)}$ and $y_{\psi(i)}$ are used on the independent set $V_{\psi(i)}$.

Note that the colors in the directed path from $x_{\phi(0)}$ to z are pairwise distinct, and that the colors in the directed path from $x_{\psi(0)}$ to $z+1$ are pairwise distinct. Also since $x_{\phi(f)}$ and $x_{\psi(g)}$ have A-labels, they cannot belong to one bi-chromatic class. So, $\phi(f) \neq \psi(g)$. By the construction of the complete k-partite graph G, there exists a subpath Q of the Hamiltonian path P that visits the independent sets in the ordering

$$V_{\phi(0)}, V_{\phi(1)}, V_{\phi(2)}, \ldots, V_{\phi(f)}, V_{\psi(g)}, V_{\psi(g-1)}, V_{\psi(g-2)}, \ldots, V_{\psi(1)}, V_{\psi(0)}$$

Let $v_{\phi(i)}$ and $v'_{\psi(j)}$ be the corresponding vertices on Q. What are the possible colors for these vertices in the backbone coloring under investigation? Vertex $v_{\phi(0)}$ is in a mono-chromatic set, and so it must get color $x_{\phi(0)}$. Vertex $v_{\phi(1)}$ is in a bi-chromatic set, and can be colored by color $x_{\phi(1)}$ or by color $y_{\phi(1)}$. However, $v_{\phi(0)}$ is adjacent to $v_{\phi(1)}$, and its color $x_{\phi(0)}$ is adjacent to $y_{\phi(1)}$. Therefore, $v_{\phi(1)}$ must be colored by $x_{\phi(1)}$. Analogous arguments show that every vertex $v_{\phi(i)}$ is colored by color $x_{\phi(i)}$, and that every vertex $v'_{\psi(i)}$ is colored by color $x_{\psi(i)}$.

Now we arrive at the desired contradiction: Vertex $v_{\phi(f)}$ is colored by color $x_{\phi(f)} = z$, vertex $v'_{\psi(g)}$ is colored by color $x_{\psi(g)} = z+1$, and hence two adjacent vertices on the backbone are colored by adjacent colors. ■

Let L denote the set of colors z for which $z+1$ is labeled A after termination. If color 1 is labeled A, then $|L| = a-1$, and otherwise $|L| = a$. In any case, $|L| \geq a-1$. No color in L can be labeled A, since this would contradict property (T3) in Lemma 5. At most b of the colors in L can be labeled B. Hence, L contains at least $a-1-b = s_1-1$ unlabeled colors, where the equation follows from (T1). None of these s_1-1 unlabeled colors can be bi-chromatic; otherwise, there would be another possible step in the second phase. Hence, these s_1-1 unlabeled colors in L must all be poly-chromatic. Among the ℓ colors used by the backbone coloring, there are s_1 mono-chromatic ones, $2s_2$ bi-chromatic ones, and at least

$s_1 - 1$ poly-chromatic ones. Therefore,

$$2s_1 + 2s_2 - 1 \leq \ell \tag{5}$$

Adding inequality (4) to inequality (5), and subtracting three times the equation in (3) yields

$$3k + s_2 - 1 \leq 2\ell \tag{6}$$

Since s_2 is non-negative, (6) implies that $\ell \geq \lceil (3k-1)/2 \rceil$. For the three cases (c) $k = 4t$, (d) $k = 4t+1$, (e) $k = 4t+2$ in Theorem 2 this already implies the claimed lower bounds (c) $\ell \geq 6t$, (d) $\ell \geq 6t+1$, and (e) $\ell \geq 6t+3$, respectively. The case (f) $k = 4t+3$ can be handled as follows: If $s_1 + s_2 \geq 3t+3$, then (5) implies $\ell \geq 6t+5$. If $s_1 + s_2 \leq 3t+2$, then subtracting three times (3) from (4) yields

$$\ell - 3k \geq -2s_1 - s_2 \geq -2(s_1 + s_2) \geq -6t - 4,$$

and hence $\ell \geq 6t+5$ as desired in statement (f).

It remains to prove the "small" cases $k \leq 6$ in statements (a) and (b) of Theorem 2. The cases $k = 1$ and $k = 2$ are trivial.

Proof of the case $\mathbf{k = 3}$. Suppose that for the case $k = 3$ there is a backbone coloring of (G, T) with $\ell \leq 4$ colors. Then the equations and inequalities (3)–(6) do not have any solution s_1, s_2, s_3 over the non-negative integers. This settles the case $k = 3$.

