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# COPS AND ROBBER WITH CONSTRAINTS*

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**Abstract.** Cops and robber is a classical pursuit-evasion game on undirected graphs, where the task is to identify the minimum number of cops sufficient to catch the robber. In this paper, we investigate the changes in problem's complexity and combinatorial properties with constraining the following natural game parameters: *fuel*, the number of steps each cop can make; *cost*, the total sum of steps along edges all cops can make; and *time*, the number of rounds of the game.

**Key words.** cops and robber games, complexity, random graphs

**AMS subject classifications.** 05C80, 05C85, 68R10

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## 1. Introduction.

The cops and robber game is a discrete variant of the classical man and lion pursuit-evasion problem attributed to Rado by Littlewood in [22]. The game is played by two players, cop and robber, on an undirected graph. The cop-player has a team of cops who attempt to capture the robber. At the beginning of the game the cop-player selects vertices and puts cops on these vertices. Then the robber-player puts the robber on a vertex. The players take turns, starting with the cop-player. At every move each of the cops can be either moved to an adjacent vertex or kept on the same vertex. Similarly, the robber-player responds by moving the robber to an adjacent vertex or by keeping him on the same vertex. The cop-player wins if at some step of the game he catches the robber, that is, puts one of his cops on a vertex occupied by the robber.

The game, with one cop, was introduced independently by Winkler and Nowakowski [26] and by Quilliot [29]. Aigner and Fromme [1] initiated the study of the problem with several cops. Different combinatorial [4, 10, 30] and algorithmic [12, 13, 19] aspects of the game were studied intensively. We refer to surveys [3, 14, 20] and the recent monograph [8] for references on different pursuit-evasion and search games on graphs.

There are two main open problems concerning the cops and robber game. The first open problem is about the upper bound on the number of cops sufficient to win in any connected graph on $n$ vertices. The famous conjecture, attributed to Meyniel by Frankl [15], is that for every $n$-vertex connected graph $O(\sqrt{n})$ cops always have a winning strategy. It was shown by Frankl that the cop number of a graph is $O(n\log\log n/\log n)$. Despite many attempts, the best known upper bound of $\frac{n}{2^{(1-o(1))(\sqrt{\log_2 n})}}$ due to Lu and Peng [23] and Scott and Sudakov [32] is still quite far from $\sqrt{n}$. See also the works of Alon and Mehrabian [2], Frieze, Krivelevich, and Loh [16], and Mehrabian [25] on extensions of Meyniel's conjecture. The bounds on

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TABLE 1.1

*Complexity classification of the cops and robber game with different constraints on resources. Constant means the corresponding parameter (fuel $\phi$, time $\tau$, and cost $\sigma$) is not part of the input. Polynomial means that the corresponding parameter is bounded by some polynomial of the input length. The last line of the table corresponds to arbitrary sizes of parameters.*

| | $\phi \leq s$ | $\sigma \leq s$ | $\tau \leq s$ |
|---|---|---|---|
| Constant | $s = 1$, NP-compl. [13]<br>$s \geq 2$, PSPACE-compl. [13] | NP-compl. | NP-compl. [6] |
| Polynomial | PSPACE-compl. [13] | PSPACE-compl. | PSPACE-compl. |
| Unbounded | PSPACE-hard [13] | PSPACE-hard | PSPACE-hard |

the cop number of random graphs were given by Bollobás, Kun, and Leader [5] and by Łuczak and Prałat [24]. It has been shown recently by Prałat and Wormald [28] that the Meyniel conjecture holds asymptotically almost surely for random graphs.

The second open problem is the computational complexity of the problem. Goldstein and Reingold in [19] conjectured that the game is EXPTIME-complete,[1] but so far it is only known to be NP-hard [12]. The complexity of several variants of the cops and robber game are better understood. Goldstein and Reingold [19] proved that the version of the cops and robbers game on *directed* graphs is EXPTIME-complete. The version of the game on undirected graphs when the cops and the robber are given their initial positions is also EXPTIME-complete [19]. The version of the game where each cop can make at most $\phi$ steps, the version with fuel constraint, is PSPACE-complete for each $\phi$ [13].

In this paper we study the cops and robber game when one of the following parameters is bounded:

- *Fuel*, the number of moves $\phi$ each cop can make during the game. In other words, each cop can pass through at most $\phi$ edges.
- *Cost*, the sum $\sigma$ of the number of steps that all cops can make in total. Thus the total number of edges passed by all cops is at most $\sigma$.
- *Time*, the number of rounds $\tau$ of the game.

The version of constraint time was studied by Bonato et al. [6] and Gavenčiak [18]. It was shown in [6] that for every fixed integer $\tau$, and integer $k$ being part of the input, deciding if at most $k$ cops can win within time $\tau$ is NP-complete. We show that similar NP-completeness result holds for fixed cost constraint $\sigma$ and $k$ being part of the input (Theorem 3.1). For fuel constraint $\phi = 1$ the problem is equivalent to the minimum dominating set problem and thus is NP-complete. For $\phi \geq 2$ the problem is PSPACE-hard [13]. In this paper, we establish PSPACE-hardness results for the other two variants of the game, when one of the parameters, either time $\tau$ or cost $\sigma$, is part of the input (Theorem 3.4). Each of the variants of the cops and robber game is PSPACE-complete when the corresponding parameter ($\tau$, $\sigma$, or $\phi$) does not exceed some polynomial of the input length (Theorem 3.7). This establish almost complete classification of the complexity landscape of the game with different constraints on the resources of the players. We summarize the complexity results in Table 1.1.

We also present a number of results for new variants of the game played on binomial random graphs. The variant with fuel constraints and the one with time constraints behave similarly from that perspective. We show that the number of cops required to catch the robber as a function of the average degree changes in a very intriguing manner. We get that asymptotically almost surely the logarithm of the cop

[1]Goldstein and Reingold in [19] call EXPTIME = DTIME($2^{O(|I|)}$), where $|I|$ is the input size; more often this class is denoted by E or ETIME.

number is asymptotic to the zig-zag function that depends on the value of $s$. (Hence, in fact, we obtain an infinite family of zig-zags.) The third variant of the game, that is, the one with cost constraints, seems to be more challenging to investigate. We provide both upper and lower bounds for this graph parameter but the shape of the zig-zag function associated with this variant still remains an open problem.

The remaining part of the paper is organized as follows. In section 2 we give basic definitions and observations. We give hardness proofs in section 3. The results about cops and robber on random graphs are in section 4.

**2. Basic definitions and preliminaries.** We consider finite undirected graphs without loops or multiple edges. The vertex set of a graph $G$ is denoted by $V(G)$ and its edge set by $E(G)$, or simply by $V$ and $E$ if this does not create confusion. If $U \subseteq V(G)$, then the subgraph of $G$ induced by $U$ is denoted by $G[U]$. For a vertex $v$, the set of vertices which are adjacent to $v$ is called the *(open) neighborhood* of $v$ and denoted by $N_G(v)$. The closed neighborhood of $v$ is the set $N_G[v] = N_G(v) \cup \{v\}$. For $Q \subseteq V(G)$, $N_G[Q] = \bigcup_{v \in Q} N_G[v]$ is the closed neighborhood of $Q$. The *distance* $\mathrm{dist}_G(u, v)$ between a pair of vertices $u$ and $v$ in a connected graph $G$ is the number of edges in a shortest $u, v$-path in $G$. For a positive integer $r$, let $N_G(v, s)$ denote the set of vertices ("ball") within distance at most $s$ from $v$; that is, $N_G(v, s) = \{u \in V(G) \colon \mathrm{dist}_G(u, v) \leq s\}$. Whenever there is no ambiguity, we omit the subscripts. All logarithms with no subscript are natural.

The cops and robber game is defined as follows. Let $G$ be a connected graph. The game is played by two players, the cop-player $\mathcal{C}$ and the robber-player $\mathcal{R}$, which make moves alternately. The cop-player $\mathcal{C}$ has a team of $k$ cops who attempt to capture the robber. At the beginning of the game this player selects vertices and put cops on these vertices. Then $\mathcal{R}$ puts the robber on a vertex. The players take turns starting with $\mathcal{C}$. At every turn each of the cops can be either moved to an adjacent vertex or kept on the same vertex. Let us note that several cops can occupy the same vertex at some point of the game. Similarly, $\mathcal{R}$ responds by moving the robber to an adjacent vertex or keeping him on the same vertex. It is said that a cop *catches* (or captures) the robber at some round if at that round they occupy the same vertex. The cop-player wins if one of his cops catches the robber. Player $\mathcal{R}$ wins if he can avoid such a situation. For a graph $G$, the minimum number $k$ of cops sufficient for $\mathcal{C}$ to win on graph $G$ is called the *cop number* of $G$ and is denoted by $c(G)$.

We consider the following variants of the game. Let $s$ be a positive integer. In the first variant of the game, each cop can make at most $s$ moves along edges (have a bounded "charge" or amount of "fuel"). Notice that a cop needs fuel only to move from one vertex to another, and even if a cop cannot move to adjacent vertex because he runs out of fuel, he is still active and the robber cannot step on a vertex occupied by such cop without being caught. We denote the minimum number of cops with fuel at most $s$ sufficient to win on $G$ by $c_{\phi \leq s}(G)$, and we refer to this variant of the game as cops and robber with fuel constraints.

