Quadratic Upper Bounds on the Erdős–Pósa Property for a Generalization of Packing and Covering Cycles

Fedor V. Fomin,[1] Daniel Lokshtanov,[2] Neeldhara Misra,[3] Geevarghese Philip,[3] and Saket Saurabh[3]

[1] *DEPARTMENT OF INFORMATICS*
UNIVERSITY OF BERGEN
N-5020 BERGEN, NORWAY
E-mail: fomin@ii.uib.no.

[2] *UNIVERSITY OF CALIFORNIA, SAN DIEGO*
LA JOLLA CA 92093-0404
E-mail: dlokshtanov@cs.ucsd.edu

[3] *THE INSTITUTE OF MATHEMATICAL SCIENCES*
CHENNAI 600113, INDIA
E-mail: {neeldhara|gphilip|saket}@imsc.res.in.

Received December 8, 2011; Revised November 23, 2012

Published online 24 January 2013 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/jgt.21720

Abstract: According to the classical Erdős–Pósa theorem, given a positive integer k, every graph G either contains k vertex disjoint cycles or a set of at most $\mathcal{O}(k \log k)$ vertices that hits all its cycles. Robertson and Seymour (J Comb Theory Ser B 41 (1986), 92–114) generalized this result in the best possible way. More specifically, they showed that if $\mathcal{H}$ is the class of all graphs that can be contracted to a fixed planar graph H, then every graph G either contains a set of k vertex-disjoint subgraphs of G, such that each of these subgraphs is isomorphic to some graph in $\mathcal{H}$ or there exists a set S of at most $f(k)$ vertices such that $G \setminus S$ contains no subgraph isomorphic to any graph in $\mathcal{H}$. However, the function f is exponential. In this note, we prove that this function becomes quadratic when $\mathcal{H}$ consists all graphs that can be contracted to a fixed planar graph θ_c. For a fixed c, θ_c is the graph with two vertices and $c \geq 1$ parallel edges. Observe that for $c = 2$ this corresponds to the classical Erdős–Pósa theorem. J. Graph Theory 74: 417–424, 2013

Keywords: *Erdos Posa property; generalization of covering and packing cycles*

1. INTRODUCTION

Given a graph G, we denote by $V(G)$ and $E(G)$ its vertex and edge set, respectively. Let G be a graph, and let $\mathcal{H}$ be a class of graphs. An *$\mathcal{H}$-packing* is a set of vertex-disjoint subgraphs of G such that each of these subgraphs is isomorphic to some graph in $\mathcal{H}$. Similarly, a subset of vertices $S \subseteq V(G)$ is called an *$\mathcal{H}$-cover* if $G \setminus S$ contains no subgraph isomorphic to any graph in $\mathcal{H}$. The class $\mathcal{H}$ is said to have the Erdős-Pósa property for some graph class $\mathcal{G}$ if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that, for every $k \geq 0$, every graph $G \in \mathcal{G}$ either contains an $\mathcal{H}$-packing of size at least k, or has an $\mathcal{H}$-cover of size at most $f(k)$.

Erdős and Pósa [7] proved that the Erdős-Pósa property holds for all graphs when $\mathcal{H}$ is the class of all cycles. The problem of identifying more general graph classes where the Erdős-Pósa property is satisfied has attracted a lot of attention [2], [4], [11], [13], [17], [18]. Extensions of this problem defined on matroids have also been investigated [9], [10].

Our discussions are concerned with graphs that are permitted to have *parallel edges*, that is, multiple edges with the same end points. While such structures are called *multigraphs* to distinguish them from *simple graphs*, we will continue to use the term "graph" with the implicit understanding that parallel edges are allowed.

The operation of *contracting* an edge $e = (u, v)$ in a graph G results in a graph G', in which u and v are replaced by a new vertex v_e and in which for every neighbor w of u or v in G, there is an edge (w, v_e) in G'. If a contraction operation results in more than one edge between a pair of vertices, then we retain all the multiple edges in the resulting graph. We say that a graph *G can be contracted to a graph H* if H can be obtained from G by a series of edge contractions. We say that *H is a minor of G* if some subgraph $\tilde{G}$ of G can be contracted to H; such a $\tilde{G}$ is called an *H-minor model* of G. A graph class $\mathcal{G}$ is *minor-closed* if any minor of a graph in $\mathcal{G}$ is again a member of $\mathcal{G}$.

