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INTRACTABILITY OF CLIQUE-WIDTH PARAMETERIZATIONS*

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Abstract. We show that EDGE DOMINATING SET, HAMILTONIAN CYCLE, and GRAPH COLORING are $W[1]$-hard parameterized by clique-width. It was an open problem, explicitly mentioned in several papers, whether any of these problems is fixed parameter tractable when parameterized by the clique-width, that is, solvable in time $g(k) \cdot n^{O(1)}$ on n-vertex graphs of clique-width k, where g is some function of k only. Our results imply that the running time $O(n^{f(k)})$ of many clique-width-based algorithms is essentially the best we can hope for (up to a widely believed assumption from parameterized complexity, namely $FPT \neq W[1]$).

Key words. parameterized complexity, clique-width, tree-width, chromatic number, edge domination, Hamiltonian cycle

AMS subject classifications. 68Q17, 68Q25, 68W40

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1. Introduction.

One of the most frequent approaches for solving graph problems is based on decomposition methods. Tree decomposition, and the corresponding parameter, the tree-width of a graph, are among the most commonly used concepts. We refer to the surveys of Bodlaender [3] and Hliněný et al. [22] for further references on tree-width and related parameters. In the quest for alternative graph decompositions that can be applied to broader classes than to those of bounded tree-width and still enjoy good algorithmic properties, Courcelle and Olariu [10] introduced the clique-width of a graph. Clique-width can be seen as a generalization of tree-width, in a sense that every graph class of bounded tree-width also has bounded clique-width [5].

In recent years, clique-width has received much attention. Corneil et al. [4] show that graphs of clique-width at most 3 can be recognized in polynomial time. Fellows et al. [16] settled a long standing open problem by showing that computing clique-width is NP-hard. Oum and Seymour [27] describe an algorithm that, for any fixed k, runs in time $O(|V(G)|^9 \log |V(G)|)$ and computes $(2^{3k+2} - 1)$ expressions for a graph G of clique-width at most k. Oum [26] improved this result by providing an algorithm computing $(8^k - 1)$ expressions in time $O(|V(G)|^3)$. Recently, Hliněný and Oum [21] obtained an algorithm running in time $O(|V(G)|^3)$ and computing $(2^{k+1}-1)$ expressions for a graph G of clique-width at most k. It is also worthwhile to mention here the related graph parameters NLC-width introduced by Wanke [30] and rank-width introduced by Oum and Seymour [27], which are equivalent to clique-width in the sense that the same classes of graph have bounded clique-width, NLC-width, and rank-width.

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By the seminal result of Courcelle [6, 9] (see also [1]), every decision problem on graphs expressible in monadic second order logic is fixed parameter tractable when parameterized by the tree-width of the input graph. For problems expressible in monadic second order logic with logical formulas that do not use edge set quantifications (so-called MS_1 logic), it is possible to extend the meta theorem of Courcelle to graphs of bounded clique-width. As was shown by Courcelle, Makowsky, and Rotics [7], all problems expressible in MS_1 logic are fixed parameter tractable when parameterized by the clique-width of a graph.

There are many problems (various problems mentioned here will be defined later) expressible in monadic second order logic that cannot be expressed in MS_1 logic. The most natural are, perhaps, Hamiltonian Cycle and Edge Dominating Set. Edge Dominating Set and Hamiltonian Cycle are expressible in monadic second order logic with edge set quantification and thus can be solved in linear time on classes of graphs of bounded tree-width. Graph Coloring or Chromatic Number is not expressible in monadic second order logic. However, for every fixed r, checking whether the vertices of a graph G can be colored with at most r colors such that no two adjacent vertices are of the same color can be expressed in monadic second order logic even without edge set quantification. Since graphs of tree-width at most t are $t+1$ colorable, this implies that Graph Coloring can be solved in linear time on graph classes of bounded tree-width. It is also known that these problems can be solved in polynomial time on each class of graph of bounded clique-width (with known upper bound) and a significant amount of the literature is devoted to algorithms for these problems and their generalizations. Polynomial time algorithms for Graph Coloring and its different generalizations including computations of chromatic and Tutte polynomials of graphs for graph classes of bounded clique-width are given in [19, 18, 20, 23, 24, 25, 28, 29]. Polynomial time algorithms for Hamiltonian Cycle can be found in [30, 13] (in terms of NLC-width). Algorithms for Edge Dominating Set are given in [23, 24]. The running time of all these algorithms on an n-vertex graph of clique-width at most k is $O(n^{f(k)})$, where f is some function of k. Since all these problems are solvable in time $O(g(k) \cdot n^c)$, or even $O(g(k) \cdot n)$, when the tree-width of the graph is at most k, the most natural question to ask is whether a similar behavior can be expected on graphs of bounded clique-width. The question on the existence of fixed parameter tractable algorithms (with clique-width being the parameter) for all these problems (or their generalizations) was asked by Gerber and Kobler [18], Kobler and Rotics [23, 24], and Makowsky et al. [25, 20].

1.1. Our results and organization of the paper.

In this paper we prove that Edge Dominating Set, Hamiltonian Cycle, and Graph Coloring are $W[1]$-hard parameterized by clique-width, thus resolving open questions raised in [18, 20, 23, 24, 25]. Our results show that the running time $O(n^{f(k)})$ of many clique-width-based algorithms [13, 19, 18, 20, 23, 24, 25, 28, 29, 30] is essentially the best we can hope for (unless the hierarchy of parameterized complexity classes collapses)—the price we pay for generality.

The remaining part of the paper is organized as follows. We provide definitions and preliminaries in section 2. In section 3 we prove the hardness of Graph Coloring. Sections 4 and 5 are devoted to the results on Edge Dominating Set and Hamiltonian cycle correspondingly.

2. Definitions and preliminary results.

Parameterized Complexity. Parameterized complexity is a two-dimensional framework for studying the computational complexity of a problem. One dimension is the input size n and another one is a *parameter* k. Formally, a parameterized problem $Q \subseteq \Sigma^* \times \mathbb{N}$, where Σ is a finite alphabet. A parameterized problem is called *fixed parameter tractable* (FPT) if it can be solved in time $f(k) \cdot n^c$, where f is a function only depending on k and c is some constant. Next we define the notion of parameterized reduction.

DEFINITION 1. *Let Q and Q' be parameterized problems over the alphabets Σ and Σ', respectively. We say that Q is (uniformly many:one)* FPT reducible *to Q' if there exist functions $f, g : \mathbb{N} \to \mathbb{N}$, a constant $\alpha \in \mathbb{N}$, and a mapping $\Phi\ : \Sigma^* \times \mathbb{N} \to \Sigma'^* \times \mathbb{N}$ such that*

1. *$\Phi(x, k)$ is computable in time $f(k)|x|^\alpha$,*
2. *if $(x', k') = \Phi(x, k)$, then $k' \leq g(k)$, and*
3. *$(x, k) \in Q$ if and only if $\Phi(x, k) \in Q'$.*

The basic complexity class for fixed parameter intractability is $W[1]$. INDEPENDENT SET and CLIQUE parameterized by solution size are two fundamental problems which are known to be $W[1]$-complete. The principal way of showing that a parameterized problem is unlikely to be fixed parameter tractable is to prove $W[1]$-hardness. To show that a problem is $W[1]$-hard, it is enough to give a parameterized reduction from a known $W[1]$-hard problem. Throughout this paper we follow this recipe to show a problem $W[1]$-hard. We refer to the books of Downey and Fellows [12] and Flum and Grohe [17] for a detailed treatment to parameterized complexity.

