Strengthening Erdős–Pósa Property for Minor-Closed Graph Classes

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Abstract: Let $\mathcal{H}$ and $\mathcal{G}$ be graph classes. We say that $\mathcal{H}$ has the Erdős–Pósa property for $\mathcal{G}$ if for any graph $G \in \mathcal{G}$, the minimum vertex covering of all $\mathcal{H}$-subgraphs of G is bounded by a function f of the maximum packing of $\mathcal{H}$-subgraphs in G (by $\mathcal{H}$-subgraph of G we mean any subgraph of G that belongs to $\mathcal{H}$). Robertson and Seymour [J Combin Theory Ser B 41 (1986), 92–114] proved that if $\mathcal{H}$ is the class of all graphs that can be contracted to a fixed planar graph H, then $\mathcal{H}$ has the Erdős–Pósa

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property for the class of all graphs with an exponential bounding function. In this note, we prove that this function becomes linear when $\mathcal{G}$ is any non-trivial minor-closed graph class. J Graph Theory 66: 235–240, 2011

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1. INTRODUCTION

Given a graph G we denote by $V(G)$ and $E(G)$ its vertex and edge set, respectively. A graph class $\mathcal{G}$ is called *non-trivial* if it does not contain all graphs. We use the notation $H \subseteq G$ to denote that H is a subgraph of G. We say that a graph G is a *$\mathcal{G}$-subgraph of a graph G'* if $G \subseteq G'$ and $G \in \mathcal{G}$.

Let $\mathcal{H}$ be a class of graphs. Given a graph G, we define the *covering number of G with respect to the class $\mathcal{H}$* as

$$\mathbf{cover}_{\mathcal{H}}(G) = \min\{k \mid \exists S \subseteq V(G) |S| \leq k \quad \text{and} \quad \forall_{H \in \mathcal{H}} H \nsubseteq G \setminus S\}.$$

In other words, $\mathbf{cover}_{\mathcal{H}}(G) \leq k$ if there is a set of at most k vertices meeting every $\mathcal{H}$-subgraph of G. The *packing number of G with respect to the class $\mathcal{H}$* is defined as

$$\mathbf{pack}_{\mathcal{H}}(G) = \max \{k \mid \exists \text{ a partition } V_1, \ldots, V_k \text{ of } V(G) \text{ such that } \forall_{i \in \{1,\ldots,k\}} \exists_{H \in \mathcal{H}}\ H \subseteq G[V_i]\}.$$

Less formally, $\mathbf{pack}_{\mathcal{H}}(G) \geq k$ if G contains k vertex-disjoint $\mathcal{H}$-subgraphs.

A graph class $\mathcal{H}$ satisfies the Erdős–Pósa property for some graph class $\mathcal{G}$ if there is a function f (depending only on $\mathcal{H}$ and $\mathcal{G}$) such that, for any graph $G \in \mathcal{G}$,

$$\mathbf{pack}_{\mathcal{H}}(G) \leq \mathbf{cover}_{\mathcal{H}}(G) \leq f(\mathbf{pack}_{\mathcal{H}}(G)) \tag{1}$$

In [6], Erdős and Pósa proved that (1) holds for all graphs when $\mathcal{H}$ is the class of all cycles. The problem of identifying more general graph classes where the Erdős–Pósa property is satisfied, has attracted a lot of attention, see [13, 3, 10, 9, 14, 1]. For further extensions and results of the same problem on matroids, see [8, 7].

We say that a graph G *can be contracted to H* if H can be obtained from G by a series of edge contractions (the *contraction* of an edge $e = (u,v)$ in G results in a graph G', in which u and v are replaced by a new vertex v_e and in which for every neighbor w of u or v in G, there is an edge (w, v_e) in G'). We say that *H is a minor of G* if some subgraph of G can be contracted to H. A graph class $\mathcal{G}$ is *minor-closed* if any minor of a graph in $\mathcal{G}$ is again a member of $\mathcal{G}$. We denote by $\mathcal{M}(H)$ the class of graphs that can be contracted to H. In [11, Proposition 8.2] Robertson and Seymour proved the following.