Another variant of constraints we study in this paper is the case when during the whole game the total number of moves (from a vertex to another vertex) of all the cops is at most $s$. In other words, if transferring of one cop to adjacent vertex costs one unit, the total cost of all cops' movements (or the distance traveled by the cops) is at most $s$. We denote the minimum number of cops that can win in these conditions by $c_{\sigma \leq s}(G)$ and call this game cops and robber with cost constraints.

Finally, we consider the game with the number of rounds at most $s$, that is, the length of the game is restricted. In other words, the cops are supposed to catch the robber within time limit $s$. Let $c_{\tau\leq s}(G)$ be the minimum number of cops $\mathcal{C}$ needed to win now and the game is called cops and robber with time constraints.

We consider the decision version of the following problem:

| |
|---|
| Cops and Robber with Fuel Constraints<br>*Input:* A connected graph $G$ and two positive integers $k, s$.<br>*Question:* Is $c_{\phi\leq s}(G) \leq k$? |

The decision versions of cops and robber with cost constraints and cops and robber with time constraints are defined similarly, with the only difference being that we ask if $c_{\sigma\leq s}(G) \leq k$ and $c_{\tau\leq s}(G) \leq k$, correspondingly.

In some of the proofs, the notion of position will be used. The *position* of a cop at a given moment of the game is the vertex of the graph occupied by this cop, and we define the *position of a team* of $k$ cops (or *position of cops*) as a multiset of the vertices $C = (v_1, v_2, \ldots, v_k)$ occupied by the cops. Notice that $C$ is a multiset, since several cops can occupy one vertex and, as we do not distinguish the cops, it is not important which cop occupies a vertex. In the case of the game with cost or time constraints the positions of cops is the pair $(C, \ell)$, where $C$ is a multiset of the vertices occupied by the cops and $\ell$ is the number of moves along edges which the cops can do or the total number of rounds left, respectively. For the *initial* position, $\ell = s$. The *position of the robber* is a vertex of the graph occupied by him.

We will use the following observation.

Observation 1. *For a connected graph $G$ and a positive integer $s$,*

$$c(G) \leq c_{\phi\leq s}(G) \leq c_{\tau\leq s}(G) \leq c_{\sigma\leq s}(G).$$

The first two inequalities of Observation 1 follow directly from the definitions of the numbers $c(G), c_{\phi\leq s}(G), c_{\tau\leq s}(G)$, and $c_{\sigma\leq s}(G)$, and the third one follows from the fact that it always can be assumed that at least one cop is moved at each step (otherwise the robber can either keep his position or improve it).

Let the *$s$-distance domination number* $\gamma_s(G)$ be the minimum cardinality of a set $D \subseteq V(G)$ with the property that every vertex $v \in V(G)$ is at distance at most $s$ from some vertex of $D$. The relation of the game and domination is given in the following simple observation.

Observation 2. *For any connected graph $G$,*
- $c_{\phi\leq 1}(G) = c_{\sigma\leq 1}(G) = c_{\tau\leq 1}(G) = \gamma_1(G)$;
- $\gamma_s(G) \leq c_{\phi\leq s}(G) \leq c_{\tau\leq s}(G) \leq c_{\sigma\leq s}(G)$ *for every positive integer $s$.*

## 3. Complexity of cops and robber with time and cost constraints.

Observation 2 indicates that all variants of the game with constraints are at least NP-hard. In this section we establish the complexity of cops and robber with time constraints and cops and robber with cost constraints by showing that they are PSPACE-hard. However, we start with the proof that when $s$ is not a part of the input, then cops and robber with cost constraints is NP-complete.

### 3.1. When cost is not a part of the input.

In this subsection we prove that when the cost is not a part of the input, the version of the problem is NP-complete.

Theorem 3.1. *For any positive integer $s$, the following problem is* NP*-complete:*

| |
|---|
| Input: *A connected graph $G$ and a positive integer $k$.*<br>Question: *Is $c_{\sigma\leq s}(G) \leq k$?* |

*Proof.* To prove hardness, we reduce the well-known NP-complete dominating set problem [17]. Given a graph $G$, we construct the graph $H$ as follows: for each vertex $v \in V(G)$, we add a path $P_v$ of length $s-1$ with one endpoint in $v$. All vertices of $P_v$ except $v$ are new. Observe that for $s = 1$, $H = G$. We prove that $G$ has a dominating set of size $k$ if and only if $c_{\sigma\leq s}(H) \leq k$. Let $S \subseteq V(G)$ be a dominating set of size at most $k$ in $G$. We put cops on the vertices of $S$ in $H$. There is a vertex $v \in V(G)$ such that the robber occupies a vertex of the path $P_v$. Since $S$ is a dominating set in $G$, we have that either $v \in S$ or $v$ is adjacent to a vertex $u \in S$. In the second case the cop from $u$ is moved to $v$ at the first step. Now in both cases the robber occupies a vertex of $P_v - v$, the vertex $v$ is occupied by a cop, and the cops can make at least $s-1$ moves along edges. It remains to observe that cops can capture the robber by moving the cop from $v$ toward the other end of $P_v$. Hence, $k$ cops have a winning strategy on $H$ and $c_{\sigma\leq s}(H) \leq k$. Consider now a winning strategy of $k$ cops on $H$. Suppose that $(C, s)$ is the initial position of the cops in it. Let $S = \{v \in V(G) | C \cap V(P_v) \neq \emptyset\}$. Clearly, $|C| \leq k$. We prove that $S$ is a dominating set in $G$. Assume that, contrary to this claim, there is a vertex $u \in V(G)$ such that $N_G[u] \cap S = \emptyset$. Let $w$ be the endpoint of $P_u$ different from $u$. Since vertices of $P_u$ are not occupied by the cops and there are no cops in the neighborhood of $u$ in $G$, all cops are at distance at least $s+1$ from $w$. Therefore, the robber can occupy this vertex at the beginning of the game and stay safely there until the end of the game. This contradicts the existence of a winning strategy for the cops from the considered initial position. Hence $S$ is a dominating set of size at most $k$ in $G$.

To complete the proof, we have to show that for fixed $\sigma$, cops and robber with cost constraints is in NP. In order to do it, we observe that a winning strategy of the cops on $G$ can be described as a directed rooted tree of all possible moves of the robber (where the first move is a choice of the initial position) and corresponding moves of the cops. Each node of the tree correspond to the current positions of the cops and robber (the root corresponds only to the initial position of the cops). The size of this rooted tree is at most $O(|V(G)|^s)$ because out-degrees of its nodes are at most $|V(G)|$ and the height of the tree is at most $s$. It remains to observe that such a tree certifies inclusion of our problem in NP because it may be checked in polynomial time whether the strategy of cops is winning. ▯

For cops and robber with time constraints, NP-completeness was proved in [6], but the reduction described above works for it and cops and robber with fuel constraints as well. Using it and the fact that dominating set is a W[2]-complete problem (see the book of Downey and Fellows [11] for an introduction to parameterized complexity), we derive the following corollary.

Corollary 3.2. *For any fixed positive integer s, the cops and robber with cost constraints (time constraints, fuel constraints, respectively) problem is* W[2]*-hard parameterized by the number of cops.*

Combined with the nonapproximability for the dominating set problem [31], the same reduction implies the following.

Corollary 3.3. *For any fixed positive integer $s$, there is a constant $c > 0$ such that there is no polynomial time algorithm to approximate $c_{\sigma\leq s}(G)$ ($c_{\tau\leq s}(G)$, $c_{\phi\leq s}(G)$, respectively) within a multiplicative factor of $c \log n$, where $n = |V(G)|$, unless $P = NP$.*

**3.2. PSPACE-hardness of cops and robber with time and cost constraints.** When $\sigma$ and $\tau$ are a part of the input then cops and robber with cost constraints and cops and robber with time constraints become PSPACE-hard.

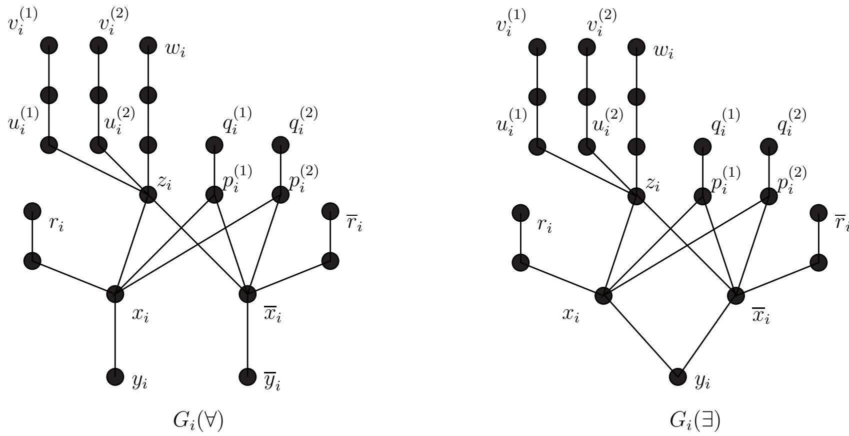



FIG. 3.1. *Graphs $G_i(\forall)$ and $G_i(\exists)$ for $s = 3$.*

The following theorem is the main result of this subsection.

