On the Complexity of Reconstructing *H*-Free Graphs from Their Star Systems

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Abstract: In the Star System problem we are given a set system and asked whether it is realizable by the multi-set of closed neighborhoods of some graph, i.e. given subsets $S_1, S_2, \ldots, S_n$ of an n-element set V does there exist a graph $G=(V,E)$ with $\{N[v]: v \in V\} = \{S_1, S_2, \ldots, S_n\}$? For a fixed graph H the H-free Star System problem is a variant of the Star System

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problem where it is asked whether a given set system is realizable by closed neighborhoods of a graph containing no *H* as an induced subgraph. We study the computational complexity of the *H*-free Star System problem. We prove that when *H* is a path or a cycle on at most four vertices the problem is polynomial time solvable. In complement to this result, we show that if *H* belongs to a certain large class of graphs the *H*-free Star System problem is NP-complete. In particular, the problem is NP-complete when *H* is either a cycle or a path on at least five vertices. This yields a complete dichotomy for paths and cycles. J Graph Theory 68: 113–124, 2011

1. INTRODUCTION

The closed neighborhood of a vertex in a graph is sometimes called the "star" of the vertex. The "star system" of a graph is then the multi-set of closed neighborhoods of all the vertices of the graph and the Star System problem is the problem of deciding whether a given system of sets is a star system of some graph. The Star System problem is a natural combinatorial problem that fits into a broader class of *realizability* problems. In a realizability problem we are given a list *P* of invariants or properties (like a sequence of vertex degrees, set of cliques, number of colorings, etc.) and the question is whether the given list is *graphical*, i.e. corresponds to the list of parameters of some graph. One of the well-studied problems of realizability is the case when *P* is a degree sequence. This can be seen as a modification of the Star System problem where, instead of stars, the list *P* contains only the sizes of the stars. In this case, graphic sequences can be characterized by the Erdős–Gallai Theorem [7].

The Star System problem (also known as the Closed Neighborhood Realization problem) is related to a number of other interesting problems. For example, it is equivalent to the question of whether a given $0-1$ matrix *A* is symmetrizable, i.e. whether by permuting rows (or columns) *A* can be turned into a symmetric matrix with all diagonal entries equal to 1. We refer to the recent survey of Boros et al. [4] for this and further problems related to the Matrix Symmetrization and Star System problems.

The question of the computational complexity of the Star System problem was first posed by Gert Sabidussi and Vera Sós at a conference in the mid-70s [9] (and since this appears to be the oldest reference to the problem, we choose to use the Star System terminology). At the same conference Babai observed that the Star System problem was at least as hard as the Graph Isomorphism problem. There are strong similarities with Graph Isomorphism, e.g. as will explained later, the Star System problem is equivalent to deciding if a given bipartite graph allows an automorphism of order 2 such that each vertex is adjacent to its image. In view of these connections to Graph Isomorphism, the NP-hardness of the Star System problem came somewhat unexpected. The proof of this fact was achieved in two steps. First, a related effort of Lubiw [12] showed that deciding whether an arbitrary graph has an automorphism of order 2 is NP-complete. Then Lalonde [11] showed that the Star System problem was NP-complete by a reduction from Lubiw's problem. This reduction came as a small surprise considering that, after Lubiw's proof, Babai had written that between Lubiw's problem and the Star System problem he "did not believe there was a deeper relationship" [3].

The result of Lalonde was rediscovered by Aigner and Triesch [1, 2], who proved it in a stronger form, and indeed discovered a subproblem which is equivalent to Graph Isomorphism. It is easy to see that the problems of reconstructing graphs from their closed neighborhood hypergraphs and from their open neighborhood hypergraphs are polynomially equivalent. It is more convenient however, to describe the results of Aigner and Triesch in the language of open neighborhoods. They proved that deciding if a set system is the open neighborhood hypergraph of a *bipartite* graph is Graph Isomorphism-complete, while deciding if the open neighborhood hypergraph of a bipartite graph can be realized by a nonisomorphic (and non-bipartite) graph becomes again NP-complete.

Since bipartite graphs (and their complements) are hereditary classes of graphs, it is natural to pay closer attention to restrictions of the Star System problem to classes of graphs defined by forbidden-induced subgraphs. The problem we investigate in this article is the following variation of the Star System problem, for a fixed graph H:

H-free Star System Problem
Input: A set system $\mathcal{S}$ over a ground set V
Question: Does there exist an H-free graph $G=(V,E)$ such that $\mathcal{S}$ is the star system of G?