Proposition 1. *Let H be a connected graph. Then $\mathcal{M}(H)$ satisfies the Erdős–Pósa property for all graphs if and only if H is planar.*

Another proof of Proposition 1 can also be found in the monograph "Graph Theory" [5], by R. Diestel (Corollary 12.4.10 and Exercise 39). According to the proof of

Proposition 1, the bounding function $f(k)$ in (1) is exponential in k. However, by the original result of Erdős–Pósa [6], when H is isomorphic to K_3, $f(k) \leq 4k \log k$. In this note, we prove that the bound becomes linear for any planar H, when graphs are restricted to some non-trivial minor-closed class. Formally, we prove the following result.

Theorem 1. *For a connected planar graph H, let $\mathcal{H}$ be the class of graphs that are contractible to H. Let also $\mathcal{G}$ be a non-trivial minor-closed graph class. Then there is a constant $\sigma_{\mathcal{G},H}$ depending only on $\mathcal{G}$ and H such that for every graph $G \in \mathcal{G}$, it holds that*

$$\mathbf{pack}_{\mathcal{H}}(G) \leq \mathbf{cover}_{\mathcal{H}}(G) \leq \sigma_{\mathcal{G},H} \cdot \mathbf{pack}_{\mathcal{H}}(G).$$

2. THE PROOF

We start with auxiliary definitions and statements. If G is a graph and $x \in V(G)$ we use the notation $G-x$ for the graph $G[V(G)-\{x\}]$.

A *tree decomposition* of a graph G is a pair $(T, \mathcal{X} = \{X_t\}_{t \in V(T)})$ such that

- $\bigcup_{u \in V(T)} = V(G)$,
- for every edge $\{x,y\} \in E(G)$ there is a $t \in V(T)$ such that $\{x,y\} \subseteq X_t$, and
- for every vertex $v \in V(G)$ the subgraph of T induced by the set $\{t \mid v \in X_t\}$ is connected.

The *width* of a tree decomposition is $\max_{t \in V(T)} |X_t| - 1$ and the *treewidth* of G, $\mathbf{tw}(G)$, is the minimum width over all tree decompositions of G.

Our first observation is the following.

Lemma 1. *For a planar graph H, let $\mathcal{H}$ be the class of graphs that are contractible to H. Let also $\mathcal{G}$ be a non-trivial minor-closed graph class. Then, there is a constant $c_{\mathcal{G},H}$, depending only on $\mathcal{G}$ and H such that for any graph $G \in \mathcal{G}$, $\mathbf{tw}(G) \leq c_{\mathcal{G},H} \cdot (\mathbf{pack}_{\mathcal{H}}(G))^{1/2}$.*

Proof. Let $k = \mathbf{pack}_{\mathcal{H}}(G)$. In this proof, for any positive integer t, we denote the $(t \times t)$-grid by Γ_t. Let

$$c_H = \min\{r \mid H \text{ is a minor of } \Gamma_r\}$$

(for a short proof that constant c_H exists, see e.g. [12, Proposition (1.5)]). Notice that if $m = \lceil k^{1/2} \rceil + 1$, then $\mathbf{pack}_{\mathcal{H}}(\Gamma_{m \cdot c_H}) > k$. We conclude that G does not contain $\Gamma_{m \cdot c_H}$ as a minor. From [4, Theorem 1], there is a constant $c_{\mathcal{G}}$ depending only on $\mathcal{G}$ such that $\mathbf{tw}(G) \leq c_{\mathcal{G}} \cdot m \cdot c_H$ and the Lemma follows. ∎

For the proof of the next Lemma, we enhance the definition of a tree decomposition $(T, \mathcal{X})$ as follows: T is a tree rooted on some node r where $X_r = \emptyset$, each of its nodes have at most two children and could be one of the following

1. *Introduce node*: a node t that has only one child t' where $X_t \supset X_{t'}$ and $|X_t| = |X_{t'}| + 1$.
2. *Forget node*: a node t that has only one child t' where $X_t \subset X_{t'}$ and $|X_t| = |X_{t'}| - 1$.
3. *Join node*: a node t with two children t_1 and t_2 such that $X_t = X_{t_1} = X_{t_2}$.
4. *Base node*: a node t that is a leaf of t is different than the root, and $X_t = \emptyset$.