THEOREM 3.4. *Cops and robber with cost constraints and cops and robber with time constraints are* PSPACE*-hard.*

*Proof.* We reduce from the PSPACE-complete quantified Boolean formula in conjunctive normal form (QBF) problem. For a set of Boolean variables $x_1, x_2, \ldots, x_n$ and a Boolean formula $F = C_1 \wedge C_2 \wedge \cdots \wedge C_m$, where $C_j$ is a clause, the QBF problem asks whether the expression

$$\Phi = Q_1 x_1 Q_2 x_2 \cdots Q_n x_n F$$

is *true*, where for every $i$, $Q_i$ is either $\forall$ or $\exists$. We assume that $n \geq 2$ is even, $Q_i = \forall$ for odd $i \in \{1, \ldots, n\}$, and $Q_i = \exists$ whenever $i$ is even. It is known that QBF remains PSPACE-complete with these restrictions [17]. Given a quantified Boolean formula $\phi$, we construct an instance $(G, n, s)$ of our problem such that $\Phi$ is true if and only if the cop-player can win on $G$ with $n$ cops for $s = n + 1$ in both games.

*Constructing $G$.* For every $Q_i x_i$ we introduce a gadget graph $G_i$ as follows:

- Construct vertices $x_i, \overline{x}_i$.
- If $Q_i = \forall$, then we construct vertices $y_i, \overline{y}_i$ and edges $x_i y_i, \overline{x}_i \overline{y}_i$.
- If $Q_i = \exists$, then we construct a vertex $y_i$ and join it with $x_i, \overline{x}_i$ by edges.
- Construct a path of length $s$ with endpoints $z_i, w_i$ and join $x_i, \overline{x}_i$ with $z_i$ by edges.
- Construct two paths of length $s - 1$ with endpoints $u_i^{(1)}, v_i^{(1)}$ and $u_i^{(2)}, v_i^{(2)}$, respectively, and join $u_i^{(1)}, u_i^{(2)}$ with $z_i$ by edges.
- Introduce two vertices $r_i$ and $\overline{r}_i$ and join them with $x_i$ and $\overline{x}_i$, respectively, by paths of length $s - 1$.
- Construct two paths of length $s - 3$ with endpoints $p_i^{(1)}, q_i^{(1)}$ and $p_i^{(2)}, q_i^{(2)}$, respectively, and join $p_i^{(1)}, p_i^{(2)}$ with $x_i, \overline{x}_i$ by edges.

For $Q_i = \forall$ and $Q_i = \exists$, we denote the obtained graphs as $G_i(\forall)$ and $G_i(\exists)$, respectively (see Figure 3.1). Let $X_i = \{x_i, \overline{x}_i\}$, $U_i = \{u_i^{(1)}, u_i^{(2)}\}$, $P_i = \{p_i^{(1)}, p_i^{(2)}\}$, $Y_i = \{y_i, \overline{y}_i\}$ for $G_i(\forall)$ and $Y_i = \{y_i\}$ for $G_i(\exists)$.

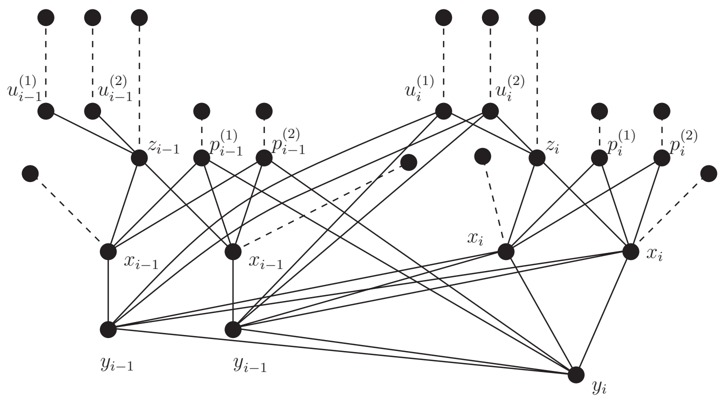



FIG. 3.2. *Connections of* $G_{i-1}(\forall)$ *and* $G_i(\exists)$.

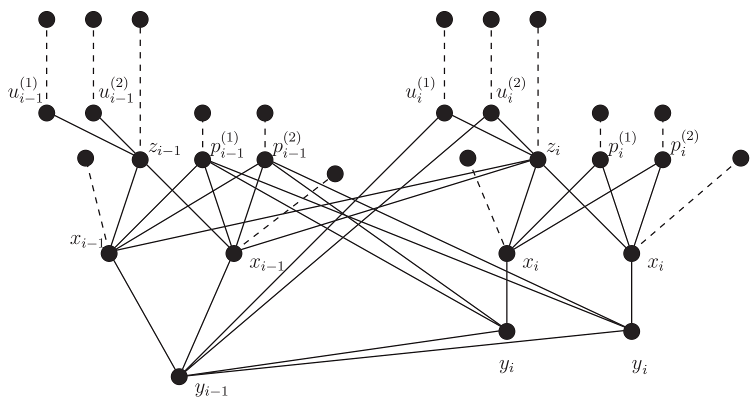



FIG. 3.3. *Connections of* $G_{i-1}(\exists)$ *and* $G_i(\forall)$.

Using these gadgets we construct $G$ as follows:

- Construct gadget graphs $G_i$ for $i \in \{1, \ldots, n\}$.
- For each $i \in \{2, \ldots, n\}$, join the vertices from the set $Y_{i-1}$ with all vertices from $Y_i$.
- For each $i \in \{2, \ldots, n\}$, join the vertices of $U_i$ with all vertices from the sets $Y_j$ for $j \in \{1, \ldots, i-1\}$.
- For each $i \in \{1, \ldots, n-1\}$, join the vertices of $P_i$ with all vertices from the sets $Y_j$ for $j \in \{i+1, \ldots, n\}$.
- For each even $i \in \{2, \ldots, n\}$, join the vertices from the set $Y_{i-1}$ with all vertices from $X_i$.
- For each odd $i \in \{3, \ldots, n-1\}$, join the vertices from the set $Y_{i-1}$ with the vertex $z_i$.
- Join $x_n$ and $\overline{x}_n$ by an edge.
- Add vertices $C_1, C_2, \ldots, C_m$ corresponding to the clauses and join them with the unique vertex of $Y_n$ (recall that $Q_n = \exists$) by edges.
- For $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, the vertex $x_i$ is joined with $C_j$ by an edge if $C_j$ contains the literal $x_i$, and $\overline{x}_i$ is joined with $C_j$ if $C_j$ contains the literal $\overline{x}_i$.

Connections between gadgets $G_{i-1}$ and $G_i$ is shown in Figures 3.2 and 3.3, and construction of $G$ is shown in Figure 3.4.

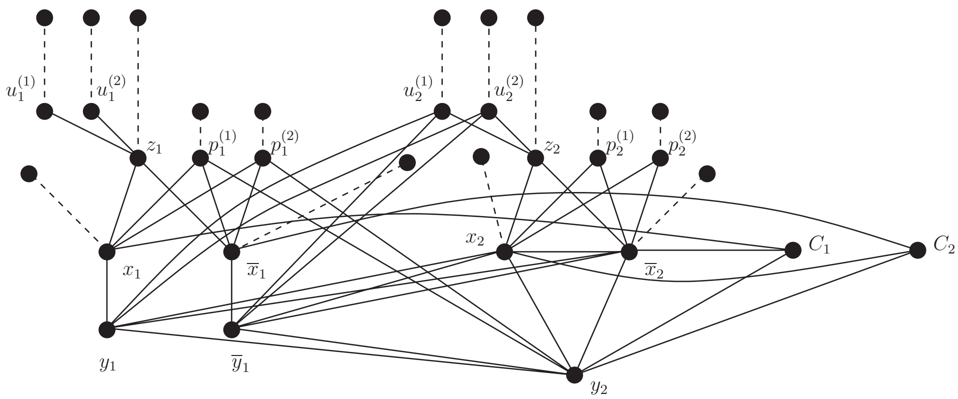



FIG. 3.4. *Construction of $G$ for $\Phi = \forall x_1 \exists x_2\ (x_1 \vee \overline{x}_2) \wedge (\overline{x}_1 \vee x_2)$, $n = 2$, and $s = 3$.*

Now we prove the following two lemmata.