Our main result is a complete dichotomy in the case when H is either a cycle C_k, or a path P_k on k vertices. We prove that the H-free Star System problem for $H\in\{C_k,P_k\}$ is polynomial time solvable when $k\leq 4$ (Section 3) and NP-complete when $k>4$ (Section 4). Our NP-completeness result for paths and cycles follows from a more general result, which shows that there exists a much larger family of graphs H such that the H-free Star System problem is NP-complete.

2. PRELIMINARIES

We use standard graph notation with $G=(V,E)$ being a simple loopless undirected graph with vertex set V and edge set E. We denote by $N[v]$ and $N(v)$ the closed and open neighborhoods of a vertex v, respectively, and by $\overline{G}$ the complement of a graph G having an edge uv iff $u\neq v$ and $uv\notin E(G)$. We also call $N[v]$ the *star of* v and say that v is the center of $N[v]$. An *automorphism* of a graph $G=(V,E)$ is an isomorphism $f:V\to V$ of the graph to itself, and it has order 2 if for every vertex x we have $f(f(x))=x$, i.e. the image of its image is itself. For a graph $G=(V,E)$, we define the $|V|$-element multiset $Stars(G)=\{N[v]:v\in V\}$. (Let us note that in this article set brackets always indicate multisets.) Similarly, we define the multiset $OpenStars(G)=\{N(v):v\in V\}$. For a fixed graph H we say that a graph G is *H-free* if G does not contain an induced subgraph isomorphic to H.

3. FORBIDDING SHORT PATHS AND CYCLES

A. Forbidding Short Paths

In this section we show that the P_k-free Star System Problem is solvable in polynomial time for $k\leq 4$ (as usual, P_k denotes the path with four vertices). For $k\leq 2$ the P_k-free Star System Problem is trivially polynomial time solvable. (For $k=1$ the realizing graph has no vertices, and for $k=2$ it has no edges.) For $k=3$ the realizing graph is a disjoint union of cliques, and in this case the problem is again trivial—for every star $S \in \mathcal{S}$, the star system $\mathcal{S}$ should contain exactly $|S|$ copies of S.

The proof that the P_4-free Star System can be solved in polynomial time occupies the remaining part of this subsection.

The graphs without induced P_4 are called *cographs*. We exploit the following two results on cographs.

Proposition 1 (Corneil et al. [6]). *A graph G is a cograph if and only if every non-trivial-induced subgraph of G either is disconnected or is the complement of a disconnected graph.*

Proposition 2 (Corneil et al. [6]). *A graph G is a cograph if and only if $\overline{G}$ is a cograph.*

Let us remark that a graph is a cograph if and only if each of its connected components is a cograph.

Given a set system $\mathcal{S}$ over a ground set V, we define the graph

$$G_{\mathcal{S}} = (V, \{uv | \exists S \in \mathcal{S} : u \in S, v \in S\}).$$

The vertex sets $V_1, \ldots, V_k$ of the connected components of $G_{\mathcal{S}}$ form a partition of V. We will call each such V_i a *component* of $\mathcal{S}$. Furthermore, for each non-empty set $S \in \mathcal{S}$, we have that there is unique V_i with $S \subseteq V_i$. We say that S *corresponds* to V_i. For each $i \in \{1, \ldots, k\}$, we define the set system $\mathcal{S}_i$ to be the set system over V_i of all sets in $\mathcal{S}$ corresponding to V_i.

Observation 3. *There is a cograph G* with *Stars(G)* $=\mathcal{S}$ *if and only if for each* $i \in \{1, \ldots, k\}$ *there is a cograph* G_i *with Stars*$(G_i) = \mathcal{S}_i$.

If $\mathcal{S}$ has more than one component, by Observation 6 the problem breaks down into independent subproblems. When $\mathcal{S}$ has exactly one component, we define the set system $\overline{\mathcal{S}} = \{V \setminus S : S \in \mathcal{S}\}$. Note that the sets corresponding to isolated vertices are empty, but are listed in the OpenStar system. Notice also that since $\mathcal{S}$ can have multiple occurrences of the same set S, so can $\overline{\mathcal{S}}$. Now, for any graph $G=(V,E)$ and set system $\mathcal{S}$ over V we have that *Stars*$(G) = \mathcal{S}$ if and only if *OpenStars*$(\overline{G}) = \overline{\mathcal{S}}$. In particular, by Proposition 2, we have the following Lemma.