Notice that, according to the above definitions, the root r of T is either a forget or a join node. It is easy to see that any tree decomposition can be transformed to one with the above requirements while maintaining the same width (see e.g. [2]). From now on, when we refer to a tree decomposition $(T,\mathcal{X})$ we presume the above requirements.

Given a tree decomposition $(T,\mathcal{X})$ and some node t of T, we define as T_t the subtree of T rooted at t. Clearly, if r is the root of T, it holds that $T_r = T$. We also define $G_t = G[\bigcup_{s\in V(T_t)} X_s]$ and $G_t^- = G_t - X_t$.

Given a graph G, we call a triple (V_1, S, V_2) *d-separation triple* of G if $|S| \leq d$ and $\{V_1, S, V_2\}$ is a partition of $V(G)$ such that there is no edge in G between a vertex in V_1 and a vertex in V_2.

Lemma 2. *For a connected planar graph H, let $\mathcal{H}$ be the class of graphs that are contractible to H. Let also $\mathcal{G}$ be a non-trivial minor-closed graph class and let $G\in\mathcal{G}$ such that $1 \leq \mathbf{pack}_{\mathcal{H}}(G) = k$. Then there is an $c_{\mathcal{G},H}\cdot\sqrt{k}$-separation triple (V_1, X, V_2) of G, where $k/3 \leq \mathbf{pack}_{\mathcal{H}}(G[V_1]) \leq 2k/3$ and $\mathbf{pack}_{\mathcal{H}}(G[V_1]) + \mathbf{pack}_{\mathcal{H}}(G[V_2]) \leq k$ ($c_{\mathcal{G},H}$ is a constant depending on $\mathcal{G}$ and H).*

Proof. Let $(\mathcal{X}, T)$ be a tree decomposition of G of width at most $c_{\mathcal{G},H}\cdot\sqrt{k}$, as in Lemma 1. We set up a labeling $p: V(T) \to \mathbb{N}\cup\{0\}$ such that

$$p(t) = \mathbf{pack}_{\mathcal{H}}(G_t^-).$$

The following observations are direct consequences of the definitions.

Observation 1. *If $t\in V(T)$ is an introduce node with t' as a child, then $p(t') = p(t)$. This holds because then $G_{t'}^- = G_t^-$.*

Observation 2. *If $t\in V(T)$ is a forget node with t' as child, then $p(t) - p(t') \in \{0,1\}$. This holds because $G_{t'}^- = G_t^- - x$ for some vertex $x \in V(G_{t'}^-)$.*

Observation 3. *If $t\in V(T)$ is a join node with t_1 and t_2 as children, then $p(t_1) + p(t_2) = p(t)$. This holds because $G_{t_1}^-$ and $G_{t_2}^-$ are disjoint graphs, $G_t^- = G_{t_1}^- \cup G_{t_2}^-$ and any graph in $\mathcal{H}$ is connected (because of the connectivity of H).*

Observation 4. *If $t\in V(T)$ is a base node, then $p(t) = 0$. This holds because then G_t is the empty graph.*

Observation 5. *$p(r) = \mathbf{pack}_{\mathcal{H}}(G)$. This holds because, $X_r = \emptyset$ and thus $G_r^- = G_r = G$.*

Let $t\in V(T)$ be the node where $p(t) > 2k/3$ and for each child t' of t, $p(t') \leq 2k/3$. From the above observations, this node exists and is unique provided that $k > 0$. Moreover, t may be either a forget node or a join node (by Observation 1 and the definition of t).