LEMMA 3.5. *If $\Phi = true$, then $n$ cops have a winning strategy on $G$ in cops and robber with cost constraints and cops and robber with time constraints.*

*Proof.* By Observation 1, it is sufficient for the proof to describe a winning strategy for the cop-player for cops and robber with cost constraints. The cops start by occupying vertices $z_1, \ldots, z_n$. If the robber occupies a vertex of some $(z_i, w_i)$-path of $G_i$, then the cop from the vertex $z_i$ moves toward the robber in this path. Since the path has length $s$, the robber will be captured by this cop. If the robber occupies a vertex of some $(u_i^{(j)}, v_i^{(j)})$-path or $(x_i, r_i)$-path or $(\overline{x}_i, \overline{r}_i)$-path of $G_i$, then the cop from the vertex $z_i$ moves to $u_i^{(j)}$ or $x_i$ or $\overline{x}_i$, respectively, and then moves toward the robber in this path. Since these paths have length $s - 1$, the robber will be captured. If the robber occupies a vertex of some $(p_i^{(j)}, q_i^{(j)})$-path of $G_i$ for $i < n$, then the cop from the vertex $z_{i+1}$ moves to $x_{i+1}$, and then he moves to $y_{i+1}$. If the robber tries to move to $x_i$ or $\overline{x_i}$, then he is captured by the cop from $z_i$ in one step. Hence, the robber has to stay on the path. Now the cop from $y_{i+1}$ moves to $p_i^{(j)}$, and after that he moves toward the robber in this path. Since the path has length $s - 3$, we have that the robber will be captured. If $i = n$, then the cop from the vertex $z_i$ moves to $x_i$. If the robber moves to $\overline{x_i}$, then he is captured by the cop from $x_i$ in one step. So, the robber stays on the path. Now the cop from $x_i$ moves to $p_i^{(j)}$, and after that he moves toward the robber in this path. Thus we assume that the robber is on a vertex of some set $Y_i$ or in $\{C_1, \ldots, C_m\}$. We consider two cases.

*Case* 1. The robber occupies a vertex of some set $Y_i$. For each $j \in \{1, \ldots, i - 1\}$, the cop from $z_j$ moves either to $x_j$ or to $\overline{x}_j$. We assume that the choice of $x_j$ corresponds to the value *true* of the variable $x_j$ and the choice of $\overline{x}_j$ corresponds to the value *false* of $x_j$. Since $\phi = true$, the variables $x_1, \ldots, x_{i-1}$ can be assigned values such that $Q_i x_i \ldots Q_n x_n F = true$. The cop from the vertex $z_j$ moves according to the value of $x_j$. If $x_j = true$, then he moves to the vertex $x_j$, otherwise he moves to $\overline{x}_j$. Now inductively for $j \in \{i, \ldots, n\}$, we assume that the robber occupies a vertex of $Y_j$ and we move the cop from $z_j$ according to the following subcases:

(a) $Q_j = \forall$. If the robber occupies the vertex $y_j$, then the cop from $z_j$ moves to $x_j$, and if the robber is on $\overline{y}_j$, then the cop moves to $\overline{x}_j$. We again suppose that a placement of a cop on $x_j$ corresponds to the value *true* of the variable $x_j$, and a moving a cop to $\overline{x}_j$ corresponds to the value *false*. Notice that now the robber choses the value of the variable $x_j$. Now the robber should make his move:

- If the robber stays in his old position, then he will be captured in one step by the cop which is either on $x_j$ or $\overline{x}_j$.
- If the robber moves to a vertex of $X_{j+1}$, then he will be captured in one step by the cop from $z_{j+1}$ in $G_{j+1}(\exists)$.
- If the robber moves to a vertex of $Y_{j-1}$, then again he will be captured in one step by the cop which is either on $x_{j-1}$ or $\overline{x}_{j-1}$ in $G_{j-1}(\exists)$.
- If the robber moves to a vertex $u_t^{(1)}$ or $u_t^{(2)}$ for $t > j$, then he will be captured in one step by the cop from $z_t$.
- If the robber moves to a vertex $p_t^{(1)}$ or $p_t^{(2)}$ for $t < j$, then he will be captured in one step by the cop from $x_t$ or $\overline{x}_t$.

Hence he should move either to a vertex of $Y_{j+1}$ or to one of the vertices $C_1, \dots, C_m$ if $j = n$ to avoid the capture.

(b) $Q_j = \exists$. Then the robber occupies $y_j$. The cop from $z_j$ moves either to $x_j$ or to $\overline{x}_j$. The vertex is chosen in such a way that it corresponds to the value of the variable $x_j$ for which (and for already assigned values of the variables $x_1, \dots, x_{j-1}$) $Q_{j+1}x_{j+1} \dots Q_n x_n F = \mathit{true}$. Then the following are similar to subcase (a):

- If the robber stays in his old position, then he will be captured in one step by the cop which is either on $x_j$ or $\overline{x}_j$.
- If the robber moves to a vertex of $X_j$ unoccupied by the cops, then he will be captured in one step either by the cop from $z_{j+1}$ in $G_{j+1}(\forall)$ if $j < n$ or by the cop from $X_j$ if $j = n$.
- If the robber moves to a vertex of $Y_{j-1}$, then again he will be captured in one step by the cop which is either on $x_j$ or $\overline{x}_j$ in $G_j(\exists)$.
- If the robber moves to a vertex $u_t^{(1)}$ or $u_t^{(2)}$ for $t > j$, then he will be captured in one step by the cop from $z_t$.
- If the robber moves to a vertex $p_t^{(1)}$ or $p_t^{(2)}$ for $t < j$, then he will be captured in one step by the cop from $x_t$ or $\overline{x}_t$.

Hence, the robber is either captured by the next step or moves to a vertex of $Y_{j+1}$ or to one of the vertices $C_1, \dots, C_m$ if $j = n$.

Finally, the robber is either captured or occupies some vertex $C_t$. Notice that if the robber is not captured yet, then the cops made exactly $n = s - 1$ moves along edges. Observe also that the cops have chosen the vertices of the sets $X_1, \dots, X_n$ such that $F = \mathit{true}$ for the corresponding values of Boolean variables. Hence there is a cop on a vertex adjacent with $C_t$ and he captures the robber by the next move.

*Case* 2. The robber occupies some vertex $C_j$ in the beginning of the game. For each $i \in \{1, \dots, n\}$ such that $C_j$ contains either $x_i$ or $\overline{x}_i$, the cop from $z_i$ moves either to $x_i$ or to $\overline{x}_i$ according to the inclusion of the literal. If $x_n, \overline{x}_n$ are not included in $C_j$, then additionally the cop from $z_n$ moves to $x_n$. Clearly, the robber should either stay on $C_j$ or move to $y_n$. In the first case he is captured by one of the cops on $x_i, \overline{x}_i$, and he is captured by the cop from $x_n$ or $\overline{x}_n$ in the second case. ∎

To complete the proof of PSPACE-hardness, it remains to prove the following lemma.

LEMMA 3.6. *If $\Phi = false$, then the robber has a winning strategy against $n$ cops on $G$ in cops and robber with cost constraints and cops and robber with time constraints.*

*Proof.* By Observation 1, now it is sufficient for the proof, to describe a winning strategy for the robber-player for cops and robber with time constraints. Assume that the cops have chosen their initial positions. If there is a path $(z_i, w_i)$-path in some

$G_i$ such that all vertices of the path are unoccupied by the cops, then we place the robber on $w_i$. Since there are no cops at distance at least $s+1$ from $w_i$, the winning strategy for the robber is trivial—he should stay on $w_i$. Suppose now that for each $(z_i, w_i)$-path, there is a cop on one of the vertices of the path. We have $n$ cops. Hence exactly one cop occupies one vertex of each path. Moreover, this cop should occupy $z_i$ if the cops do not want the robber to win by a trivial strategy, since otherwise the vertices $v_i^{(1)}, v_i^{(2)}, r_i, \overline{r}_i$ have no cops at distance at least $s+1$, and the robber can safely stay on one of these vertices. Denote this cop by $P_i$. The robber is placed on a vertex of $Y_1$. The choice of the vertex and further moves of the robber are described inductively for $i \in \{1, \dots, n\}$. Suppose that for each $i \le j \le n$, the cop $P_j$ is on the vertex $z_j$, and each $1 \le j < i$, $P_j$ is either on $x_i$ or $\overline{x}_i$. Assume also that values of the variables $x_1, \dots, x_{i-1}$ are already defined and $Q_i x_i \dots Q_n x_n = false$ for this assignment.