Lemma 4. *Given a set system $\mathcal{S}$ over V, there exists a cograph $G=(V,E)$ such that Stars$(G) = \mathcal{S}$, if and only if there is a cograph G' such that OpenStars$(G') = \overline{\mathcal{S}}$.*

This means that given a set system $\mathcal{S}$, our problem is equivalent to checking whether there exists a cograph whose *open* star system realizes $\overline{\mathcal{S}}$. The next lemmata will give us all tools needed to solve this equivalent problem.

Lemma 5. *Let $V_1, \ldots, V_k$ be the components of $\mathcal{S}$ and suppose that OpenStars$(G) = \mathcal{S}$. Then for each $i \in \{1, \ldots, k\}$,*

- *either $G[V_i]$ is a connected component of G and $G[V_i]$ is not bipartite*;
- *or $G[V_i]$ is an independent set and there is $j \in \{1, \ldots, k\}$ such that $G[V_i \cup V_j]$ is a connected component of G and $G[V_i \cup V_j]$ is bipartite.*

Proof. We claim that if $OpenStars(G)=\mathcal{S}$, then two elements u and v of V are in the same component of $\mathcal{S}$ if and only if there is an even length walk between them in G. Indeed, suppose that there is an even length walk $u,w_1,w_2,w_3,\ldots,w_{2k+1},v$ in G. Then $N(w_1)$ contains both u and w_2, $N(w_3)$ contains both w_2 and w_4 , etc., and $N(w_{2k+1})$, contains w_{2k} and v. Hence u and v are in the same connected component of $G_{\mathcal{S}}$ and thus the same component of $\mathcal{S}$. Now, suppose that u and v are in the same component of $\mathcal{S}$. Then there is a path $P=u,w_2,w_4,w_6,\ldots,w_{2k},v$ from u to v in $G_{\mathcal{S}}$. This means that there is a sequence of vertices $P'=u,w_1,w_2,w_3,w_4,w_5,w_6,\ldots,w_{2k},w_{2k+1},v$ such that $N(w_1)$ contains both u and w_2, $N(w_3)$ contains both w_2 and w_4, etc., and $N(w_{2k+1})$, contains w_{2k} and v. But then P' is an even length walk from u to v in G.

We prove that there is an even length walk between two vertices u and v in G if and only if they either appear in the same connected non-bipartite component of G or if they appear in the same bipartition class of a bipartite connected component of G. If there is an even length walk between two vertices u and v in G, then u and v must appear in the same connected component of G. Furthermore, if this component is bipartite, u and v must appear in the same bipartition class. In the other direction, if u and v appear in the same bipartition class of a bipartite component of G, any path between u and v forms an even walk. If u and v appear in a non-bipartite component of G, this component must contain an odd length cycle C. Let W_1 be a walk from u to v that passes through a vertex x in C. Construct the walk W_2 from W_1 by walking around C upon the first visit to x. Since C is an odd cycle at least one of W_1 and W_2 must be an even walk from u to v. This concludes the proof. ■

The following observation is folklore.

Observation 6. *Every connected bipartite cograph is a complete bipartite graph.*

Proof. Let $G=(V_1\cup V_2,E)$ be a bipartite connected cograph. Then G is complete because otherwise its complement $\overline{G}$ is connected, which is a contradiction to Proposition 1. ■

We say that a component V_i of $\mathcal{S}$ is *normal* if every set in $\mathcal{S}_i$ is a proper subset of V_i and $|\mathcal{S}_i|=|V_i|$. For positive integers a and b we say that a component V_i is an (a,b)-component if $|V_i|=a$, $|\mathcal{S}_i|=b$ and for every set $S\in\mathcal{S}_i$ we have that $S=V_i$.

Lemma 7. *There is a cograph G with* OpenStars$(G)=\mathcal{S}$ *if and only if all of the following conditions are satisfied*:

- *Every component of $\mathcal{S}$ is either a normal component or an (a,b)-component for some positive integers a, b.*
- *For every normal component V_i of $\mathcal{S}$ there is a cograph G_i with $OpenStars(G_i)=\mathcal{S}_i$.*
- *For every pair of integers a, b with $a\neq b$ and $b\neq 0$, the number of (a,b)-components is equal to the number of (b,a)-components.*
- *For every integer a the number of (a,a)-components is even.*
- *The number of empty sets in $\mathcal{S}$ is equal to the number of isolated vertices of $G_{\mathcal{S}}$ not contained in any set in $\mathcal{S}$.*

Proof. Suppose that there is a cograph G with $OpenStars(G)=\mathcal{S}$. By Lemma 5, for each i, either $G[V_i]$ is a non-bipartite connected component of G or there is j such that V_i and V_j are bipartition classes of a bipartite connected component $G[V_i\cup V_j]$

of G. Thus, for every component $G[C]$ of G, either $C = V_i$ for some i or $C = V_i \cup V_j$ for some i, j.