For a fixed graph H, the class $\mathcal{H} = \mathcal{M}H$ consists of graphs that can be contracted to H. For a fixed c, let θ_c be the graph with two vertices and $c \geq 1$ parallel edges. Observe that for $H = \theta_1$, and $H = \theta_2$, $\mathcal{H} = \mathcal{M}H$ consists of all graphs that contain at least one edge and all graphs that contain at least one cycle, respectively. Robertson and Seymour [15, Proposition 8.2] proved the following seminal result.

Proposition 1. *$\mathcal{M}H$ satisfies the Erdős-Pósa property for all graphs if and only if H is planar.*

See the monograph "Graph Theory" by R. Diestel [5, Corollary 12.4.10 and Exercise 39] for an alternate proof of Proposition 1, with the additional assumption that H is connected. The bounding function $f(k)$ in the Erdős-Pósa property, as obtained in the different proofs of Proposition 1, is exponential in k. Fomin et al. [8] showed that the bound becomes linear for any connected planar graph H when the graph class $\mathcal{G}$ is any non trivial minor-closed class, and a result of Birmelé et al. demonstrates a quadratic bound for the class of all graphs when H is a cycle of fixed length [2]. Also, the classical result of Erdős-Pósa [7] shows that the class $\mathcal{H} = \mathcal{M}\theta_2$ (the family of all cycles) has the Erdős-Pósa property with $f(k) = O(k \log k)$ when $\mathcal{G}$ is the set of all graphs. In this note, we prove a quadratic bound for the case when $\mathcal{G}$ consists of all graphs and $\mathcal{H}$ consists of all graphs which can be contracted to a fixed planar graph θ_c. Observe that for $c = 2$ this corresponds to classical Erdős–Pósa theorem, albeit with a larger bound. Given a graph G and a vertex subset $S \subseteq V(G)$, we call a set S a *θ_c-hitting set* if $G \setminus S$ does not contain θ_c as a minor. The main result of this article is:

Theorem 1 [Erdős-Pósa property for θ_c]. *For any fixed $c \in \mathbb{N}$, every graph G either contains k vertex-disjoint θ_c-minor models, or has a θ_c-hitting set of size at most $f(k) = O(k^2)$.*

2. THE ERDŐS-PÓSA PROPERTY FOR θ_c

In this section, we give the proof of Theorem 1. Toward this we need following definitions.

Let G be a graph. A *tree decomposition* of a graph G is a pair $(T, \mathcal{X} = \{X_t\}_{t \in V(T)})$ such that

- $\cup_{t \in V(T)} X_t = V(G)$,
- for every edge $(x, y) \in E(G)$ there is a $t \in V(T)$ such that $x, y \subseteq X_t$, and
- for every vertex $v \in V(G)$ the subgraph of T induced by the set $\{t \mid v \in X_t\}$ is connected.

The *width* of a tree decomposition is $\left(\max_{t \in V(T)} |X_t|\right) - 1$ and the *treewidth* of G is the minimum width over all tree decompositions of G. A tree decomposition $(T, \mathcal{X})$ is called a *nice tree decomposition* if T is a tree rooted at some node r where $X_r = \emptyset$, each node of T has at most two children, and each node is of one of the following kinds:

1. *Introduce node*: a node t that has only one child t' where $X_t \supset X_{t'}$ and $|X_t| = |X_{t'}| + 1$.
2. *Forget node*: a node t that has only one child t' where $X_t \subset X_{t'}$ and $|X_t| = |X_{t'}| - 1$.
3. *Join node*: a node t with two children t_1 and t_2 such that $X_t = X_{t_1} = X_{t_2}$.
4. *Base node*: a node t that is a leaf of t, is different than the root, and $X_t = \emptyset$.