Graphs. We consider only finite undirected graphs without loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$ and its edge set by $E(G)$. A set $S \subseteq V(G)$ of pairwise adjacent vertices is called a *clique*. For $v \in V(G)$, we denote by $E(v)$ the set of edges incident with v.

Tree-width. A *tree decomposition* of a graph G is a pair (X, T) where T is a tree whose vertices we will call *nodes* and $X = (\{X_i \mid i \in V(T)\})$ is a collection of subsets of $V(G)$ such that

1. $\bigcup_{i \in V(T)} X_i = V(G)$,
2. for each edge $(v, w) \in E(G)$, there is an $i \in V(T)$ such that $v, w \in X_i$, and
3. for each $v \in V(G)$, the set of nodes $\{i \mid v \in X_i\}$ forms a subtree of T.

The *width* of a tree decomposition $(\{X_i \mid i \in V(T)\}, T)$ equals $\max_{i \in V(T)}\{|X_i| - 1\}$. The *tree-width* of a graph G is the minimum width over all tree decompositions of G. We use notation $\mathbf{tw}(G)$ to denote the tree-width of a graph G.

Clique-width. Let G be a graph and k be a positive integer. A *k-graph* is a graph whose vertices are labeled by integers from $\{1, 2, \ldots, k\}$. We call the k-graph consisting of exactly one vertex labeled by some integer from $\{1, 2, \ldots, k\}$ an initial k-graph. The *clique-width* $\mathbf{cwd}(G)$ is the smallest integer k such that G can be constructed by means of repeated application of the following four operations on k-graphs: (1) *introduce*: construction of an initial k-graph labeled by i and denoted by $i(v)$ (that is, $i(v)$ is a k-graph with v as a single vertex and label i), (2) *disjoint union* (denoted by $\oplus$), (3) *relabel*: changing all labels i to j (denoted by $\rho_{i\to j}$), and (4) *join*: connecting all vertices labeled by i with all vertices labeled by j by edges (denoted by $\eta_{i,j}$). Using the symbols of these operations, we can construct well-formed expressions. An expression is called *k-expression* for G if the graph produced by performing these operations, in the order defined by the expression, is isomorphic to G when labels are removed, and $\mathbf{cwd}(G)$ is the minimum k such that there is a k-expression for G.

It is convenient for us to associate with a k-expression, the *k-expression tree.* This allows us to easily describe modifications to k-expressions in our hardness reductions while showing upper bounds on the clique-width of the graphs in question. A k-expression tree (or simply expression tree if the parameter is clear) of a graph G is the syntactic tree of a k-expression. It is a rooted labeled tree T of the following form:

- The nodes of T are of four types i, $\oplus$, η, and ρ.
- Introduce nodes $i(v)$ are leaves of T, corresponding to initial k-graphs with vertices v, which are labeled i.
- A union node $\oplus$ stands for a disjoint union of k-graphs associated with its children (since disjoint union is commutative, we need not distinguish a left child from a right child).
- A relabel node $\rho_{i\to j}$ has one child and is associated with the k-graph, which is the result of relabeling operation for the k-graph corresponding to the child.
- A join node $\eta_{i,j}$ has one child and is associated with the k-graph, which is the result of join operation for the k-graph corresponding to the child.
- The graph G is isomorphic to the graph associated with the root of T (with all labels removed).

A graph G has $\mathbf{cwd}(G) \le k$ if and only if it is possible to construct a k-expression tree T of G.

Hliněný and Oum [21] obtained an algorithm running in time $O(|V(G)|^3)$ and computing $(2^{k+1}-1)$ expressions for a graph G of clique-width at most k. Hence, the algorithm of Hliněný and Oum [21] only approximates the clique-width but does not provide an algorithm to construct an optimal k-expression tree for a graph G of clique-width at most k. But this approximation is usually sufficient for algorithmic purposes.

It is well known that the clique-width of a graph is bounded in terms of its tree-width by means of a fixed function, as recalled in Theorem 1 below.

THEOREM 1 (see [5]). *If graph G has tree-width at most t, then $\mathbf{cwd}(G)$ is at most $k = 3 \cdot 2^{t-1}$. Moreover, a k-expression tree for G of width at most k can be constructed in time $f(t) \cdot |V(G)|^{O(1)}$ from the tree decomposition of G.*

The second claim in Theorem 1 is not given explicitly in [5]. However, it can be shown since the upper bound proof in [5] is constructive (see also [8, 14]). Note that if a graph has bounded tree-width, then the corresponding tree decomposition can be constructed in linear time [2].

3. Graph coloring—chromatic number. In this section, we prove that GRAPH COLORING is $W[1]$-hard when parameterized by clique-width. Recall that a *coloring* of a graph G is an assignment $c\colon V(G) \to \mathbb{N}$ of a positive integer (*color*) to each vertex of G. The coloring c is *proper* if adjacent vertices receive distinct colors. The *chromatic number* $\chi(G)$ of a graph G is the smallest number of colors of a proper coloring of G.

> GRAPH COLORING (or CHROMATIC NUMBER): Given a graph G and a positive integer r, decide whether $\chi(G) \le r$.

For a fixed r, checking whether the vertices of a graph G can be properly colored with at most r colors is definable in MS_1.

Our reduction is from the EQUITABLE COLORING problem parameterized by the number r of colors used and the tree-width of the input graph. In the EQUITABLE COLORING problem one is given a graph G on n vertices and integer r and asked whether G can be properly r-colored in such a way that the number of vertices in any two color classes differs by at most 1 (such coloring is called an *equitable r-coloring*). Notice that if n is divisible by r this implies that all color classes must contain the same number of vertices. In our reduction we will assume that in the instance we reduce from, n is divisible by r. For a justification of this assumption, if r does not divide n we can add a clique of size $r - (n - \lfloor \frac{n}{r} \rfloor r)$ to G. We reduce from the exact version of EQUITABLE COLORING, that is, the version where we are looking for an equitable coloring of G with exactly r colors.

THEOREM 2 (see [15]). EQUITABLE COLORING *is $W[1]$-hard when parameterized by the tree-width t of the input graph and the number r of colors.*

Reduction. On input (G, r) to EQUITABLE COLORING, we construct an instance (G', r') of GRAPH COLORING as follows. Let n denote the number of vertices of G. We start with a copy of G and let $r' = r + nr$. We now add a clique P of size r' to G'. The clique P will function as a *palette* in our reduction, as we have to use all r' available colors to properly color it. We partition P into $r+1$ parts as follows: $P = P^M \cup P_1 \cup P_2 \cdots \cup P_r$, where P^M has size r and P_i has size n for every i. We call P^M the main palette and denote the vertices in P^M by p_i for $1 \le i \le r$. We add edges between every vertex of $P \setminus P^M$ and every vertex of the copy of G. For each vertex $u \in V(G)$, we assign a vertex $u_{P_i} \in P_i$ for every i. Now, for every $1 \le i \le r$, we add a set S_i of n vertices which contains copies of all vertices of G. For each vertex $u \in V(G)$, we denote the copy of u in S_i by u_{S_i} for every $1 \le i \le r$, and make u_{S_i} adjacent to u and the entire palette P *except for* u_{P_i} and p_i. We conclude the construction by adding a clique C_i of $n\frac{r-1}{r}$ vertices and making every vertex of C_i adjacent to all of the vertices of S_i and the entire palette except for P_i. See Figure 3.1 for an illustration.