Let us consider first the case when $C = V_i$, and thus $G[C]$ is not bipartite. Then for every set S in $\mathcal{S}_i$ there is $v \in C = V_i$ such that $S = N(v)$. Since $N(v) \subset V_i$ for every $v \in V_i$, it follows that V_i is a normal component of $\mathcal{S}$ and that $OpenStars(G[C]) = \mathcal{S}_i$.

Similarly, when $G[C]$ is bipartite, then $C = V_i \cup V_j$ for some i,j. Furthermore, by Observation 6, $G[C]$ is a complete bipartite graph. Thus, for every $u \in V_i$, $N(u) = V_j$ whereas for every $v \in V_j$, $N(v) = V_i$. Hence V_i is a $(|V_i|, |V_j|)$-component and V_j is a $(|V_j|, |V_i|)$-component. Since every vertex of G is in some connected component of G, we have that every component of $\mathcal{S}$ is either a normal component or an (a,b)-component.

Finally, every bipartite component of G with bipartition (V_i, V_j) forms exactly two components of $\mathcal{S}$: $(|V_j|, |V_i|)$-component and $(|V_i|, |V_j|)$-component. Thus for every pair of integers a, b with $a \neq b$ the number of (a,b)-components is equal to the number of (b,a)-components, and for every integer a the number of (a,a)-components is even. Every isolated vertex v of G is also isolated in $G_{\mathcal{S}}$. Furthermore, $N(v) = \emptyset$ and for every non-isolated vertex u, we have that $N(u) \neq \emptyset$.

In the other direction, suppose that every component of $\mathcal{S}$ is either a normal component, or an (a,b)-component, for every normal component V_i of $\mathcal{S}$ there is a cograph G_i with $OpenStars(G_i) = \mathcal{S}_i$, the number of (a,b)-components is equal to the number of (b,a)-components, the number of (a,a)-components is even, and the number of empty sets in $\mathcal{S}$ is equal to the number of isolated vertices of $G_{\mathcal{S}}$ not contained in any set in $\mathcal{S}$.

We construct the graph G as follows. For every normal component V_i, G has G_i as a connected component. Then we match every (a,b)-component V_i with some (b,a)-component V_j. Since the number of (a,b)-components is equal to the number of (b,a)-components and the number of (a,a)-components is even, we can find such a matching. For each pair i,j, the graph $G[V_i \cup V_j]$ is a complete bipartite graph. For every empty set in $\mathcal{S}$, we create an isolated vertex of G.

Finally, it follows from the construction that $OpenStars(G) = \mathcal{S}$. ■

We now have the required toolkit to give an algorithm for the P_4-free Star System problem.

Theorem 8. *There is an $O(n^4)$ algorithm for the P_4-free Star System problem.*

Proof. If $\mathcal{S}$ is a system with a single set on a single element we can safely answer "Yes". Now, if $\mathcal{S}$ has more than one component, we apply Observation 6 to run the algorithm recursively on each $\mathcal{S}_i$ separately. If $\mathcal{S}$ has one component, we count the number of components of $\overline{\mathcal{S}}$. If $\overline{\mathcal{S}}$ has one component, the algorithm answers "No". This step is correct because if there is a cograph G with $Stars(G) = \mathcal{S}$, and equivalently $OpenStars(\overline{G}) = \overline{\mathcal{S}}$, then by Observation 6, G is connected and by Lemma 5, we conclude that $\overline{G}$ is connected, contradicting Proposition 1. If on the other hand, $\overline{\mathcal{S}}$ has more than one component, we apply Lemma 4 and check whether there is a cograph G' such that $OpenStars(G') = \overline{\mathcal{S}}$ by using Lemma 7. To do this we need to verify that for every normal component V_i of $\overline{\mathcal{S}}$ there is a cograph G'_i such that $OpenStars(G'_i) = \overline{\mathcal{S}}_i$. This can be done by running the algorithm StarSystem recursively on $\mathcal{S}_i = \overline{\overline{\mathcal{S}}}_i$. The correctness follows immediately from Lemmas 3 and 7 together with Proposition 1.