Notice that, according to the above definition, the root r of T is either a forget node or a join node. It is well known that any tree decomposition of G can be transformed into a nice tree decomposition maintaining the same width [12]. We use G_t to denote the graph induced on the vertices $\cup_{t'} X'_t$, where t' ranges over all descendants of t, including t. We use H_t to denote $G_t[V(G_t) \setminus X_t]$.

We prove Theorem 1 by establishing the following two lemmas.

Lemma 1. *If the treewidth of a graph G is at least $2c^2k^2$, then G contains at least k vertex-disjoint θ_c-minor models.*

Lemma 2. *If the treewidth of G is at most $2c^2k^2$ and G does not contain k vertex-disjoint θ_c-minor models, then G contains a θ_c-hitting set of size at most $\eta k^2 = O(k^2)$, where the constant η depends only on c.*

The proof of Theorem 1 follows from the above two lemmas.

Proof of Theorem 1. Suppose graph G does not contain k vertex-disjoint θ_c-minor models. Then by Lemma 1, G has treewidth at most $2c^2k^2$. Now by applying Lemma 2, we have that G contains a θ_c-hitting set of size $O(k^2)$. □

We now define some terms which we use in the proof of Lemma 1. A *bramble* is a set of connected subgraphs, called the *elements* of the bramble, any two of which either intersect or are linked by at least one edge. A *hitting set* of a bramble is a set of vertices which meets every element of the bramble. The *order* of a bramble is the minimum cardinality of a hitting set of the bramble. The maximum order of a bramble in a graph is its *bramble number*. Brambles and tree decompositions are dual structures in the following sense.

Proposition 2 [16]**.** *The tree-width of any graph is exactly one less than its bramble number.*

Our proof of Lemma 1 uses some ideas from the proof of Lemma 3.2 in Reed and Wood's recent work [14] on grid-like minors.

Lemma 3 [3]**.** *Let $\mathcal{B}$ be a bramble in a graph G. Then G contains a path that intersects every element of $\mathcal{B}$.*

Now we are ready to give a proof of Lemma 1.

Proof of Lemma 1. We show that if the treewidth of a graph G is at least $2c^2k^2$, then G contains at least k vertex-disjoint θ_c-minor models. If the treewidth of G is at least $2c^2k^2$, then by Proposition 2, G contains a bramble (call it $\mathcal{B}$) of order at least $2c^2k^2 + 1$. By Lemma 3, there exists a path that visits every element of the bramble at least once. Let P be such a path, and let $v_1, \ldots, v_t$ be the vertices of P (stated in the order of their appearance in P). Note that $t \geq 2c^2k^2 + 1$, as otherwise P would be a hitting set of $\mathcal{B}$ with fewer vertices than the order of $\mathcal{B}$.

For $1 \leq i \leq t$, let B_i denote the set of all elements of $\mathcal{B}$ which contain the vertex v_i. Note that for $1 \leq i \leq t$, $\cup_{j=1}^{i} B_j$ is a bramble. That is, the family

$$\mathcal{T}_i = \bigcup_{j=1}^{i} B_j = \{B | B \in \mathcal{B} \text{ and } B \cap \{v_1, \ldots, v_i\} \neq \emptyset\}$$

is a bramble. Let O_i denote the order of this bramble. Let s be the smallest number such that $O_s = c^2k^2$. The existence of such s is guaranteed by the fact that $O_1 = 1, O_t > 2c^2k^2$, and for $1 \leq i \leq t-1$, $O_{i+1} \leq O_i + 1$. Let $\mathcal{B}_1 = \mathcal{T}_s$, and let $\mathcal{B}_2 = \mathcal{B} \setminus \mathcal{B}_1$. Since the value of O_i increases by at most one in a single step, we have that the order of $\mathcal{B}_2$ is at least c^2k^2—indeed, if not, combining a smallest hitting set for $\mathcal{B}_1$ with one for $\mathcal{B}_2$ would give us a hitting set of $\mathcal{B}$ which is smaller than the order of $\mathcal{B}$, a contradiction. Let P_1 be the subpath of P starting at v_1 and ending at v_s, and P_2 the subpath starting at v_{s+1} and ending at v_t. By the above argument, P_1 and P_2 contain at least c^2k^2 vertices each.