(a) $Q_i = \forall$. Since $Q_i x_i \dots Q_n x_n = false$, there is a value of $x_i$ for which $Q_{i+1} x_{i+1} \dots Q_n x_n = false$. If this value is *true*, then the robber is placed on $y_i$ and he is placed on $\overline{y}_i$ otherwise. Observe, that the value of $x_i$ is chosen by the robber-player. In the following cases the robber has a straightforward winning strategy:
- One of the cops $P_j$ for $1 \le j < i$ leaves the vertex of $X_j$. If $j \ne i-1$ or $j = i-1$ and the cop from $z_i$ does not move simultaneously to $X_{i-1}$, then the robber chooses one of the vertices $p_j^{(t)}$ not occupied by $P_j$, moves to $p_j^{(t)}$, and then moves by his subsequent moves to $q_j^{(t)}$. Observe that in this case the cops already made at least two moves and any cop would have to make at least two further moves to reach the vertex $p_j^{(t)}$. Since this $(p_j^{(t)}, q_j^{(t)})$-path is not occupied by the cops when $P_j$ leaves $X_j$, this is a winning strategy for the robber.
- One of the cops $P_j$ for $i < j \le n$ leaves the vertex $z_j$. Then the robber chooses one of the vertices $u_j^{(t)}$ not occupied by $P_j$ and moves to $u_j^{(t)}$ and then moves by his subsequent moves to $v_j^{(t)}$. Since this $(u_j^{(t)}, v_j^{(t)})$-path is not occupied by the cops when $P_j$ leaves $z_j$, this is a winning strategy for the robber.
- The cop $P_i$ moves to the vertex of $X_i$ not adjacent with the robber's position. Then the robber moves to the vertex of $X_i$ adjacent with his current position and then moves by his subsequent moves to either $r_i$ or $\overline{r}_i$ and wins.
- The cop $P_i$ moves from $z_i$ to a vertex of $X_{i-1}$, but the cop from $X_{i-1}$ does not go to $z_i$ by the same step. Then the robber again moves to the vertex of $X_i$ adjacent with his current position and then moves by his subsequent moves to either $r_i$ or $\overline{r}_i$ and wins.

The cop-player has the following remaining possibilities:
- Exactly two cops $P_{i-1}$ and $P_i$ move: the cop $P_{i-1}$ moves from his current position in $X_{i-1}$ to $z_i$ and the cop $P_i$ moves to $X_{i-1}$. We call a move of this type *switching*, and we assume that these cops exchange their names $P_{i-1}$ and $P_i$ after this move. In this case the robber stays in his current position.
- Exactly one cop $P_i$ moves to the vertex of $X_i$ adjacent with the robber's position. Then the robber moves to a vertex of $Y_{i+1}$ or to some vertex of $\{C_1, \dots, C_m\}$ if $i = n$.

(b) $Q_j = \exists$. The robber is placed on $y_i$. In the following cases the robber wins directly.
- One of the cops $P_j$ for $1 \le j < i$ leaves the vertex of $X_j$. Then the robber chooses one of the vertices $p_j^{(t)}$ not occupied by $P_j$, moves to $p_j^{(t)}$, and then moves by his subsequent moves to $q_j^{(t)}$. Since this $(p_j^{(t)}, q_j^{(t)})$-path is not occupied by the cops when $P_j$ leaves $X_j$, this is a winning strategy for the robber.
- One of the cops $P_j$ for $i < j \le n$ leaves the vertex $z_j$. Then the robber chooses one of the vertices $u_j^{(t)}$ not occupied by $P_j$, moves to $u_j^{(t)}$, and then moves by his subsequent moves to $v_j^{(t)}$. Since this $(u_j^{(t)}, v_j^{(t)})$-path is not occupied by the cops when $P_j$ leaves $z_j$, this is a winning strategy for the robber.
- The cop $P_i$ moves from $z_i$ to either $u_i^{(1)}$ or $u_i^{(2)}$. Then the robber moves to the vertex of $X_i$ adjacent to his current position and then moves by his subsequent moves to either $r_i$ or $\overline{r}_i$.

Only one case remains: exactly one cop $P_i$ moves to the vertex of $X_i$. If he moves to $x_i$, then we let $x_i = true$ and $x_i = false$ otherwise. Observe that now the cop-player chooses the value of $x_i$. Then the robber moves to a vertex of $Y_{i+1}$ or to some vertex of $\{C_1, \dots, C_m\}$ if $i = n$.

It remains to define the strategy for the case when the robber moves to a vertex of $\{C_1, \dots, C_m\}$. Now we can assume that the variables $x_1, \dots, x_n$ have values for which $F = false$. Hence, there is a clause $C_j = false$. Observe that the total number of steps is at least $n = s - 1$. Moreover, if the cops made at least one switching step, then the total number of steps is $s$ and the game is over. It follows that by each step exactly one cop was moved and they occupy vertices $x_i, \overline{x}_i$ according to the values of the corresponding variables. Therefore, the cops cannot move by the last step to $C_j$. The robber moves to this vertex and stays there. ∎

Now the proof of the theorem follows from Lemmata 3.5 and 3.6. ∎

**3.3. Inclusion in PSPACE.** Theorem 3.4 establishes only PSPACE-hardness of cops and robber with cost constraints and cops and robber with time constraints. In what follows, we prove that these problems are in PSPACE when the parameter $s$ bounded by some polynomial of the input size or if it is assumed that these integers are encoded in unary.

THEOREM 3.7. *For every integers $s, k \ge 1$ and an $n$-vertex graph $G$, it is possible to decide whether $c_{\sigma \le s}(G) \le k$ ($c_{\tau \le s}(G) \le k$, respectively) by making use of space $O(s \cdot n^{O(1)})$.*

*Proof.* The proof is constructive. We describe a recursive algorithm which solves cops and robber with cost constraints. Note that we can consider only strategies of the cop-player such that at least one cop is moved to an adjacent vertex. Otherwise, if all cops are staying in old positions, the robber can only improve his position.

Our algorithm uses a recursive procedure $W(P, u)$, which for a position of the cops $P = (C, l)$, $C = (v_1, \dots, v_k)$ such that $l \le s$, and a vertex $u \in V(G)$, returns *true* if $k$ cops can win starting from the position $P$ against the robber which starts from the vertex $u$, and the procedure returns $false$ otherwise. Clearly, $k$ cops can capture the robber on $G$ if and only if there is an initial position $P_0$ such that for any $u \in V(G)$, $W(P_0, u) = true$ for $l = s$.

If $l = 0$, then $W(P, u) = true$ if and only if $u = v_i$ for some $1 \le i \le k$. Suppose that $l > 0$. Then $W(P, u) = true$ in the following cases:

- $u = v_i$ for some $1 \le i \le k$,
- $u \in N_G(v_i)$ for some $1 \le i \le k$,
- there is a position $P' = (C', l')$, where $C' = (v'_1, \dots, v'_k)$ and $l' < l$, such that the cops can go from $P$ to $P'$ in one step in such a way that exactly $l - l'$ cops move to their new positions along edges, and for any $u' \in N_G[u]$, $W(P', u') = true$.

Since all positions can be listed (without storing them) by using polynomial space, the number of possible moves of the robber is at most $n$ and the depth of the recursion is at most $s$, the algorithm uses space $O(s \cdot n^{O(1)})$.

The algorithm for cops and robber with time constraints is almost the same. The only difference is that for $l > 0$, $W(P, u) = true$ if and only if

- $u = v_i$ for some $1 \le i \le k$, or
- $u \in N_G(v_i)$, or
- there is a position $P' = (C', l - 1)$, where $C' = (v'_1, \dots, v'_k)$, such that the cops can go from $P$ to $P'$ in one step, and for any $u' \in N_G[u]$, $W(P', u') = true$. ∎

**4. Random graphs.** In this section, we present asymptotic results for the game of cops and robber played on a binomial random graph $\mathcal{G}(n, p)$. The section is organized as follows. In subsection 4.1 we define the probability space we are interested in and briefly describe results for the original game played on a random graphs. In subsection 4.2 we describe our main results for the the variant with fuel constraints as well as the one with time constraints. Upper and lower bounds are investigated separately in subsections 4.3 and 4.4, respectively. The proofs are based on techniques and ideas from [24, 28]. Finally, in subsection 4.5 we discuss the third variant of the game, namely the one with cost constraints, in the context of random graphs. A nontrivial upper bound is provided for this case but the behavior of this variant of the game remains to be investigated.

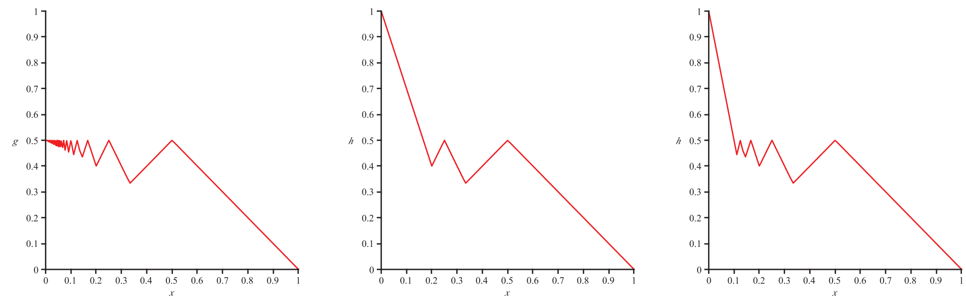
**4.1. Definitions and the original game.** Let us recall the classic model of random graphs that we study in this paper. The binomial random graph $\mathcal{G}(n, p)$ is defined as a random graph with vertex set $[n] = \{1, 2, \dots, n\}$ in which a pair of vertices appears as an edge with probability $p$, independently for each such a pair. As typical in random graph theory, we shall consider only asymptotic properties of $\mathcal{G}(n, p)$ as $n \to \infty$, where $p = p(n)$ may and usually does depend on $n$. We say that an event in a probability space holds *asymptotically almost surely* (*a.a.s.*) if its probability tends to one as $n$ goes to infinity.