We show that this algorithm terminates in $O(n^4)$ time. Building $G_{\mathcal{S}}$ and $G_{\overline{\mathcal{S}}}$ and finding the components of $\mathcal{S}$ and $\overline{\mathcal{S}}$ can be done in $O(n^3)$ time. Except for the recursive steps, checking the conditions of Lemma 7 can be done in $O(n^3)$ time as well. The number of nodes in the recursion tree is $O(n)$. Thus, the total amount of work is bounded by $O(n^4)$. ■

B. Forbidding C_3 and C_4

In this subsection we show that the C_3-free and C_4-free Star System Problems are solvable in polynomial time.

Theorem 9. *The C_3-free Star System problem is solvable in $O(n^3)$ time.*

Proof. Let $\mathcal{S}$ be a set system on a ground set V. The crucial observation is that if $\mathcal{S}$ is a star system of a C_3-free graph $G=(V,E)$, then for every edge $uv \in E$ there are exactly two sets containing u and v. In fact, since $uv \in E$, we have that u and v should be in at least two stars, one of which is centered in u and one centered in v. Let S_u and S_v be these stars. If there is a third star S containing u and v, then the center of this star, $x \neq u,v$ is adjacent to u and v, and thus xuv forms a C_3 in G, which is a contradiction.

Let us assume that the system $\mathcal{S}$ is connected, i.e. for every two elements u and v there is a sequence of elements $u=u_1,u_2,\ldots,u_k=v$ such that for every $i \in \{1,\ldots,k-1\}$ there is a set $S \in \mathcal{S}$ containing u_i and u_{i+1}. (If $\mathcal{S}$ is not connected, then we apply our arguments for each connected component of $\mathcal{S}$.)

Assume that we have correctly guessed the star $S_v \in \mathcal{S}$ of a vertex v in some C_3-free graph G with $Stars(G)=\mathcal{S}$. Then each $x \in S_v$, $x \neq v$, is adjacent to v in G. Thus there is a unique star $S_x \neq S_v$ containing both v and x, and vertex x should be the center of S_x. Now every vertex y from S_x should have a unique star containing x and y, and so on. Since $\mathcal{S}$ is connected, we thus have that after guessing the star for the first vertex v we can uniquely assign stars to the remaining vertices. There are at most n guesses to be made for the first vertex and we can in $O(n^2)$ time check the correctness of the guess, i.e. check if the star system of the constructed graph corresponds to $\mathcal{S}$. This proves the theorem. ■

Theorem 10. *The C_4-free Star System Problem is solvable in $O(n^4)$ time.*

Proof. The proof is based on the following observation. Let $G=(V,E)$ be a C_4-free graph and let $x,y \in V$. Let $S_1,S_2,\ldots,S_t$ be the set of stars of G containing both x and y. If $xy \in E$, then

$$2 \leq \left|\bigcap_{i=1}^{t} S_i\right| \leq t. \tag{1}$$

Indeed, when $xy \in E$, x and y have $t-2$ common neighbors. Every vertex $v \in \bigcap_{i=1}^{t} S_i \setminus \{x,y\}$ is adjacent to x and y, thus v is the center of the star S_i for some $i \in \{1,\ldots,t\}$ and (1) follows.

If $xy \notin E$, then

$$t=0 \quad \text{or} \quad \left|\bigcap_{i=1}^{t} S_i\right| \geq t+2. \tag{2}$$

In fact, if $xy \notin E$ and $t>0$, then x and y have t neighbors in common. Moreover, because G is C_4-free, these neighbors form a clique in G. Thus $\bigcap_{i=1}^{t} S_i$ contains all these t vertices plus the vertices x and y which yields (2).

Given a set system $\mathcal{S}$ on ground set V, the algorithm checking if $\mathcal{S}$ is a star system of some C_4-free graph works as follows. We already have shown that if $\mathcal{S}$ is a star system of some C_4-free graph G, then every pair of adjacent vertices x,y of G should satisfy (1), and for every pair of non-adjacent vertices x,y, (2) holds. Thus if $\mathcal{S}$ is a star system of some C_4-free graph, it should also be the star system of the C_4-free graph $G=(V,E)$ constructed by taking $xy\in E$ when the sets of $\mathcal{S}$ containing both x and y satisfy (1) and by taking $xy\notin E$ when the sets of $\mathcal{S}$ containing both x and y satisfy (2). Thus if we fail to construct such a graph G, we conclude that $\mathcal{S}$ is not a star system of a C_4-free graph. For constructed graph G, we check if $Stars(G)=\mathcal{S}$ and that G is C_4-free. If this is the case, then the answer is "Yes". Otherwise the answer is "No".