Now, there must exist a collection, say $\mathcal{P}$, of at least c^2k^2 vertex-disjoint paths that begin in P_1 and end in P_2. If not, then by Menger's theorem, there exists a P_1–P_2 separator, say S, of size less than c^2k^2. Note that S cannot be a hitting set of the brambles $\mathcal{B}_1$ or $\mathcal{B}_2$, since the order of each of these is at least c^2k^2. So there exist elements $A \in \mathcal{B}_1, B \in \mathcal{B}_2$ such that $A \cap S = \emptyset = B \cap S$. But since A and B are connected subgraphs which either intersect or are linked by an edge—being elements of $\mathcal{B}$—and $A \cap P_1 \neq \emptyset, B \cap P_2 \neq \emptyset$, S cannot be a P_1–P_2 separator.

We now show that $\mathcal{P} \cup P_1 \cup P_2$ contains k vertex-disjoint θ_c-minor models. Let V_p be the set of vertices that form the end points (on P_1 and P_2) of the paths in $\mathcal{P}$. For $i \in \{1, 2\}$, let $Q_i = P_i \cap V_p$. We label both Q_1 and Q_2 with a common index set $[M]$, where $M = |Q_1| = |Q_2|$. Let $f : [M] \to [M]$ be the following bijection: $f(i) = j$ if and only if there is a path in $\mathcal{P}$ that begins in i and ends in j. We say that a subset of paths $C \subseteq \mathcal{P}$ is *cross-free* under this labeling if there does not exist $i, i' \in Q_1 \cap C; i < i'$ and $f(i) > f(i')$.

Note that since the paths in $\mathcal{P}$ are vertex-disjoint, the numbers $f(1), f(2), \ldots, f(M)$ form a permutation of M, and by the Erdős–Szekeres Theorem [6], the sequence $\langle f(1), f(2), \ldots, f(M)\rangle$ contains a monotonically increasing or decreasing subsequence of length at least t, where t is $\sqrt{|M|} = ck$. Let a witness subsequence be $\langle f(s_1), f(s_2), \ldots, f(s_t)\rangle$. Let $Q_1' = \{s_1, s_2, \ldots, s_t\}$ and $Q_2' = \{f(s_1), f(s_2), \ldots, f(s_t)\}$. Then the paths in $\mathcal{P}$ that have their end points in Q_1', Q_2' form a cross-free collection. These paths together with P_1, P_2 contain at least k vertex-disjoint θ_c-minor models. □

Before proving Lemma 2, we define the notion of a *good labeling function*. Given a nice tree decomposition $(T, \mathcal{X} = \{X_t\}_{t\in V(T)})$ of a graph G, a function $g : V(T) \to \mathbb{N}$ is called a *good labeling function* if it satisfies the following properties:

- if t is a base node then $g(t) = 0$;
- if t is an introduce node, then $g(t) = g(s)$, where s is the child of t;
- if t is a join node, then $g(t) = g(s_1) + g(s_2)$, where s_1 and s_2 are the children of t; and
- if t is a forget node, then $g(t) \in \{g(s), g(s) + 1\}$, where s is the child of t.

Now we are ready to prove the covering lemma—Lemma 2.

Proof of Lemma 2. Here, we show that if G has treewidth at most $2c^2k^2$ and does not have more than $k - 1$ disjoint minor models of θ_c, then there exists a set $S \subseteq V(G)$, $|S| = O(k^2)$, such that $G \setminus S$ does not contain θ_c as a minor.

Consider a nice tree decomposition $(T, \mathcal{X} = \{X_t\}_{t\in V(T)})$ of the graph of width at most $2c^2k^2$. Recall that for $t \in V(T)$, G_t is the graph induced on the vertices $\cup_{t'} X_{t'}$, where t' ranges over all descendants of t including t, and H_t is $G_t \setminus X_t$.

Let $P_{\theta_c}(G)$ denote the maximum number of vertex-disjoint θ_c-minor models in G. We use k' to denote $P_{\theta_c}(G)$, and note that $k' \leq k - 1$.