Let us first briefly describe some known results on the cop number of $\mathcal{G}(n, p)$. Bonato, Hahn, and Wang [7] started investigating such games in $\mathcal{G}(n, p)$ random graphs. Bonato, Pralat, and Wang [9] generalized these results to sparser random graphs and their generalizations were used to model complex networks with a power-law degree distribution. From their results it follows that if $2 \log n / \sqrt{n} \le p < 1 - \varepsilon$ for some $\varepsilon > 0$, then a.a.s.

$$c(\mathcal{G}(n, p)) = \Theta(\log n / p).$$

A simple argument using dominating sets shows that $c(\mathcal{G}(n, p)) = o(\log n)$ a.a.s. if $p$ tends to 1 as $n$ goes to infinity (see [27] for this and stronger results). Recently, Bollobás, Kun, and Leader [5] showed that for $p(n) \ge 2.1 \log n / n$, we get that a.a.s.

$$\frac{1}{(np)^2} n^{(\log\log(np) - 9)/(2\log\log(np))} \le c(\mathcal{G}(n, p)) \le 160000\sqrt{n} \log n\,.$$



(a) $g(x)$ (b) $h_3(x)$ (c) $h_5(x)$

FIG. 4.1. *The "zigzag" functions.*

From these results, if $np \geq 2.1 \log n$ and either $np = n^{o(1)}$ or $np = n^{1/2+o(1)}$, then a.a.s. $c(\mathcal{G}(n,p)) = n^{1/2+o(1)}$. Somewhat surprisingly, between these values it was shown by Łuczak and Pralat [24] that the cop number has more complicated behavior. It follows that a.a.s. $\log_n c(\mathcal{G}(n, n^{x-1}))$ is asymptotic to the function $g(x)$ shown in Figure 4.1(a).

Below we precisely state the result.

THEOREM 4.1 (see [24]). *Let $0 < \alpha < 1$, let $j \geq 1$ be integer, and let $d = d(n) = (n-1)p = n^{\alpha+o(1)}$.*

1. *If $\frac{1}{2j+1} < \alpha < \frac{1}{2j}$, then a.a.s.*

$$c(\mathcal{G}(n,p)) = \Theta(d^j)\,.$$

2. *If $\frac{1}{2j} < \alpha < \frac{1}{2j-1}$, then a.a.s.*

$$\frac{n}{d^j} = O\big(c(\mathcal{G}(n,p))\big) = O\left(\frac{n}{d^j}\log n\right)\,.$$

3. *If $\alpha = 1/(2j)$ or $\alpha = 1/(2j+1)$, then a.a.s. $c(\mathcal{G}(n,p)) = d^{j+o(1)}$.*

In particular, it follows from [5, 24] that the cop number is a.a.s. of order $O(\sqrt{n}\log n)$, provided that $d \geq 2.1 \log n$. Recently, Pralat and Wormald removed unnecessary $\log n$ factor and extended the result to sparser graphs and random $d$-regular graphs, proving Meyniel's conjecture for random graphs [28].

**4.2. Main results for the two variants of the game.** In this subsection, we describe the behavior of the cop number with two constraints we consider in this paper: the variant with fuel constraints as well as the one with time constraints. From the point of view of random graphs, asymptotic behaviors of $c_{\phi\leq s}(\mathcal{G}(n,p))$ and $c_{\tau\leq s}(\mathcal{G}(n,p))$ are essentially the same. All our proofs work for both variants of the game, and we state the result for the fuel constraints variant only. The variant with cost constraints is discussed in subsection 4.5. This variant is still not fully investigated.

It follows from Observation 1 that for any $s \in \mathbb{N}$ and any graph $G$ we have that $c_{\tau\leq s}(G) \geq c_{\phi\leq s}(G) \geq c(G)$. However, it turns out that introducing constraints in the random graph case does not affect substantially the cop number, provided that the graph is dense enough; that is, the average degree is at least $n^{1/(2s)}$. With a $\log n$ times more cops (or sometimes just $O(1)$ times more) we can catch the robber in a few steps only, which provides an upper bound of $c_{\tau\leq s}(G)$. On the other hand, for sparse random graphs the distance $s$-domination number $\gamma_s(\mathcal{G}(n,p))$ is a.a.s. large which implies that the constrained cop number has to go up. We will show that a.a.s. $c_{\tau\leq s}(G)$ (and thus $c_{\phi\leq s}(G)$ as well) has the same order as this obvious lower bound.

In particular for $s = 3, 5$, it follows that a.a.s. $\log_n c_{\tau\leq s}(\mathcal{G}(n, n^{x-1}))$ (and $\log_n c_{\phi\leq s}(\mathcal{G}(n, n^{x-1}))$ as well) is asymptotic to the function $h_s(x)$ shown in Figure 4.1(b) and (c), respectively. Here is a precise result for any $s \in \mathbb{N}$.

THEOREM 4.2. *Let $0 < \alpha < 1$, let $s \geq 1$ be integer, and let $d = d(n) = (n-1)p = n^{\alpha+o(1)}$.*

(i) *If $\alpha < \frac{1}{2s-1}$, then a.a.s.*

$$c_{\phi\leq s}(\mathcal{G}(n,p)) = \Theta\left(\frac{n}{d^s}\log n\right).$$

(ii) *If $\frac{1}{2j+1} < \alpha < \frac{1}{2j}$ for some integer $j < s$, then a.a.s.*

$$c_{\phi\leq s}(\mathcal{G}(n,p)) = \Theta(d^j).$$

(iii) *If $\frac{1}{2j} < \alpha < \frac{1}{2j-1}$ for some integer $j < s$, then a.a.s.*

$$\frac{n}{d^j} = O\big(c_{\phi\leq s}(\mathcal{G}(n,p))\big) = O\left(\frac{n}{d^j}\log n\right).$$

(iv) *If $\alpha = 1/(2j)$ or $\alpha = 1/(2j+1)$ for some integer $j < s$, then a.a.s. $c_{\phi\leq s}(\mathcal{G}(n,p)) = d^{j+o(1)}$.*

*Exactly the same statement holds for $c_{\tau\leq s}(\mathcal{G}(n,p))$.*

**4.3. Upper bound.** In this section, we provide an upper bound for the main theorem of this section, Theorem 4.2. The result is actually slightly stronger than the one stated in the main theorem. First of all, we prove it for all random graphs but quite sparse ones (the average degree at least $\log^3 n$) whereas the main theorem focuses on dense graphs (the average degree $n^{\alpha+o(1)}$ for some $\alpha \in (0,1)$). Second of all, we allow $s = s(n)$ to be a function of $n$ whereas the main theorem assumes this to be arbitrarily large but fixed constant.

We will use the following expansion-type property of random graphs that is proved in [28].

LEMMA 4.3 (see [28]). *Suppose that $d = p(n-1) \geq \log^3 n$. Let $G = (V,E) \in G(n,p)$. Then the following property holds a.a.s. Let $Q \subseteq V$ be any set of $q = |Q|$ vertices, and let $s = s(n) \in \mathbb{N}$. Then*

$$\left|\bigcup_{v\in Q} N(v,s)\right| \geq \frac{1}{3}\min\{qd^s, n\}.$$

With this tool in hand we are ready to prove the following theorem.

THEOREM 4.4. *Let $s = s(n) \geq 1$ be integer and let $d = d(n) = (n-1)p$.*

(i) *If $n^{1/(2j+1)} \leq d \leq (n\log n)^{1/(2j)}$ for some integer $j < s$, then a.a.s.*

$$c_{\phi\leq s}(\mathcal{G}(n,p)) = O(d^j\beta),$$

*where $\beta = \max\{n\log n/d^{2j+1}, 1\} = O(\log n)$.*

(ii) *If $(n\log n)^{1/(2j+2)} \leq d \leq n^{1/(2j+1)}$ for some integer $j < s$, then a.a.s.*

$$c_{\phi\leq s}(\mathcal{G}(n,p)) = O\left(\frac{n}{d^{j+1}}\log n\right).$$

(iii) *If $d \leq (n\log n)^{1/(2s)}$, then a.a.s.*

$$c_{\phi\leq s}(\mathcal{G}(n,p)) = O\left(\frac{n}{d^s}\log n\right).$$

Before we start the proof, let us mention that (i) actually works for any value of $d$. This will provide a nontrivial upper bound for the third variant of the game we discuss in subsection 4.5. Here, we put restriction on $d$ to focus on the range that gives the best upper bound. Note that when $d = n^{1/(2j+1)}$ for some $j < s$, then both (i) and (ii) give the same bound of $O(n^{j/(2j+1)} \log n)$. Similarly, if $d = (n \log n)^{1/(2j)}$ for some $j < s$, then both (i) and (ii) give a bound of $O(\sqrt{n \log n})$.

*Proof.* We show that the upper bounds hold for an arbitrary graph $G$ possessing the properties stated in Lemma 4.3; then that lemma implies the theorem.

We start with (i). The team of cops is determined by independently choosing each vertex of $v \in V$ to be occupied by a cop with probability $Cd^j\beta/n$, where $C$ is a constant to be determined soon. The total number of cops is $(1 + o(1))Cd^j\beta$ a.a.s.