The running time of the algorithm is proportional to the time required to construct graph G by making use of (1) and to check if $Stars(G)=\mathcal{S}$ and that G is C_4-free. All these operations can be done in time $O(n^4)$. ■

4. FORBIDDING LONG PATHS AND CYCLES

In this section we show that there exists an infinite family of graphs H for which the H-free Star System problem is NP-complete. In particular both P_k and C_k, with $k>4$, belong to it.

Definition 11. For an arbitrary graph H, we define $B(H)$ to be its bipartite neighborhood graph, i.e. the bipartite graph with both bipartition classes having $|V(H)|$ vertices labelled by $V(H)$ and having an edge between a vertex labelled u in one bipartition class and a vertex labelled v in the other one if and only if $uv\in E(H)$.

For example, for the cycle on 5 vertices C_5, we have $\overline{C_5}=C_5$ and $B(\overline{C_5})=C_{10}$ (see Fig. 1). Our main NP-completeness result is that the H-free Star System problem is NP-complete whenever $B(\overline{H})$ has a cycle or two vertices of degree larger than two in the same connected component. For a bipartite graph $G=(V,E)$ with bipartition classes V_1,V_2 we say that an automorphism $f:V\to V$ is *side-switching* if $f(V_1)=V_2$ and $f(V_2)=V_1$. Consider the following two problems.

AUT-BIP-2SS
Input: A bipartite graph G
Question: Does G have an automorphism of order 2 that is side-switching?

AUT-BIP-2SS-NA
Input: A bipartite graph G
Question: Does G have an automorphism of order 2 that is side-switching and such that every vertex is non-adjacent to its image?

Lalonde [11] has shown that the AUT-BIP-2SS problem is NP-complete. Together with Sabidussi he also reduced AUT-BIP-2SS to AUT-BIP-2SS-NA. The proof of our main NP-completeness result is a (nontrivial) refinement of the reduction of Lalonde-Sabidussi, which will ensure that AUT-BIP-2SS-NA remains NP-complete for various restricted classes of bipartite graphs.

To relate NP completeness of AUT-BIP-2SS-NA to the Star System Problem, we use the following lemma.

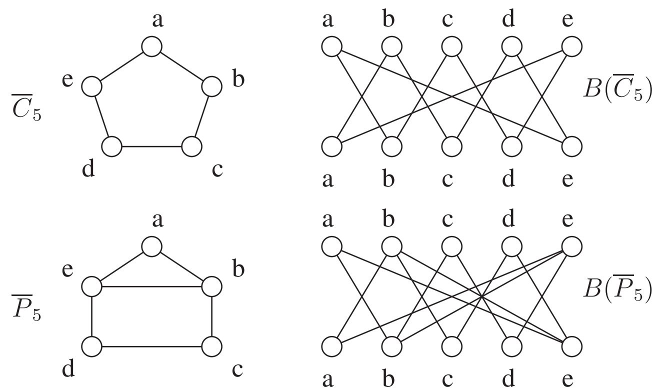

FIGURE 1. Example of bipartite neighborhood graphs for the complement of a C_5, that remains a C_5, and the complement of a P_5, that is equivalent to a *house*.

Lemma 12. *If AUT-BIP-2SS-NA is NP-complete on bipartite $B(\overline{H})$-free graphs, then the H-free Star System Problem is NP-complete.*

Proof. We reduce the first problem, which takes as input a bipartite $B(\overline{H})$-free graph F, to the second one, which takes as input a set system $\mathcal{S}$. We may assume that the two partition classes of F are of equal size, since otherwise an automorphism switching the two sides cannot exist. Let the vertices of one partition class of F be $V_1=\{v_1,v_2,\ldots,v_n\}$ and of the other one $V_2=\{w_1,w_2,\ldots,w_n\}$. We construct a set system $\mathcal{S}=\{S_1,S_2,\ldots,S_n\}$ over V_2 where $S_i=\{w_j : v_iw_j \notin E(F)\}$, i.e. the non-neighbors of v_i in V_2.

As already noted by Babai [3], it is not hard to see that F is a Yes-instance of AUT-BIP-2SS-NA iff there exists a graph G with $Stars(G)=\mathcal{S}$. Let us give the argument.