LEMMA 1. *If G has an equitable r-coloring ψ, then G' has a proper r'-coloring ϕ.*

Proof. We construct a coloring ϕ of G' as follows. The coloring ϕ colors the copy of G in G' in the same way that ψ colors G. We color the palette, assigning a unique color to each vertex and making sure that the main palette P^M is colored using the same colors that are used to color the vertices of G. For every vertex u_{S_i} we color u_{S_i} with $\phi(p_i)$ if $\phi(u) \neq \phi(p_i)$ and with $\phi(u_{P_i})$ if $\phi(u) = \phi(p_i)$. We color every vertex of C_i with some color from P_i (a color used to color a vertex of P_i). To do this we need $n\frac{r-1}{r}$ different colors from P_i. Since exactly n/r vertices of G are colored with $\phi(p_i)$, exactly $n\frac{r-1}{r}$ of S_i are colored with $\phi(p_i)$ and thus n/r vertices of S_i are colored with colors of P_i. Hence there are $n\frac{r-1}{r}$ colors of P_i available to color C_i. Thus, ϕ is a proper r'-coloring of G concluding the proof. ∎

LEMMA 2. *If G' has a proper r'-coloring ϕ, then G has an equitable r-coloring ψ.*

Proof. We prove that the restriction of ϕ to the copy of G in G' in fact is an equitable r-coloring of G. Since ϕ can use only the colors of P^M, ϕ is a proper r-coloring of G. It remains to prove that for any i between 1 and r, at most n/r vertices of G are colored with $\phi(p_i)$. Suppose for contradiction that there is an i such that more than n/r vertices of G are colored with $\phi(p_i)$. Then there are more than n/r vertices of S_i that are colored with colors of P_i. Since each such vertex must take a different color from P_i, there are less than $n\frac{r-1}{r}$ different colors of P_i available to color the vertices of C_i. However, since C_i is a clique on $n\frac{r-1}{r}$ vertices that must be colored with colors of P_i, this is a contradiction. ∎

LEMMA 3. *If the tree-width of G is t, then the clique-width of G' is at most $k = 3 \cdot 2^{t-1} + 7r + 2$.*

Proof. By Theorem 1, we can compute an expression tree for G of width at most $3 \cdot 2^{t-1}$. Our strategy is as follows. We first show how to modify the expression tree to give a width k expression tree for $G' \setminus (P^M \bigcup_{i=1}^{r} C_i)$. Then we change this tree into an expression tree for G'. In order to give an expression tree for G' we introduce the following extra labels:

- For every $1 \le i \le r$ the labels α_i, α_i^L, and α_i^R for vertices in P_i.
- For every $1 \le i \le r$ the labels β_i, β_i^L, and β_i^R for vertices in S_i.

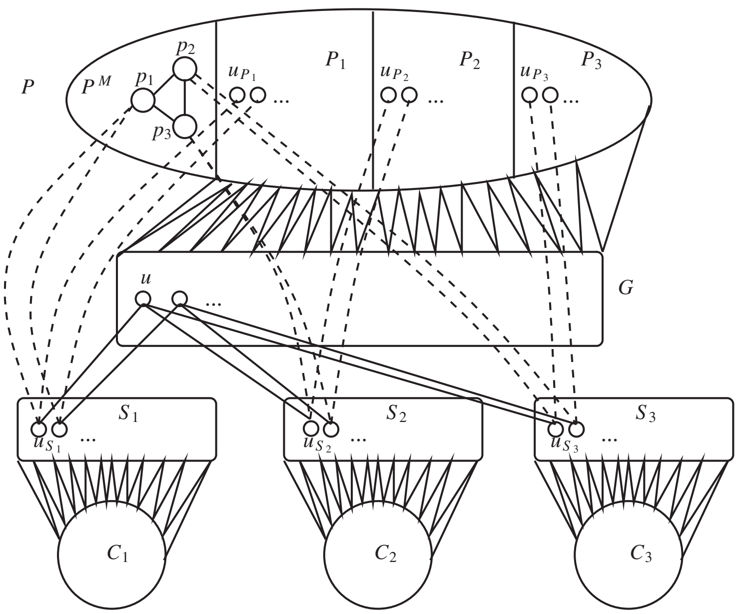

FIG. 3.1. *The figure shows the construction of G' for $r = 3$. The edges between vertices of S_i and P and between C_i and P are not shown. The dotted lines indicate nonedges.*

- For every $1 \leq i \leq r$ the label ζ_i for vertices in C_i.
- A "work" label – γ^W and a label γ^M for each vertex that belongs to P_M.

In the expression tree for G, we replace every introduce node $i(v)$ with a small expression tree $T_i(v)$. In $T_i(v)$, the vertex v is introduced with label γ^W and the vertices $v_{P_1}, \ldots, v_{P_r}$ and $v_{S_1}, \ldots, v_{S_r}$ are introduced with labels $\alpha_1, \ldots, \alpha_r$ and $\beta_1, \ldots, \beta_r$, respectively. Also, the vertices labeled γ^W are joined with the vertices labeled $\beta_1, \ldots, \beta_r$, and for every p, the vertices with the label β_p are joined with the vertices labeled by α_q for every $q \neq p$. Also, for every $p \neq q$, the vertices with the label α_p are joined with the vertices labeled α_q. Finally, the vertices labeled γ^W are relabeled by i (i.e., v receives the color used for it in the expression tree for G).

Now, for every union node in the expression tree (not the union nodes inside the T_i's), we add extra nodes on the edges incident to this node. On the edge from the node to its left child, we add nodes that relabel the vertices labeled α_p by α_p^L and the vertices labeled β_p by β_p^L for every p. Similarly, on the edge from the union node to its right child, we add nodes that relabel the vertices labeled α_p by α_p^R and the vertices labeled β_p by β_p^R for every p. Finally, on the edge from the union node to its parent we add nodes that first join every vertex labeled α_p^L with every vertex labeled β_q^R or α_q^R, join every vertex labeled α_p^R with every vertex with the label β_q^L, and then relabel every vertex labeled α_p^L or α_p^R to α_p and every vertex labeled β_p^L or β_p^R by β_p.

To conclude the construction of $G' \setminus (P^M \bigcup_{i=1}^r C_i)$, we need to add some extra nodes above the root of the expression tree. We add the edges between $P \setminus P^M$ and G by joining every vertex labeled α_p with all vertices labeled by the labels used for constructing G.

We now need to add the construction of P^M and $\bigcup_{i=1}^r C_i$ to our expression tree. We start by making C_p for every p between 1 and r. For every p, we add a clique on $n^{\frac{r-1}{r}}$ vertices labeled ζ_p. Every vertex with the label ζ_p is joined with the vertex labeled β_p and for every pair $p \neq q$, the vertices labeled ζ_p are joined with the vertices labeled α_q.

Finally, we add the construction of P^M. For every i, we introduce the vertex p_i with label γ^W, join the vertices labeled γ^W with the vertices labeled α_j and ζ_j for every j, the vertices labeled γ^W with the vertices labeled β_j for every $j \neq i$, and finally join the vertices labeled γ^W with the vertices labeled γ^M and relabel the vertices with the label γ^W by γ^M. This concludes the construction of G'. Notice that this expression tree for G' uses $k = 3 \cdot 2^{t-1} + 7r + 2$ labels. ∎

By Lemmas 1, 2, and 3, we have the following result.