Proof of the case $\mathbf{k = 4}$. Suppose that for the case $k = 4$ there is a backbone coloring of (G, T) with $\ell \leq 6$ colors. Then the equations and inequalities (3)–(6) have $s_1 = 3$, $s_2 = 0$, $s_3 = 1$ as unique solution over the non-negative integers. Up to symmetric cases Lemma 5.(T3) only allows $C_1 = \{1\}$, $C_2 = \{3\}$, $C_3 = \{5\}$, and $C_1 = \{1\}$, $C_2 = \{3\}$, $C_3 = \{6\}$ as mono-chromatic color sets. In the first case $C_4 = \{2, 4, 6\}$ and in the second case $C_4 = \{2, 4, 5\}$. There exists a vertex $v \in V_4$ that is adjacent to vertices from C_2 and from C_3 on the Hamiltonian path P. In either case, there is no feasible color for this vertex v, and we arrive at the desired contradiction.

Proof of the case $\mathbf{k = 5}$. Suppose for the sake of contradiction that for the case $k = 5$ there is a backbone coloring of (G, T) with $\ell \leq 7$ colors. Then the equations and inequalities (3)–(6) have $s_1 = 4$, $s_2 = 0$, $s_3 = 1$ as unique solution over the non-negative integers. By Lemma 5.(T3), the only possible mono-chromatic color sets are $C_1 = \{1\}$, $C_2 = \{3\}$, $C_3 = \{5\}$, $C_4 = \{7\}$. Hence, the poly-chromatic color set must be $C_5 = \{2, 4, 6\}$. But there exists a vertex $v \in V_5$ that is adjacent to vertices from C_2 and from C_3 on the Hamiltonian path P. Hence, there is no feasible color for v and we arrive at the desired contradiction.

Proof of the case k = 6. Suppose that for the case $k = 6$ there is a backbone coloring of (G, T) with $\ell \leq 9$ colors. Then the equations and inequalities (3)–(6) have only two solutions over the non-negative integers: $s_1 = 5$, $s_2 = 0$, $s_3 = 1$, or $s_1 = 4$, $s_2 = 1$, $s_3 = 1$. Using Lemma 5.(T3), the first solution yields only one possibility for the mono-chromatic color sets, with colors 1, 3, 5, 7, 9, respectively. Since there exists a vertex v in the poly-chromatic set that is adjacent to vertices with colors 3 and 7 in P, there is no feasible color for v. We continue with the second solution. Suppose the colors c_1, c_2, c_3, and c_4 for the mono-chromatic color sets C_1, C_2, C_3, C_4 are chosen in increasing order, and let C_5 and C_6 denote the bi-chromatic and poly-chromatic color set, respectively. For a vertex $v_5 \in V_5$ and a vertex $v_6 \in V_6$ that are adjacent to vertices with colors c_2 and c_4 on P, we have no feasible color within the set $\{c_1, c_2 - 1, c_2, c_2 + 1, c_3, c_4 - 1, c_4\}$ of different colors, and we obtain an extra forbidden color if $c_4 \neq 9$. We conclude that $c_4 = 9$, and by symmetry (using c_3 and c_1) that $c_1 = 1$. If $c_3 \neq c_2 + 2$, then by considering two vertices from V_5 and V_6 that are adjacent to vertices with colors c_2 and c_3 on P, we obtain the eight forbidden colors $1, c_2 - 1, c_2, c_2 + 1, c_3 - 1, c_3, c_3 + 1$, and 9, so we cannot color both of these vertices. Hence, $c_3 = c_2 + 2$. There remain two possibilities, up to symmetry: $c_2 = 3$ (or 5) or $c_2 = 4$.

If $c_2 = 4$, we have mono-chromatic colors 1, 4, 6, 9; we obtain a contradiction in the following way: considering vertices $v_5 \in V_5$ and $v_6 \in V_6$ adjacent to vertices with colors 1 and 6 in P, we deduce that colors 3 and 8 are not in the same set; similarly with colors 4 and 6, we deduce that colors 2 and 8 are in different sets; finally with colors 6 and 9, we obtain that colors 2 and 3 are in different sets, which is absurd.

We are left with the case that $c_2 = 3$, and with mono-chromatic colors 1, 3, 5, 9. Using colors 3 and 5 as in the previous case, we conclude that colors 7 and 8 cannot be in the same set (V_5 or V_6); using colors 3 and 9, the same holds for colors 6 and 7; using colors 5 and 9, the same holds for colors 2 and 7. The only possibility is a bi-chromatic set $C_5 = \{4, 7\}$ and a poly-chromatic set $C_6 = \{2, 6, 8\}$. Now consider a subpath Q of P on four vertices visiting the sets in the order V_2, V_5, V_6, V_2. Since V_2 has color 3, the only possible color on Q in V_5 is 7, and we cannot find a feasible color on Q in V_6, our final contradiction.