The robber appears at some vertex $v \in V$. We will show below that a.a.s. for each vertex $v \in V$ it is possible to assign distinct cops to all vertices $u$ in $N(v, j)$ such that a cop assigned to $u$ is within distance $(j + 1)$ of $u$. (Note that here, a.a.s. refers to the randomness in distributing the cops; the graph $G$ is fixed.) If this can be done, then after the robber appears these cops can begin moving straight to their assigned destinations in $N(v, j)$. Since the first move belongs to the cops, they have $j + 1 \le s$ steps, after which the robber must still be in $N(v, j)$, which is fully occupied by cops. He is caught and the game ends after at most $s$ rounds.

In order to show that the assignments we require exist a.a.s. for every vertex $v$, we show that the random placement of cops a.a.s. satisfies the Hall condition for matchings in bipartite graphs. Set

$$q_0 = \max\{q : qd^{j+1} < n\}.$$

Fix any vertex $v \in V$ and let $Q \subseteq N(v, j)$ with $|Q| = q \le q_0$. It follows by applying the condition in Lemma 4.3 to bound the size of $\bigcup_{u \in Q} N(u, j+1)$ that the number of cops occupying this set of vertices can be stochastically bounded from below by the binomial random variable $\mathrm{Bin}(\lfloor qd^{j+1}/3 \rfloor, Cd^j\beta/n)$ with expected value asymptotic to $\frac{C}{3} qd^{2j+1}\beta/n \ge \frac{C}{3} q \log n$. We next use a consequence of Chernoff's bound (see, e.g., [21, Cor. 2.3, p. 27]), that

$$\mathbb{P}(|X - \mathbb{E}X| \ge \varepsilon \mathbb{E}X)) \le 2 \exp\left( -\frac{\varepsilon^2 \mathbb{E}X}{3} \right)$$

for $0 < \varepsilon < 3/2$. This implies that the probability that there are fewer than $q$ cops in this set of vertices is less than $\exp(-4q \log n)$ when $C$ is a sufficiently large constant. Hence, the probability that the necessary condition in the statement of Hall's theorem fails for at least one set $Q$ with $|Q| \le q_0$ is at most

$$\sum_{q=1}^{q_0} \binom{|N(v, j)|}{q} \exp(-4q \log n) \le \sum_{q=1}^{q_0} n^q \exp(-4q \log n) = o(n^{-1}).$$

Let $Q \subseteq N(v, j)$ with $q_0 < |Q| = q \le |N(v, j)| \le 2d^j$. (The last inequality is a rough estimation. In fact, it follows from (4.1) below that $|N(v, j)| = (1+o(1))d^j$.) It follows from Lemma 4.3 that the size of $\bigcup_{u \in Q} N(u, j+1)$ is at least $n/3$, so we expect at least $\frac{C}{3} d^j$ cops in this set. Using Chernoff's bound again, we get that the number of cops is at least $2d^j \ge |N(v, j)| \ge |Q|$ with probability at least $1 - \exp(-4d^j)$, by taking the constant $C$ to be large enough. Since

$$\sum_{q=q_0+1}^{|N(v,j)|} \binom{|N(v, j)|}{q} \exp(-4d^j) \le 2d^j 2^{2d^j} \exp(-4d^j) = o(n^{-1}),$$

the necessary condition in Hall's theorem holds with probability $1 - o(n^{-1})$. Hence, the assignment can be done with at least this probability for each possible starting vertex $v \in V$. It follows that the strategy is a winning one for the cops a.a.s. This finishes the proof of (i).

One can adjust the proof of (i) to get (ii). Note that in this case for any set $Q \subseteq N(v,j)$, it follows from Lemma 4.3 that the size of $\bigcup_{u \in Q} N(u, j+1)$ is at least $|Q|d^{j+1}/6$, since $|Q| \leq 2d^j$ and so $|Q|d^{j+1} \leq 2n$. Therefore, the "bottleneck" when checking Hall's condition is not for sets of large cardinality anymore; this time we need to adjust the number of cops so that the condition holds for small sets. The team of cops is determined by independently choosing each vertex of $v \in V$ to be occupied by a cop with probability $C \log n / d^{j+1}$, where $C$ is a large enough constant. The expected number of cops in $\bigcup_{u \in Q} N(u, j+1)$ is at least $C|Q| \log n/6$, and we are fine.

Finally, let us investigate (iii). The game has to end after $s$ steps (at least in the case of the game with time constraints). Hence, we need to restrict ourselves and search for cops in a ball of radius $s$. Similarly as before, the team of cops is determined by independently choosing each vertex of $v \in V$ to be occupied by a cop with probability $C \log n / d^s$, where, as usual, $C$ is a large enough constant. For any set $Q \subseteq N(v, s-1)$, we expect $|Q| \log n/3$ cops in $\bigcup_{u \in Q} N(u, s)$ and the proof works exactly as before. ∎

**4.4. Lower bound.** In this section, we provide a lower bound of the cop number for the main theorem of this section, Theorem 4.2. In order to do it we need another expansion-type property of random graphs. Lemma 4.3 provides a lower bound for the size of a union of neighborhoods; this time we are interested in finding an upper bound for it. In order to prove the following lemma, an adjustment of the proof from [28] is needed.

LEMMA 4.5. *Suppose that $d = p(n-1) \geq \log^3 n$. Let $G = (V,E) \in \mathcal{G}(n,p)$. Then the following property holds a.a.s. Let $Q \subseteq V$ be any set of $q = |Q|$ vertices, and let $s = s(n) \in \mathbb{N}$ be such that $qd^s \leq \frac{1}{2} n \log n$. Then*

$$\left| \bigcup_{v \in Q} N(v,s) \right| < n.$$

*In particular, a.a.s. $\gamma_s(\mathcal{G}(n,p)) > \frac{n \log n}{2d^s}$.*

*Proof.* Let $Q \subseteq V$, $q = |Q|$, and consider the random variable $X = X(Q) = |N[Q]|$. We will bound $X$ from below in a stochastic sense. There are two things that need to be estimated: the expected value of $X$ and the concentration of $X$ around its expectation.

It is clear that

$$\begin{aligned}
\mathbb{E}X &= n - \left(1 - \frac{d}{n-1}\right)^q (n-q) \\
&= n - \exp\left(-\frac{dq}{n}(1 + O(d/n))\right)(n-q) \\
&= dq(1 + O(\log^{-1} n))
\end{aligned}$$

provided $dq \leq n/\log n$. It follows from Chernoff's bound that the expected number of sets $Q$ that have $\big||N[Q]| - d|Q|\big| > \varepsilon d|Q|$ and $|Q| \leq n/(d \log n)$ is for $\varepsilon = 2/\log n$

at most

$$\sum_{q\geq 1} n^q \exp\left(-\frac{\varepsilon^2 q \log^3 n}{3}\right) = \sum_{q\geq 1} \exp\left(q\log n - \frac{4}{3} q \log n\right) = o(1).$$

So a.a.s. if $|Q| \leq n/(d\log n)$, then $|N[Q]| = d|Q|(1 + O(1/\log n))$, where the bound in $O()$ is uniform. We may assume this statement holds.

Given this assumption, we have good bounds on the ratios of the cardinalities of $N[Q]$, $N[N[Q]] = \bigcup_{v\in Q} N(v,2)$, and so on. We consider this up to the $s$th iterated neighborhood provided $qd^s \leq n/\log n$ and thus $s = O(\log n/\log\log n)$. Then the cumulative multiplicative error term is $(1 + O(\log^{-1} n))^s = (1 + o(1))$, that is,

$$\left|\bigcup_{v\in Q} N(v,s)\right| = (1 + o(1))qd^s \tag{4.1}$$

for all $q$ and $s$ such that $qd^s \leq n/\log n$. This establishes the property in this case.

Suppose now that $qd^s = cn\log n$ with $1/\log^2 n < c = c(n) \leq 1/2$. Using (4.1), we have that $U = \bigcup_{v\in Q} N(v, s-1)$ has cardinality $(1+o(1))qd^{s-1} = O(n/\log^2 n) = o(n)$. Now $V \setminus N[U]$ has expected size

$$e^{-c\log n} n(1 + o(1)) \geq \sqrt{n}(1 + o(1)),$$

since $c \leq 1/2$. Chernoff's bound can be used again in the same way as before to show that with high probability $|V \setminus N[U]|$ is concentrated near its expected value and hence that a.a.s. $|V \setminus N[U]| > \frac{1}{2}\sqrt{n}$ for all $q$ and $s$ in this case. Thus the statement holds also in this case. ∎

Lemma 4.5 implies easily the following general lower bound that is tight for sparse graphs with the expected degree $d \leq n^{1/(2s-1)}$.