THEOREM 3. *The* GRAPH COLORING *problem is* $W[1]$*-hard when parameterized by clique-width.*

Proof. Lemmas 1, 2, and 3 together give a parameterized reduction from EQUITABLE COLORING parameterized by the tree-width t of the input graph and the number of colors r to GRAPH COLORING parameterized by the clique-width. Lemmas 1 and 2 ensure the correctness of the reduction while Lemma 3 shows that if an input (G, r) of EQUITABLE COLORING has tree-width at most t, then the input (G', r') constructed for the GRAPH COLORING has clique-width at most $f(t) = 3 \cdot 2^{t-1} + 7r + 2$. By Theorem 2, EQUITABLE COLORING parameterized by the tree-width t of the input graph and the number of colors r is W[1]-hard, and hence GRAPH COLORING parameterized by the clique-width is W[1]-hard. ∎

4. Edge dominating set. An *edge dominating set* of a graph G is a set $X \subseteq E(G)$ such that every edge of G is either in X or adjacent to at least one edge of X.

> EDGE DOMINATING SET: Given a graph G and a positive integer r, decide whether there exists an edge dominating set of G of size at most r.

In this section, we show that EDGE DOMINATING SET is $W[1]$-hard when parameterized by clique-width.

Our reduction is from a variant of CAPACITATED DOMINATING SET problem.

4.1. Exact saturated capacitated dominating set. A *capacitated graph* is a pair (G, c), where G is a graph and $c\colon V(G) \to \mathbb{N}$ is a *capacity* function such that $1 \leq c(v) \leq \deg(v)$ for every vertex $v \in V(G)$ (sometimes we simply say that G is a capacitated graph if the capacity function is clear from the context). A set $S \subseteq V(G)$ is called a *capacitated dominating set* if there is a *domination mapping* $f\colon V(G) \setminus S \to S$ which maps every vertex in $V(G) \setminus S$ to one of its neighbors in such a way that the total number of vertices mapped by f to any vertex $v \in S$ does not exceed its capacity $c(v)$. We say that for a vertex $u \in S$, vertices in the set $f^{-1}(u)$ are *dominated by* u. The CAPACITATED DOMINATING SET problem is formulated as follows: Given a capacitated graph (G, c) and a positive integer k, determine whether there exists a capacitated dominating set S for G containing at most k vertices. It was proved by Dom et al. [11] that this problem is $W[1]$-hard when parameterized by tree-width.

THEOREM 4 ([11]). CAPACITATED DOMINATING SET *is* $W[1]$*-hard when parameterized by the tree-width* t *of the input graph and the solution size* k*.*

For the intractability proof of EDGE DOMINATING SET, we need a special variant of CAPACITATED DOMINATING SET problem which we call EXACT SATURATED CAPACITATED DOMINATING SET. Given a capacitated dominating set S and a domination mapping f, we say that f *saturates* a vertex $v \in S$ if $|f^{-1}(v)| = c(v)$. A capacitated dominating set $S \subseteq V(G)$ is called *saturated* if there is a domination mapping f which saturates all vertices of S. In EXACT SATURATED CAPACITATED DOMINATING SET, a capacitated graph (G, c) and a positive integer k is given, and the objective is to check whether G has a saturated capacitated dominating set S with exactly k vertices.

LEMMA 4. *The* EXACT SATURATED CAPACITATED DOMINATING SET *problem is $W[1]$-hard when parameterized by clique-width.*

Proof. We reduce from EXACT CAPACITATED DOMINATING SET, an exact version of the CAPACITATED DOMINATING SET problem parameterized by the tree-width of the input graph. In the exact capacitated dominating set problem, the question is to determine whether there exists a capacitated dominating set of size exactly k.

CLAIM 1. EXACT CAPACITATED DOMINATING SET *is $W[1]$-hard when parameterized by the tree-width t of the input graph and the solution size k.*

Proof. We give an easy reduction from the CAPACITATED DOMINATING SET problem. By Theorem 4, we know that the CAPACITATED DOMINATING SET problem is $W[1]$-hard when parameterized by tree-width. Given a capacitated graph (G, c) and a positive integer k, an instance of CAPACITATED DOMINATING SET, we get an instance of EXACT CAPACITATED DOMINATING SET by taking (G, c) and a positive integer k itself. If G has a capacitated dominating set S of size at most k, then we can make it exactly equal to k by adding $k-|S|$ vertices from $V(G)\setminus S$ arbitrarily. In the other direction, if G has a capacitated dominating set of size exactly k, then it is also a capacitated dominating set of size at most k. This concludes the proof of the claim. ∎

Let r be a positive integer and $H_r(u)$ denote a capacitated graph rooted at vertex u. The graph $H_r(u)$ is constructed as follows. Its vertex set is given by $\{u, v, x_1, \ldots, x_r, y_1, \ldots, y_r\}$ and the edges are given by making u adjacent to all vertices x_i, making v adjacent to all vertices y_i, and finally adding edges x_iy_j, $1 \le i, j \le r$. We define the capacity function as follows: $c(v) = r - 1$, $c(x_i) = r + 1$, and $c(y_i) = i$ for all $i \in \{1, 2, \ldots, r\}$ (note that the capacity function is not defined for the root u).

Let (G, c) be a capacitated graph, $u \in V(G)$, and $r \ge \max\{3, c(u)+1\}$. We add a copy of $H_r(u)$ to G with u being its root. Let G' be the resulting capacitated graph. We need two auxiliary claims about the graph G'.

CLAIM 2. *Any capacitated dominating set S, with the domination mapping f, of G can be extended to a capacitated dominating set S' of G' in such a way that all vertices of $H_r(u)$ are saturated.*

Proof. Let S be a capacitated dominating set in G with the domination mapping f. We define s to be $|f^{-1}(u)|$ if $u \in S$ and $c(u)$ otherwise. Let $S' = S \cup \{v, y_j\}$ where $j = r - c(u) + s$. The mapping f is extended as follows: $f(x_i) = u$ for $1 \le i \le c(u) - s$, $f(x_i) = y_j$ for $i > c(u) - s$, and $f(y_i) = v$ for all $i \ne j$. ∎

CLAIM 3. *Every saturated capacitated dominating set in G' contains exactly two vertices from $V(H_r(u)) \setminus \{u\}$.*

Proof. Let S' be a saturated capacitated dominating set in G' and f be its corresponding domination mapping. We first show that S' does not contain any x_i's. Suppose that some vertex x_i is included in S'. Then because of capacity constraint that $c(x_i) = r + 1$, it implies that $y_1, y_2, \ldots, y_r \notin S'$ and $f(y_j) = x_i$ for all these vertices. Therefore $v \in S'$ but clearly this vertex cannot be saturated. Hence, $x_1, x_2, \ldots, x_r \notin S'$. Now we show that v must be in S'. Assume to the contrary that $v \notin S'$. Then $y_1, y_2, \ldots, y_r \in S'$, as they need to be dominated. But these vertices cannot be saturated since $\sum_{i=1}^{r} c(y_i) = 1 + \cdots + r = \frac{r(r+1)}{2} > r + 1$. This means that $v \in S'$. The capacity of v is $r - 1$; hence at most one vertex y_i can be included in S'. On the other hand, since $c(u) < r$, there exists at least one vertex x_j such that $f(x_j) \ne u$. Hence to dominate this vertex we need a vertex $y_i \in S'$. This concludes the proof. ∎

Now we are ready to complete the proof of Lemma 4. Let (G, c) be a capacitated graph with the vertex set $\{u_1, u_2, \ldots, u_n\}$, $r = \max\{c(v)\colon v \in V(G)\} + 2$. For every vertex u_i, we add a copy of $H_r(u_i)$ to G with u_i being its root. Let H be the resulting capacitated graph. By applying Claims 2 and 3 we conclude that G has a capacitated dominating set of the size k if and only if H has an exact saturated dominating set of the size $k + 2n$.