First, assume a graph G on the vertex set V_2 satisfies $Stars(G)=\mathcal{S}$. Let g: $\{1,2,\ldots,n\}\to\{1,2,\ldots,n\}$ be a mapping such that $w_{g(i)}$ is the center of the star S_i, i.e. $S_i=N_G[w_{g(i)}]$. Clearly, g is a permutation of $\{1,2,\ldots,n\}$. We define a mapping $f:V(F)\to V(F)$ as follows

$$f(v_i)=w_{g(i)} \quad \text{for } v_i\in V_1,$$

$$f(w_j)=v_{g^{-1}(j)} \quad \text{for } w_j\in V_2.$$

It is obvious that f is a side-switching bijection of order 2. Since $w_{g(i)}\in S_i$ for every i, each vertex is mapped onto a non-adjacent one. It remains to show that f is an automorphism of F. To see this we only need to check pairs v_iw_j since F is bipartite. For such a pair we see that $v_iw_j\in E(F)$ iff $w_j\notin S_i$ iff $w_j\notin N_G[w_{g(i)}]$ iff $w_jw_{g(i)}\notin E(G)$ iff $w_{g(i)}\notin N_G[w_j]$ iff $w_{g(i)}\notin S_{g^{-1}(j)}$ iff $w_{g(i)}v_{g^{-1}(j)}\in E(F)$ iff $f(v_i)f(w_j)\in E(F)$.

Secondly, assume that $f:V_1\cup V_2\to V_1\cup V_2$ is a side-switching automorphism of F of order 2 such that $xf(x)\notin E(F)$ for all $x\in V(F)$. Construct the graph G on the vertex set V_2 by making $w_{f(i)}$ adjacent to all vertices of S_i (except itself), for $i=1,2,\ldots,n$. Since f maps every vertex on a nonadjacent one, we have $w_{f(i)}\in S_i$ and hence $S_i\subseteq N_G[w_{f(i)}]$. We could only have $S_i\neq N_G[w_{f(i)}]$ if there is a $w_{f(j)}$ such that $w_{f(i)}\in S_j$ and $w_{f(j)}\notin S_i$.

But since f is an order 2 automorphism, $w_{f(i)}v_j \notin E(F)$ (which is equivalent to $w_{f(i)} \in S_j$) implies $v_i w_{f(j)} \notin E(F)$ and indeed $w_{f(j)} \in S_i$, a contradiction. Thus $S_i = N_G[w_{f(i)}]$ for every $i = 1,2,\ldots,n$ and $\mathcal{S} = Stars(G)$.

It remains to show that if F is $B(\overline{H})$-free and $Stars(G) = \mathcal{S}$, then G must be H-free. Suppose for the contrary that $G[\{w_j : j \in I\}] \cong H$ for some $I \subseteq \{1,2,\ldots,n\}$. Let w_j be the center of $S_{f(j)}$. Then $v_{f(i)}w_i \notin E(F)$ for all $i \in I$ and $v_{f(i)}w_j \in E(F)$ iff $w_j \notin S_{f(i)}$ iff $w_j w_i \notin E(G)$ for all $i,j \in I$. Thus $F[\{w_i, v_{f(i)} : i \in I\}] \cong B(\overline{H})$, a contradiction. ■

Definition 13. Let D_p be the class of bipartite graphs of girth larger than p such that the distance of any two vertices of degree greater than two is at least p.

Theorem 14. *For any integer p the problem AUT-BIP-2SS-NA is NP-complete even when restricted to graphs in D_p.*

Proof. We reduce from the NP-complete AUT-BIP-2SS problem and adapt the construction given by Lalonde and Sabidussi [11] for our purposes.

Given a bipartite graph $G = (V,E)$ with bipartition classes A and B we describe how to construct $F \in D_p$ with the property that G is a yes-instance of AUT-BIP-2SS if and only if F is a yes-instance of AUT-BIP-2SS-NA. Note first that we can assume G has no vertex v of degree 1 since if we remove each such v (simultaneously) and add a cycle of length $2k$, where k is greater than the maximum cycle length in G, attached to the unique neighbor of v, then G has a side-switching automorphism of order 2 if and only if the new graph has one.

Let p' be the smallest even integer at least as large as p. Let F be the graph obtained by replacing each edge of G by two paths of length $p'+1$. Note that the inner vertices of these paths are then the only vertices of degree 2 in F. Moreover, we have $F \in D_p$ and the two bipartition classes of F respect A and B.