4. COMPLEXITY RESULTS

This section is devoted to a proof of Theorem 3.

We start with the following straightforward observation that is useful throughout this section.

Observation 6. *Let $G = (V, E)$ be a graph, let $f, g : V \to \{1, \ldots, k\}$ be two colorings of V such that $f(v) + g(v) = k + 1$ for all $v \in V$. Then for any spanning tree T of G, coloring f is a backbone coloring of (G, T) if and only if g is a backbone coloring of (G, T).*

We start with the positive result in statement (a). The cases where $\ell \leq 2$ are trivial. Now let $\ell \geq 3$ and let $G = (V, E)$ be a graph with a spanning tree $T = (V, E_T)$. Let $V = V_0 \cup V_1$ be the bipartition of the vertex set induced by T. Then in any backbone coloring with colors $\{1, 2, 3\}$, the color 2 cannot be used at all. Consider some fixed vertex $v \in V_0$. By Observation 6, we may assume without loss of generality that the color of v is 1. Then all vertices in V_0 must be colored by 1, and all vertices in V_1 must be colored by 3. Hence, $\text{BBC}(G, T) = 3$ if and only if G is bipartite.

Next, consider the case of backbone colorings with $\{1, 2, 3, 4\}$. Consider some fixed vertex $v \in V_0$. By Observation 6, we may assume without loss of generality that the color of v is in $\{1, 2\}$. Then all vertices in V_0 must be colored by colors $\{1, 2\}$, and all vertices in V_1 must be colored by colors $\{3, 4\}$. Hence, $\text{BBC}(G, T) \leq 4$ if and only if the two subgraphs of G that are induced by V_0 and by V_1 are both bipartite with the additional condition that none of the edges of E_T has end vertices with color 2 in V_0 and color 3 in V_1. Checking these conditions can be modeled as a 2-SAT problem, as follows. We introduce a Boolean variable x_v for each vertex $v \in V(G)$, where we let the two literals x_v and $\overline{x}_v$ correspond to assigning color 1 or 2 to v, if $v \in V_0$, and to assigning color 3 or 4 to v, if $v \in V_1$. Now $G[V_0]$ is bipartite if and only if there is a satisfying truth assignment for $(x_u \vee x_v) \wedge (\overline{x}_u \vee \overline{x}_v)$ for each edge $uv \in E(G[V_0])$. The same statement holds for $G[V_1]$. Finally, an edge $uv \in E_T$ with $u \in V_0$ is properly colored according to a backbone 4-coloring if and only if there is a satisfying truth assignment for $x_u \vee \overline{x}_v$. Since 2-SAT is polynomially solvable (see Garey & Johnson [12]), this completes the proof of the statement in (a).

Now let us prove the negative result in statement (b) of Theorem 3. The reduction is done from the NP-complete classical ℓ-coloring problem (see Garey & Johnson [12] for more information): Given a graph $H = (V_H, E_H)$, does there exist a proper ℓ-coloring of H?

Let $H = (V_H, E_H)$ be an instance of ℓ-coloring, and let $v_1, v_2, \ldots, v_n$ be an enumeration of the vertices in V_H. We create $3(n-1)$ new vertices a_i, b_i, c_i with $1 \leq i \leq n-1$. For every $i = 1, \ldots, n-1$ we introduce the new edges $v_i a_i$, $a_i b_i$, $b_i c_i$, and $c_i v_{i+1}$. The graph that results from adding these $3(n-1)$ new vertices and these $4(n-1)$ new edges to H is denoted by G. The vertices $v_1, a_1, b_1, c_1, v_2, a_2, b_2, \ldots, c_{n-1}, v_n$ form a Hamiltonian path P in G. We claim that $\chi(H) \leq \ell$ if and only if $\text{BBC}(G, P) \leq \ell$.

Indeed, assume that $\text{BBC}(G, P) \leq \ell$ and consider such a backbone ℓ-coloring. Then the restriction to the vertices in V_H yields a proper ℓ-coloring of H. Next assume that $\chi(H) \leq \ell$, and consider a proper ℓ-coloring $f : V_H \to \{1, \ldots, \ell\}$. We extend f to a backbone ℓ-coloring of (G, P): Every vertex b_i receives color 3. If $f(v_i) \leq 3$, then a_i is colored ℓ, and otherwise it is colored 1. If $f(v_{i+1}) \leq 3$, then c_i is colored ℓ, and otherwise it is colored 1. This completes the proof of Theorem 3.