Corollary 4.6. *Suppose that $d = p(n-1) \geq \log^3 n$. Let $G = (V,E) \in G(n,p)$ and $s = s(n) \in \mathbb{N}$. Then a.a.s.*

$$c_{\phi\leq s}(G) = \Omega\left(\frac{n\log n}{d^s}\right).$$

*Proof.* Suppose that $q = \frac{n\log n}{2d^s}$ cops try to catch the robber. It follows from Lemma 4.5 that a.a.s. for every starting configuration for cops, there is a vertex that is at the distance at least $s+1$ from any of the cops. The robber starts the game on this vertex and remains safe to the end of the game. In short, we use the fact that $c_{\phi\leq s}(G) \geq \gamma_s(G)$. ∎

For denser graphs ($d > n^{1/(2s-1)}$), a lower bound for the classic game is useful, since $c_{\phi\leq s}(G) \geq c(G)$. It turns out that these bounds are tight. In other words, introducing more constraints on cops is not affecting substantially the cop number. The following results have been shown in [24] for the classic cop number, which provide a lower bounds for our variants of the game.

Theorem 4.7 ([24]). *Let $\frac{1}{2j+1} < \alpha < \frac{1}{2j}$ for some natural $j \geq 1$, $c = c(j,\alpha) = \frac{3}{1-2j\alpha}$, and $d = d(n) = (n-1)p = n^{\alpha+o(1)}$. Let $s = s(n) \in \mathbb{N}$. Then a.a.s.*

$$c_{\phi\leq s}(G(n,p)) \geq c(G(n,p)) \geq \left[\frac{d}{3cj}\right]^j.$$

THEOREM 4.8 ([24]). *Let $\frac{1}{2j} < \alpha < \frac{1}{2j-1}$ for some natural number $j \geq 2$, $\bar{c} = \bar{c}(\alpha) = \frac{3}{1-(2j-1)\alpha}$, and $d = d(n) = (n-1)p = n^{\alpha+o(1)}$. Let $s = s(n) \in \mathbb{N}$. Then a.a.s.*

$$c_{\phi \leq s}(G(n,p)) \geq c(G(n,p)) \geq \left[\frac{d}{3\bar{c}j}\right]^j \frac{n}{\bar{c}d^{2j}}.$$

Note that in the above result, the case when $np = n^{1/k+o(1)}$ for some natural $k$ is skipped. It was done for technical reasons. However, it was verified that up to a factor of $\log^{O(1)} n$ the result extends to the case $np = n^{1/k+o(1)}$ as well.

Finally, as we already mentioned, very dense random graphs were investigated in [9, 27]. We know that if $np = n^{\alpha+o(1)}$ with $\alpha \in [1/2, 1]$, then a.a.s. $c(\mathcal{G}(n,p)) = n^{1-\alpha+o(1)}$. Specifically, to get an upper bound it was proved that a.a.s. there is a dominating set of cardinality $n^{1-\alpha+o(1)}$. If this is the case, then one move is enough to catch the robber. We have that a.a.s.

$$c_{\sigma \leq s}(\mathcal{G}(n, n^{\alpha-1})) = n^{1-\alpha+o(1)}$$

for any $s \geq 1$ and $1/2 \leq \alpha \leq 1$. This implies that the same statement holds for $c_{\phi \leq s}(\mathcal{G}(n, n^{\alpha-1}))$ and $c\tau \leq s(\mathcal{G}(n, n^{\alpha-1}))$, which gives us the whole picture for a wide range of $d \geq \log^3 n$.

**4.5. Cops and robber with cost constraints.** In this section, we discuss the last variant of the game, the one with cost constraints. From the perspective of random graphs, this variant seems to be the most challenging one. When the total distance is bounded by $n^{\eta+o(1)}$ for some $\eta < 1/2$, our upper bound sometimes does not match the lower bound of the original cop number. We still do not know the shape of the function associated with this variant of the game.

Let us note that the proof of Theorem 4.4(i) provides an upper bound for the game with cost constraints as well. We restricted the range of $d$ before for the best outcome, but, as we already mentioned, the statement holds for any value of $d$. Let $j \in \mathbb{N}$. For $d \geq (n \log n)^{1/(2j+1)}$, the strategy requiring $O(d^j)$ cops was presented; each cop made at most $(j+1)$ moves for a total distance traveled by cops of $(1+o(1))(j+1)d^j$. For $d \leq (n \log n)^{1/(2j+1)}$, more cops were needed ($O\left(\frac{n}{d^{j+1}} \log n\right)$) but only a small fraction of them were actually moving ($(1+o(1))d^j$). Again, each "active" cop moved exactly $(j+1)$ times for a total distance of $(1+o(1))(j+1)d^j$. We get the following corollary.

COROLLARY 4.9. *Let $j \geq 1$ be integer, let $d = d(n) = (n-1)p$, and consider a game played on $\mathcal{G}(n,p)$. Then, a.a.s. cops have a winning strategy requiring*

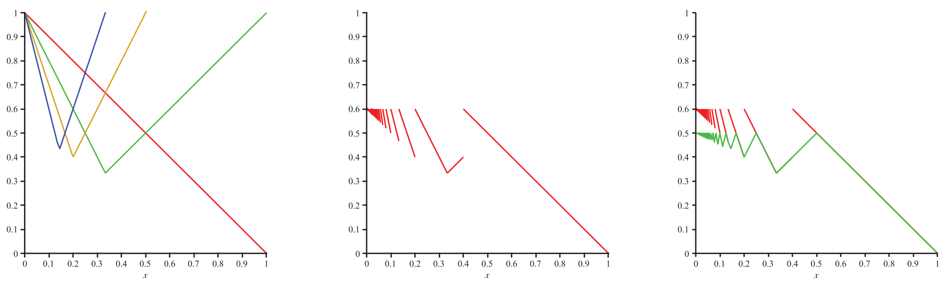
(i) *$O(d^j\beta)$ cops, where $\beta = \max\{n \log n / d^{2j+1}, 1\}$,*

(ii) *$(1+o(1))(j+1)d^j$ moves in total.*

*In particular, if $s \geq (1+\varepsilon)(j+1)d^j$ for some $j \in \mathbb{N}$ and $\varepsilon > 0$, then a.a.s. $c_{\sigma \leq s}(\mathcal{G}(n,p)) = O(d^j\beta)$, where $\beta = \max\{n \log n/d^{2j+1}, 1\}$.*

Upper bounds corresponding to $j = 1, 2, 3$ as well as the one corresponding to the strategy based on the domination number that works for any $s \geq 1$ are presented in Figure 4.2(a). (Although it has nothing to do with $j$ from the previous corollary, let us say that this universal bound corresponds to $j = 0$.)

Note that if the total distance is bounded by $n^{\eta+o(1)}$ for some $\eta \geq 1/2$, the additional restriction does not prevent us from choosing the optimal value of $j$ and we get exactly the same zigzag as before (see Figure 4.1(a)). The upper bound matches the lower bound for the original game. Unfortunately, it is not the case when $\eta < 1/2$. This time we need to choose the largest value of $j$ such that $s \geq (1+\varepsilon)(j+1)d^j$ for some



(a) strategies for $j \in \{0, 1, 2, 3\}$ (b) $l_{\frac{2}{5}}(x)$ (c) $l_{\frac{2}{5}}(x)$ and the lower bound

Fig. 4.2. *More "zigzag" functions.*

$\varepsilon > 0$. If no such $j$ exists, we need to use the universal strategy ($j = 0$). The function associated with an upper bound is not continuous anymore and bounds do not match for many densities. However, it would not be surprising if this is the behavior of the cop number for this variant of the game. This remains an open problem.

In order to illustrate the behavior of the upper bound, we consider the following example: suppose that $s = n^{\eta+o(1)}$ with $\eta = \frac{2}{5}$ and the graph has average degree of $d = n^{x+o(1)}$ for some $x \in (0, 1)$. To get the best outcome, we choose the largest value of $j$ such that $j < \frac{\eta}{x}$; that is, $j = \lceil \frac{\eta}{x} \rceil - 1$. In particular, for $x \geq \frac{2}{5}$ we take $j = 0$ to get a bound of $n^{1-x+o(1)}$. For $\frac{1}{5} \leq x < \frac{2}{5}$, we select $j = 1$ but a bound changes at $x = \frac{1}{3}$: we have $n^{x+o(1)}$ for $\frac{1}{3} \leq x \leq \frac{2}{5}$ and $n^{1-2x+o(1)}$ for $\frac{1}{5} \leq x < \frac{1}{3}$. This is the only "lower peak" we get for this value of $\eta$. If $\frac{2}{5(k+1)} \leq x < \frac{2}{5k}$, then we take $j = k$ and we get a bound of $n^{1-(k+1)x+o(1)}$. The function $l_{\frac{2}{5}}(x)$ associated with the case of $\eta = \frac{2}{5}$ is presented in Figure 4.2(b). The function is repeated in Figure 4.2(c) together with the lower bound.

For $\eta \in (0, \frac{1}{2})$, the shape of $l_\eta(x)$ is similar. If $\eta > \frac{2}{5}$ we will see more "lower peaks" (but still a finite number); however, there will be none of them for $\eta \leq \frac{1}{3}$. For a sparse graphs ($x$ small enough), $l_\eta(x) = 1 - \lceil \frac{\eta}{x} \rceil x$ and so $\lim_{x \to 0} l_\eta(x) = 1 - \eta$.

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