It remains to prove that if the tree-width of G is bounded, then the clique-width of H is bounded. Let $\mathbf{tw}(G) \leq t$. By Theorem 1 $\mathbf{cwd}(G) \leq 3 \cdot 2^{t-1}$. We prove that $\mathbf{cwd}(H) \leq \mathbf{cwd}(G) + 4$. Assume that the construction of the labeled graph G uses labels from the set $\{\alpha_1, \ldots, \alpha_w\}$ where $w = \mathbf{cwd}(G)$. To construct H from G we use additional labels $\{\beta_1, \beta_2, \beta_3, \beta_4\}$.

When a vertex u having a label α_j is introduced, we do the following sequence of operations: first the vertex introductions denoted by $\alpha_j(u)$, $\beta_1(x_i)$, and $\beta_2(y_i)$ for all $i \in \{1, \ldots, r\}$, and $\beta_3(v)$. After this we apply the following operations: η_{α_j,β_1}, η_{β_1,β_2}, η_{β_2,β_3}, and $\rho_{\beta_i \to \beta_4}$ for $i = 1, 2, 3$. We omit the union operations in this description: It is assumed that if some vertex is introduced, then this operation is automatically performed. Join, union, and relabel operations with labels $\{\alpha_1, \ldots, \alpha_w\}$ are done as it is done for the expression tree of G. This concludes the construction of the expression tree for H. ∎

4.2. Intractability of edge dominating set problem. In this section we show that EDGE DOMINATING SET is $W[1]$-hard when parameterized by clique-width by giving a reduction from EXACT SATURATED DOMINATING SET. We start with descriptions of auxiliary gadgets.

Auxiliary gadgets. Let $s \leq t$ be positive integers. We construct a graph $F_{s,t}$ with the vertex set $\{x_1, \ldots, x_s, y_1, \ldots, y_s, z_1, \ldots, z_t\}$ and edges $x_i y_i$, $1 \leq i \leq s$, and $y_i z_j$, $1 \leq i \leq s$, and $1 \leq j \leq t$. Basically we have a complete bipartite graph between the y_i's and the z_j's with pendent vertices attached to y_i's. The vertices $z_1, z_2, \ldots, z_t$ are called the *roots* of $F_{s,t}$.

Graph $F_{s,t}$ has the following property.

LEMMA 5. *Any set of s edges incident with vertices $y_1, \ldots, y_s$ forms an edge dominating set in $F_{s,t}$. Furthermore, let G be a graph obtained by the union of $F_{s,t}$ with some other graph H such that $V(F_{s,t}) \cap V(H) = \{z_1, \ldots, z_t\}$. Then every edge dominating set of G contains at least s edges from $F_{s,t}$.*

The proof of Lemma 5 follows from the fact that every edge dominating set includes at least one edge from $E(y_i)$ for $i \in \{1, \ldots, s\}$.

Reduction. Let (G, c) be a capacitated graph with the vertex set $\{u_1, \ldots, u_n\}$, and k be a positive integer. For every vertex u_i, the set U_i with $c(u_i)$ vertices is introduced, and then vertex sets $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ are added. For every edge $u_i u_j \in E(G)$, all vertices of U_i are joined with v_j and all vertices of U_j are joined with v_i by edges. Then every vertex v_i is joined to its counterpart w_i and to every vertex v_i we add one additional leaf (a pendent vertex). Now vertex sets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are constructed, and vertices a_i are made adjacent to all vertices of U_i, w_i, and b_i. For every vertex b_i, a set R_i of $c(u_i)+1$ vertices is added and b_i is made adjacent to all the vertices in R_i. Then we add to every vertex of $R_1 \cup R_2 \cup \cdots \cup R_n$ a path of length two. Let X be the set of middle vertices of these paths. We denote the obtained graph by G' (see Figure 4.1). Finally, we introduce three copies of $F_{s,t}$:

- a copy of $F_{n-k,n}$ with roots $\{a_1, \ldots, a_n\}$,
- a copy of $F_{k,n}$ with roots $\{b_1, \ldots, b_n\}$, and a
- a copy of $F_{n,r}$ where $r = \sum_{i=1}^{n} c(u_i)$ with roots in X.

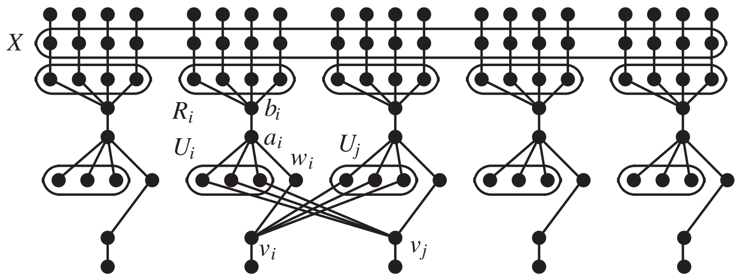

FIG. 4.1. *Graph G'.*

Let H be this final resulting graph.

LEMMA 6. *A capacitated graph (G, c) on n vertices has an exact saturated dominating set of size k if and only if H has an edge dominating set of cardinality at most $2n + r$. Here, $r = \sum_{v \in V(G)} c(v)$.*

Proof. Let S be an exact saturated dominating set of the size k in G and f be its corresponding domination mapping. For convenience (without loss of a generality) we assume that $S = \{u_1, \ldots, u_k\}$. We construct the edge dominating set of H as follows. First we select a specific edge emanating from every vertex in the set $\{v_1, \ldots, v_n\}$. For every vertex v_i, $1 \le i \le k$, the edge $v_i w_i$ is selected. Now let us assume that $k < i \le n$ and $f(u_i) = u_j$. We choose a vertex u in U_j which is not incident with already chosen edges and add the edge uv_i to our set. Notice that we always have such a choice of $u \in U_j$ as $c(u_j) = |U_j|$. We observe that these edges already dominate all the edges in the sets $E(v_i)$, $1 \le i \le n$, and in sets $E(u)$ for $u \in U_1 \cup \cdots \cup U_k \cup \{w_1, \ldots, w_k\}$. Now we add $n - k$ edges from $F_{n-k,n}$ which are incident with vertices in $\{a_{k+1}, \ldots, a_n\}$ and k edges from $F_{k,n}$ which are incident with $\{b_1, \ldots, b_k\}$. Then $r - n$ matching edges joining vertices of $R_{k+1}, \ldots, R_n$ to the vertices of X are included in the set. Finally, we add n edges from $F_{n,r}$ which are incident with vertices of X which are adjacent to vertices of $R_1, \ldots, R_k$. Since S is an exact capacitated dominating set, $\sum_{i=1}^{k}(c(u_i) + 1) = n$, and from our description it is clear that the resulting set is an edge dominating set of size $2n + r$ for H.