If $f : V(G) \to V(G)$ is an order-two side-switching automorphism of G, then define $g : V(F) \to V(F)$ as follows:

- $g(v) = f(v)$ for every $v \in A \cup B$,
- for the newly added vertices of degree 2, let $u, uv^1_1, uv^1_2, \ldots, uv^1_{p'}, v$ and $u, uv^2_1, uv^2_2, \ldots, uv^2_{p'}, v$ be the two paths joining u and v, and let $x, xy^1_1, xy^1_2, \ldots, xy^1_{p'}, y$ and $x, xy^2_1, xy^2_2, \ldots, xy^2_{p'}, y$ be the two paths joining $x = f(v)$ and $y = f(u)$. Then set $g(uv^i_j) = xy^{3-i}_{p'+1-j}$ for $i = 1,2$ and $j = 1,2,\ldots,p'$.

It is straightforward to see that g is an order-two side-switching automorphism. The only place where $ug(u)$ might be an edge would be in the middle of a path $u, uv^1_1, uv^1_2, \ldots, uv^1_{p'}, v$ when $f(u) = v$, but note that the vertices of one path are mapped onto vertices of the other one and $xg(x) \notin E(F)$ is fulfilled.

On the other hand, suppose $g : V(F) \to V(F)$ is an order-two side-switching automorphism of F. Since the original vertices of G have degrees greater than 2 in F, the restriction of g to $V(G)$ is a correctly defined mapping $g : V(G) \to V(G)$. Since the paths of length $p'+1$ uniquely correspond to edges of G, this restriction of g is an automorphism of G. It is obviously of order 2, and since the sides of F respect the sides of G, it is side-switching. (Note that we even did not need to assume that $ug(u) \notin E(F)$ for this implication.) ■

Definition 15. *Let H be a graph. We define a function $f(H)$ from graphs to integers and infinity. If $B(\overline{H})$ is acyclic with no connected component having two vertices of degree larger than two then we set $f(H)=\infty$. Otherwise, let $f(H)$ be the smallest of* (i) *the length of the smallest induced cycle of $B(\overline{H})$, and* (ii) *the length of the shortest path between any two vertices of degree larger than two in $B(\overline{H})$.*

For example, for the cycle on 5 vertices C_5, we have $B(\overline{C_5})=C_{10}$ and thus $f(C_5)=10$. Note that if $f(H)\neq\infty$ then $D_{f(H)}$ is contained in the class of bipartite $B(\overline{H})$-free graphs. We therefore have the following Corollary of Lemma 12 and Theorem 14.

Corollary 16. *The H-free Star System Problem is NP-complete whenever $f(H)\neq\infty$. Moreover, if $\mathcal{F}$ is a set of graphs for which there exists an integer p such that for any $H\in\mathcal{F}$ we have $f(H)\leq p$, then the $\mathcal{F}$-free Star System Problem (i.e. deciding on an input S if there is a graph having no induced subgraph isomorphic to any graph in $\mathcal{F}$) is NP-complete.*

Since $B(\overline{C_k})$ contains a cycle for any $k\geq 5$ we have the corollary.

Corollary 17. *For any $k\geq 5$, the C_k-free Star System problem is NP-complete.*

Similarly, $B(\overline{P_k})$ is connected and contains at least 2 vertices of degree greater or equal to 3 for any $k\geq 5$. Hence we also have the following corollary.

Corollary 18. *For any $k\geq 5$, the P_k-free Star System problem is NP-complete.*

5. CLOSING REMARKS

In this article we obtained a complete dichotomy for the H-free Star System problem when the forbidden graph H is either a path or a cycle. Moreover, our NP-completeness result holds for H taken from a much larger family of graphs, and thus the remaining cases in which the problem might not be NP-complete are very restricted. It is tempting to ask if the H-free Star System problem has a P vs NP-completeness dichotomy in general, i.e. whether for any graph H the H-free Star System problem is either polynomial-time solvable or NP-complete (and thus presumably not Graph Isomorphism-complete). See [5, 10] for a discussion of such dichotomy results.

A closely related question is on the complexity of the Star System problem restricted to graph classes defined by several forbidden-induced subgraphs as in Corollary 16. By the result of Aigner and Triesch [1, 2] (see also [4]), we do not have dichotomy in general, as there are classes of graphs defined by an infinite set of forbidden-induced subgraphs (like forbidding the complements of odd cycles) such that the Star System problem is Graph Isomorphism complete on these classes. However, we do not know whether there is a graph class characterized by a *finite* set of forbidden induced subgraphs such that the Star System problem on this class is Graph Isomorphism complete, or if instead dichotomy may hold in this case.

And finally, two concrete questions: What is the complexity of the H-free Star System problem for $H=K_4$ and for $H=K_4-e$?

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