Consider the function $\mu : V(T) \to [k']$, defined as follows: $\mu(t) = P_{\theta_c}(H_t)$. The function μ is a good labeling function because:

- If t is a base node then $\mu(t) = 0$ as H_t is an empty graph.
- If t is an introduce node, then $\mu(t) = \mu(s)$, where s is the child of t. Indeed, this follows from the fact that the graphs H_t and H_s are exactly the same.

- If t is a join node, then $\mu(t) = \mu(s_1) + \mu(s_2)$, where s_1 and s_2 are the children of t. This follows from the fact that the bag X_t is a separator of G_t and $V(H_{s_1}) \cap V(H_{s_2}) = \emptyset$.
- If t is a forget node, then $\mu(t) \in \{\mu(s), \mu(s) + 1\}$, where s is the child of t. This is because H_t has at most one vertex more than H_s, which can add at most one to the number of vertex-disjoint θ_c-minor models.

By definition, and by the convention that the bag X_r corresponding to the root r is $\emptyset$ we have that $\mu(r) = k'$. To find the desired θ_c-hitting set, we give a recursive algorithm. We find a bag X in the given tree decomposition such that its removal allows us to decompose the graph into two parts such that there are no edges from one part to another and the number of vertex-disjoint minor models of θ_c in each part is essentially a constant fraction of the original. After this we find a θ_c-hitting set in each of these graphs and then take the union of these sets, together with the bag we removed to get these graphs, to get the desired hitting set for the whole graph. Let $t \in V(T)$ be the node where $\mu(t) > 2k'/3$ and for each child t' of t, $\mu(t') \leq 2k'/3$. From the definition of good labeling function, this node exists and is unique provided that $k' > 0$. Moreover, observe that t could either be a forget node or a join node. We distinguish these two cases.

- *Case 1*. If t is a forget node, we set $V_1 = V(H_{t'})$ and $V_2 = V(G) \setminus (V_1 \cup X_{t'})$ and observe that $P_{\theta_c}(G[V_i]) \leq \lfloor 2k'/3 \rfloor$, $i = 1, 2$. Also we set $X = X_{t'}$.
- *Case 2*. If t is a join node with children t_1 and t_2, we have that $\mu(t_i) \leq 2k'/3$, $i = 1, 2$. However, as $\mu(t_1) + \mu(t_2) > 2k'/3$, we also have that either $\mu(t_1) \geq k'/3$ or $\mu(t_2) \geq k'/3$. Without loss of generality we assume that $\mu(t_1) \geq k'/3$ and we set $V_1 = V(H_{t_1})$, $V_2 = V(G) \setminus (V_1 \cup X_{t_1})$ and $X = X_{t_1}$.

Algorithm 1. HIT-SET (G)

1: Compute $(T, \mathcal{X} = \{X_t\}_{t \in V(T)})$, a nice tree decomposition of G. Now compute the function $\mu : V(T) \to [k']$, defined as follows: $\mu(t) = P_{\theta_c}(H_t)$.
2: **if** $(\mu(r) = 0)$ **then**
3: Return $\emptyset$.
4: **else**
5: Find the partitioning of the vertex set $V(G)$ into V_1, V_2, and X (a bag corresponding to a node in T) as described in Cases 1 and 2.
6: **end if**
7: Return $\left(X \bigcup \text{HIT-SET}(G[V_1]) \bigcup \text{HIT-SET}(G[V_2])\right)$.

We present a detailed algorithm to find a θ_c-hitting set in Algorithm 1. The algorithm HIT-SET(G) takes as input a graph G and returns a θ_c-hitting set for G. Now we bound the size of the θ_c-hitting set returned by the algorithm. Let $\mathcal{S}(G, P_{\theta_c}(G)) = \mathcal{S}(G, k')$ be the size of the θ_c-hitting set returned by HIT-SET(G), where the second parameter denotes the number of minor models of θ_c in G. Then the value of $\mathcal{S}(G, P_{\theta_c}(G))$ is upper bounded by the following recurrence:

$$\mathcal{S}(G, k') \leq \max_{1/3 \leq \alpha \leq 2/3} \{\mathcal{S}(G[V_1], \alpha k') + \mathcal{S}(G[V_2], (1-\alpha)k') + 2c^2k^2\}.$$

Note that $P_{\theta_c}(G[V_1]) + P_{\theta_c}(G[V_2]) \leq 2P_{\theta_c}(G)/3$. Recalling that $k' \leq k$, it is easy to see that the above recurrence solves to $O(k^2)$ using the Akra–Bazzi Theorem [1]. This concludes the proof.