We distinguish two cases:

Case 1. If t is a forget node, we set $V_1 = V(G_{t'}^-)$ and $V_2 = V(G) - (V_1 \cup X_{t'})$ and observe that $\mathbf{pack}_{\mathcal{H}}(G_i) \leq \lfloor 2k/3 \rfloor, i = 1,2$ (by Observation 2 and the definition of t). Also we set $X = X_{t'}$.

Case 2. If t is a join node with children t_1 and t_2, we have that $p(t_i) \leq 2k/3, i = 1,2$ (by Observation 3 and definition of t). However, as $p(t_1) + p(t_2) > 2k/3$, we also have that either $p(t_1)\geq k/3$ or $p(t_2)\geq k/3$. W.l.o.g. we assume that $p(t_1)\geq k/3$ and we set $V_1=V(G^{-}_{t_1})$, $V_2=V(G)-(V_1\cup X_{t_1})$ and $X=X_{t_1}$.

We set $k_i=\mathbf{pack}_{\mathcal{H}}(G[V_i]), i=1,2$. We conclude that in both cases, $k/3\leq k_1\leq 2k/3$ and $k_1+k_2\leq k$. Therefore (V_1,X,V_2) is the required $c_{\mathcal{G},H}\cdot\sqrt{k}$-separation triple. ■

Proof of Theorem 1. We only prove the right hand inequality as the left hand one is trivial. In fact, we prove that

$$\mathbf{cover}_{\mathcal{H}}(G)\leq\alpha\cdot c_{\mathcal{G},H}\cdot\mathbf{pack}_{\mathcal{H}}(G)-\beta\cdot c_{\mathcal{G},H}\sqrt{\mathbf{pack}_{\mathcal{H}}(G)} \tag{2}$$

for some constants α, β (where $\alpha-1\geq\beta>2$) that will be determined later.

Clearly, (2) holds trivially when $\mathbf{pack}_{\mathcal{H}}(G)=0$ and assume that it holds when $\mathbf{pack}_{\mathcal{H}}(G)<k$ for some $k\geq 1$. Let G be a graph such that $\mathbf{pack}_{\mathcal{H}}(G)=k\geq 1$. According to Lemma 2, G contains a $c_{\mathcal{G},H}\cdot\sqrt{k}$-separation triple (V_1,X,V_2), where $k/3\leq\mathbf{pack}_{\mathcal{H}}(G[V_1])\leq 2k/3$ and $\mathbf{pack}_{\mathcal{H}}(G[V_1])+\mathbf{pack}_{\mathcal{H}}(G[V_2])\leq k$. Notice that $\mathbf{cover}_{\mathcal{H}}(G)\leq\mathbf{cover}_{\mathcal{H}}(G[V_1])+\mathbf{cover}_{\mathcal{H}}(G[V_2])+|X|$. Using the induction hypothesis, we obtain that, for some $\delta\in[1/3,2/3]$,

$$\begin{aligned}\mathbf{cover}_{\mathcal{H}}(G[V_i]) &\leq \alpha\cdot c_{\mathcal{G},H}\cdot\delta\cdot k-\beta\cdot c_{\mathcal{G},H}\cdot\sqrt{\delta\cdot k}\\ &\quad+\alpha\cdot c_{\mathcal{G},H}\cdot(1-\delta)\cdot k-\beta\cdot c_{\mathcal{G},H}\cdot\sqrt{(1-\delta)\cdot k}+c_{\mathcal{G},H}\cdot\sqrt{k}\end{aligned}$$

which is upper bounded by $\alpha\cdot c_{\mathcal{G},H}\cdot k-\beta\cdot c_{\mathcal{G},H}\cdot\sqrt{k}$, if we choose $\alpha=3.54$ and $\beta=2.54$. Therefore, Theorem holds for $\sigma_{\mathcal{G},H}=3.54\cdot c_{\mathcal{G},H}$. ■

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