We proceed by proving the other direction of the equivalence. Let L be an edge dominating set of H of cardinality at most $2n + r$. The set L is forced to contain at least one edge from every $E(v_i)$, at least $n - k$ edges from $F_{n-k,n}$, at least k edges from $F_{k,n}$, and at least one edge from $E(x)$ for all $x \in X$ because of pendent edges. This implies that $|L| = 2n + r$, and L contains exactly one edge from every $E(v_i)$, exactly $n - k$ edges from $F_{n-k,n}$, exactly k edges from $F_{k,n}$, and exactly one edge from $E(x)$ for all $x \in X$. Every edge $a_i b_i$ needs to be dominated by some edge of L; in particular it must be dominated from an edge of either $F_{n-k,n}$ or $F_{k,n}$. Let $I = \{i : a_i \text{ is incident to an edge from } L \cap E(F_{n-k,n})\}$ and $J = \{j : b_j \text{ is incident to an edge from } L \cap E(F_{k,n})\}$. The above constraints on the set L implies that $|I| = n - k$, $|J| = k$, and these sets form a partition of $\{1, \ldots, n\}$. The edges which join vertices b_i and R_i for $i \in I$ are not dominated by edges from $L \cap E(F_{k,n})$. Hence to dominate these edges we need at least $\sum_{i \in I} |R_i|$ edges which connect sets R_i and X. Since at least n edges of $F_{n,r}$ are included in L, we have that $\sum_{i \in I} |R_i| \le r - n$ and $\sum_{j \in J} |R_j| = r - \sum_{i \in I} |R_i| \ge r - (r - n) \ge n$. Let $S = \{u_j : j \in J\}$. Clearly, $|S| = k$. Now we show that S is a saturated capacitated dominating set. For $j \in J$, edges which join a vertex a_j to U_j and w_j are not dominated by edges from $L \cap E(F_{n-k,k})$, and hence they have to be dominated by edges from sets $E(v_i)$. Since $n \le \sum_{j\in J} |R_j| = \sum_{j\in J}(|U_j| + 1)$, there are exactly n such edges, and every such edge must be dominated by exactly one edge from L. An edge $a_j w_j$ can be dominated only by edge $v_j w_j$. We also know that $L \cap E(v_i) \ne \emptyset$ for all $i \in \{1, \ldots, n\}$, and hence for every v_i, $i \notin J$, there is exactly one edge which joins it with some vertex $u \in U_j$ for some $j \in J$. Furthermore, all these edges are not adjacent; that is, they form a matching. We define $f(u_i) = u_j$ for $i \notin J$. From our construction it follows that f is a domination mapping for S and S is an exact saturated dominating set in G. □

Lemma 7 shows that if the graph G we started with has clique-width at most k, then H has clique-width bounded by some function of k.

LEMMA 7. *If* $\mathbf{cwd}(G) \le t$, *then* $\mathbf{cwd}(H) \le 2t + 16$.

Proof. The graph G is of clique-width at most t. Suppose that the expression tree for G uses t labels $\{\alpha_1, \ldots, \alpha_t\}$. To construct the expression tree for H we need the following additional labels:

- Labels $\beta_1, \ldots, \beta_t$ for the vertices in $U_1, \ldots, U_n$.
- Labels ξ_1, ξ_2, and ξ_3 for attaching $F_{n-k,n}$, $F_{k,n}$, and $F_{n,r}$, respectively.
- Labels $\zeta_1, \ldots, \zeta_4$ for marking some vertices like $w_1, \ldots, w_n$.
- Auxiliary labels $\gamma_1, \ldots, \gamma_9$.

When a vertex $u_i \in V(G)$ labeled α_j is introduced, we perform the following set of operations. First we introduce the following vertices with some working labels: v_i with label γ_1, $c(u_i)$ vertices of U_i with label γ_2, the vertex w_i with label γ_3, and the additional vertex (the leaf attached to v_i) with label γ_4. Now we join the vertex labeled with γ_1 to vertices labeled with γ_3 and γ_4 (basically joining v_i with w_i and its pendent leaf). Finally, we relabel γ_4 to ζ_1 and γ_1 to β_j. Now we introduce vertices a_i and b_i with labels γ_5 and γ_6, respectively. Then we join the vertex labeled γ_4 (a_i) with all the vertices labeled with γ_2, γ_3, and γ_6 (U_i, w_i, b_i). The join operation is followed by relabeling γ_3 to ζ_2, γ_2, to α_j, and γ_5 with ξ_1.

Now we want to make the vertices of R_i and the paths attached to it. To do so we perform the following operations $c(u_i)+1$ times: (a) introduce three nodes labeled with γ_7, γ_8, and γ_9; (b) join γ_6 with γ_7, γ_7 with γ_8, and γ_8 with γ_9; and (c) finally we relabel γ_6 to ξ_2, γ_7 to ζ_3, γ_8 to ξ_3, and γ_9 to ζ_4. We omit the union operations from the description and assume that if some vertex is introduced, then this operation is performed.

If in the expression tree of G, we have join operation between two labels, say α_i and α_j, then we simulate this by applying join operations between α_i and β_j and α_j and β_i. The relabel operation in the expression tree of G, that is, relabel α_i to α_j, is replaced by relabel α_i to α_j and relabel β_i to β_j. Union operations in the expression tree are done as before.

Finally to complete the expression tree for H, we need to add $F_{n-k,n}$, $F_{k,n}$, and $F_{n,r}$. Notice that all the vertices in $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$, and X are labeled ξ_1, ξ_2, and ξ_3, respectively. From here we can easily add $F_{n-k,n}$, $F_{k,n}$, and $F_{n,r}$ with root vertices $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$, and X, respectively, by using working labels. This concludes the description for the expression tree for H. □

Lemmas 6 and 7 imply the following result.

THEOREM 5. *The* EDGE DOMINATING SET *problem is* $W[1]$*-hard when parameterized by clique-width.*

Proof. Lemmas 6 and 7 together give a parameterized reduction from EXACT SATURATED CAPACITATED DOMINATING SET parameterized by the clique-width t of the input graph to EDGE DOMINATING SET parameterized by the clique-width. Lemma 6 ensures the correctness of the reduction while Lemma 7 shows that if an input (G, c) of EXACT SATURATED CAPACITATED DOMINATING SET has clique-width at most t, then the input H constructed for the EDGE DOMINATING SET has clique-width at most $f(t) = 2t + 16$. By Lemma 4, EXACT SATURATED CAPACITATED DOMINATING SET parameterized by the clique-width t of the input graph is W[1]-hard and hence EDGE DOMINATING SET parameterized by the clique-width is W[1]-hard. ◻

5. Hamiltonian cycle problem. In this section we show that the HAMILTONIAN CYCLE, which is defined as

HAMILTONIAN CYCLE: Given a graph G, decide whether there exists a cycle passing through every vertex of G,

is $W[1]$-hard when parameterized by clique-width.

Our reduction is from the CAPACITATED DOMINATING SET problem described in section 4.1 and shown to be $W[1]$-hard in Theorem 4. We need auxiliary gadgets.

Auxiliary gadgets. We denote by L_1 the graph with the vertex set $\{x, y, z, a, b, c, d\}$ and the edge set $\{xa, ab, bc, cd, dy, bz, cz\}$. Let P_1 be the path $xabzcdy$ and $P_2 = xabcdy$. (See Figure 5.1.)

We abstract a property of this graph in the following lemma.