3. CONCLUSION

In this short note we obtained a quadratic upper bound on the Erdős-Pósa property of a generalization of packing and covering cycles. It follows from tight Erdős-Pósa bound on cycles that the quadratic upper bound obtained in this article on a generalization of packing and covering cycles can not be improved beyond $O(k \log k)$. We believe that even for θ_c, the correct upper bound on the size of a minimum hitting set when a graph G does not have k vertex disjoint θ_c-minor models is $O(k \log k)$. An interesting question will be to classify those planar graphs H, such that $\mathcal{M}H$ has Erdős-Pósa property with a polynomial function on all graphs.

REFERENCES

[1] M. Akra and L. Bazzi, On the solution of linear recurrence equations, Comput Optim Appl 10(2) (1998), 195–210.
[2] E. Birmelé, J. A. Bondy, and B. A. Reed, The Erdős-Pósa property for long circuits, Combinatorica 27(2) (2007), 135–145.
[3] E. Birmelé, J. A. Bondy, and B. A. Reed, Brambles, prisms, and grids, In: Graph Theory in Paris, Trends Math., Birkhäuser Verlag, 2007, pp. 37–44.
[4] I. J. Dejter and V. N. Lara, Unboundedness for generalized odd cyclic transversality, In: Combinatorics (Eger, 1987), Editors: A. Hajnal, L. Lovász and V. T. Sós, volume 52 of Colloq. Math. Soc. János Bolyai. North-Holland, Amsterdam, 1988, pp. 195–203.
[5] R. Diestel, Graph Theory, volume 173 of Graduate Texts in Mathematics, 3rd edn., Springer-Verlag, Berlin, 2005.
[6] P. Erdös and G. Szekeres, A combinatorial problem in geometry, Compositio Math 2 (1935), 463–470.
[7] P. Erdős and L. Pósa, On independent circuits contained in a graph, Canad J Math 17 (1965), 347–352.
[8] F. V. Fomin, S. Saurabh, and D. M. Thilikos, Strengthening Erdős–Pósa property for minor-closed graph classes, J Graph Theory 66(3) (2011), 235–240.
[9] J. F. Geelen, A. M. H. Gerards, and G. Whittle, Disjoint cocircuits in matroids with large rank, J Combin Theory Ser B 87(2) (2003), 270–279.
[10] J. Geelen and K. Kabell, The Erdős-Pósa property for matroid circuits, J Combin Theory Ser B 99(2) (2009), 407–419.
[11] K.-I. Kawarabayashi and A. Nakamoto, The Erdős-Pósa property for vertex- and edge-disjoint odd cycles in graphs on orientable surfaces, Discrete Math 307(6) (2007), 764–768.

[12] T. Kloks, Treewidth. Computations and Approximations, vol. 842. Lecture Notes in Computer Science, Springer, 1994.
[13] D. Rautenbach and B. Reed, The Erdős-Pósa property for odd cycles in highly connected graphs, Combinatorica 21(2) (2001), 267–278. Paul Erdős and his mathematics (Budapest, 1999).
[14] B. A. Reed and D. R. Wood, Polynomial treewidth forces a large grid-like-minor, Eur. J. Comb. 33(3) (2012), 374–379.
[15] N. Robertson and P. D. Seymour, Graph minors. V. Excluding a planar graph, J Comb Theory Ser B 41 (1986), 92–114.
[16] P. D. Seymour and R. Thomas, Graph searching and a min-max theorem for tree-width, J Combin Theory Ser B 58(1) (1993), 22–33.
[17] C. Thomassen, On the presence of disjoint subgraphs of a specified type, J Graph Theory 12(1) (1988), 101–111.
[18] C. Thomassen, The Erdős-Pósa property for odd cycles in graphs of large connectivity, Combinatorica 21(2) (2001), 321–333. Paul Erdős and his mathematics (Budapest, 1999).