5. CONCLUSION

In this paper, we have analyzed the combinatorics and the complexity of backbone colorings of graphs where the backbone is formed by a Hamiltonian path or by a spanning tree. We have investigated the relation of the backbone coloring number to the classical chromatic number, and we proved that the general problem is NP-complete.

Since this area is new, it contains many open problems. We suppose that it would be interesting to investigate the relation between the backbone coloring number and the chromatic number for different classes of perfect graphs. We did it for *split* graphs.

A *split graph* is a graph whose vertex set can be partitioned into a *clique* (i.e., a set of mutually adjacent vertices) and an *independent set* (i.e., a set of mutually non-adjacent vertices), with possibly edges in between. Split graphs were introduced by Hammer & Földes [16]; see also the book [13] by Golumbic. Split graphs are perfect graphs, and hence satisfy $\chi(G) = \omega(G)$, where $\omega(G)$ is the size of a largest clique in G. It is known (see [5] for detailed information), that for every spanning tree T in a split graph G, $\mathrm{BBC}(G, T) \leq \chi(G) + 2$. Also if $\omega(G) \neq 3$, then for every Hamiltonian path P in G, $\mathrm{BBC}(G, P) \leq \chi(G) + 1$, and if $\omega(G) = 3$, then $\mathrm{BBC}(G, P) \leq 5$. All these bounds are tight.

What about more interesting classes of perfect graphs like *chordal* graphs? It can be shown that $\mathrm{BBC}(G, P) \leq \chi(G) + 4$ whenever G is chordal and P is a Hamiltonian path of G. We just use induction and the existence of a simplicial vertex v (its neighbor set S induces a clique in G). Observe that the graph edges and Hamiltonian path edges forbid at most $|S| + 4$ colors for v, while $\chi(G) \geq |S| + 1$. Does this result carry over to arbitrary spanning trees, that is, does $\mathrm{BBC}(G, T) \leq \chi(G) + c$ hold for any chordal graph G with spanning tree T?

Another question is what can be said about *triangle-free* graphs G? Does there exist a small constant c such that $\mathrm{BBC}(G, T) \leq \chi(G) + c$ holds for all triangle-free graphs G?

Finally, what about *planar* graphs? The four-color theorem together with Theorem 1 implies that $\mathrm{BBC}(G, T) \leq 7$ holds for any planar graph G with spanning tree T. However, this bound 7 is probably not best possible. Can it be improved to 6? The planar graph G^* in Figure 1 demonstrates that this bound cannot be improved to

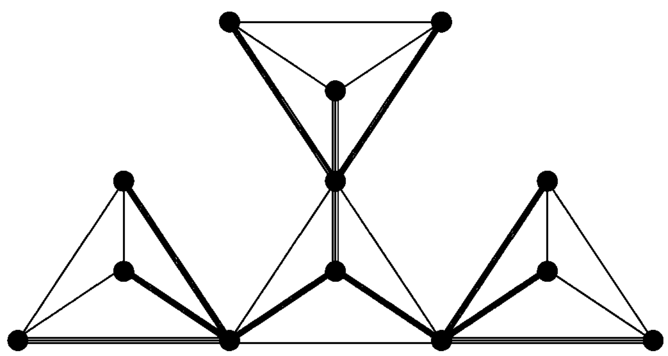

FIGURE 1. A planar graph G^* with a spanning tree T^* (bold edges) such that $\mathrm{BBC}(G^*, T^*) = 6$.

5: Note that graph G^* consists of four copies of K_4 that all have a $K_{1,3}$ as spanning tree. In any backbone coloring of such a K_4 with only five colors, the central vertex of the $K_{1,3}$ must either receive color 1 or color 5. With this observation, it is easy to see that $\text{BBC}(G^*, T^*) \geq 6$ $(= 6)$.

An example of a planar graph G with a Hamiltonian path P such that $\text{BBC}(G, P) = 6$ appears in [6].

Another open question for planar graphs is how to prove (these) upper bounds for the backbone coloring number of a planar graph without using the four-color theorem.

We finish the paper with a final open problem on general graphs. One may consider requiring a larger separation between the colors along the backbone, say $|f(u) - f(v)| \geq \lambda$ for some fixed integer λ. Would one still get a tight bound in the analogues of the results for tree and path backbones? For matching and star backbones the answer is affirmative by results that appear in [6].

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