LEMMA 8. *Let G be a Hamiltonian graph such that $G[V']$ is isomorphic to L_1. If all edges in $E(G) \setminus E(G[V'])$ incident with V' are incident with the copies of the vertices x, y, and z in V', then every Hamiltonian cycle in G includes either the path P_1 or the path P_2 as a segment.*

Our second auxiliary gadget is the graph L_2. This graph has $\{x, y, z, s, t, a, b, c, d, e, f, g, h\}$ as its vertex set. We first include the following $\{xa, ab, bz, cz, cd, dy, se, ef, fb, ch, hg, gt\}$ in its edge set. Then x, y path $xw_1 \cdots w_9 y$ of length 10 is added, and edges $fw_3, w_1w_6, w_4w_9, w_7h$ are included in the set of edges. Let $P = xabzcdy$, $R_1 = sefbaxw_1w_2 \dots w_9ydchgt$, and $R_2 = sefw_3w_2w_1w_6w_5w_4w_9w_8w_7hgt$. (See Figure 5.1.) This graph has the following property.

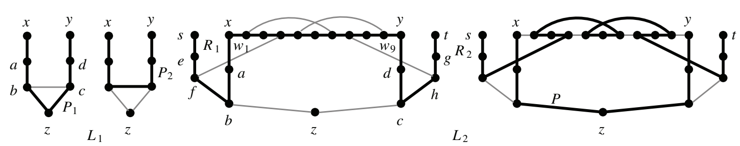

FIG. 5.1. *Graphs L_1 and L_2. Paths P_1, P_2, R_1, R_2, and P are shown by thick lines.*

LEMMA 9. *Let G be a Hamiltonian graph such that $G[V']$ is isomorphic to L_2. If all edges in $E(G) \setminus E(G[V'])$ incident with V' are incident with the copies of the vertices x, y, z, s, t in V', then every Hamiltonian cycle in G includes either the path R_1 or two paths P and R_2 as segments.*

Lemma 9 easily follows from the presence of degree 2 vertices in the graph L_2, since for any such vertex, it and adjacent vertices have to belong to one segment of a Hamiltonian path.

Reduction. Let (G, c) be a capacitated graph with the vertex set $\{v_1, \dots, v_n\}$ and m edges, and let k be a positive integer. For every vertex v_i, four vertices a_i, b_i, c_i, and w_i are introduced, and the vertices b_i and c_i are joined by $c(v_i)+1$ paths of length two. Let C_i denote the set of middle vertices of these paths, and $X_i = C_i \cup \{a_i, b_i, c_i\}$. Then a copy L_2^i of the graph L_2 with $z = w_i$ is added, and vertices x and y of this gadget are joined by edges to a_i and b_i, respectively. By s_i and t_i we denote the vertices s and t of L_2^i. For every ordered pair $\{v_i, v_j\}$ such that $v_iv_j \in E(G)$, a copy

L_2^{ij} of L_2 is attached with $z = w_j$ and vertices x and y made adjacent to all the vertices of C_i. The vertices corresponding to s and t are called s_{ij} and t_{ij} in L_2^{ij}. Furthermore, let x_{ij} and y_{ij} denote the vertices corresponding to x and y in L_2^{ij}. The paths corresponding to P in L_2^i is called P^i. Similarly, the paths corresponding to P, R_1, and R_2 are called P^{ij}, R_1^{ij}, and R_2^{ij}, respectively, in L_2^{ij}. Denote the obtained graph by $G'(c)$. (See Figure 5.2 for an illustration.)

In the next step we add two vertices g and h which are joined by $\sum_{i=1}^{n}(c(v_i) + 4) + n + 2m + 1$ paths of length two. Let Y be the set of middle vertices of these paths. All vertices s_i, t_i, s_{ij}, and t_{ij} are joined by edges with all vertices of Y. For every vertex r such that $r \in X_i$ (recall $X_i = C_i \cup \{a_i, b_i, c_i\}$), $i \in \{1, \ldots, n\}$, a copy L_1^r of L_1 with $z = r$ is attached and the vertices x and y of this gadget are joined to all vertices of Y. We let x_r and y_r denote the vertices corresponding to x and y in L_1^r. Similarly, P_1^r and P_2^r denote paths in L_1^r corresponding to P_1 and P_2, respectively.

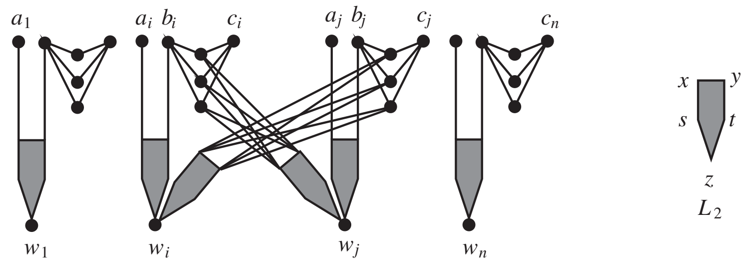

FIG. 5.2. *Graph* $G'(c)$.

Finally we add $k+1$ vertices, namely $\{p_1, \ldots, p_{k+1}\}$, and make them adjacent to all the vertices $\{a_i, c_i : 1 \leq i \leq n\}$ and to g and h. Let H be this resulting graph. The construction of H can easily be done in time polynomial in n and m.

LEMMA 10. *A graph* (G, c) *has a capacitated dominating set of size at most* k *if and only if* H *has a Hamiltonian cycle.*

Proof. Let S be a capacitated dominating set of size at most k in (G, c) with the corresponding dominating mapping f. Without loss of a generality we assume that $|S| = k$ and $S = \{v_1, \ldots, v_k\}$. The Hamiltonian cycle we are trying to construct is naturally divided into $k+1$ parts by the vertices $\{p_1, \ldots, p_{k+1}\}$. We construct the Hamiltonian cycle starting from the vertex p_1. Assume that the part of the cycle up to the vertex p_i is already constructed. We show how to construct the part from p_i to p_{i+1}. We include the edge $p_i a_i$ in it. We add to the cycle the path P^i and two edges, which join the endpoints of P_i with a_i and b_i. Let $J = \{j : f(v_j) = v_i\}$. If $J = \emptyset$, then a b_i, c_i path of length two which goes through one vertex of C_i is included in the cycle. Otherwise all paths P^{ij} for $j \in J$ are included in the cycle as follows. We consider the paths P^{ij} in the increasing order of indices in J and add them to the cycle. We take the first path, say $P^{ij'}$, and attach $x_{ij'}$ and $y_{ij'}$ to a pair of vertices $\{j_1, j_2\}$ in C_i. Suppose iteratively we have included first $l \geq 1$ paths in J, and the lth path is incident to some $\{j_l, j_{l+1}\}$ in C_i; now we attach the $(l+1)$th path by attaching x_{it} of this to j_{l+1} and y_{it} of this to j_{l+2}, where j_{l+2} is a new vertex not incident to any previously included paths. We can always find such a vertex as $|J| \leq c(v_i) = |C_i| - 1$. Now we include the edge $b_i j_1$ and $j_{|J|+1} c_i$. Finally we include the edge $c_i p_{i+1}$.

When the vertex p_{k+1} is reached, we move to the set Y. Note that at this stage all vertices $\{w_1, \ldots, w_n\}$ are already included in the cycle. We start by including the edge $p_{k+1}g$. We will add the following segments to the cycle and connect them appropriately.

- For every L_2^i we add the path R_1^i to the cycle if P^i was not included with it, and include the path R_2^i otherwise. The number of such paths is n.
- Similarly, for every L_2^{ij}, the path R_1^{ij} is added to the cycle if P^{ij} was not included with it, else the path R_2^{ij} is added. Note that $2m$ such paths are included with the cycle.
- For every vertex r such that $r \in X_i$ for some $i \in \{1, \ldots, n\}$, the path P_2^r is included in the cycle if r is already included in the constructed part of the cycle, else the path P_1^r is added. Clearly, we add $\sum_{i=1}^{n}(c(v_i)+4)$ paths.

Finally the total number of paths we will add is $\sum_{i=1}^{n}(c(v_i)+4)+n+2m = |Y|-1$. We add the segments of the paths mentioned with the help of vertices in Y, in the way we added the paths P^{ij} with the help of vertices in C_i. Let q_1 and q_2 be the endpoints of the resultant joined path. Notice that (a) $q_1, q_2 \in Y$ and (b) this path includes all the vertices of Y. Now we add edges gq_1, q_2h, and hp_1. This completes the construction of the Hamiltonian cycle.

For the reverse direction of the proof, we assume that we have been given C, a Hamiltonian cycle in H. Let $S = \{v_i \mid p_j a_i \in E(C), a_i p_s \notin E(C), j \neq s$, for some $j \in \{1, 2, \ldots, k+1\}\}$. We prove that S is a capacitated dominating set in G of cardinality at most k. We first argue about the size of S; clearly its size is at most $k+1$. To argue that it is at most k, it is enough to observe that by Lemmas 8 and 9 either $p_j g$ or $p_j h$ must be in $E(C)$ for some $j \in \{1, \ldots, k+1\}$. Now we show that S is indeed a capacitated dominating set. Our proof is based on the following observations.

- Every vertex w_j appears in either a vertex segment, that is P^j, or an edge segment, that is, P^{ij} for some $j \in \{1, \ldots, n\}$ in C.
- If some P^{ij} appear as a segment in C, then from the gadgets $L_1^{b_i}$ and $L_1^{c_i}$ the paths $P_2^{b_i}$ and $P_2^{c_i}$ are part of C. Hence the only way to include b_i in C is by using the edge incident to it from the gadget L_2^i. This implies that from the gadget L_2^i we use the path P^i and two edges, which join the endpoints of P_i with a_i and b_i.
- By Lemma 8 the cycle contains the edge which joins a_i to some vertex in $\{p_1, \ldots, p_{k+1}\}$.

Now given $v_j \in V(G) \setminus S$, for the domination function f, we assign $f(v_j) = v_i$ where P^{ij} is a segment in C. Clearly $v_i \in S$ as by above observation there exists a $j \in \{1, 2, \ldots, k+1\}$ such that $p_j a_i \in E(C)$, $a_i p_s \notin E(C)$, and $j \neq s$. For every $v_i \in S$, the set $f^{-1}(v_i)$ contains at most $c(v_i)$ vertices as $|C_i| = c(v_i)+1$. This concludes the proof. ∎

Lemma 11 provides an upper bound to the clique-width of the resulting graph H.

LEMMA 11. *If* $\mathbf{tw}(G) \leq t$, *then* $\mathbf{cwd}(H) \leq 9 \cdot 2^{\max\{2t,24\}} + 12$.

Proof. We define $c'(v_i) = 0$ for all $i \in \{1, 2, \ldots, n\}$ and consider the graph $G'(c')$. It is easy to see that $\mathbf{tw}(G'(c')) \leq \max\{2t+1, |V(L_2)|+3\} = \max\{2t+1, 25\}$. By Theorem 1 $\mathbf{cwd}(G'(c')) \leq 3 \cdot 2^{\max\{2t,24\}}$; i.e., we can construct the labeled graph $G'(c')$ by using at most $l = 3 \cdot 2^{\max\{2t,24\}}$ labels $\alpha_1, \ldots, \alpha_l$. Using $l+1$ additional labels $\beta_1, \ldots, \beta_l$ and γ_1, we can ensure that all vertices s_i, t_i, s_{ij}, and t_{ij} are labeled by the label γ_1, and only these vertices have label γ_1 in the following way. At the moment when such a vertex r labeled, e.g., j is introduced, we label it by the label β_j, and then these labels are used in the operations in the same way as labels α_j. Finally, all vertices labeled by these labels are relabeled γ_1. Similarly, by using $l+1$ more labels we assume that all vertices a_i and c_i are labeled by the label γ_2, and this label is used only for these vertices. Denote by d_i the only vertex in the set C_i in $G'(c')$.

The graph $G'(c)$ can be obtained from $G'(c')$ by the substitution of d_i by $c(v_i)+1$ vertices with the same neighborhoods. This operation does not change clique-width, and $\mathbf{cwd}(G'(c)) \le 3l+2$.

Recall that for every vertex $r \in X_i$ we add a copy of L_1 with $z = r$. We show how to construct the obtained graph using no more than $|V(L_1)|+1 = 8$ additional labels, in such a way that vertices x_r and y_r are labeled by the label γ_1. When a vertex r is introduced, we construct a copy of L_1 using $|V(L_1)|$ extra labels, making sure that the z in this copy gets r's label. Then we relabel x and y by γ_1, and the remaining $|V(L_1)|-3$ vertices are relabeled by an additional label ζ which acts as a "waste" label. We use two labels to construct the vertices g and h with $|Y|$ paths of length two between them. Additionally, we ensure that at the end of this construction g and h are labeled with γ_2 and that the vertices of Y are labeled by γ_3.

Now, the join operation is done for vertices labeled γ_1 and γ_3. Now, by using one more label γ_4, the vertices $p_1, p_2, \ldots, p_{k+1}$ are introduced, and the join operation is performed on the labels γ_2 and γ_4. We used no more than $3l+12$ labels to construct H, and $\mathbf{cwd}(H) \le 3l+12 \le 9 \cdot 2^{\max\{2t,24\}} + 12$. ◻

Lemmas 10 and 11 together imply the following result.

THEOREM 6. *The* HAMILTONIAN CYCLE *problem is* $W[1]$*-hard when parameterized by clique-width.*

Proof. Lemmas 10 and 11 together give a parameterized reduction from CAPACITATED DOMINATING SET parameterized by the tree-width t of the input graph and the solution size k to HAMILTONIAN CYCLE parameterized by the clique-width. Lemma 10 ensures the correctness of the reduction while Lemma 7 shows that if an input (G, c) of CAPACITATED DOMINATING SET has tree-width at most t then the input H constructed for the HAMILTONIAN CYCLE has clique-width at most $f(t) = 9 \cdot 2^{\max\{2t,24\}} + 12$. Now by Theorem 4, we know that CAPACITATED DOMINATING SET parameterized by the clique-width t of the input graph is W[1]-hard, and hence HAMILTONIAN CYCLE parameterized by the clique-width is W[1]-hard. ◻

6. Conclusions.

In this article, we settled the computational complexity of several important problems parameterized by the clique-width of the input graph. Our results show that the existing algorithms for EDGE DOMINATING SET, HAMILTONIAN CYCLE, and GRAPH COLORING on graphs of bounded clique-width essentially are the best one can hope for, unless an unlikely collapse in parameterized complexity occurs. It is an interesting open problem to investigate complexity of other graph problems when parameterized by the clique-width of the input graph.

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